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Analysis of Stock-Dependent Arrival Process in a Retrial Stochastic Inventory System with Server Vacation

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Abstract: The present study deals with the stock-dependent Markovian demand of a retrial queueing system with a single server and multiple server vacation. The items are restocked under a continuous review (s, Q) ordering policy. When there is no item in the system, the server goes on vacation. Further, any arrival demand permits entry into an infinite orbit whenever the server is on vacation. In the Matrix geometric approach with the Neuts-Rao truncation technique, the steady-state joint distribution of the number of customers in orbit, the server status, and the inventory level is obtained. Under the steady-state conditions, some significant system performance measures, including the long-run total cost rate, are derived, and the Laplace-Stieltjes transform is also used to investigate the waiting time distribution. According to various considerations of uncontrollable parameters and costs, the merits of the proposed model, especially the important characteristics of the system with stock dependency over non-stock dependency, are explored. Ultimately, the important facts and ideas behind this model are given in conclusion.

Keywords: Markovian demand; stock-dependent demand rate; vacation; infinite orbit; lead time

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1. Introduction

Normally, allowing a vacation for a server helps to maintain the working efficiency and increase the life span of the server (machine). Due to continuously working in the system, even a human server may encounter physical or mental stress, which reduces their efficiency. As a result, the vacation allows the human server to de-stress and re-energize. After completing the vacation, the server can maintain the productivity of the system without any hardship. One can gain a deep understanding of a single server vacation under the queuing system by referring to Doshi [1], Tian and Zhang [2], and Ke et al. [3].

The concept of vacation was first initiated in an inventory system by Daniel, and Ramanarayanan [4] who applied the server vacation during the stock-out time. Sivakumar [5] extended the multiple vacation policy in a retrial queuing inventory system. Jeganathan [6] analyzed a finite queuing inventory system with multiple vacations of a single server and impatient customers whose reneging time is assumed to be exponential. Yadavalli and Jeganathan [7] studied a retrial perishable inventory finite queuing system with two heterogeneous servers in which one server is assumed to take multiple vacations. Again, Jeganathan et al. [8] explored the significance of heterogeneous servers over homogeneous servers on a finite retrial inventory system with server vacations.

Either during a period of stock out or a server vacation, any arriving customer's demand is not fulfilled. Rather, some of their customers may revisit the shop/stall to meet their demand in the future due to the credibility of either of their products or services. This retrial concept plays an important role in the development of either queuing or inventory theory. The retrial concept with an inventory system was first studied by Artaljeo et al. [9]. Paul Manuel et al. [10] analyzed a retrial perishable inventory system with negative demands. Generally, we notice that the retrial of any individual customer does not depend on the other customers in orbit. However, the rate of retrial customers is directly proportional to the number of customers in orbit. An inventory system with this retrial policy is known as the classical retrial inventory system (CRIS). The analytical approach to CRIS was first established by Ushakumari [11]. Krishnamoorthy and Jose [12] studied a retrial inventory system in which the capacity of orbital customers is assumed to be infinite. Jeganathan et al. [13], investigated a $M|M|1$ retrial inventory system connected to a finite capacity waiting hall, where the service rate is queue dependent and a classical retrial policy is used for orbital customers. Recently, Jeganathan et al. [14] studied a multi-server queueing inventory system with a classical retrial facility.

Dong-YuhYang et al. [15] discussed a retrial queueing system with a single server that takes multiple optional vacations in a finite order under which batch arrivals and a constant retrial rate are also considered. Dong-Yuh Yang and Chia-Huang Wub [16] described a finite capacity classical retrial queue with server breakdown in which the server is further encountered with working vacation under Bernoulli trail. Reshmi and Jose [17] investigated a perishable inventory system with a classical retrial policy under a matrix analytic approach in which both primary and retrial customers were considered to enter into an orbit with independent Bernoulli's schedules. Dhanya Shajin et al. [18] deeply studied the marked Markovian arrival demand of a retrial inventory system with additional items and two kinds of customers where services are provided according to preemptive priority and threshold-based inventory, and classical retail policy is used for low priority customers with the Bernoulli approach. Jothivel Kathiresan et al. [19] worked on a finite buffer inventory system with two kinds of services, and the nature of the service is assumed by Bernoulli distribution.

Most researchers have studied an inventory system where customers' arrival policies are independent of stock levels in the system. Nevertheless, in a real-life scenario, the higher rate of customers entering happens in a system where the stock is kept in large quantities. This can be found at any marketplace where all dealers have kept their products in a line. Further, customers are inspired by a dealer who stocks their products in different sizes and modes in some conditions. For instance, in a food exhibition, a stall displays a food product with different varieties that attract different sections of people according to their age and diets. It is because the product choice varies according to customer preferences. Hence, customer preferences are highly satisfied by such a system with a large stock level. This will be seen on a regular basis at food exhibitions, book exhibitions, mobile showrooms, and car showrooms, among other places.

The following literature reviews show how the stock level influences the demand pattern of customers. Levin et al. [20], Silver and Peterson [21] both considered the number of customers' arrivals functionally related to the quantity of displayed stock over a period of time. Gupta and Vrat [22] defined the consummation rate of items as dependent upon the inventory size. Baker and Urban [23] studied an inventory system with a deterministic approach where the demand is estimated as a polynomial function with the reference of stock level over the time interval. Badmanabhan and Vrat [24] determined the optimum ordering quantity where the demand rate is assumed to be stock-dependent.

Further, Urban [25] delineated and analyzed two kinds of demand rates, one of which is dependent upon the stock out period, and the other is dependent upon the stock-in period. Rathod and Bhathawala [26] studied a stock-dependent inventory system with varying holding costs and shortages. Alfares [27] deeply analyzed an inventory system where demand and storage time are assumed to be correlated with stock level and holding cost, respectively. Sudhir Kumar Sahu et al. [28] discussed a perishable inventory system in which the demand rate is dependent upon the present stock level. Sandeep Kumar [29] furnished the optimum ordering quantity and cycle time of a stock-dependent inventory system with shortages and variable holding costs.

Gabi Hanukov et al. [30] also analyzed the stock-dependent Markovian demand with two servers. In this model, the preliminary service inventory is also made during the servers' idle time. Jeganathan et al. [31] analyzed the comparative study in their queueing-inventory system in which they assumed that the arrival process of a customer was dependent on the current stock level of the system. Recently, Abdul Reiyas and Jeganathan [32] discussed stock-dependent arrivals in the base stock queueing-inventory system. Mostly, many authors applied a positive service time in their respective models, whereas Paul Manual et al. [10], Sivakumar [33], Sivakumar [34], and Jeganathan et al. [31] assumed that the inventory in the system was depleted at the instant of the arrival of a customer.

These observations strongly motivated us to do further research on an inventory system with stock-dependent arrivals. In the extension of Sivakumar [5], the arrival of both primary and retrial is incorporated with the dependency of stock level, which makes the novelty of this study. In addition, we employ the classical retrial policy for a retrial customer. The rest of the paper is designed as follows. In the next section, the mathematical formulation of the model is explained. The details of the mathematical approach of the model with the steady-state analysis are presented in Section 3. Furthermore, the analysis of waiting time is done in Section 4. Some key system performance measures and sensitive analysis of the model are achieved in Sections 5 and 6, respectively. Furthermore, the conclusion is given in the last section.

2. Explanation of System

This paper investigates a continuous review inventory to explore a stock-dependent arrival process for a customer and two different tasks for the server. This system can hold a maximum of S items. The server availability can be either in vacation mode or in normal mode (not on vacation). In this connection, the arriving customer receives an item immediately whenever the inventory is positive. More clearly, the customer's service time is assumed to be instantaneous. In the event of a zero stock level, the server goes on vacation. If the server finds a positive inventory at the end of the vacation, then only he will return from the vacation; otherwise, (zero stock level), he will take another vacation. This is called the "multiple vacation policy".

Any primary arrival of the system is assumed to be a non-homogeneous Poisson process. This is because the primary arrival to the system is dependent on the current stock level. The intensity rate of a primary arrival is defined as λ_j where $1 \leq j \leq S$. As we stated earlier, during the stock-out period, the server goes into vacation mode. In such a period (the server is on vacation), an arriving primary customer enters into an infinite orbit with an intensity α. The customer from orbit can approach the system at any time. However, the successful retrial of a customer happens only when the inventory is not empty and the server is in normal mode. The time between two successful retrials is assumed to be exponentially distributed. The retrial process of a customer is dependent on the current stock level as well as the number of customers in the orbit. The intensity of a retrial customer is defined as $k\theta_j$, where k is the number of customers in the orbit and $1 \leq j \leq S$. Further, the replenishment process of the system will be started immediately if the inventory level falls to the predetermined stock level s under the (s, Q) ordering policy. The lead time follows an exponential distribution and its intensity is denoted as β.

Description of Stock-Dependent Parameters

θ_j: mean retrial rate of orbital customers is given by $\theta_j = \theta j^a$, $1 \le j \le S$, $\theta > 0$, $0 \le a \le 1$.
λ_j: mean arrival rate of primary customers is given by $\lambda_j = \lambda j^b$, $1 \le j \le S$, $\lambda > 0$, $0 \le b \le 1$.

$$\begin{array}{lcl} \delta_{ij} & : & \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{otherwise} \end{cases} \\ \bar{\delta}_{ij} & : & 1 - \delta_{ij} \\ H(i-j) & : & \begin{cases} 1, & \text{if } i \ge j, \\ 0, & \text{if } i < j \end{cases} \end{array}$$

3. Analysis of the System

This system can be referred by triplets $(U(t), V(t), W(t))$, where $U(t), V(t)$ and $W(t)$ represent orbital customers' size, server status and inventory level at time t, respectively.

The status of the server is defined by

$$V(t) = \begin{cases} 0 & \text{if the server is on vacation mode at time } t, \\ 1 & \text{if the server is not on vacation mode at time } t. \end{cases}$$

Based on the assumptions of the given model, the continuous time discrete state random process $\{X(t),\ t \ge 0\} = \{(U(t), V(t), W(t)),\ t \ge 0\}$ is said to follow Markov process with the state space E is determined by

$$E = \{(u,0,0) \cup (u,0,Q) \cup (u,1,w) \mid u = 0,1,2,\dots; w = 1,2,\dots,S\},$$

and its infinitesimal generator transition rate matrix M can be framed as :

$$M = \begin{pmatrix} M_{00} & M_{01} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ M_{10} & M_{11} & M_{01} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & M_{20} & M_{22} & M_{01} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & M_{k0} & M_{kk} & M_{01} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & M_{(k+1)0} & M_{(k+1)(k+1)} & M_{01} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

Suppose (u_2, v_2, w_2) be a transition state from a given state (u_1, v_1, w_1), then the following transition sub matrices are determined as follows:

Case (i):

The following sub-block holds the transition of a primary customer entering into orbit if the server is on vacation mode.

$$[M_{01}]_{(v_1 w_1)(v_2 w_2)} = \begin{cases} \alpha & v_2 = v_1, \quad v_1 = 0; \quad w_2 = w_1, \quad w_1 = 0, Q \\ 0 & \text{otherwise.} \end{cases}$$

Case (ii):

The following sub-block holds the transition of a retrial customer who purchases the product successfully if the server is in normal mode.

For $u_1 = 1,2,3,\dots; w_1 = 1,2,\dots,S$

$$[M_{u_1 0}]_{(v_1 w_1)(v_2 w_2)} = \begin{cases} u_1 \theta_w & v_2 = v_1, v_1 = 1; \quad w_2 = w_1 - 1, w_1 = w. \\ 0 & \text{otherwise.} \end{cases}$$

Case (iii):

The following sub-block holds the transition of the reorder level, vacation return, primary and retrial customer is purchasing.
For $u_1 = 0, 1, 2, \ldots; w = 1, 2, \ldots, S$

$$[M_{u_1 u_1}]_{(v_1 w_1)(v_2 w_2)} = \begin{cases} \beta & \begin{array}{ll} v_2 = v_1 & v_1 = 0 \\ w_2 = w_1 + Q, & w_1 = 0 \end{array} \\ \beta & \begin{array}{ll} v_2 = v_1 & v_1 = 1 \\ w_2 = w_1 + Q, & w_1 = 1, \ldots, s \end{array} \\ \gamma & \begin{array}{ll} v_2 = v_1 + 1 & v_1 = 0 \\ w_2 = w_1, & w_1 = Q \end{array} \\ u_1\theta_1 + \lambda_1 & \begin{array}{ll} v_2 = 0 & v_1 = 1 \\ w_2 = 0, & w_1 = 1 \end{array} \\ \bar{\delta}_{0u_1} u_1 \theta_{w_1} + \lambda_{w_1} & \begin{array}{ll} v_2 = v_1 & v_1 = 1 \\ w_2 = w_1 - 1, & w_1 = 2, 3, \ldots, S \end{array} \\ -(\alpha + \beta) & \begin{array}{ll} v_2 = v_1 & v_1 = 0 \\ w_2 = w_1, & w_1 = 0 \end{array} \\ -(\alpha + \gamma) & \begin{array}{ll} v_2 = v_1 & v_1 = 0 \\ w_2 = w_1, & w_1 = Q \end{array} \\ -(\bar{\delta}_{0u_1} u_1 \theta_{w_1} + \lambda_{w_1} + H(s - w_1)\beta) & \begin{array}{ll} v_2 = v_1 & v_1 = 1 \\ w_2 = w_1, & w_1 = w \end{array} \\ 0, & \text{otherwise.} \end{cases}$$

It is noted that the above matrices are all square matrices of order $S + 2$.

3.1. *Matrix Geometric Approximation*

Steady-State Analysis

Consider k to be the cutoff point for the matrix-geometric approximation in the truncation process. Since the solving procedures of a classical retrial system have some analytical difficulty, we apply the Neuts-Rao truncation method. The classical retrial system under consideration is terminated at the truncation point k. After such truncation point, the system admits a constant retrial policy for a retrial customer. This concept is called the Neuts-Rao truncation method. We assume $M_{i0} = M_{k0}$ and $M_{ii} = M_{kk}$ for all $i \geq k$. The modified generator matrix of the truncated system $X(t)$ is shown below.

$$\hat{M} = \begin{pmatrix} M_{00} & M_{01} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ M_{10} & M_{11} & M_{01} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & M_{20} & M_{22} & M_{01} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & M_{k0} & M_{kk} & M_{01} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & M_{k0} & M_{kk} & M_{01} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

3.2. *Analysis of Steady-State Behavior*

Let $N = M_{k0} + M_{kk} + M_{01}$. Then N can also be determined by

$$[N]_{v_1 v_2} = \begin{cases} N_1, & v_2 = v_1, \quad v_1 = 0, \\ N_2, & v_2 = v_1 + 1, \quad v_1 = 0, \\ N_3, & v_2 = v_1 - 1, \quad v_1 = 1, \\ N_4, & v_2 = v_1, \quad v_1 = 1, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[N_1]_{w_1w_2} = \begin{cases} \beta & w_2 = w_1 + Q, \quad w_1 = 0 \\ -\beta & w_2 = w_1, \qquad\quad w_1 = 0 \\ -\gamma & w_2 = w_1, \qquad\quad w_1 = Q \\ 0, & \text{otherwise,} \end{cases}$$

$$[N_2]_{w_1w_2} = \begin{cases} \gamma & w_2 = w_1, \quad w_1 = Q \\ 0, & \text{otherwise,} \end{cases}$$

$$[N_3]_{w_1w_2} = \begin{cases} k\theta_{w_1} + \lambda_{w_1} & w_2 = w_1 - 1, \quad w_1 = 1 \\ 0, & \text{otherwise,} \end{cases}$$

$$[N_4]_{w_1w_2} = \begin{cases} \beta & w_2 = w_1 + Q, \quad w_1 = 1, 2, \ldots, s \\ k\theta_{w_1} + \lambda_{w_1} & w_2 = w_1 - 1, \quad w_1 = 2, 3, \ldots, S \\ -(k\theta_{w_1} + \lambda_{w_1} + \beta) & w_2 = w_1, \qquad\quad w_1 = 1, \ldots, s \\ -(k\theta_{w_1} + \lambda_{w_1}) & w_2 = w_1, \qquad\quad w_1 = s+1, \ldots, S \\ 0, & \text{otherwise,} \end{cases}$$

Clearly N is a square matrix of order $S+2$ and the sub-matrices N_1, N_2, N_3 and N_4 are all matrices of orders $2 \times 2, 2 \times S, S \times 2$ and $S \times S$, respectively.

Lemma 1. *The steady-state probability vector* $\mathbf{\Pi} = (\boldsymbol{\pi}^{(\mathbf{0})}, \boldsymbol{\pi}^{(\mathbf{1})})$ *where* $\boldsymbol{\pi}^{(\mathbf{0})}$=$(\pi^{(0,1)}, \pi^{(0,Q)})$ *and* $\boldsymbol{\pi}^{(\mathbf{1})}$=$(\pi^{(1,1)}, \pi^{(1,2)}, \ldots, \pi^{(1,S)})$ *corresponding to the generator N is given by*

$$\begin{aligned} \pi^{(0,w_1)} &= \pi^{(1,1)} a_{w_1}, \quad w_1 = 0, Q. \\ \pi^{(1,w_1+1)} &= \pi^{(1,1)} b_{w_1}, \quad w_1 = 1, \ldots, S-1. \end{aligned}$$

where,

$$a_{w_1} = \begin{cases} \frac{k\theta_1 + \lambda_1}{\beta}, & w_1 = 0 \\ \frac{k\theta_1 + \lambda_1}{\gamma}, & w_1 = Q \end{cases}$$

$$b_{w_1} = \begin{cases} \frac{k\theta_{w_1} + \lambda_{w_1} + \beta}{k\theta_{w_1+1} + \lambda_{w_1+1}}, & w_1 = 1 \\ \frac{b_{w_1-1}(k\theta_{w_1} + \lambda_{w_1} + \beta)}{k\theta_{w_1+1} + \lambda_{w_1+1}} & w_1 = 2, \ldots, s \\ \frac{b_{w_1-1}(k\theta_{w_1} + \lambda_{w_1})}{k\theta_{w_1+1} + \lambda_{w_1+1}} & w_1 = s+1, \ldots, Q-1 \\ \frac{\sum_{j_1=w_1+1}^{S} b_{j_1-Q}\beta}{k\theta_{w_1+1} + \lambda_{w_1+1}}, & w_1 = Q, \ldots, S-1 \end{cases}$$

and $\pi^{(1,1)}$ *can be obtained by solving equation*
$\gamma\pi^{(0,Q)} - (k\theta_Q + \lambda_Q)\pi^{(1,Q)} + (k\theta_{Q+1} + \lambda_{Q+1})\pi^{(1,Q+1)} = 0$ *and* $\mathbf{\Pi e} = 1$.

Proof. Let $\mathbf{\Pi}$ be the steady-state probability vector of N. That is, $\mathbf{\Pi}$ satisfies $\mathbf{\Pi}N = \mathbf{0}$, $\mathbf{\Pi e} = 1$.

The equation $\mathbf{\Pi}N = \mathbf{0}$ of the above yields the following set of equations:

$$\boldsymbol{\pi}^{(\mathbf{0})} N_1 + \boldsymbol{\pi}^{(\mathbf{1})} N_2 = 0, \tag{1}$$

$$\boldsymbol{\pi}^{(\mathbf{0})} N_3 + \boldsymbol{\pi}^{(\mathbf{1})} N_4 = 0, \tag{2}$$

Now, expanding the Equations (1) and (2) explicitly, we obtain the following set of equations,

$$
\begin{aligned}
-\beta\pi^{(0,0)} + (k\theta_1 + \lambda_1)\pi^{(1,1)} &= 0, \\
\beta\pi^{(0,0)} - \gamma\pi^{(0,Q)} &= 0, \\
-(k\theta_{w_1} + \lambda_{w_1} + \beta)\pi^{(1,w_1)} + (k\theta_{w_1+1} + \lambda_{w_1+1})\pi^{(1,w_1+1)} &= 0, \\
w_1 &= 1,\ldots,s \\
-(k\theta_{w_1} + \lambda_{w_1})\pi^{(1,w_1)} + (k\theta_{w_1+1} + \lambda_{w_1+1})\pi^{(1,w_1+1)} &= 0, \\
w_1 &= s+1,\ldots,Q-1 \\
\gamma\pi^{(0,Q)} - (k\theta_Q + \lambda_Q)\pi^{(1,Q)} + (k\theta_{Q+1} + \lambda_{Q+1})\pi^{(1,Q+1)} &= 0, \\
\beta\pi^{(1,w_1-Q)} - (k\theta_{w_1-Q} + \lambda_{w_1-Q})\pi^{(1,w_1)} + (k\theta_{w_1+1} + \lambda_{w_1+1})\pi^{(1,w_1+1)} &= 0, \\
w_1 &= Q+1,\ldots,S-1 \\
\beta\pi^{(1,s)} - (k\theta_S + \lambda_S)\pi^{(1,S)} &= 0.
\end{aligned}
$$

Solving the above system of equations recursively and using the normalizing condition, we get the stated result. □

Next, we derive the condition under which the system is stable.

Lemma 2. *The stability condition of the system under study is given by*

$$(a_0 + a_Q)\alpha < k\theta_1 + k\sum_{w_1=1}^{S-1} b_{w_1}\theta_{w_1+1} \tag{3}$$

Proof. From the well known result of Neuts [35] on the positive recurrence of M we have

$$\mathbf{\Pi} M_{01}\mathbf{e} < \mathbf{\Pi} M_{k0}\mathbf{e}$$

and by exploiting the structure of the matrices M_{01} and M_{k0}, and $\mathbf{\Pi}$ the stated result follows. □

It can be seen from the structure of the rate matrix M and from the Lemma 2, that the Markov process $\{(U(t), V(t), W(t)), t \geq 0\}$ with the state space E is regular. Hence the limiting probability distribution

$$\mathbf{\Phi}^{(u,v,w)} = \lim_{t\to\infty} Pr[U(t) = u, V(t) = v, W(t) = w \mid U(0) = 0, V(0) = 0, W(0) = 0],$$

exists and is independent of the initial state. Let $\mathbf{\Phi} = \left(\mathbf{\Phi^{(0)}}, \mathbf{\Phi^{(1)}}, \ldots,\right)$ satisfies

$$\mathbf{\Phi} M = \mathbf{0}, \quad \mathbf{\Phi e} = 1.$$

We can partition the vector $\mathbf{\Phi^{(u)}}, \mathbf{u} \geq \mathbf{0}$ as

$$\mathbf{\Phi^{(u)}} = \left(\Phi^{(u,0,0)}, \Phi^{(u,0,Q)}, \Phi^{(u,1,1)}, \ldots, \Phi^{(u,1,S)}\right).$$

3.3. R Matrix Calculation

For analyzing the QBD process, suppose the steady-state probability vector can be determined by the relation

$$R^2 M_{k0} + R M_{kk} + M_{01} = \mathbf{0}. \tag{4}$$

The rate matrix, R, is the smallest non-negative solution to the quadratic equation above. Because the matrix M_{01} only has two non-zero rows, the structure of the unknown rate matrix R also has two non-zero rows, resulting in a rate matrix R with only non-zero entries in the first two rows and only zero elements in the remaining rows:

$$R = \begin{pmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1S+1} \\ y_{21} & y_{22} & y_{23} & \cdots & y_{2S+1} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Apply R to the Equation (4), we get the following set of equations:
For $i = 1, 2$,

$$(y_{i1}y_{13} + y_{i1}y_{23})k\theta_1 - y_{i1}(\alpha + \beta) + y_{i3}\lambda_1 + \alpha\delta_{i1} = 0,$$

$$y_{i1}\beta - y_{i2}(\alpha + \gamma) + \alpha\delta_{i2} = 0.$$

For $i = 1, 2; j = 3, \cdots, S + 2$,

$$\begin{aligned} \delta_{(S+2)j}(y_{i1}y_{1j+1} + y_{i2}y_{2j+1})k\theta_{j-1} - y_{ij}(k\theta_{j-2} + \lambda_{j-2} + H(s+1-j)\beta) + \\ (1 - \delta_{(S+2)j})y_{ij+1}\lambda_{j-i} + \delta_{(S-1)j}y_{i2}\gamma + H(j-S)y_{i(j+3-S)}\beta = 0. \end{aligned}$$

From the above non-linear equations, the matrix R can be determined explicitly by using Gauss–Seidel iterative technique.

Theorem 1. *The vector* $\mathbf{\Phi}$ *can be determined by*

$$\mathbf{\Phi}^{(i+k-1)} = \mathbf{\Phi}^{(k-1)}R^i; \quad i \geq 0 \tag{5}$$

due to the special structure of M and where R is as in Equation (4) and the vector $\mathbf{\Phi}^{(i)}, i \geq 0$

$$\mathbf{\Phi}^{(i)} = \begin{cases} \sigma X^{(0)} \prod_{j=i}^{k} M_{j0}(-M_{j-1}), & 0 \leq i \leq k-1 \\ \sigma X^{(0)} R^{(i-k)}, & i \geq k \end{cases} \tag{6}$$

where

$$\sigma = [1 + X^{(0)} \sum_{i=0}^{k-1} \prod_{j=i}^{k} M_{j0}(-M_{j-1})\mathbf{e}]^{-1} \tag{7}$$

and $X^{(0)}$ *can be computed by set of equations*

$$X^{(0)}[M_k + RM_{k0}] = \mathbf{0}$$
$$X^{(0)}(I - R)^{-1}\mathbf{e} = 1.$$

Proof. The sub vector $(\mathbf{\Phi}^{(0)}, \mathbf{\Phi}^{(1)}, \ldots, \mathbf{\Phi}^{(k-1)})$ and the block partitioned matrix of $\hat{M}$ gives the set of equations

$$\begin{aligned} \mathbf{\Phi}^{(0)}M_{00} + \mathbf{\Phi}^{(1)}M_{10} &= \mathbf{0} \\ \mathbf{\Phi}^{(i-1)}M_{01} + \mathbf{\Phi}^{(i)}M_{ii} + \mathbf{\Phi}^{(i+1)}M_{(i+1)0} &= \mathbf{0}; \quad 1 \leq i \leq k-1. \end{aligned} \tag{8}$$

using Equation (8),

$$\mathbf{\Phi}^{(0)} = \mathbf{\Phi}^{(1)}M_{10}(-M_0)^{-1}$$

again using (8),

$$\mathbf{\Phi}^{(1)} = \mathbf{\Phi}^{(2)} M_{20}(-M_1)^{-1},$$

where $M_1 = (M_{11} + M_{10}(-M_0)^{-1}M_{01}), M_0 = M_{00}$.
Next,

$$\mathbf{\Phi}^{(2)} = \mathbf{\Phi}^{(3)} M_{30}(-M_2)^{-1},$$

where $M_2 = M_{22} + M_{20}(-M_1)^{-1}M_{01}$.
On continuing this procedure up to $k-1$ times, we get,

$$\mathbf{\Phi}^{(i)} = \mathbf{\Phi}^{(i+1)} M_{(i+1)0}(-M_i)^{-1}, \quad 0 \leq i \leq k-1 \tag{9}$$

where

$$M_i = \begin{cases} M_{i0}, & i = 0 \\ M_{ii} - M_{i0}(-M_{i-1})^{-1}M_{01}, & 1 \leq i \leq k \end{cases}$$

We use the block Gaussian elimination method to find the vectors $(\mathbf{\Phi}^{(k)}, \mathbf{\Phi}^{(k+1)}, \mathbf{\Phi}^{(k+2)} \ldots)$. The sub vector $(\mathbf{\Phi}^{(k)}, \mathbf{\Phi}^{(k+1)}, \mathbf{\Phi}^{(k+2)} \ldots)$ satisfies the following relation,

$$(\mathbf{\Phi}^{(k)}, \mathbf{\Phi}^{(k+1)}, \mathbf{\Phi}^{(k+2)} \ldots) \begin{pmatrix} M_k & M_{01} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ M_{k0} & M_{kk} & M_{01} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & M_{k0} & M_{kk} & M_{01} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \mathbf{0}. \tag{10}$$

Assume,

$$\begin{aligned} \sigma &= \sum_{i=k}^{\infty} \mathbf{\Phi}^{(i)}\mathbf{e} \\ X^{(i)} &= \sigma^{-1}\mathbf{\Phi}^{(k+i)}, \quad i \geq 0. \end{aligned}$$

From (10) we get

$$\begin{aligned} \mathbf{\Phi}^{(k)} M_k + \mathbf{\Phi}^{(K+1)} M_{k0} &= \mathbf{0} \\ \mathbf{\Phi}^{(k+i)} &= \mathbf{\Phi}^{(k+i-1)} R, \quad i \geq 1. \end{aligned}$$

This can be written as

$$\begin{aligned} X^{(0)} M_k + X^{(1)} M_{K0} &= \mathbf{0} \\ X^{(i)} &= X^{(i-1)} R, \quad i \geq 1 \end{aligned} \tag{11}$$

(11) becomes

$$X^{(0)}[M_k + RM_{k0}] = \mathbf{0}. \tag{12}$$

Since $\sum_{i=0}^{\infty} X^{(i)}\mathbf{e} = 1$, then

$$X^{(0)}(I - R)^{-1}\mathbf{e} = 1. \tag{13}$$

As a result, $X^{(0)}$ is the only solution to the Equations (12) and (13).
Hence,

$$\mathbf{\Phi}^{(i)} = \sigma X^{(0)} R^{(i-k)}, \quad i \geq k. \tag{14}$$

Again by (9) and (14), we get (6). Since $\sum_{i=0}^{\infty} \mathbf{\Phi}^{(i)}\mathbf{e} = 1$ and using (6),

$$\sigma X^{(0)} \sum_{i=0}^{k-1} \prod_{j=i}^{k} M_{j0}(-M_{j-1})\mathbf{e} + \sigma X^{(0)} \sum_{k}^{\infty} R^{(i-k)}\mathbf{e} = 1$$

which gives σ as in (7). □

4. Waiting Time Analysis

Waiting time (WT) is the time interval between an epoch when demand approaches the orbit and the moment when their time of operation completion occurs. Using the Laplace-Stieltjes transform (LST), we look at the WT of demand in orbit (LST). To find the orbital demand's waiting period, we naturally limit the orbit to a finite size. The concept of restriction of orbit size is followed from Lopez Herrero [36]. The continuous-time random variable W_o represents the waiting time of an orbital customer.

WT of Orbital Customers

Theorem 2. *An orbital demand does not wait with probability*

$$P\{W_o = 0\} = 1 - \omega_o \tag{15}$$

where $\omega_o = \sum_{u=1}^{L-1} \sum_{w=1}^{S} \Phi^{(u,1,w)} + \sum_{u=1}^{L-1} [\Phi^{(u,0,0)} + \Phi^{(u,0,Q)}]$

Proof. Since the total probability of zero and positive waiting time is 1, we have

$$P\{W_o = 0\} + P\{W_o > 0\} = 1. \tag{16}$$

Clearly, the probability of positive WT of orbital demand can be determined as

$$P\{W_o > 0\} = \sum_{u=1}^{L-1} \sum_{w=1}^{S_2} \Phi^{(u,0,w)}. \tag{17}$$

The Equation (17) can be found easily using Theorem 1 and substitute in Equation (16) we get the stated result as desired in (15). □

To enable the distribution of W_o, we shall define some complimentary variables. Suppose that the queueing-inventory system is at state (u,v,w), $u > 0$ at an arbitrary time t,

1. $W_o(u,v,w)$ be the time until chosen demand become satisfied.
2. LST of $W_o(u,v,w)$ is ${}^*W_o(u,v,w)(y)$ and we denote W_o by ${}^*W_o(y)$.
3. ${}^*W_o(y) = E[e^{yW_o}]$ LST of unconditional waiting time(UWT).
4. ${}^*W_o(u,v,w)(y) = E[e^{yW_o(u,v,w)}]$ LST of conditional waiting time(CWT).

Theorem 3. *The LST* $\{{}^*W_o(u,v,w)(y), (u,v,w) \in H^*$ *where* $H^* = E \cup \{*\}\}$ *satisfy the following system*

$$Z_o(y)^*W_o(y) = -\theta_w \mathbf{e}(u,v,w), (u,v,w) \in E. \tag{18}$$

$Z_o(y) = (P - yI)$, *the matrix P is derived from E by deleting the following state* $(0,0,0)$, $(0,0,Q)$, $(0,1,w)$, $1 \le w \le S$ *and* $\{*\}$ *be the absorbing state and the absorption appears if the orbital demand finds the positive commodities and the server is in the system.*

Proof. To analyze the CWT, we apply first step analysis as follows:
For $1 \le u \le L$,

$$(y+\bar{\delta}_{uL}\alpha+\beta)^{*}W_o(u,0,0)(y)=\bar{\delta}_{uL}\alpha^{*}W_o(u+1,0,0)(y)+\beta\,^{*}W_o(u,0,Q)(y). \tag{19}$$

For $1 \le u \le L$,

$$(y+\bar{\delta}_{uL}\alpha+\gamma)^{*}W_o(u,0,Q)(y)=\bar{\delta}_{uL}\alpha^{*}W_o(u+1,0,0)(y)+\gamma\,^{*}W_o(u,1,Q)(y). \tag{20}$$

For $1 \le u \le L, 1 \le w \le S$,

$$\begin{aligned}(y+\lambda_w+H(s-w)\beta+u\theta_w)^{*}W_o(u,1,w)(y)=\lambda_w^{*}W_o(u,1,w-1)\\+H(s-w)\beta\,^{*}W_o(u,1,w+Q)(y)+(u-1)\theta_w^{*}W_o(u-1,1,w-1)(y)+\theta_w.\end{aligned} \tag{21}$$

From the Equations (19)–(21) we attain a co-efficient matrix of the unknowns as a block tri-diagonal yields a stated result as in (18). □

Theorem 4. *The nth moments of conditional waiting time is given by*

$$Z_o(y)\frac{d^{n+1}}{dy^{n+1}}\,^{*}W_o(y)-(n+1)\frac{d^{n+1}}{dy^{n+1}}\,^{*}W_o(y)=0 \tag{22}$$

and

$$\frac{d^{n+1}}{dy^{n+1}}\,^{*}W_o(y)|_{y=0}=E[W_o^{n+1}(u,v,w)(y)],(u,v,w)\in H^{*} \tag{23}$$

Proof. Linear equations which are obtained in Theorem 3, we get a recursive algorithm to find a conditional and unconditional waiting times.

Now, we differentiate the Equations (19)–(21) for $(n+1)$ times and setting at $y=0$, we have,
For $1 \le u \le L$

$$(\bar{\delta}_{uL}\alpha+\beta)E[W_o^{n+1}(u,0,0)]=\bar{\delta}_{uL}\alpha E[W_o^{n+1}(u+1,0,0)]+\beta E[W_o^{n+1}(u,0,Q)]. \tag{24}$$

For $1 \le u \le L$

$$(\bar{\delta}_{uL}\alpha+\gamma)E[W_o^{n+1}(u,0,Q)]=\bar{\delta}_{uL}\alpha E[W_o^{n+1}(u+1,0,0)]+\gamma E[W_o^{n+1}(u,1,Q)]. \tag{25}$$

For $1 \le u \le L, 1 \le w \le S$

$$\begin{aligned}(\lambda_w+H(s-w)\beta+u\theta_w)E[W_o^{n+1}(u,1,w)]=\lambda_w E[W_o^{n+1}(u,1,w-1)]+\\H(s-w)\beta E[W_o^{n+1}(u,1,w+Q)]+(u-1)\theta_w E[W_o^{n+1}(u-1,1,w-1)].\end{aligned} \tag{26}$$

With reference to Equations (24)–(26), one can determine the unknowns $E[W_p^{n+1}(u,v,w,x)]$ in terms of moments of one order less. On setting $n=0$, we obtain the desired moments of particular order in an algorithmic way. □

Theorem 5. *The LST of UWT of orbital demand is given by*

$$^{*}W_o(y)=1-\omega_o+\omega_o\,^{*}W_o(u+1,v,w)(y) \tag{27}$$

Proof. Using PASTA property, one can obtain the LS transform of W_o as follows:

$$^{*}W_o(y)=\Phi^{(u)}\,^{*}W_o(u,v,w)(y),\quad 0\le u\le L,\quad 0\le v\le 1,\quad 0\le w\le S \tag{28}$$

using the expressions (28), we get the stated result. By considering Euler and Post-Widder algorithms in Abatt and Whitt [37] for the numerical inversion of (27), we obtain the desired result. □

Theorem 6. *The nth moments of UWT, using the Theorem 5, is given by*

$$E[W_o^n] = \delta_{0n} + (1-\delta_{0n}) \sum_{u=0}^{L-1} \sum_{v=0}^{1} \sum_{w=0}^{S} \Phi^{(u,v,w)} E[W_o^n(u+1,v,w)] \tag{29}$$

Proof. To determine the nth moments of UWT in terms of the CWT of the same order, we differentiate the expression (27) n times and calculate at $y = 0$ to obtain the desired result. □

Theorem 7. *The expected waiting time of an orbital demand is defined by*

$$E[W_o] = \sum_{u=0}^{L-1} \sum_{v=0}^{1} \sum_{w=0}^{S} \Phi^{(u,v,w)} E[W_o(u+1,v,w)] \tag{30}$$

Proof. Using Equation (29) in Theorem 6 and substitute $n = 1$, we get the desired result as in (30). □

5. Measures of Various Activities of the System

In this section, the following measures of corresponding activities are used to obtain the expected total cost under the steady-state transitions.

1. Expected inventory level is

$$E_i = \sum_{u=0}^{\infty} Q\Phi^{(u,0,Q)} + \sum_{u=0}^{\infty} \sum_{w=1}^{S} w\Phi^{(u,1,w)} \tag{31}$$

2. Expected reorder rate of commodity is

$$E_r = \sum_{u=0}^{\infty} (\lambda_{s+1} + u\theta_{s+1})\Phi^{(u,1,s+1)} \tag{32}$$

3. Expected number of customers enters into the orbit is

$$E_e = \sum_{u=0}^{\infty} \alpha[\Phi^{(u,0,0)} + \Phi^{(u,0,Q)}] \tag{33}$$

4. Expected number of customers in the orbit is

$$E_o = \sum_{u=1}^{\infty} \sum_{w=1}^{S} u\Phi^{(u,1,w)} + \sum_{u=1}^{\infty} u[\Phi^{(u,0,0)} + \Phi^{(u,0,Q)}] \tag{34}$$

5. Expected number of overall retrial customers is

$$E_{or} = \sum_{u=1}^{\infty} \sum_{w=1}^{S} u\theta_w\Phi^{(u,1,w)} + \sum_{u=1}^{\infty} u\theta_Q\Phi^{(u,0,Q)} \tag{35}$$

6. Expected number of successful retrial customers is

$$E_{sr} = \sum_{u=1}^{\infty} \sum_{w=1}^{S} u\theta_w\Phi^{(u,1,w)} \tag{36}$$

7. Fraction of successful retrial rate is

$$F_{sr} = \frac{E_{sr}}{E_{or}} \tag{37}$$

8. Expected total cost is determined by

$$Tc(S,s) = C_h E_i + C_s E_r + C_w E_o \tag{38}$$

where

C_h: Carrying cost per unit item of commodity per unit time.
C_s: Ordering cost of commodity per order.
C_w: Waiting cost of an orbiting customer per unit time.

6. Numerical Investigation

The author's real-life experience is studied to provide a numerical picture for the readers regarding the contemplated recommended model. One day, the author went to a mobile store and observed how it operated. It also sells clients' pen drives. Those who come into the shop to make a purchase are instantly served. They will not let a new customer into the system if there's a zero-stock situation. They go outside and perform some personal work in such a situation. They return after some time to buy the pen-drive if it is still available. Suppose the shop has more pen drives. They start displaying or advertising them. When customers or people see the advertisement, they start purchasing. When the current stock level reaches some fixed quantity, the server makes a call to the supplier to furnish the replenishment. If there is no pen drive available currently, the server will close its service and take a rest. Once the ordered pen drive comes, the server will continue his service. From this experience, the author wanted to use the pen-drive sales functions as a mathematical model. Since the arrival occurs according to the displayed stock level, a stock-dependent arrival process is considered in this paper. At zero stock level, the server's rest situation is considered a vacation, and new customers are not allowed. To provide a numerical illustration, an arrival rate (positive stock), $\lambda = 19.2$, a reorder rate, $\beta = 1.47$, an arrival rate (zero stock), $\alpha = 3.8$, a vacation completion rate, $\gamma = 9.98$, scale factors, $a = 0.6, b = 0.5$, a retrial rate, $\theta = 4.5$, a holding cost, $C_h = 0.0046$, a setup cost, $C_s = 5.4$, and awaiting cost per unit in the orbit, $C_w = 0.032$ are assumed.
Case (i):

The two dimensional, local convexity of the expected total cost curve(Tc) is obtained when the maximum number of items(number of pen-drive) S lies in the integer interval [30,40] with regard to pre-fixed reorder level $s = 6$, $s = 7$ and $s = 8$ which are shown in Figures 1–3 respectively. Each figure depicts the convexity of the Tc curve under the classification of stock-dependent (SD) and non-stock-dependent (NSD). Each curve in those diagrams has a minimum point, which is referred to as the optimum point. That is, the estimated overall cost of pen-drive sales should be kept to a minimum (optimized). These curves show the total cost of the pen-drive sales when the arrival is wholly SD ($a = 0.21, b = 0.1$) and NSD ($a = 0, b = 0$) or partially SD ($a = 0$, $b = 0.1$ and $a = 0.21$, $b = 0$). Because the optimum total cost is determined for both fully SD (or NSD) and partially SD arrivals, the organizer can pick for either the purely SD (or NSD) or partially SD arrival approach to boost pen-drive sales profits. The contrasting results of both solely and partially SD arrival processes are also shown in the Figures 1–3. This will be valuable to all readers as well as business tycoons who are in the inventory business (electric and electronic items, home appliances, etc.) and apply any of the arrival policies depending on their business plan. The optimal predicted total cost of the pen-drive store is attained when the store follows a partially SD arrival method, as shown in all three figures. However, if the s varies, one can see that the best-reduced cost varies continually. They will determine the critical reorder level to attain the optimal S, as indicated in Figures 1–3. Overall, at the middle reorder level ($s = 7$), the optimum estimated total cost of the pen-drive business exists. The overall expenditure of the store may be regulated with the help of this shown case, which shows the optimal predicted total cost of the pen-drive business and the best fit of the reorder point.

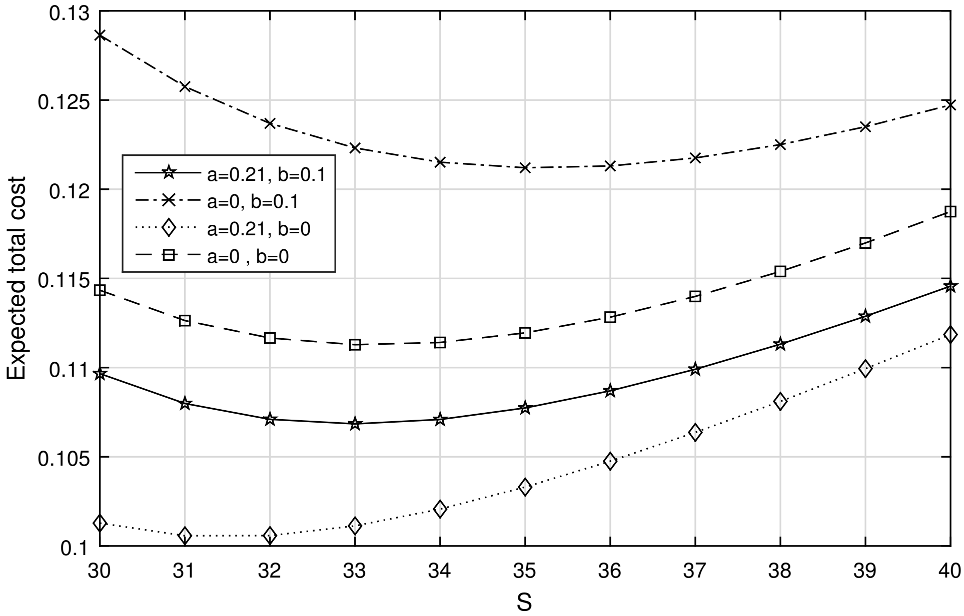

Figure 1. $Tc(S)$ with $s = 6$.

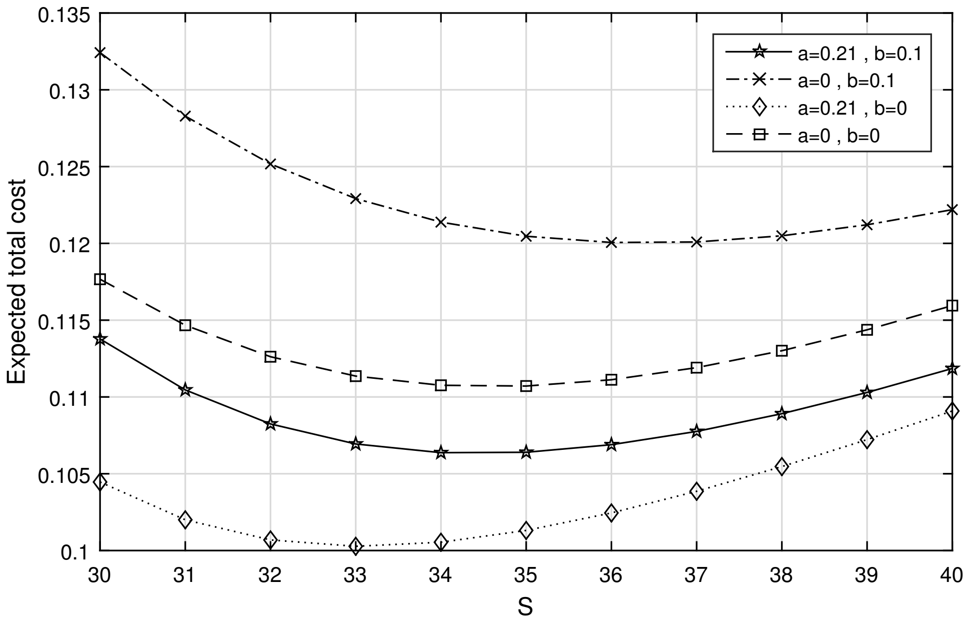

Figure 2. $Tc(S)$ with $s = 7$.

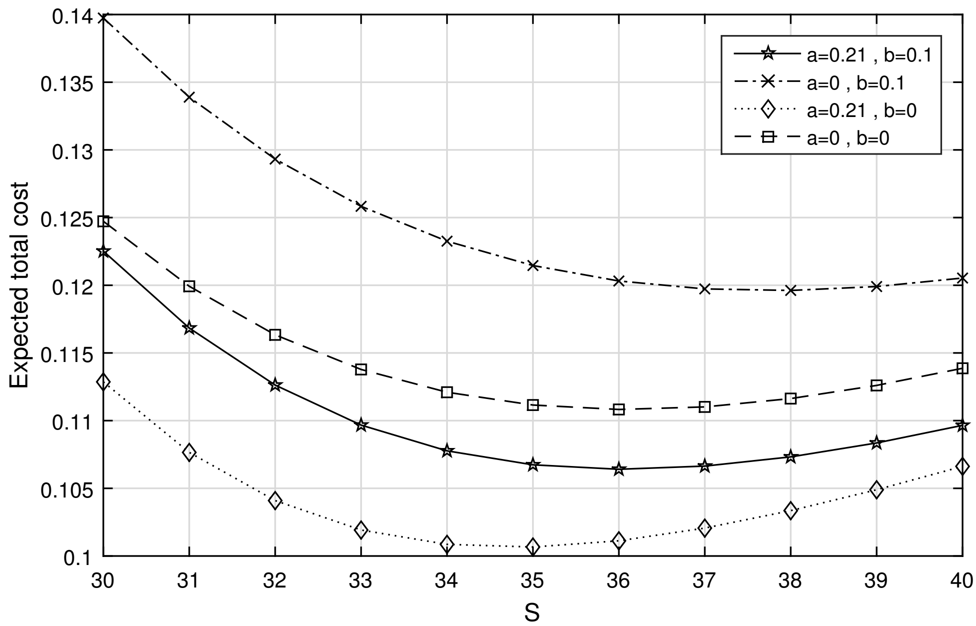

Figure 3. $Tc(S)$ with $s = 8$.

Case (ii):

This case explores the expected number of existing pen drives, its re-order rate, the number of customers in orbit, and the total and successful rate of retrial. With respect to purely SD (or NSD) and partially SD classifications. The increase in β shows that the average time between two consecutive reorders has decreased. Figure 4 demonstrates that the expected number of current stock levels has increased if the lead time decreased. The owner of the pen-drive store will have more products in the storage system (if β is increased). That is, the shop's displayed stock level has been increased. When individuals look at the objects on showcase, they may be tempted to them. They become customers and begin purchasing the displayed things if they are intrigued with them. The information concerning about expected reorder rate is shown in Figure 5 which demonstrates that the re-order intensity rate is always proportional to its expected rate. As a result, as shown in Figure 5, the merchant ensures that the appropriate steps are followed to obtain a prompt replacement. Figures 6–8 show the results as the lead time is inversely proportional to the number of customers in the orbit, the expected total, and the success rate of retrial customers, respectively. As previously stated, if Q products are restocked quickly (i.e., if β is increased), the shop's displayed stock level is likewise boosted. The boosting of the current stock level indicates the estimated number of consumers in orbit, the expected total retrial rate, and the expected successful retrial rate respectively. In the same way, the vacation will soon come to an end. As illustrated in Figures 9–13 when the parameter γ is increased, current stock level, expected reorder rate, average customer in the orbit, and their total and successful rate of retrial decrease. That means, if the pen-drive store owner spends as much time as possible on vacation, the measured metrics of the store's performance will be impacted, as we predicted. This will assist them in deciding whether to lengthen or shorten their vacation time based on their preferences.

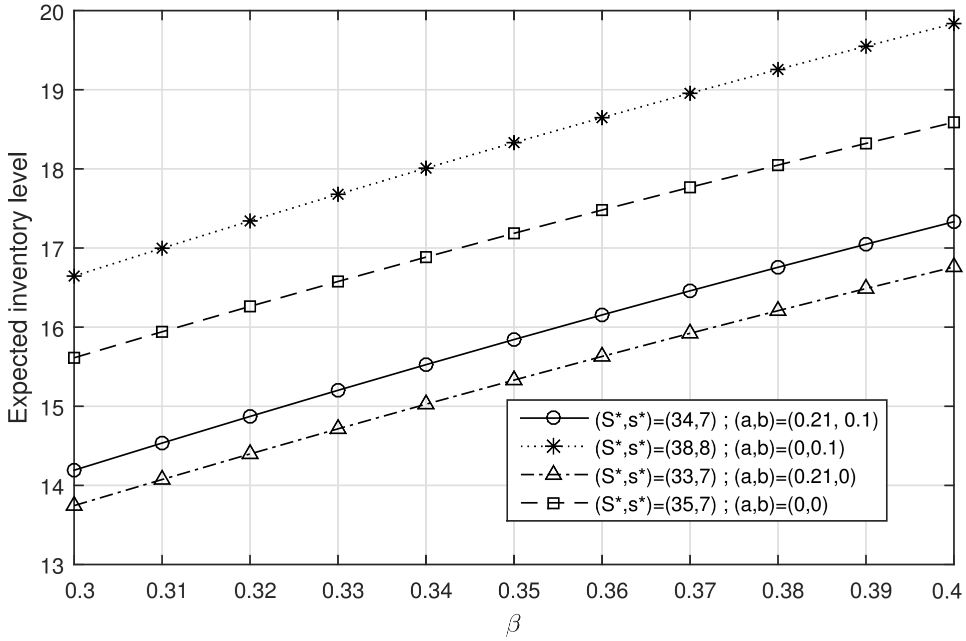

Figure 4. E_i vs. *Lead time*.

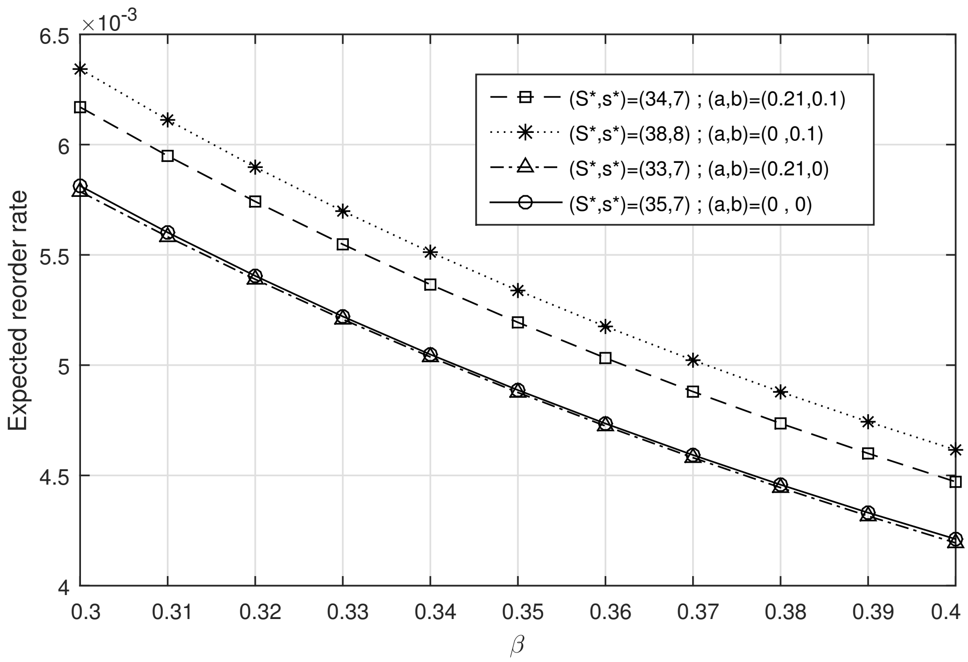

Figure 5. E_r vs. *Lead time*.

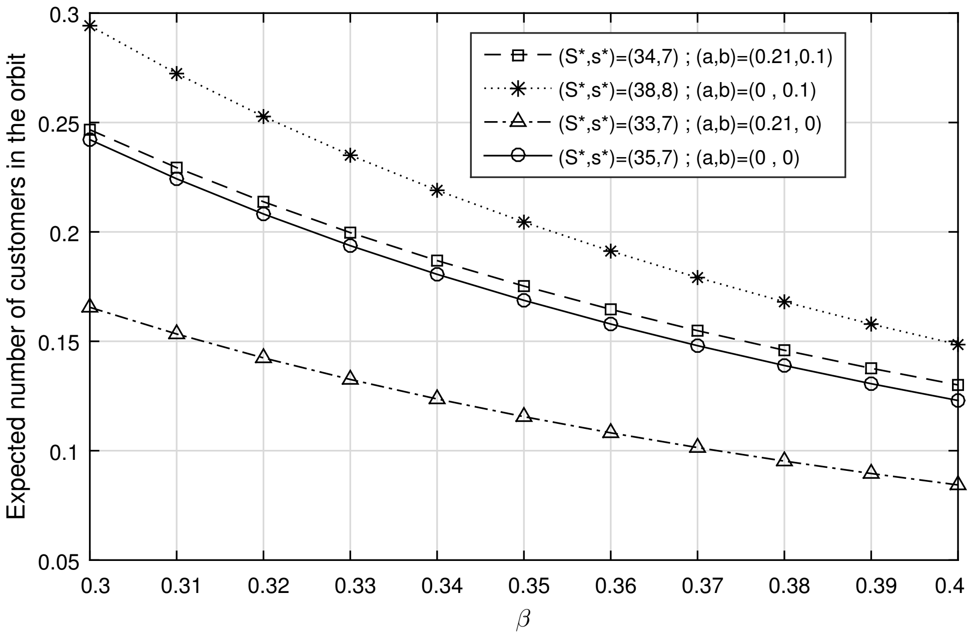

Figure 6. E_o vs. *Lead time*.

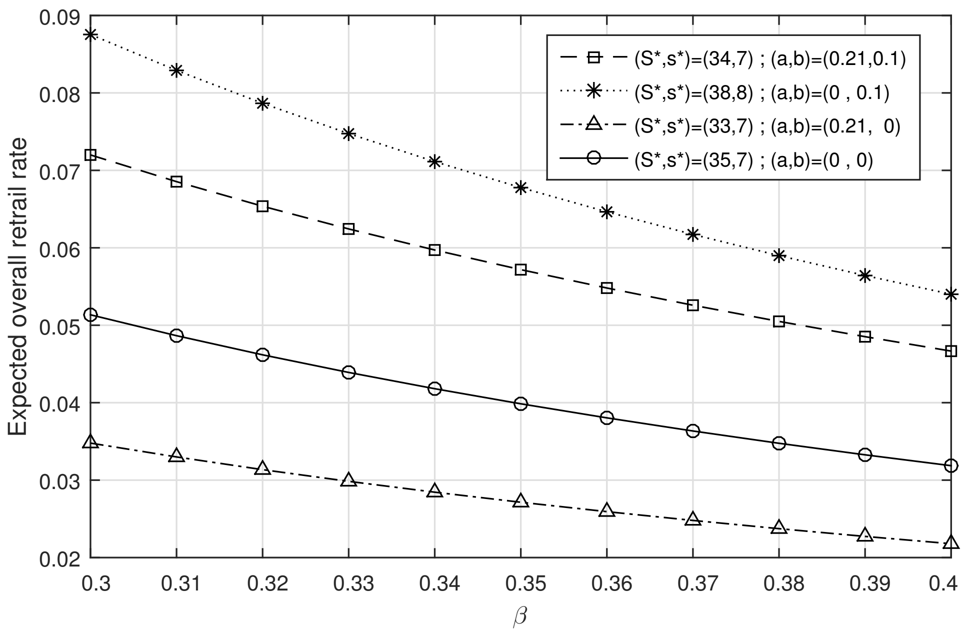

Figure 7. E_{or} vs. *Lead time*.

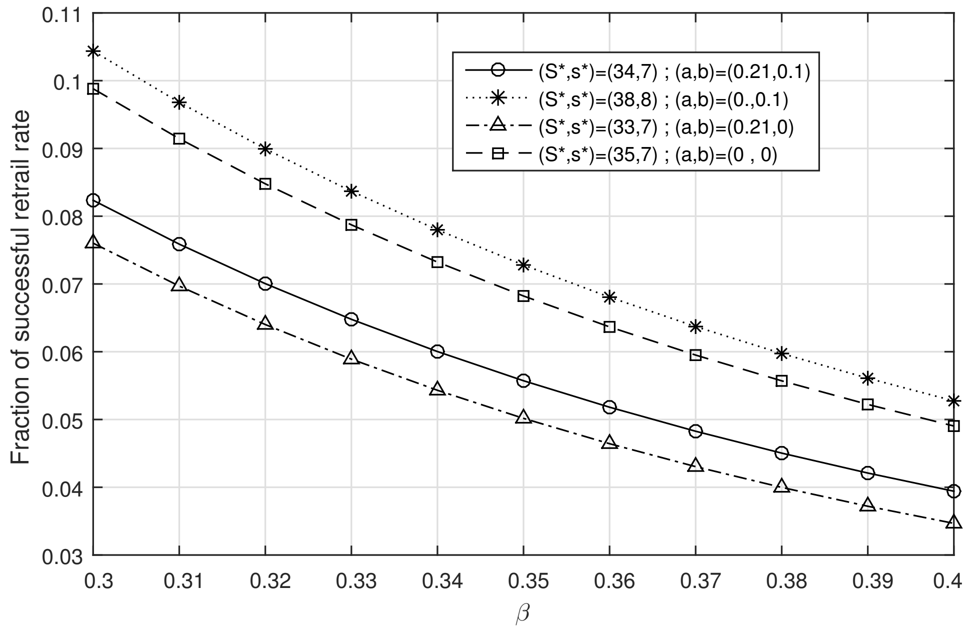

Figure 8. F_{sr} vs. *Lead time*.

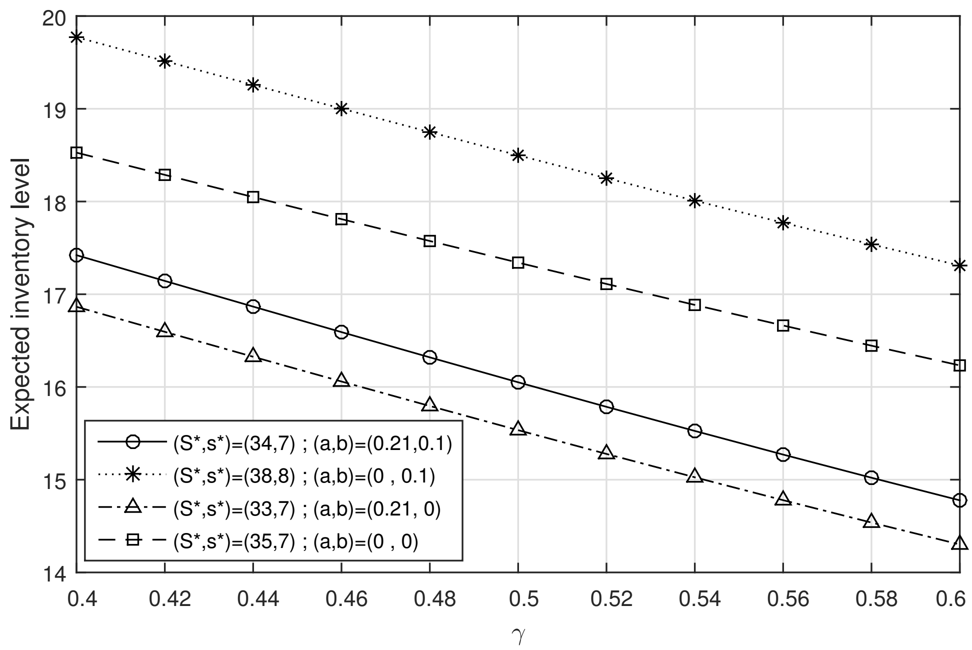

Figure 9. E_i vs. *Vacation time*.

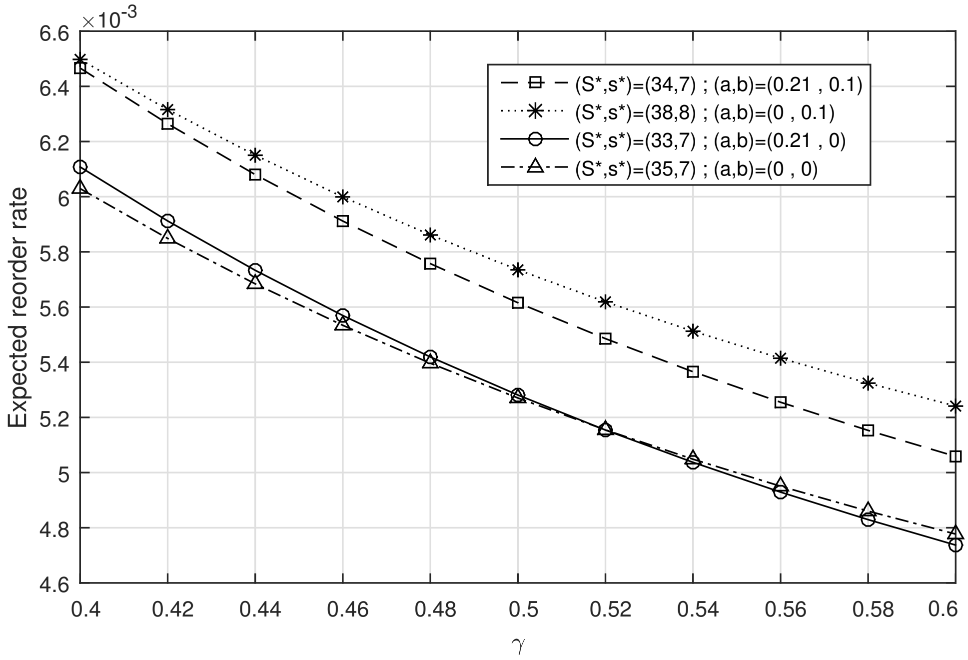

Figure 10. E_r vs. *Vacation time.*

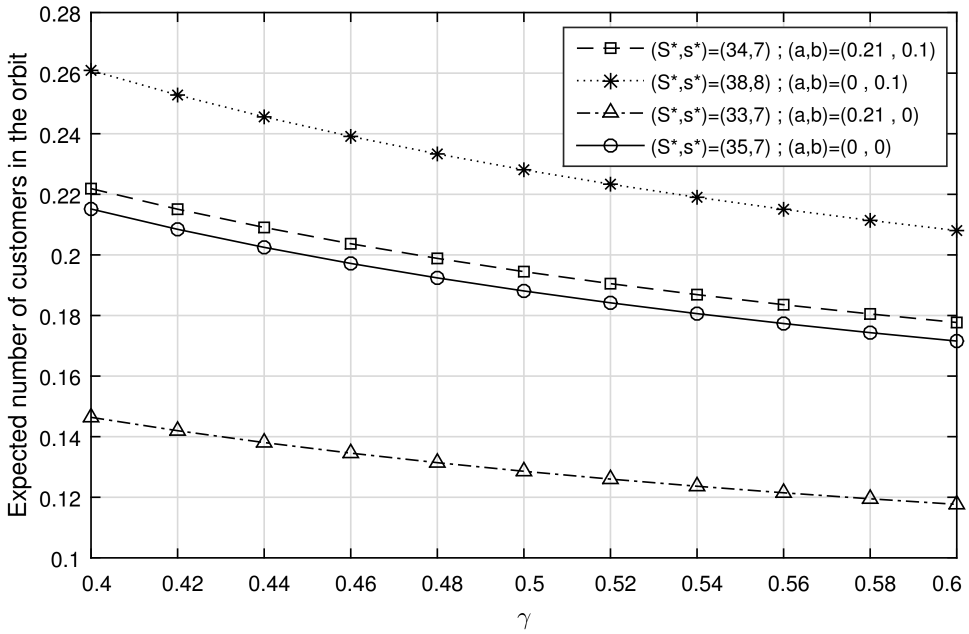

Figure 11. E_o vs. *Vacation time.*

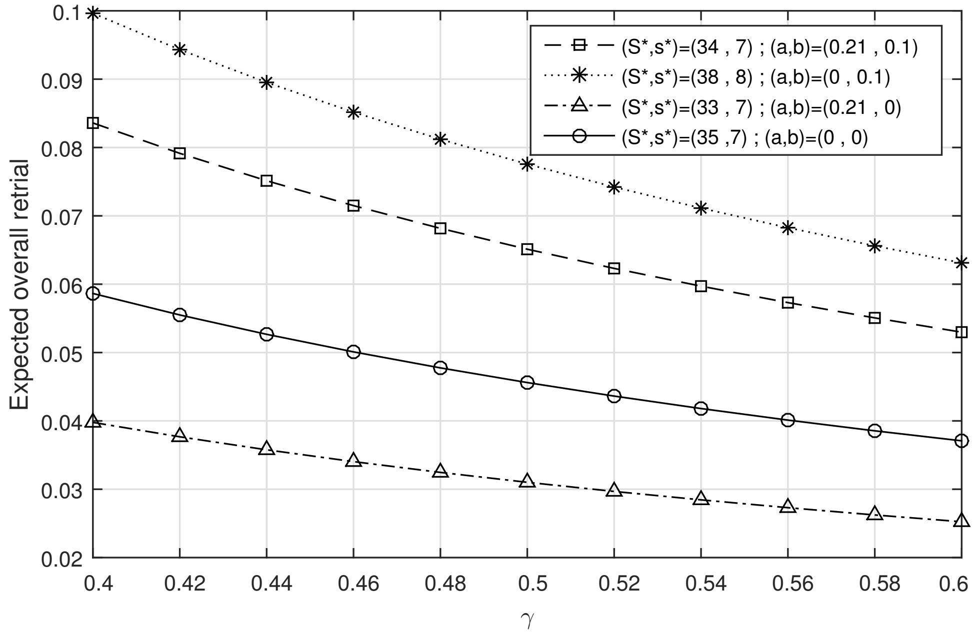

Figure 12. E_{or} vs. *Vacation time*.

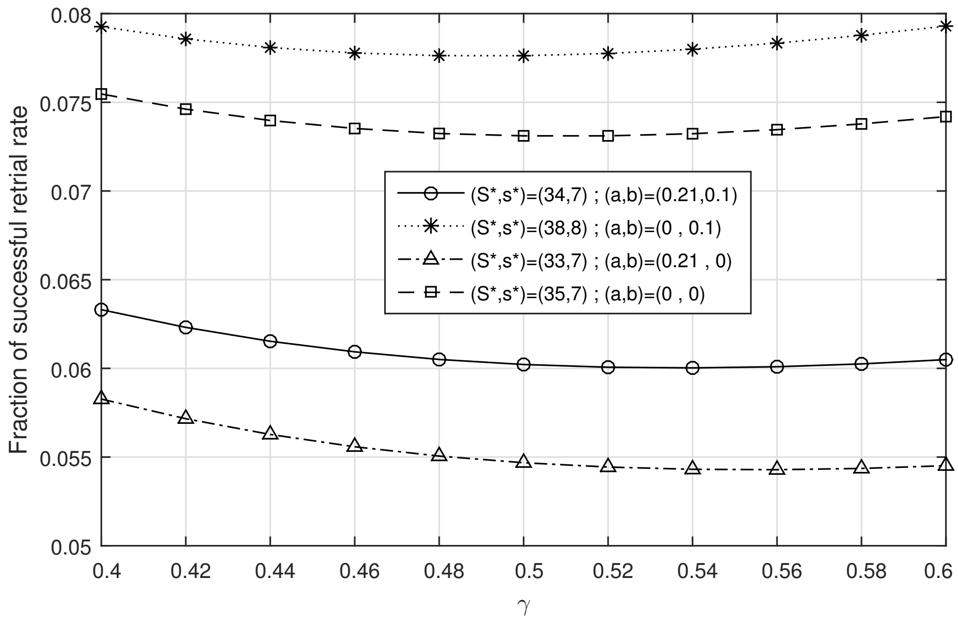

Figure 13. F_{sr} vs. *Vacation time*.

Case (iii):

The ideal total cost, optimal stock level, and optimal reorder level of the store are achieved by varying the cost values of waiting cost per customer, holding cost per item, and set-up cost per order together. From a business standpoint, cost functions are critical to making a profit. This explanation illustrates how changes in respective cost values affect the optimum number of pen drives that have to be stored, its re-order level, and predicted total cost. The store owner notices that the findings produced in Table 1 demonstrate that the waiting cost per customer in the store and the set-up cost per order are both increased, and that the optimal stock, re-order levels, and predicted total cost are also increased. Based on these findings, the model's assumed cost structure indicates that the store or firm should focus on calculating their cost values. If they lose control over the cost values, their total spending costs will rise and their profit will decrease. As a result of the cost function variation, the findings in Table 1 show that if the store owner raises the holding cost per pen-drive, the calculated total cost rises. Similarly, the remaining set-up cost of the pen-drive for every order, as well as the waiting expenses per client in the orbit, drive up the total cost.

Table 1. Optimum ordering policy with different combinations of various costs.

		$C_s = 4.4$			$C_s = 5.4$			$C_s = 6.4$		
C_w	C_h	S^*	s^*	Tc^*	S^*	s^*	Tc^*	S^*	s^*	Tc^*
0.022	0.0036	35	7	0.083579	36	7	0.088457	36	7	0.09299
	0.0046	32	6	0.098958	34	7	0.104504	35	7	0.109522
	0.0056	31	6	0.11338	32	6	0.119566	33	6	0.125273
0.032	0.0036	36	8	0.085174	37	8	0.08987	38	8	0.094231
	0.0046	34	7	0.101007	34	7	0.106373	35	7	0.111305
	0.0056	31	6	0.115923	33	7	0.12186	34	7	0.127264
0.042	0.0036	38	9	0.086534	39	9	0.091091	40	9	0.095338
	0.0046	35	8	0.102671	36	8	0.10787	37	8	0.112674
	0.0056	33	7	0.117893	34	7	0.123767	36	8	0.129104

Case (iv):

In Tables 2–5, under the parameter variation, optimum stock level, reorder level, total cost and expected inventory level, re-order rate, the average customer in the orbit are discussed. When we raise the arrival rate (both zero and positive stock or λ and α) and the scale factors a and b and the retrial rate, θ are proportional to optimum stock level, reorder level, total cost and expected, re-order rate, the average customer in the orbit separately. The number of customers entering the system per unit time has increased as the average arrival rate has increased. The system must stock a larger quantity of products in order to satisfy or give service to all of the arriving clients. If the current stock level falls below the reorder level quickly, the replenishment of Q goods may be replaced promptly. If the replenishment does not take place right away, the current stock level will be depleted. Any arriving customer who discovers that the inventory level is empty may be sent into orbit. The scaling factors a and b perform the same function as λ. On the other hand, the re-order intensity rate, β influences the measures optimum stock level, reorder level, total cost and expected inventory level, re-order rate, the average customer in the orbit in the inverse direction. This is because the existing stock level of the system is increased if the replenishment time is reduced. Since the re-ordered quantities arrive quickly, the store owner can provide the service as much as possible. If the service completion is to be done fast, the estimated measures taken into this case are to be decreased. Similarly, the vacation completion time (if γ is increased) ends as soon as the server starts his work immediately. Suppose the customer finds that the server is in working mode, the number of the customer going to orbit will reduce. On following that the optimum stock and reorder level also decreased. In addition, the orbital customers also get the opportunity to get a quicker service. So that the customers expected total and successive retrial rates are reduced. This illustration inspires the readers to develop the stock-dependent arrival strategy in their business. Many businesses nowadays use social media to display their products in the most efficient way in order to enhance client traffic. The expansion and development of the inventory industry will be aided by such arrival dependencies.

Table 2. Response of retrial factors (θ,a) on S^*, s^*, Tc^*, E_i, E_r and E_o.

a	θ	S^*	s^*	Tc^*	E_i	E_r	E_o
0.4	4.0	130	5	23.2558	18.3919	0.0046	1.2573
	4.5	121	5	21.3804	17.0708	0.0037	1.1522
	5.0	113	5	19.8516	15.8955	0.0032	1.0693
0.6	4	120	6	19.5224	16.7811	0.0024	1.0170
	4.5	112	6	18.0376	15.6050	0.0020	0.9371
	5	107	7	16.8157	14.7228	0.0018	0.8683
0.8	4	111	8	15.8538	15.1655	0.0015	0.8312
	4.5	104	8	14.7204	14.1356	0.0013	0.7693
	5	98	8	13.7805	13.2526	0.0011	0.7186

Table 3. Response of primary arrival factors (λ, b) on S^*, s^*, Tc^*, E_i, E_r and E_o.

b	λ	S^*	s^*	Tc^*	E_i	E_r	E_o
0.3	17.12	98	9	13.6277	13.1061	0.0010	0.6677
	19.12	105	9	14.7002	14.1365	0.0011	0.7202
	21.12	111	9	15.7336	15.0195	0.0012	0.7742
0.6	17.12	106	7	16.6518	14.5755	0.0018	0.8598
	19.12	112	6	18.0376	15.6050	0.0020	0.9371
	21.12	120	6	19.3804	16.7816	0.0023	1.0060
0.9	17.12	114	5	20.5399	16.0404	0.0038	1.1177
	19.12	122	4	22.3593	17.3623	0.0042	1.2197
	21.12	130	4	24.1368	18.5364	0.0050	1.3217

Table 4. Response of vacation and lead time parameters (γ, β) on S^*, s^*, Tc^*, E_i, E_r and E_o.

γ	β	S^*	s^*	Tc^*	E_i	E_r	E_o
8.98	1.37	117	6	19.3946	16.9227	0.00228	1.0027
	1.47	107	6	18.9158	16.5227	0.00231	0.9771
	1.57	99	6	18.4859	16.2495	0.00232	0.9515
9.98	1.37	124	7	18.5065	16.0521	0.00211	0.9601
	1.47	112	6	18.0376	15.6050	0.00204	0.9371
	1.57	103	6	17.6158	15.2519	0.00207	0.9146
10.98	1.37	130	7	17.7551	15.3398	0.00187	0.9236
	1.47	119	7	17.2952	14.9881	0.00188	0.8980
	1.57	109	7	16.8813	14.5788	0.00192	0.8778

Table 5. Response of orbit entrance and lead time parameters (α, β) on S^*, s^*, Tc^*, E_i, E_r and E_o.

α	β	S^*	s^*	Tc^*	E_i	E_r	E_o
3.4	1.37	114	7	17.0140	14.6825	0.00167	0.8860
	1.47	103	7	16.5873	14.1349	0.00164	0.8691
	1.57	95	6	16.2031	13.9961	0.00165	0.8431
3.8	1.37	124	7	18.5065	16.0521	0.00211	0.9601
	1.47	112	6	18.0376	15.6050	0.00204	0.9371
	1.57	103	6	17.6158	15.2519	0.00207	0.9146
4.2	1.37	134	7	19.9756	17.4208	0.00262	1.0322
	1.47	121	6	19.4645	16.9269	0.00253	1.0074
	1.57	112	6	19.0050	16.6643	0.00259	0.9791

7. Conclusions

This model accounts for both primary and retrial arrivals, each with its own exponential time and different rates that depend on the current stock level. Furthermore, the research considers the independent exponential times of server vacation and lead-time. The matrix geometric technique is used to generate the steady-state joint distribution with its components, orbit size, server status, and inventory level at any time t. This steady-state behavior is used to investigate the nature of the expected total cost and various system operations, including waiting time. A numerical analysis of the various metrics derived in Section 5 is offered to improve the considered model. We looked at how stock-dependent and non-stock-dependent arrival rules affect the optimal ordering policy for both main and retrial consumers. We also experimented with the properties of different measures by changing the lead time and vacation factors. Then, for the various cost values and parameters, we looked at the best ordering strategy and optimum total cost. All of the numerical visualizations are illustrated with examples from a pen-drive store. By glancing at each illustration, the business owner can understand more about the suggested model. From the examination of the ideal total cost in case (i), the store owner or any businessman will understand the unreliability of the optimal total expenditure cost as well as the optimal stock level and re-order level. On the other hand, the influence of the lead time and vacation completion time on the system performance measures suggests that every businessperson makes a strategy to reduce the average lead and vacation time, as illustrated in case (ii). However, by case (iii), one can conclude that the holding cost of the product and the setup cost per order of the product, and the waiting cost per customer are offset by having determined the optimum stock, re-order, and predicted total cost, there is considerable variation as they are changed. Therefore, fixing the cost values of the inventory sales service process will have a great contribution to the profit of the business. Case (iv) investigates the impact of arrival rate, retrial rate, and scale factors, along with vacation completion time and lead time, on system performance. In all the above cases, the results are presented for both SD and NSD clients. Readers can apply the proposed model whenever a suitable inventory sales business is executed. According to the results obtained in the numerical section, they can implement their new strategies for further growth and development. In the future, this considered model will be extended with positive service time.

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