Supplementary Materials: Robust Multi-stage Nonlinear Model Predictive Control using Sigma Points

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1. Formulation of robust multi-stage NMPC

1.1. Multi-stage NMPC based on the vertex over-approximation

The entire formulation of the multi-stage NMPC based on the vertex approximation optimization problem solved at time t (s^{th} sampling time) reads as:

$$\min_{\boldsymbol{x}_k^j,\boldsymbol{u}_k^j,\boldsymbol{d}^j} \sum_{k=s}^{s+N_r-1} \sum_{j=1}^{N_b^{k-s+1}} \omega_{k+1}^j l(\boldsymbol{x}_{k+1}^j, \Delta\boldsymbol{u}_k^j) + \sum_{k=s+N_r}^{s+N_p-1} \sum_{j=1}^{N_b^{N_r}} \omega_{k+1}^j l(\boldsymbol{x}_{k+1}^j, \Delta\boldsymbol{u}_k^j), \quad \forall (j,k+1) \in I_{st}, \tag{1a}$$

subject to

$$\boldsymbol{x}_{k+1}^j = \boldsymbol{f}(\boldsymbol{x}_k^{p(j)}, \boldsymbol{u}_k^j, \boldsymbol{d}^{r(j)}), \quad \forall\, (j,k+1) \in I_{st}, \boldsymbol{d}^{r(j)} \in \mathcal{D}, \tag{1b}$$

$$\boldsymbol{g}(\boldsymbol{x}_{k+1}^j, \boldsymbol{u}_k^j) \le 0\,, \quad \forall\, (j,k+1) \in I_{st}, \tag{1c}$$

$$\boldsymbol{u}_k^j = \boldsymbol{u}_k^l \text{ if } \boldsymbol{x}_k^{p(j)} = \boldsymbol{x}_k^{p(l)}, \quad \forall\, (j,k),(l,k) \in I_{st}, \tag{1d}$$

$$\underline{\boldsymbol{u}} \le \boldsymbol{u}_k^j \le \overline{\boldsymbol{u}}, \quad \forall\, (j,k) \in I_{st}, \tag{1e}$$

$$\boldsymbol{x}_s^1 = \boldsymbol{x}_s^m \tag{1f}$$

$$\underline{\boldsymbol{d}} = \boldsymbol{d}_0 - \text{diag}^{\frac{1}{2}}(\boldsymbol{P}_0),\ \overline{\boldsymbol{d}} = \boldsymbol{d}_0 + \text{diag}^{\frac{1}{2}}(\boldsymbol{P}_0), \tag{1g}$$

$$\mathcal{D} = \boldsymbol{d}_0 \cup \mathbb{C}_{vp}(\underline{\boldsymbol{d}}, \overline{\boldsymbol{d}}), \tag{1h}$$

where $N_b = 2^{n_d} + 1$.

1.2. Multi-stage NMPC based on the box over-approximation of the reachable set of states

The entire formulation of the multi-stage NMPC based on the box over-approximation of reachable set of states optimization problem solved at time t (s^{th} sampling time) reads as:

$$\min_{\boldsymbol{x}_k^j,\boldsymbol{u}_k^j,\boldsymbol{d}^j,\boldsymbol{x}_{m,k}^j,\boldsymbol{X}_{c,k}^j} J_{so}(\mathcal{X}_{k+1}^q, \Delta\boldsymbol{u}_k^j) + \sum_{k=s+N_r}^{s+N_p-1} \sum_{j=1}^{N_b^{N_r}} \omega_{k+1}^j l(\boldsymbol{x}_{k+1}^j, \Delta\boldsymbol{u}_k^j), \quad \forall (j,k+1) \in I_{st}, q \in I_b(k), \tag{2a}$$

subject to:

$$\boldsymbol{x}_{k+1}^j = \boldsymbol{f}(\boldsymbol{x}_k^{p(j)}, \boldsymbol{u}_k^j, \boldsymbol{d}^{r(j)}), \quad \forall\, (j,k+1) \in I_{st}, \boldsymbol{d}^{r(j)} \in \mathcal{D}, \tag{2b}$$

$$\boldsymbol{g}(\boldsymbol{x}_{k+1}^j, \boldsymbol{u}_k^j) \le 0\,, \quad \forall\, (j,k+1) \in I_{st}, \tag{2c}$$

$$\boldsymbol{u}_k^j = \boldsymbol{u}_k^l \text{ if } \boldsymbol{x}_k^{p(j)} = \boldsymbol{x}_k^{p(l)}, \quad \forall\, (j,k),(l,k) \in I_{st}, \tag{2d}$$

$$\underline{\boldsymbol{u}} \le \boldsymbol{u}_k^j \le \overline{\boldsymbol{u}}, \quad \forall\, (j,k) \in I_{st}, \tag{2e}$$

$$\boldsymbol{x}_s^1 = \boldsymbol{x}_s^m \tag{2f}$$

$$\mathcal{D} = \mathbb{S}(\boldsymbol{d}_0, \boldsymbol{P}_0), \tag{2g}$$

$$\left(\boldsymbol{x}_{m,k+1}^q, \boldsymbol{X}_{c,k+1}^q\right) = \mathbb{U}(\kappa_{x,k}, \boldsymbol{x}_{k+1}^{[(q-1)N_b+1:qN_b]}), \quad \forall k \in I_{br}, q \in I_b(k), \tag{2h}$$

$$\underline{\boldsymbol{x}}_{m,k+1}^q = \boldsymbol{x}_{m,k+1}^q - \text{diag}^{\frac{1}{2}}(\boldsymbol{X}_{c,k+1}^q), \quad \forall k \in I_{br}, q \in I_b(k), \tag{2i}$$

$$\overline{\boldsymbol{x}}^q_{m,k+1} = \boldsymbol{x}^q_{m,k+1} + \mathrm{diag}^{\frac{1}{2}}(\boldsymbol{X}^q_{c,k+1}), \qquad \forall k \in I_{br}, q \in I_b(k), \tag{2j}$$

$$\mathcal{X}^q_{k+1} = \boldsymbol{x}^q_{m,k+1} \cup \mathbb{C}_{vp}(\underline{\boldsymbol{x}}^q_{m,k+1}, \overline{\boldsymbol{x}}^q_{m,k+1}), \qquad \forall k \in I_{br}, q \in I_b(k), \tag{2k}$$

$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}_k^{(q-1)N_b+1}) \leq 0\,, \qquad \forall k \in I_{br}, q \in I_b(k), \boldsymbol{x} \in \mathcal{X}^q_{k+1}. \tag{2l}$$

where $N_b = 2n_d + 1$, J_{so} is given as

$$J_{so}(\mathcal{X}^q_{k+1}, \Delta \boldsymbol{u}^j_k) = \sum_{k=s}^{s+N_r-1} \sum_{q=1}^{N_b^{k-s+1}} \sum_{i=1}^{2^{n_x}+1} \omega^{i+q-1}_{k+1} L(\boldsymbol{x}^i, \Delta \boldsymbol{u}_k^{(q-1)N_b+1}), \tag{2m}$$

1.3. *Multi-stage NMPC based on the box over-approximation of the reachable set of the constraint function*

The entire formulation of the multi-stage NMPC based on the box over-approximation of reachable constraint function set optimization problem solved at time t (s^{th} sampling time) reads as:

$$\min_{\boldsymbol{x}^j_k, \boldsymbol{u}^j_k, \boldsymbol{d}^j, \boldsymbol{g}^q_{m,k}, \boldsymbol{G}^q_{c,k}} \sum_{k=s}^{s+N_r-1} \sum_{j=1}^{N_b^{k-s+1}} \omega^j_{k+1} l(\boldsymbol{x}^j_{k+1}, \Delta \boldsymbol{u}^j_k) + \sum_{k=s+N_r}^{s+N_p-1} \sum_{j=1}^{N_b^{N_r}} \omega^j_{k+1} l(\boldsymbol{x}^j_{k+1}, \Delta \boldsymbol{u}^j_k), \;\; \forall (j, k+1) \in I_{st}, \tag{3a}$$

subject to:

$$\boldsymbol{x}^j_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k^{p(j)}, \boldsymbol{u}^j_k, \boldsymbol{d}^{r(j)}), \qquad \forall\, (j, k+1) \in I_{st}, \boldsymbol{d}^{r(j)} \in \mathcal{D}, \tag{3b}$$

$$\boldsymbol{g}(\boldsymbol{x}^j_{k+1}, \boldsymbol{u}^j_k) \leq 0\,, \qquad \forall\, (j, k+1) \in I_{st}, \tag{3c}$$

$$\boldsymbol{u}^j_k = \boldsymbol{u}^l_k \text{ if } \boldsymbol{x}_k^{p(j)} = \boldsymbol{x}_k^{p(l)}, \qquad \forall\, (j,k), (l,k) \in I_{st}, \tag{3d}$$

$$\underline{\boldsymbol{u}} \leq \boldsymbol{u}^j_k \leq \overline{\boldsymbol{u}}, \qquad \forall\, (j,k) \in I_{st}, \tag{3e}$$

$$\boldsymbol{x}^1_s = \boldsymbol{x}^m_s \tag{3f}$$

$$\mathcal{D} = \mathbb{S}(\boldsymbol{d}_0, \boldsymbol{P}_0), \tag{3g}$$

$$\boldsymbol{g}^j_{n,k+1} = \boldsymbol{g}(\boldsymbol{x}^j_{k+1}, \boldsymbol{u}^j_k), \qquad \forall\, (j, k+1) \in I_{st}, \tag{3h}$$

$$\left(\boldsymbol{g}^q_{m,k+1}, \boldsymbol{G}^q_{c,k+1}\right) = \mathbb{U}(\kappa_{c,k}, \boldsymbol{g}_{n,k+1}^{[(q-1)N_b+1:qN_b]}), \qquad \forall k \in I_{br}, q \in I_b(k), \tag{3i}$$

$$\boldsymbol{g}^q_{m,k+1} + \mathrm{diag}^{\frac{1}{2}}(\boldsymbol{G}^q_{c,k+1}) \leq 0, \qquad \forall k \in I_{br}, q \in I_b(k), \tag{3j}$$

where $N_b = 2n_d + 1$.

2. Formulation of adaptive robust multi-stage NMPC

2.1. *Adaptive multi-stage NMPC*

The entire formulation of the adaptive multi-stage NMPC optimization problem solved at time t (s^{th} sampling time) reads as:

$$\min_{\boldsymbol{x}^j_k, \boldsymbol{u}^j_k, \boldsymbol{d}^j} \sum_{k=s}^{s+N_r-1} \sum_{j=1}^{N_b^{k-s+1}} \omega^j_{k+1} l(\boldsymbol{x}^j_{k+1}, \Delta \boldsymbol{u}^j_k) + \sum_{k=s+N_r}^{s+N_p-1} \sum_{j=1}^{N_b^{N_r}} \omega^j_{k+1} l(\boldsymbol{x}^j_{k+1}, \Delta \boldsymbol{u}^j_k), \quad \forall (j, k+1) \in I_{st}, \tag{4a}$$

subject to

$$\boldsymbol{x}^j_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k^{p(j)}, \boldsymbol{u}^j_k, \boldsymbol{d}^{r(j)}), \qquad \forall\, (j, k+1) \in I_{st}, \boldsymbol{d}^{r(j)} \in \mathcal{D}, \tag{4b}$$

$$\boldsymbol{g}(\boldsymbol{x}^j_{k+1}, \boldsymbol{u}^j_k) \leq 0\,, \qquad \forall\, (j, k+1) \in I_{st}, \tag{4c}$$

$$\boldsymbol{u}^j_k = \boldsymbol{u}^l_k \text{ if } \boldsymbol{x}_k^{p(j)} = \boldsymbol{x}_k^{p(l)}, \qquad \forall\, (j,k), (l,k) \in I_{st}, \tag{4d}$$

$$\underline{\boldsymbol{u}} \leq \boldsymbol{u}_k^j \leq \overline{\boldsymbol{u}}, \qquad \forall\, (j,k) \in I_{st}, \tag{4e}$$

$$\boldsymbol{x}_s^1 = \boldsymbol{x}_s^m \tag{4f}$$

$$\underline{\boldsymbol{d}}_{[o]} = \underline{\boldsymbol{d}}_{s,[o]},\, \overline{\boldsymbol{d}}_{[o]} = \overline{\boldsymbol{d}}_{s,[o]}, \qquad \forall o \in I_d, \tag{4g}$$

$$\mathcal{D} = \mathbb{C}_a(\underline{\boldsymbol{d}}, \boldsymbol{d}_s, \overline{\boldsymbol{d}}). \tag{4h}$$

where $N_b = 2^{n_d} + 1$.

2.2. Adaptive multi-stage NMPC based on the vertex over-approximation

The entire formulation of the adaptive multi-stage NMPC based on the vertex approximation optimization problem solved at time t (s^{th} sampling time) reads as:

$$\min_{\boldsymbol{x}_k^j, \boldsymbol{u}_k^j, \boldsymbol{d}^j} \sum_{k=s}^{s+N_r-1} \sum_{j=1}^{N_b^{k-s+1}} \omega_{k+1}^j l(\boldsymbol{x}_{k+1}^j, \Delta \boldsymbol{u}_k^j) + \sum_{k=s+N_r}^{s+N_p-1} \sum_{j=1}^{N_b^{N_r}} \omega_{k+1}^j l(\boldsymbol{x}_{k+1}^j, \Delta \boldsymbol{u}_k^j), \quad \forall (j, k+1) \in I_{st}, \tag{5a}$$

subject to

$$\boldsymbol{x}_{k+1}^j = \boldsymbol{f}(\boldsymbol{x}_k^{p(j)}, \boldsymbol{u}_k^j, \boldsymbol{d}^{r(j)}), \qquad \forall\, (j, k+1) \in I_{st}, \boldsymbol{d}^{r(j)} \in \mathcal{D}, \tag{5b}$$

$$\boldsymbol{g}(\boldsymbol{x}_{k+1}^j, \boldsymbol{u}_k^j) \leq 0\,, \qquad \forall\, (j, k+1) \in I_{st}, \tag{5c}$$

$$\boldsymbol{u}_k^j = \boldsymbol{u}_k^l \text{ if } \boldsymbol{x}_k^{p(j)} = \boldsymbol{x}_k^{p(l)}, \qquad \forall\, (j,k), (l,k) \in I_{st}, \tag{5d}$$

$$\underline{\boldsymbol{u}} \leq \boldsymbol{u}_k^j \leq \overline{\boldsymbol{u}}, \qquad \forall\, (j,k) \in I_{st}, \tag{5e}$$

$$\boldsymbol{x}_s^1 = \boldsymbol{x}_s^m \tag{5f}$$

$$\underline{\boldsymbol{d}}_{[o]} = \underline{\boldsymbol{d}}_{s,[o]},\, \overline{\boldsymbol{d}}_{[o]} = \overline{\boldsymbol{d}}_{s,[o]}, \qquad \forall o \in I_d, \tag{5g}$$

$$\mathcal{D} = \boldsymbol{d}_s \cup \mathbb{C}_{vp}(\underline{\boldsymbol{d}}, \overline{\boldsymbol{d}}). \tag{5h}$$

where $N_b = 2^{n_d} + 1$.

2.3. Adaptive multi-stage NMPC based on the box over-approximation of the reachable set of states

The entire formulation of the adaptive multi-stage NMPC based on the box over-approximation of reachable set of states optimization problem solved at time t (s^{th} sampling time) reads as:

$$\min_{\boldsymbol{x}_k^j, \boldsymbol{u}_k^j, \boldsymbol{d}^j, \boldsymbol{x}_{m,k}^j, \boldsymbol{X}_{c,k}^j} J_{so}(\mathcal{X}_{k+1}^q, \Delta \boldsymbol{u}_k^j) + \sum_{k=s+N_r}^{s+N_p-1} \sum_{j=1}^{N_b^{N_r}} \omega_{k+1}^j l(\boldsymbol{x}_{k+1}^j, \Delta \boldsymbol{u}_k^j), \quad \forall (j, k+1) \in I_{st}, q \in I_b(k), \tag{6a}$$

subject to:

$$\boldsymbol{x}_{k+1}^j = \boldsymbol{f}(\boldsymbol{x}_k^{p(j)}, \boldsymbol{u}_k^j, \boldsymbol{d}^{r(j)}), \qquad \forall\, (j, k+1) \in I_{st}, \boldsymbol{d}^{r(j)} \in \mathcal{D}, \tag{6b}$$

$$\boldsymbol{g}(\boldsymbol{x}_{k+1}^j, \boldsymbol{u}_k^j) \leq 0\,, \qquad \forall\, (j, k+1) \in I_{st}, \tag{6c}$$

$$\boldsymbol{u}_k^j = \boldsymbol{u}_k^l \text{ if } \boldsymbol{x}_k^{p(j)} = \boldsymbol{x}_k^{p(l)}, \qquad \forall\, (j,k), (l,k) \in I_{st}, \tag{6d}$$

$$\underline{\boldsymbol{u}} \leq \boldsymbol{u}_k^j \leq \overline{\boldsymbol{u}}, \qquad \forall\, (j,k) \in I_{st}, \tag{6e}$$

$$\boldsymbol{x}_s^1 = \boldsymbol{x}_s^m \tag{6f}$$

$$\boldsymbol{P}_s^\star = (1 - \phi(1-\phi)(\boldsymbol{d}_s - \boldsymbol{d}_{s-1}^\star)^T \boldsymbol{P}_s^{-1} \hat{\boldsymbol{F}}^{-1} (\boldsymbol{P}_{s-1}^\star)^{-1} (\boldsymbol{d}_s - \boldsymbol{d}_{s-1}^\star)) \hat{\boldsymbol{F}}^{-1}, \tag{6g}$$

$$\boldsymbol{d}_s^\star = \hat{\boldsymbol{F}}^{-1} (\phi (\boldsymbol{P}_{s-1}^\star)^{-1} \boldsymbol{d}_{s-1}^\star + (1-\phi) \boldsymbol{P}_s^{-1} \boldsymbol{d}_s), \tag{6h}$$

$$\mathcal{D} = \mathbb{S}(\boldsymbol{d}_s^\star, \boldsymbol{P}_s^\star), \tag{6i}$$

$$\left(\boldsymbol{x}_{m,k+1}^q, \boldsymbol{X}_{c,k+1}^q\right) = \mathbb{U}(\kappa_{x,k}, \boldsymbol{x}_{k+1}^{[(q-1)N_b+1:qN_b]}), \qquad \forall k \in I_{br}, q \in I_b(k), \tag{6j}$$

$$\underline{\boldsymbol{x}}^q_{m,k+1} = \boldsymbol{x}^q_{m,k+1} - \mathrm{diag}^{\frac{1}{2}}(\boldsymbol{X}^q_{c,k+1}), \qquad \forall k \in I_{br}, q \in I_b(k), \tag{6k}$$

$$\overline{\boldsymbol{x}}^q_{m,k+1} = \boldsymbol{x}^q_{m,k+1} + \mathrm{diag}^{\frac{1}{2}}(\boldsymbol{X}^q_{c,k+1}), \qquad \forall k \in I_{br}, q \in I_b(k), \tag{6l}$$

$$\mathcal{X}^q_{k+1} = \boldsymbol{x}^q_{m,k+1} \cup \mathbb{C}_{vp}(\underline{\boldsymbol{x}}^q_{m,k+1}, \overline{\boldsymbol{x}}^q_{m,k+1}), \qquad \forall k \in I_{br}, q \in I_b(k), \tag{6m}$$

$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}_k^{(q-1)N_b+1}) \leq 0\,, \qquad \forall k \in I_{br}, q \in I_b(k), \boldsymbol{x} \in \mathcal{X}^q_{k+1}. \tag{6n}$$

where $N_b = 2n_d + 1$, $\hat{\boldsymbol{F}} = \phi(\boldsymbol{P}^\star_{s-1})^{-1} + (1-\phi)\boldsymbol{P}_s^{-1}$, and J_{so} is given as

$$J_{so}(\mathcal{X}^q_{k+1}, \Delta\boldsymbol{u}^j_k) = \sum_{k=s}^{s+N_r-1} \sum_{q=1}^{N_b^{k-s+1}} \sum_{i=1}^{2^{n_x}+1} \omega^{i+q-1}_{k+1} L(\boldsymbol{x}^i, \Delta\boldsymbol{u}_k^{(q-1)N_b+1}), \tag{6o}$$

2.4. Adaptive multi-stage NMPC based on the box over-approximation of the reachable set of the constraint function

The entire formulation of the adaptive multi-stage NMPC based on the box over-approximation of reachable constraint function set optimization problem solved at time t (s^{th} sampling time) reads as:

$$\sum_{k=s}^{s+N_r-1} \sum_{j=1}^{N_b^{k-s+1}} \omega^j_{k+1} l(\boldsymbol{x}^j_{k+1}, \Delta\boldsymbol{u}^j_k) + \sum_{k=s+N_r}^{s+N_p-1} \sum_{j=1}^{N_b^{N_r}} \omega^j_{k+1} l(\boldsymbol{x}^j_{k+1}, \Delta\boldsymbol{u}^j_k), \quad \forall (j, k+1) \in I_{st}, \tag{7a}$$

subject to:

$$\boldsymbol{x}^j_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k^{p(j)}, \boldsymbol{u}^j_k, \boldsymbol{d}^{r(j)}), \qquad \forall\, (j, k+1) \in I_{st}, \boldsymbol{d}^{r(j)} \in \mathcal{D}, \tag{7b}$$

$$\boldsymbol{g}(\boldsymbol{x}^j_{k+1}, \boldsymbol{u}^j_k) \leq 0\,, \qquad \forall\, (j, k+1) \in I_{st}, \tag{7c}$$

$$\boldsymbol{u}^j_k = \boldsymbol{u}^l_k \text{ if } \boldsymbol{x}_k^{p(j)} = \boldsymbol{x}_k^{p(l)}, \qquad \forall\, (j,k), (l,k) \in I_{st}, \tag{7d}$$

$$\underline{\boldsymbol{u}} \leq \boldsymbol{u}^j_k \leq \overline{\boldsymbol{u}}, \qquad \forall\, (j,k) \in I_{st}, \tag{7e}$$

$$\boldsymbol{x}^1_s = \boldsymbol{x}^m_s \tag{7f}$$

$$\boldsymbol{P}^\star_s = (1 - \phi(1-\phi)(\boldsymbol{d}_s - \boldsymbol{d}^\star_{s-1})^T \boldsymbol{P}_s^{-1} \hat{\boldsymbol{F}}^{-1} (\boldsymbol{P}^\star_{s-1})^{-1} (\boldsymbol{d}_s - \boldsymbol{d}^\star_{s-1})) \hat{\boldsymbol{F}}^{-1}, \tag{7g}$$

$$\boldsymbol{d}^\star_s = \hat{\boldsymbol{F}}^{-1}(\phi(\boldsymbol{P}^\star_{s-1})^{-1}\boldsymbol{d}^\star_{s-1} + (1-\phi)\boldsymbol{P}_s^{-1}\boldsymbol{d}_s), \tag{7h}$$

$$\mathcal{D} = \mathbb{S}(\boldsymbol{d}^\star_s, \boldsymbol{P}^\star_s), \tag{7i}$$

$$\boldsymbol{g}^j_{n,k+1} = \boldsymbol{g}(\boldsymbol{x}^j_{k+1}, \boldsymbol{u}^j_k), \qquad \forall\, (j, k+1) \in I_{st}, \tag{7j}$$

$$\left(\boldsymbol{g}^q_{m,k+1}, \boldsymbol{G}^q_{c,k+1}\right) = \mathbb{U}(\kappa_{c,k}, \boldsymbol{g}_{n,k+1}^{[(q-1)N_b+1:qN_b]}), \qquad \forall k \in I_{br}, q \in I_b(k), \tag{7k}$$

$$\boldsymbol{g}^q_{m,k+1} + \mathrm{diag}^{\frac{1}{2}}(\boldsymbol{G}^q_{c,k+1}) \leq 0, \qquad \forall k \in I_{br}, q \in I_b(k), \tag{7l}$$

where $N_b = 2n_d + 1$, $\hat{\boldsymbol{F}} = \phi(\boldsymbol{P}^\star_{s-1})^{-1} + (1-\phi)\boldsymbol{P}_s^{-1}$.

3. Tuning of the scaling factor κ of multi-stage NMPC based on the box over-approximation of the reachable set of the constraint function

The scaling factor κ can be computed such that the box over-approximation of the scaled constraint ellipsoids tightly over-approximates the entire reachable set of the constraint function by solving n_c optimization problems given below, where n_c gives the number of constraints, $j \in I_c := \{1, \cdots, n_c\}$,

$$\hat{\boldsymbol{\kappa}}_{c,[j]} = \max_{\kappa, \boldsymbol{x}_0^{\text{wc}}, \boldsymbol{u}_0^{\text{wc}}, \boldsymbol{d}^{\text{wc}} \in \mathbb{D}, \boldsymbol{d}^i \in \mathbb{S}(\boldsymbol{d}_0, \boldsymbol{P}_0)} \frac{\left| g^{\text{wc}}_{1,[j]} - g_{m,1,[j]} \right|}{\sqrt{G^p_{c,1,[j,j]}}}, \tag{8a}$$

subject to

$$
\begin{aligned}
& \boldsymbol{g}_{n,1}^{i} = \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}_0^{\mathrm{wc}}, \boldsymbol{u}_0^{\mathrm{wc}}, \boldsymbol{d}^i), \boldsymbol{u}_0^{\mathrm{wc}}), && \forall\, i \in I_{sp}, \boldsymbol{d}^i \in \mathbb{S}(\boldsymbol{d}_0, \boldsymbol{P}_0), && \text{(8b)} \\
& \left(\boldsymbol{g}_{m,1}, \boldsymbol{G}_{c,1}^{p}\right) = \mathbb{U}(1, \boldsymbol{g}_{n,1}^{[1:2n_d+1]}), && && \text{(8c)} \\
& \boldsymbol{g}_1^{\mathrm{wc}} = \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}_0^{\mathrm{wc}}, \boldsymbol{u}_0^{\mathrm{wc}}, \boldsymbol{d}^{\mathrm{wc}}), \boldsymbol{u}_0^{\mathrm{wc}}), && && \text{(8d)} \\
& \boldsymbol{g}(\boldsymbol{x}_0^{\mathrm{wc}}, \boldsymbol{u}_0^{\mathrm{wc}}) \leq 0, && && \text{(8e)} \\
& \boldsymbol{G}_{c,1,[j,j]}^{p} > 0, && && \text{(8f)}
\end{aligned}
$$

The objective function (8a) gives the maximum value by which the constraint ellipsoids should be scaled such that its box over-approximation encloses the reachable set of j^{th} constraint function. The predicted value of the constraint function obtained using the sigma points is given by (8b). The mean ($\boldsymbol{g}_{m,1}$) and unscaled constraint covariance matrix ($\boldsymbol{G}_{c,1}^{p}$) is given by (8c). The vector $\boldsymbol{g}_1^{\mathrm{wc}}$ in (8d) denotes the constraint value prediction obtained in the whole operating region of the plant that results in the maximum value of the objective function. Vectors $\boldsymbol{x}_0^{\mathrm{wc}}$ and $\boldsymbol{u}_0^{\mathrm{wc}}$ are bounded by operating region of the plant using (8e). The constraint (8f) makes sure that the optimization problem becomes infeasible if the sensitivity of the constraints to the uncertain parameters is 0. The scaling factor corresponding to different constraint is given by

$$
\hat{\boldsymbol{\kappa}}_{c,[j]}^{\star} = \begin{cases} 0, & \text{if (8) is infeasible} \\ \hat{\boldsymbol{\kappa}}_{c,[j]}, & \text{otherwise} \end{cases}, \qquad \forall j \in I_c \tag{9}
$$

where $\hat{\boldsymbol{\kappa}}_c^{\star} \in \mathbb{R}^{n_c}$. The scaling factor is set to 0 for the constraint which is not influenced by the uncertainty considered in the scenario tree. The scaling factor $\kappa_c^{\star}$ is given by $||\hat{\boldsymbol{\kappa}}_c^{\star}||_{\infty}$. This makes sure that the box over-approximation of the scaled constraint ellipsoids encloses the reachable set of the constraint function.