

## Article

# Asymptotic Estimates for the One-Year Ruin Probability under Risky Investments

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**Abstract:** Motivated by the EU Solvency II Directive, we study the one-year ruin probability of an insurer who makes investments and hence faces both insurance and financial risks. Over a time horizon of one year, the insurance risk is quantified as a nonnegative random variable  $X$  equal to the aggregate amount of claims, and the financial risk as a  $d$ -dimensional random vector  $Y$  consisting of stochastic discount factors of the  $d$  financial assets invested. To capture both heavy tails and asymptotic dependence of  $Y$  in an integrated manner, we assume that  $Y$  follows a standard multivariate regular variation (MRV) structure. As main results, we derive exact asymptotic estimates for the one-year ruin probability for the following cases: (i)  $X$  and  $Y$  are independent with  $X$  of Fréchet type; (ii)  $X$  and  $Y$  are independent with  $X$  of Gumbel type; (iii)  $X$  and  $Y$  jointly possess a standard MRV structure; (iv)  $X$  and  $Y$  jointly possess a nonstandard MRV structure.

**Keywords:** asymptotics; Breiman's theorem; max-domain of attraction; multivariate regular variation; ruin probability

**MSC:** primary 62P05; secondary 60G70, 62E20

## 1. Introduction

Ruin theory, as one of the most developed areas in risk theory, mainly focuses on the ultimate ruin probability (RP) as a measurement of solvency of an insurance business. However, property and casualty insurance companies mainly sell short-term insurance contracts such as auto insurance, home insurance, and so on. Then the policyholders do not really care about the ultimate RP, but they are satisfied as long as the insurer is able to cover all qualified claims during the contract year. Moreover, the estimation of the RP depends on valuations of assets and liabilities. As insurers usually prepare their balance sheets and close their books annually, it is natural to check ruin on a yearly time grid. As an important application, the one-year RP is used in evaluation of Solvency Capital Requirement. The Solvency II Directive (2009/138/EC)<sup>1</sup> states that “the Solvency Capital Requirement should be determined as the economic capital to be held by insurance and reinsurance undertakings in order to ensure that ruin occurs no more often than once in every 200 cases or, alternatively, that those undertakings will still be in a position, with a probability of at least 99.5%, to meet their

<sup>1</sup> Available online: <http://eur-lex.europa.eu/legal-content/en/TXT/?uri=CELEX:32009L0138>.

obligations to policy holders and beneficiaries over the following 12 months.” See [Bauer et al. \(2012\)](#) and [Christiansen and Niemeyer \(2014\)](#) for in-depth discussions on the issue of calculating Solvency Capital Requirement from the Solvency II Directive.

Consider a one-year insurance risk model. Let  $u > 0$  represent the initial wealth of the insurer at time  $t = 0$ , which includes a regulatory initial capital and premiums collected on policies with coverage from  $t = 0$  to  $t = 1$ . Suppose that there are  $d$  risk-free or risky assets available for the insurer to make investments, each with an annual return rate  $R_i > -1$ ,  $i = 1, \dots, d$ . The restriction  $R_i > -1$  is to exclude the worst scenario of losing everything invested in the  $i$ th asset. Suppose that the insurer invests a proportion  $w_i \in [0, 1]$  of its initial wealth in the  $i$ th asset,  $i = 1, \dots, d$ , where we have excluded short positions. Thus,

$$\mathbf{w} \in \Sigma^d := \left\{ \mathbf{w} \in [0, 1]^d : w_1 + \dots + w_d = 1 \right\}.$$

The value of the investment portfolio at time  $t = 1$  becomes  $u(1 + \mathbf{w}^\top \mathbf{R})$ . Further assume that all insurance claims and associated expenses, totaled by a nonnegative random variable  $X$ , are paid at time  $t = 1$ . Then the insurer’s wealth at time  $t = 1$  becomes  $u(1 + \mathbf{w}^\top \mathbf{R}) - X$ .

The insurer becomes insolvent if its wealth runs too low. Naturally, the one-year RP is defined by

$$RP = P(u(1 + \mathbf{w}^\top \mathbf{R}) - X < 0), \quad u > 0, \mathbf{w} \in \Sigma^d. \quad (1)$$

We point out that, although this may not exactly define the probability of ruin, it indeed measures the likelihood that the insurer is in the insolvency state. This definition is consistent with most of recent works in this literature; see, e.g., [Eling et al. \(2009\)](#) and [Asanga et al. \(2014\)](#).

From the risk management point of view, it is more customary to look at discount factors than returns. Denote by  $Y_1, \dots, Y_d$  the discount factors of the  $d$  individual assets and by  $Y_{\mathbf{w}}$  the overall discount factor of the investment portfolio, which are, respectively,

$$Y_i = \frac{1}{1 + R_i}, \quad i = 1, \dots, d,$$

and

$$Y_{\mathbf{w}} = \frac{1}{1 + \mathbf{w}^\top \mathbf{R}} = \left( \sum_{i=1}^d w_i Y_i^{-1} \right)^{-1}. \quad (2)$$

In this way, the RP in (1) is reduced to the tail probability of the product of  $X$  and  $Y_{\mathbf{w}}$ ,

$$RP = P(XY_{\mathbf{w}} > u). \quad (3)$$

As calculated by [Tang and Tsitsiashvili \(2003\)](#), the random variable  $X$  quantifies the insurance risk and the random variables  $Y_i$ ,  $i = 1, \dots, d$ , and  $Y_{\mathbf{w}}$  quantify the financial risks.

For calculating the RP in (1) and its equivalent expression in (3), a closed-form solution is available only under some ideal but usually unrealistic model assumptions on the marginal distributions and the dependence structure. Alternatively, one can employ the crude Monte Carlo (CMC) method to estimate it. However, the true value of the RP must be very small, such as 0.5% as required by the Solvency II Directive, and correspondingly  $u$  must be very large. When the RP is about 0.5%, a sample of size as large as  $n = 10^6$  is needed for the coefficient of variation of the CMC estimate to be about 1–2%. See [Tang and Yuan \(2012\)](#) for a related discussion on this issue.

In this paper, we aim at asymptotic estimates for the RP for the case with a large initial wealth  $u$ . Throughout the paper we assume that  $\mathbf{Y}$  possesses a multivariate regular variation (MRV) structure. Exact asymptotic formulas for the RP are derived for the following cases, which essentially cover all important scenarios with regularly varying insurance and financial risks:

- (i)  $X$  and  $Y$  are independent with  $X$  of Fréchet type;
- (ii)  $X$  and  $Y$  are independent with  $X$  of Gumbel type;
- (iii)  $X$  and  $Y$  jointly possess a standard MRV structure;
- (iv)  $X$  and  $Y$  jointly possess a nonstandard MRV structure.

In the rest of this paper, first we highlight some preliminaries of max-domains of attraction (MDA) and MRV in Section 2, and then we derive asymptotic formulas of the RP for cases (i) and (ii) in Section 3, and for cases (iii) and (iv) in Section 4.

## 2. Preliminaries

We need to collect some preliminaries before we are able to state our main results.

### 2.1. Max-Domain of Attraction (MDA)

In this subsection, we highlight some preliminaries of max-domain of attraction (MDA), a basic concept in univariate extreme value theory, to be used in deriving our main results. Standard textbook treatments of this concept in the context of insurance, finance, and risk management are given by Embrechts et al. (1997) and McNeil et al. (2015), among others.

A distribution function  $H$  on  $\mathbb{R}$  is said to belong to the MDA of a non-degenerate distribution function  $H_0$ , denoted by  $H \in \text{MDA}(H_0)$ , if

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} |H^n(c_n x + d_n) - H_0(x)| = 0$$

holds for some normalizing constants  $c_n > 0$  and  $d_n \in \mathbb{R}$ ,  $n \in \mathbb{N}$ . By the classical Fisher–Tippett–Gnedenko theorem, only three choices for  $H_0$  are possible, namely the Fréchet, Gumbel, and Weibull distributions, which are denoted by  $\Phi$ ,  $\Lambda$ , and  $\Psi$ , respectively.

The following results give equivalent descriptions of the membership of the three MDAs. These results are well known; the reader is referred to Section 3.3 of Embrechts et al. (1997) for more details. A distribution function  $H$  belongs to  $\text{MDA}(\Phi_\gamma)$  for  $\gamma > 0$  if and only if its upper endpoint  $\hat{x} = \sup\{x \in \mathbb{R} : H(x) < 1\}$  is infinite and its survival function  $\bar{H} = 1 - H$  is regularly varying at  $\infty$  with index  $-\gamma$ , denoted as  $\bar{H} \in \text{RV}_{-\gamma}$ , namely, the relation

$$\lim_{x \uparrow \infty} \frac{\bar{H}(xy)}{\bar{H}(x)} = y^{-\gamma}, \quad y > 0,$$

holds. A distribution function  $H$  belongs to  $\text{MDA}(\Lambda)$  if and only if the relation

$$\lim_{x \uparrow \hat{x}} \frac{\bar{H}(x + ya(x))}{\bar{H}(x)} = e^{-y}, \quad y \in \mathbb{R},$$

holds for some positive auxiliary function  $a(\cdot)$  on  $(-\infty, \hat{x})$ , where the upper endpoint  $\hat{x}$  can be finite or infinite. The auxiliary function  $a(\cdot)$  is unique up to asymptotic equivalence and a commonly used choice for  $a(\cdot)$  is the mean excess function,

$$a(x) = E[Z - x | Z > x], \quad x < \hat{x},$$

where  $Z$  is a random variable distributed by  $H$ . Moreover, a distribution function  $H$  belongs to  $\text{MDA}(\Psi_\gamma)$  for  $\gamma > 0$  if and only if its upper endpoint  $\hat{x}$  is finite and  $\bar{H}$  is regularly varying at  $\hat{x}$  with index  $\gamma$ , namely,

$$\lim_{x \downarrow 0} \frac{\bar{H}(\hat{x} - xy)}{\bar{H}(\hat{x} - x)} = y^\gamma, \quad y > 0.$$

## 2.2. Multivariate Regular Variation (MRV)

In this subsection, we highlight some preliminaries of MRV, an important concept in multivariate extreme value theory, to be used in deriving our main results. Since its introduction by [de Haan and Resnick \(1981\)](#), this concept has been extensively applied to many topics in insurance, finance, and risk management. To deal with multivariate extreme risks, one needs to model both the possibly enormous sizes of and the dependence between the risks. In this regard, MRV provides an ideal modeling framework, which models both marginal tails and asymptotic dependence in a unified manner and provides an explicit approximation to the tail of the joint distribution. For thorough theoretical discussions on MRV, we refer the reader to [Resnick \(1987\)](#), [Resnick \(2007\)](#), and [Rüschendorf \(2013\)](#).

Consider a random vector  $\mathbf{Z} = (Z_1, \dots, Z_d)$  consisting of  $d$  nonnegative risk variables. Assume that the marginal distributions  $H_i, i = 1, \dots, d$ , are tail equivalent in the sense that the relations

$$\lim_{x \rightarrow \infty} \frac{\overline{H}_i(x)}{\overline{H}(x)} = c_i, \quad i = 1, \dots, d, \quad (4)$$

hold for some distribution function  $H$  on  $[0, \infty)$  and some positive numbers  $c_i$ . The vector  $\mathbf{Z}$  is said to have a multivariate regularly varying tail if there exist a positive normalizing function  $b(\cdot)$  monotonically increasing to  $\infty$  and a limit Radon measure  $\nu$  not identically 0 such that, as  $x \rightarrow \infty$ ,

$$xP\left(\frac{\mathbf{Z}}{b(x)} \in \cdot\right) \xrightarrow{v} \nu(\cdot) \quad \text{on } [0, \infty] \setminus \{\mathbf{0}\}. \quad (5)$$

In this relation, the notation  $\xrightarrow{v}$  stands for vague convergence; in other words, for every Borel set  $A$  in  $[0, \infty]$  away from  $\mathbf{0}$  with boundary  $\partial A$  of  $\nu$  measure zero,  $\nu(\partial A) = 0$ , we have

$$\lim_{x \rightarrow \infty} xP\left(\frac{\mathbf{Z}}{b(x)} \in A\right) = \nu(A).$$

The definition of MRV implies that the limit measure  $\nu$  is homogeneous in the sense that there exists some  $0 < \gamma < \infty$  such that the relation

$$\nu(A_\lambda) = \lambda^{-1} \nu(A), \quad (6)$$

with  $A_\lambda = \lambda^{1/\gamma} A$ , holds for any  $\lambda > 0$  and any Borel set  $A \subset [0, \infty] \setminus \{\mathbf{0}\}$ ; see [Resnick 2007](#), p. 178).

The normalizing function  $b(\cdot)$  is not unique, but different choices may result in limit measures that differ by a constant factor. Commonly,  $b(x)$  is chosen to be  $H^\leftarrow(1 - 1/x)$ , based on which relation (5) can be rewritten as

$$xP\left(\frac{\mathbf{Z}}{H^\leftarrow(1 - 1/x)} \in \cdot\right) \xrightarrow{v} \nu(\cdot) \quad \text{on } [0, \infty] \setminus \{\mathbf{0}\}. \quad (7)$$

Here and hereafter, for a non-decreasing function  $f$  on  $\mathbb{R}$ , its càglàd inverse is defined by

$$f^\leftarrow(x) = \inf\{y \in \mathbb{R} : f(y) \geq x\},$$

where we follow the usual convention  $\inf \emptyset = \infty$ . Furthermore, as discussed by [Tang and Yuan \(2013\)](#), relation (7) is equivalent to

$$\frac{1}{\overline{H}(x)} P\left(\frac{\mathbf{Z}}{x} \in \cdot\right) \xrightarrow{v} \nu(\cdot) \quad \text{on } [0, \infty] \setminus \{\mathbf{0}\}. \quad (8)$$

We shall follow the style of relation (8) in defining a standard MRV structure, and we denote it by  $\mathbf{Z} \in \text{MRV}_{-\gamma}(\nu, \overline{H})$  depending on the context.

Some comments on the limit measure  $\nu$  follow. First, the information of asymptotic dependence in the upper-right tail of  $\mathbf{Z}$  is contained in the limit measure  $\nu$ . Plugging the set  $A = (\mathbf{1}, \infty]$  into relation (8) yields

$$\lim_{x \rightarrow \infty} \frac{P(Z_1 > x, \dots, Z_d > x)}{\bar{H}(x)} = \nu(\mathbf{1}, \infty].$$

To capture the common impact on insurance and financial risks of a certain external macroeconomic environment, it is often assumed that  $\nu(\mathbf{1}, \infty] > 0$ , which means that the components of  $\mathbf{Z}$  exhibit large joint movements or, in other words, are asymptotically dependent. Second, it is easy to see that the constants  $c_i$  in (4) can be expressed as

$$c_i = \nu(A_i), \quad i = 1, \dots, d,$$

where  $A_i = [0, \infty] \setminus [0, 1_i]$  and  $\mathbf{1}_i$  denotes the vector with the  $i$ th element being 1 and the other elements being  $\infty$ .

Next we introduce the concept of nonstandard MRV. A nonnegative random vector  $\mathbf{Z} = (Z_1, \dots, Z_d)$  is said to possess a nonstandard MRV structure if there exist normalizing functions  $b_i(\cdot)$  monotonically increasing to  $\infty$ ,  $i = 1, \dots, d$ , and a limit Radon measure  $\nu$  not identically 0 such that, as  $x \rightarrow \infty$ ,

$$xP\left(\left(\frac{Z_1}{b_1(x)}, \dots, \frac{Z_d}{b_d(x)}\right) \in \cdot\right) \xrightarrow{v} \nu(\cdot) \quad \text{on } [0, \infty] \setminus \{\mathbf{0}\}. \quad (9)$$

The nonstandard MRV given by relation (9), in comparison to the standard MRV given by (5), allows different normalizing functions for different components, and hence enables to model the case with multiple risks having different tails. By the way, the normalizing functions  $b_i(\cdot)$ ,  $i = 1, \dots, d$ , are necessarily regularly varying at  $\infty$  but with different indices. See Section 6.5.6 of Resnick (2007) for more discussions on the concept of nonstandard MRV.

The following lemma is excerpted from Tang and Xiao (2017), which establishes the homogeneity of the limit measure  $\nu$  of the nonstandard MRV. For two vectors  $\mathbf{x}$  and  $\mathbf{y}$  in  $\mathbb{R}^d$ , the operation  $\mathbf{x} \circ \mathbf{y}$  represents their Hadamard product with elements given by  $(\mathbf{x} \circ \mathbf{y})_i = x_i y_i$ ,  $i = 1, \dots, d$ .

**Lemma 1.** Suppose that the nonnegative random vector  $\mathbf{Z} = (Z_1, \dots, Z_d)$  possesses a nonstandard MRV structure with a limit measure  $\nu$  and normalizing functions  $b_i(\cdot) \in \text{RV}_{1/\gamma_i}$  for some  $\gamma_i > 0$ ,  $i = 1, \dots, d$ . The relation

$$\nu(A_\lambda) = \lambda^{-1} \nu(A),$$

with  $A_\lambda = \left\{ \left( \lambda^{1/\gamma_1}, \dots, \lambda^{1/\gamma_d} \right) \circ \mathbf{x} : \mathbf{x} \in A \right\}$ , holds for any  $\lambda > 0$  and any Borel set  $A \subset [0, \infty] \setminus \{\mathbf{0}\}$ .

Clearly, if all  $\gamma_i$ ,  $i = 1, \dots, d$ , are identical to some  $\gamma > 0$ , then the homogeneity described by Lemma 1 reduces to that of a standard MRV structure as described by (6) above.

### 3. Independent Cases

In the rest of this paper, unless otherwise stated, all limit relations are according to  $u \rightarrow \infty$  or  $x \rightarrow \infty$  depending on the context. For two positive functions  $f(\cdot)$  and  $g(\cdot)$ , we write  $f(\cdot) \sim g(\cdot)$  if  $\lim f(\cdot)/g(\cdot) = 1$ . Following the notation in Section 1, denote by  $X$  the aggregate amount of claims, by  $Y_i$  the discount factor of the  $i$ th individual asset,  $i = 1, \dots, d$ , and by  $Y_{\mathbf{w}}$ ,  $\mathbf{w} \in \Sigma^d$ , the overall discount factor of the investment portfolio. Furthermore, denote by  $F$ ,  $G_i$ , and  $G_{\mathbf{w}}$  their distribution functions, respectively. We simply call  $X$  the insurance risk and call  $Y_i$ ,  $i = 1, \dots, d$ , and  $Y_{\mathbf{w}}$  the financial risks.

In this section we consider the case with independent insurance and financial risks. This independence assumption can be justified by the fact that the occurrence of perils, such as natural catastrophes and car accidents, is in general uncorrelated with events in the broad economy, such as stock market and interest rate movements. Actually, such an assumption has been widely

made in the literature; see, e.g., [Tang and Tsitsiashvili \(2003\)](#), [Nowak and Romaniuk \(2013\)](#), and [Asanga et al. \(2014\)](#).

### 3.1. The Fréchet Case

In the following theorem we assume  $F \in \text{MDA}(\Phi_\alpha)$  for some  $\alpha > 0$  or  $\mathbf{Y} \in \text{MRV}_{-\beta}$  for some  $\beta > 0$ , indicating a univariate regularly varying tail of the insurance risk or a multivariate regularly varying tail of the financial risks, respectively. For  $\mathbf{Y} \in \text{MRV}_{-\beta}$ , we consider the asymptotically dependent case, which as stated before reflects the common impact on financial risks of the external macroeconomic environment.

**Theorem 1.** Assume that  $X$  and  $\mathbf{Y}$  are independent of each other, and let  $\mathbf{w} \in \Sigma^d$  be arbitrarily fixed.

(a) If  $F \in \text{MDA}(\Phi_\alpha)$  and  $E[Y_i^\beta] < \infty$  for some  $0 < \alpha < \beta$  and all  $i = 1, \dots, d$ , then

$$\lim_{u \rightarrow \infty} \frac{RP}{\overline{F}(u)} = E[Y_{\mathbf{w}}^\alpha]. \quad (10)$$

(b) If  $\mathbf{Y} \in \text{MRV}_{-\beta}(\nu_Y, \overline{G})$  with  $\nu_Y$  satisfying  $\nu_Y(\mathbf{1}, \infty] > 0$  and  $E[X^\alpha] < \infty$  for some  $0 < \beta < \alpha$ , then  $G_{\mathbf{w}} \in \text{MDA}(\Phi_\beta)$  and

$$\lim_{u \rightarrow \infty} \frac{RP}{\overline{G}(u)} = \nu_Y(A_{\mathbf{w}}) E[X^\beta],$$

$$\text{where } A_{\mathbf{w}} = \left\{ \mathbf{y} \in [0, \infty) : \left( \sum_{i=1}^d w_i y_i^{-1} \right)^{-1} > 1 \right\}.$$

**Proof.** (a) Recall relation (2) and notice the convexity of the function  $f(x) = x^{-\beta}$  for  $x \in (0, \infty)$ . We have

$$E[Y_{\mathbf{w}}^\beta] = E \left[ \left( \sum_{i=1}^d w_i Y_i^{-1} \right)^{-\beta} \right] \leq E \left[ \sum_{i=1}^d w_i Y_i^\beta \right] = \sum_{i=1}^d w_i E[Y_i^\beta] < \infty.$$

Then relation (10) follows straightforwardly by applying the well-known Breiman's theorem to relation (3). See [Breiman \(1965\)](#) for the original version of this theorem and see [Cline and Samorodnitsky \(1994\)](#) and Proposition 7.5 of [Resnick \(2007\)](#) for restatements.

(b) Starting from relation (2), one sees that

$$\overline{G_{\mathbf{w}}}(u) = P(Y_{\mathbf{w}} > u) = P\left(\frac{\mathbf{Y}}{u} \in A_{\mathbf{w}}\right).$$

For every  $\mathbf{w} \in \Sigma^d$ , by Lemma A.1 of [Shi et al. \(2017\)](#), the boundary  $\partial A_{\mathbf{w}}$  has  $\nu_Y$  measure zero. Thus, it follows from the assumption  $\mathbf{Y} \in \text{MRV}_{-\beta}(\nu_Y, \overline{G})$  that

$$\overline{G_{\mathbf{w}}}(u) \sim \overline{G}(u) \nu_Y(A_{\mathbf{w}}),$$

where  $\nu_Y(A_{\mathbf{w}}) > 0$  is guaranteed by the condition  $\nu_Y(\mathbf{1}, \infty] > 0$ . Since  $\overline{G} \in \text{RV}_{-\beta}$  necessarily holds by  $\mathbf{Y} \in \text{MRV}_{-\beta}(\nu_Y, \overline{G})$ , so does  $\overline{G_{\mathbf{w}}}$ . Thus,  $G_{\mathbf{w}} \in \text{MDA}(\Phi_\beta)$ . Finally, applying Breiman's theorem to relation (3) again, we obtain

$$RP \sim \overline{G_{\mathbf{w}}}(u) E[X^\beta] \sim \overline{G}(u) \nu_Y(A_{\mathbf{w}}) E[X^\beta].$$

This completes the proof.  $\square$

We give a remark on the application of Theorem 1(a) in the mean-RP optimization problem, where the insurer attempts to make an investment choice—seeking the lowest RP for a given expected return or seeking the highest expected return for a given RP. Given an i.i.d. sample  $\{(X_i, Y_i), i = 1, \dots, n\}$  of the random vector  $(X, Y)$ , relation (10) proposes an asymptotic estimate of the RP, that is,

$$\widehat{RP} = \frac{1}{n} \bar{F}(u) \sum_{i=1}^n (\mathbf{w}^T \mathbf{Y}_i^{-1})^{-\alpha},$$

which is infinitely differentiable and convex as a function of the investment strategy  $\mathbf{w}$ . Hence, after replacing the RP by its asymptotic estimate  $\widehat{RP}$  in the mean-RP optimization problem, standard techniques of convex optimization can be applied straightforwardly to find the optimal investment strategy.

### 3.2. The Gumbel Case

In this subsection, we assume that  $F \in \text{MDA}(\Lambda)$ . As described by Embrechts et al. (1997),  $\text{MDA}(\Lambda)$  is a really large class of distributions with very different tail behaviors, ranging from moderately heavy (such as lognormal) to light (such as normal), or even with a bounded support. Hence, it serves as an ideal distribution class for modeling purposes in insurance and finance.

Moreover, we allow the investment portfolio to be more heterogeneous. Under modern insurance regulatory frameworks, insurers usually invest a large proportion of their wealth into short-term low-risk assets such as money market funds and sovereign bonds. Graph 14 of EIOPA 2011 report<sup>2</sup> on the fifth quantitative impact study for Solvency II displays a typical decomposition of the investment portfolio of an insurer in Europe. In view of this, we assume that the investment portfolio consists of both risk-free assets with deterministic nonnegative returns and risky assets with stochastic returns such that the corresponding discount factors possess an MRV structure.

Precisely, denote by  $I$  the index set of those risk-free assets and by  $J = \{1, \dots, d\} \setminus I$  the index set of those risky assets. To avoid triviality, assume that both  $I$  and  $J$  are non-empty and that at least one of  $w_i, i \in I$ , is nonzero. Each risk-free asset  $i \in I$  yields a deterministic annual return rate  $r_i \geq 0$ , while each risky asset  $j \in J$  yields a stochastic annual return rate  $R_j > -1$ . It follows that

$$Y_{\mathbf{w}} = \frac{1}{1 + \mathbf{w}^T \mathbf{R}} = \frac{1}{\sum_{i \in I} w_i(1 + r_i) + \sum_{j \in J} w_j(1 + R_j)}, \quad \mathbf{w} \in \Sigma^d.$$

Then  $Y_{\mathbf{w}}$  is bounded from above by  $\hat{y}_{\mathbf{w}} = (\sum_{i \in I} w_i(1 + r_i))^{-1}$ . Further assume that the vector of corresponding financial risks  $\mathbf{Y}_J = (Y_j, j \in J) = ((1 + R_j)^{-1}, j \in J)$  possesses  $\text{MRV}_{-\beta}(v_Y^*, \bar{G})$  with  $v_Y^*$  satisfying  $v_Y^*(1, \infty] > 0$ .

**Theorem 2.** Assume that  $X$  and  $\mathbf{Y}$  are independent of each other, that  $X$  is distributed by  $F \in \text{MDA}(\Lambda)$  with an auxiliary function  $a(\cdot)$ , and that the above-mentioned conditions on the investment portfolio are in force. Then it holds for every  $\mathbf{w} \in \Sigma^d$  that

$$\lim_{u \rightarrow \infty} \frac{RP}{\bar{F}\left(\frac{u}{\hat{y}_{\mathbf{w}}}\right) \bar{G}\left(\frac{u/\hat{y}_{\mathbf{w}}^2}{a(u/\hat{y}_{\mathbf{w}})}\right)} = \frac{\Gamma(\beta + 1)v_Y^*(A_{\mathbf{w}}^*)}{\hat{y}_{\mathbf{w}}^{2\beta}},$$

where  $A_{\mathbf{w}}^* = \left\{ \mathbf{y}_J \in [0, \infty)^{|J|} : \left( \sum_{j \in J} w_j y_j^{-1} \right)^{-1} > 1 \right\}$ .

<sup>2</sup> Available online: [https://eiopa.europa.eu/Publications/Reports/QIS5\\_Report\\_Final.pdf](https://eiopa.europa.eu/Publications/Reports/QIS5_Report_Final.pdf).

**Proof.** First, we show  $G_{\mathbf{w}} \in \text{MDA}(\Psi_{\beta})$  with an upper endpoint  $\hat{y}_{\mathbf{w}}$ . For this purpose we derive

$$\begin{aligned} P(Y_{\mathbf{w}} > \hat{y}_{\mathbf{w}} - u^{-1}) &= P\left(\frac{1}{\hat{y}_{\mathbf{w}}^{-1} + \sum_{j \in J} w_j Y_j^{-1}} > \hat{y}_{\mathbf{w}} - u^{-1}\right) \\ &= P\left(\left(\sum_{j \in J} w_j Y_j^{-1}\right)^{-1} > \left((\hat{y}_{\mathbf{w}} - u^{-1})^{-1} - \hat{y}_{\mathbf{w}}^{-1}\right)^{-1}\right). \end{aligned}$$

The proof of Theorem 1(b) shows that  $\mathbf{Y}_J \in \text{MRV}_{-\beta}(\nu_Y^*, \bar{G})$  implies  $\left(\sum_{j \in J} w_j Y_j^{-1}\right)^{-1} \in \text{MDA}(\Phi_{\beta})$ . In addition,

$$\left((\hat{y}_{\mathbf{w}} - u^{-1})^{-1} - \hat{y}_{\mathbf{w}}^{-1}\right)^{-1} \sim \hat{y}_{\mathbf{w}}^2 u.$$

It follows that

$$\begin{aligned} P(Y_{\mathbf{w}} > \hat{y}_{\mathbf{w}} - u^{-1}) &\sim \hat{y}_{\mathbf{w}}^{-2\beta} P\left(\left(\sum_{j \in J} w_j Y_j^{-1}\right)^{-1} > u\right) \\ &= \hat{y}_{\mathbf{w}}^{-2\beta} P(\mathbf{Y}_J \in u A_{\mathbf{w}}^*) \\ &\sim \hat{y}_{\mathbf{w}}^{-2\beta} \nu_Y^*(A_{\mathbf{w}}^*) \bar{G}(u). \end{aligned} \quad (11)$$

In the last step of (11), the verification of  $\nu_Y^*(\partial A_{\mathbf{w}}^*) = 0$  can be done by using Lemma A.1 of Shi et al. (2017), and  $\nu_Y^*(A_{\mathbf{w}}^*) > 0$  is guaranteed by the condition  $\nu_Y^*(1, \infty] > 0$ . This shows that  $G_{\mathbf{w}} \in \text{MDA}(\Psi_{\beta})$ , as desired.

Next, we apply Theorem 3.1(a) of Hashorva et al. (2010) to obtain

$$RP = P\left(X \frac{Y_{\mathbf{w}}}{\hat{y}_{\mathbf{w}}} > \frac{u}{\hat{y}_{\mathbf{w}}}\right) \sim \Gamma(\beta + 1) P\left(\frac{Y_{\mathbf{w}}}{\hat{y}_{\mathbf{w}}} > 1 - \frac{a(u/\hat{y}_{\mathbf{w}})}{u/\hat{y}_{\mathbf{w}}}\right) \bar{F}\left(\frac{u}{\hat{y}_{\mathbf{w}}}\right).$$

Note that, by relation (11),

$$P\left(\frac{Y_{\mathbf{w}}}{\hat{y}_{\mathbf{w}}} > 1 - \frac{a(u/\hat{y}_{\mathbf{w}})}{u/\hat{y}_{\mathbf{w}}}\right) = P\left(Y_{\mathbf{w}} > \hat{y}_{\mathbf{w}} - \frac{a(u/\hat{y}_{\mathbf{w}})}{u/\hat{y}_{\mathbf{w}}^2}\right) \sim \hat{y}_{\mathbf{w}}^{-2\beta} \nu_Y^*(A_{\mathbf{w}}^*) \bar{G}\left(\frac{u/\hat{y}_{\mathbf{w}}^2}{a(u/\hat{y}_{\mathbf{w}})}\right).$$

Plugging this into the above ends the proof.  $\square$

#### 4. Dependent Cases

Recently, discussions about the convergence of the insurance and financial markets have emerged in the insurance literature; see Cummins and Weiss (2009) and references therein. For example, to hedge against catastrophe risk, insurers and reinsurers now securitize their insurance risk and transferring it to the capital market using insurance-linked securities such as catastrophe bonds and industry loss warranties. This yields interconnection between the insurance and financial markets, and hence expedites the convergence of the two markets. Motivated by this, in this section we assume that the insurance risk variable  $X$  and the financial risk vector  $\mathbf{Y}$  jointly possess a standard or nonstandard MRV structure so as to allow for asymptotic dependence between them.

##### 4.1. A Standard MRV Case

**Theorem 3.** Assume that the  $1 + d$  dimensional risk vector  $(X, \mathbf{Y})$  possesses  $\text{MRV}_{-\alpha}(\nu, \bar{F})$ , and let  $\mathbf{w} \in \Sigma^d$  be arbitrarily fixed. Then

$$\lim_{u \rightarrow \infty} \frac{RP}{\bar{F}(\sqrt{u})} = \nu(A_{\mathbf{w}}), \quad (12)$$

where  $A_w = \left\{ (x, \mathbf{y}) \in (0, \infty) \times (0, \infty)^d : x \left( \sum_{i=1}^d w_i y_i^{-1} \right)^{-1} > 1 \right\}$ .

**Proof.** Starting from relation (3), we derive

$$RP = P \left( X \left( \sum_{i=1}^d w_i Y_i^{-1} \right)^{-1} > u \right) = P \left( \frac{(X, \mathbf{Y})}{\sqrt{u}} \in A_w \right).$$

As before, the verification of  $\nu(\partial A_w) = 0$  can be done by using Lemma A.1 of Shi et al. (2017). Thus, by relation (8), we have

$$\lim_{u \rightarrow \infty} \frac{RP}{\bar{F}(\sqrt{u})} = \lim_{u \rightarrow \infty} \frac{1}{\bar{F}(\sqrt{u})} P \left( \frac{(X, \mathbf{Y})}{\sqrt{u}} \in A_w \right) = \nu(A_w).$$

This completes the proof.  $\square$

It is possible that  $\nu(A) = 0$ , for which case relation (12), while still valid, can no longer capture the asymptotic behavior of the RP. Nevertheless, this happens only if the components of  $(X, \mathbf{Y})$  are asymptotically independent, that is,

$$\lim_{x \rightarrow \infty} \frac{P(X > x, Y_1 > x, \dots, Y_d > x)}{\bar{F}(x)} = \nu(\mathbf{1}, \infty] = 0.$$

In other words, for the asymptotically dependent case (that is,  $\nu(\mathbf{1}, \infty] > 0$ ), which is of particular interest for our purpose, relation (12) gives an asymptotic formula for the RP.

#### 4.2. A Nonstandard MRV Case

The standard MRV assumption in the preceding subsection implies equivalent tails of the insurance risk  $X$  and the financial risks  $Y_i$ ,  $i = 1, \dots, d$ , which is not necessarily true in practice. In this subsection, we extend the study to a nonstandard MRV structure, which allows  $X$  and  $\mathbf{Y}$  to have different tails.

**Theorem 4.** Assume that the  $1 + d$  dimensional risk vector  $(X, \mathbf{Y})$  possesses the following nonstandard MRV structure: for some limit measure  $\nu$  on  $[0, \infty]^{1+d} \setminus \{\mathbf{0}\}$  and some distribution functions  $F$  and  $G$  on  $[0, \infty)$  with unbounded supports, as  $x \rightarrow \infty$ ,

$$xP \left( \left( \frac{X}{F^{\leftarrow}(1 - 1/x)}, \frac{\mathbf{Y}}{G^{\leftarrow}(1 - 1/x)} \right) \in \cdot \right) \xrightarrow{v} \nu(\cdot).$$

Then it holds for every  $\mathbf{w} \in \Sigma^d$  that

$$\lim_{u \rightarrow \infty} u_* RP = \nu(A_w),$$

where  $A_w$  is identical to the one in Theorem 3 and  $u_*$  solves

$$F^{\leftarrow} \left( 1 - \frac{1}{u_*} \right) G^{\leftarrow} \left( 1 - \frac{1}{u_*} \right) \sim u. \quad (13)$$

**Proof.** Still starting from (3) we derive, for arbitrarily fixed  $0 < \varepsilon < 1$  and all large  $u$ ,

$$\begin{aligned} RP &= P\left(\frac{X}{F^{\leftarrow}(1-1/u_*)} \frac{Y_{\mathbf{w}}}{G^{\leftarrow}(1-1/u_*)} > \frac{u}{F^{\leftarrow}(1-1/u_*)G^{\leftarrow}(1-1/u_*)}\right) \\ &\leq P\left(\frac{X}{F^{\leftarrow}(1-1/u_*)} \frac{Y_{\mathbf{w}}}{G^{\leftarrow}(1-1/u_*)} > 1 - \varepsilon\right) \\ &= P\left(\left(\frac{X}{F^{\leftarrow}(1-1/u_*)}, \frac{\mathbf{Y}}{G^{\leftarrow}(1-1/u_*)}\right) \in \sqrt{1-\varepsilon}A_{\mathbf{w}}\right). \end{aligned}$$

The same as in the proof of Theorem 3, the set  $\sqrt{1-\varepsilon}A_{\mathbf{w}}$  has a boundary of  $\nu$  measure 0. Thus,

$$\limsup_{u \rightarrow \infty} u_* RP = \limsup_{u_* \rightarrow \infty} u_* RP \leq \nu\left(\sqrt{1-\varepsilon}A_{\mathbf{w}}\right).$$

By Lemma 1 and the arbitrariness of  $\varepsilon$ , we have

$$\limsup_{u \rightarrow \infty} u_* RP \leq \nu(A_{\mathbf{w}}).$$

The other inequality can be established similarly and this completes the proof.  $\square$

Two remarks on Theorem 4 follow. First, the assumed nonstandard MRV structure implies that the components of  $\mathbf{Y}$  have marginal tails equivalent to  $\bar{G}$ . Second, both normalizing functions  $F^{\leftarrow}(1-1/x)$  and  $G^{\leftarrow}(1-1/x)$  are necessarily regularly varying, though we do not need to specify their indices here. Thus, the solution  $u_*$  to the asymptotic Equation (13) exists and is unique in the asymptotic sense.

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