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Optimal Reinsurance Policies under the VaR Risk Measure When the Interests of Both the Cedent and the Reinsurer Are Taken into Account

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Abstract: Optimal forms of reinsurance policies have been studied for a long time in the actuarial literature. Most existing results are from the insurer's point of view, aiming at maximizing the expected utility or minimizing the risk of the insurer. However, as pointed out by Borch (1969), it is understandable that a reinsurance arrangement that might be very attractive to one party (e.g., the insurer) can be quite unacceptable to the other party (e.g., the reinsurer). In this paper, we follow this point of view and study forms of Pareto-optimal reinsurance policies whereby one party's risk, measured by its value-at-risk (VaR), cannot be reduced without increasing the VaR of the counter-party in the reinsurance transaction. We show that the Pareto-optimal policies can be determined by minimizing linear combinations of the VaRs of the two parties in the reinsurance transaction. Consequently, we succeed in deriving user-friendly, closed-form, optimal reinsurance policies and their parameter values.

Keywords: optimal reinsurance treaties; value at risk; Pareto optimality

1. Introduction

Reinsurance is a transaction whereby one insurance company (the reinsurer) agrees to indemnify another insurance company (the reinsured, cedent or primary company) against all or part of the loss that the latter sustains under a policy or policies that it has issued. For this service, the ceding company pays the reinsurer a premium, and there are many premium calculation principles (e.g., [1,2]; see also [3,4]).

Mathematically, let X be the loss for an insurer from a policy or a group of policies. Assume that under a reinsurance treaty, a reinsurer covers the ceded part of the loss, say $f(X)$, where $0 \leq f(X) \leq X$, for a premium P_f . The primary insurer's retained loss is denoted by $I_f(X) = X - f(X)$. Commonly-used forms of reinsurance treaties are the excess-of-loss treaty, where $f(X) = (X - d)_+$ with deductible level (attaching point) $d > 0$; and the quota-share treaty, where $f(X) = aX$ with a constant (share) $0 \leq a \leq 1$.

Optimal forms of reinsurance have been studied extensively in the literature. Most of the results obtained are from the cedent's point of view. That is, the question asked is: for a given premium principle, what is the optimal functional form and/or parameter values of the ceded function f , such that the cedent's expected utility is maximized or its risk minimized? For example, by maximizing the cedent's expected utility, Arrow [5] concluded that "given a range of alternative possible reinsurance contracts, the reinsured would prefer a policy offering complete coverage beyond a deductible." Borch [6] showed that for a fixed premium and expected reinsurance payments, the variance of

the cedent's losses is minimized by the excess-of-loss reinsurance policy. In recent years, various solutions to the optimal reinsurance problem have been obtained where the value-at-risk (VaR) and the tail-value-at-risk (TVaR) have been used to measure the cedent's risk level (e.g., [7–13] and the references therein).

Borch [14] argues that “there are two parties to a reinsurance contract, and that an arrangement which is very attractive to one party may be quite unacceptable to the other.” However, as pointed out by [15], optimal forms of ceded functions considering both the cedent and the reinsurer had scarcely been discussed until quite recently. For example, Ignatov et al. [16] study the optimal reinsurance contracts under which the finite horizon joint survival probability of the two parties is maximized. Kaishev and Dimitrova [17] derive explicit expressions for the probability of joint survival up to a finite time of the cedent and the reinsurer, under an excess of loss reinsurance contract with a limiting and a retention level. Golubin [18] studies the problem of designing the Pareto-optimal reinsurance policy by maximizing a weighted average of the expected utility of the insurer and the reinsurer. Dimitrova and Kaishev [19] introduce an efficient frontier type approach to setting the limiting and the retention levels, based on the probability of joint survival. Cai et al. [20] analyse the optimal reinsurance policies that maximize the joint survival probability and the joint profitable probability of the two parties and derive sufficient conditions for optimal reinsurance contracts within a wide class of reinsurance policies and under a general reinsurance premium principle. Using the results of [20], Fang and Qu [21] derive optimal retentions of combined quota-share and excess-of-loss reinsurance that maximize the joint survival probability of the two parties. Cai et al. [22] study the optimal forms of reinsurance policies that minimize the convex combination of the VaRs of the cedent and the reinsurer under two types of constraints that describe the interests of the two parties. For the determination of the optimal excess of loss contract considering the dependency between the losses of the insurer and the reinsurer, we refer to [23] and the references therein.

A closely-related problem to optimal reinsurance is the so-called optimal transfer of risks among partners, where everybody's interests are considered simultaneously. The usual approach is to identify Pareto-optimal treaties, whereby no agent can be made better off without making another agent worse off. For results in this area, we refer to, e.g., [6,7,24,25] and the references therein.

In this paper, we determine Pareto-optimal reinsurance policies under which one party's risk, measured by its VaR, cannot be reduced without increasing that of the other party in the reinsurance contract. We consider two classes of ceded functions:

$$\mathcal{C}^1 := \{f : f \text{ is convex, non-decreasing and } 0 \leq f(x) \leq x \text{ for all } x\}$$

and:

$$\mathcal{C}^2 := \{f : f \text{ and } I_f \text{ are non-decreasing and } 0 \leq f(x) \leq x \text{ for all } x\}.$$

Note the inclusion $\mathcal{C}^1 \subset \mathcal{C}^2$, which has been verified by [13]. Furthermore, for every $f \in \mathcal{C}^2$, both f and I_f are Lipschitz continuous, and they are comonotonic.

The requirements that the ceded function f is non-decreasing and that the bounds $0 \leq f(x) \leq x$ hold for all x are needed in \mathcal{C}^1 and \mathcal{C}^2 to avoid the moral hazard problem in reinsurance. The additional requirement of the convexity of f in \mathcal{C}^1 essentially requires that $f(x)$ approaches infinity linearly when $x \rightarrow \infty$ and thus disallows the popular layered reinsurance policies. Nevertheless, this class includes the important quota-share and the excess-of-loss reinsurance policies. Note also that both classes are of interest in the more general context of economic theory with two agents having conflicting interests. Optimal reinsurance problems with admissible classes \mathcal{C}^1 and \mathcal{C}^2 have been studied extensively in the literature, and we refer to [13] for an informative review.

For simplicity of discussion, we assume that the reinsurance premiums are determined by the expected premium principle:

$$P_f = (1 + \theta)\mathbf{E}[f(X)] \tag{1}$$

where $\theta > 0$ is the safety loading. Hence, the cedent's total loss becomes:

$$C_f = X - f(X) + (1 + \theta)\mathbf{E}[f(X)],$$

and the reinsurer's total loss under the reinsurance contract is:

$$R_f = f(X) - (1 + \theta)\mathbf{E}[f(X)].$$

In this paper, we use VaR to measure the insurer's and reinsurer's risk level. A natural starting point for measuring the (joint) risk of the cedent and the reinsurer is a bivariate risk measure, such as the bivariate VaR ([26]) of the pair C_f and R_f . However, since the ceded loss $f(X)$ and the retained loss $I_f(X)$ are comonotonic (see [27,28] for a very detailed discussion of the concept of comonotonicity with applications), the set of values of the bivariate VaRs of C_f and R_f is determined by values of the univariate VaR of C_f and R_f . Therefore, the Pareto-optimal reinsurance policies could be determined by minimizing a linear combination of the univariate VaRs of C_f and R_f . We note in this regard that the optimization criterion of minimizing linear combinations of the risks of the cedent and the reinsurer was adopted by [7,22]. Our arguments provide an additional economic meaning to such criteria.

Although VaR is not sub-additive in general, it was shown that it is sub-additive in the deep right tail in many cases of interest (e.g., [29]). General results related to optimal forms of reinsurance (risk exchanges) using the so-called distortion risk measures exist in the literature, and we refer to [7,8,25]. The distortion risk measures are very general and include VaR, TVaR and proportional hazards transforms as special cases. The feature of the current paper is that we extend the geometric approach of [12] to our optimization problem that considers the interests of the two parties. The geometric proofs facilitate intuition and enable us to avoid lengthy and complex mathematical arguments. We derive closed-form and user-friendly formulas for the optimal reinsurance policies and thus provide a convenient route for practical implementation of our results.

The rest of the paper is organized as follows. Section 2 provides preliminaries and shows (cf. [25]) that the form of Pareto-optimal reinsurance policies can be determined by minimizing linear combinations of the cedent's and the reinsurer's risks. In Sections 3 and 4, we determine optimal reinsurance forms and derive the corresponding optimal parameters when the feasible classes of ceded functions are \mathcal{C}^1 and \mathcal{C}^2 , respectively. There, we also provide illustrative numerical examples. Section 5 provides further insights regarding the results of our numerical examples. Section 6 concludes the paper.

2. Preliminaries

Let F_X and S_X denote the cumulative distribution function (c.d.f.) and the survival function of X , respectively. Furthermore, let F_{C_f} and F_{R_f} denote the c.d.f.'s of C_f and R_f , respectively. Then, the individual VaRs of the cedent and the reinsurer under the reinsurance contract are:

$$\text{VaR}_\alpha(C_f) = \inf\{x : F_{C_f}(x) \geq \alpha\}$$

and:

$$\text{VaR}_\alpha(R_f) = \inf\{x : F_{R_f}(x) \geq \alpha\},$$

respectively. To consider the risk of both the cedent and the reinsurer, we propose to use the bivariate lower orthant VaR introduced by [26], which is:

$$\underline{\text{VaR}}_\alpha(C_f, R_f) = \partial\{(y, z) \in \mathbf{R}^2 : F_{C_f, R_f}(y, z) \geq \alpha\}.$$

For any ceded function $f \in \mathcal{C}^2$, the random variables C_f and R_f are comonotonic, and so:

$$\begin{aligned} \underline{\text{VaR}}_{\alpha}(C_f, R_f) &= \partial\{(y, z) \in \mathbf{R}^2 : \min\{F_{C_f}(y), F_{R_f}(z)\} \geq \alpha\} \\ &= \partial\{(y, z) \in \mathbf{R}^2 : F_{C_f}(y) \geq \alpha, F_{R_f}(z) \geq \alpha\}. \end{aligned}$$

Therefore, when the “joint” risk of the cedent and the reinsurer is measured by their bivariate lower orthant VaR, one could work with the marginal VaRs of C_f and R_f , instead of the much more complicated joint VaR.

In the following, we assume that the probability levels in the VaRs used by the cedent and the reinsurer are possibly different, say α_c and α_r , respectively, and then determine the Pareto-optimal reinsurance policies (ceded functions f) in the sense that one party’s risk, measured by its VaR, cannot be reduced without increasing the other party’s VaR. Mathematically, let f^* denote a ceded function in an admissible set \mathcal{C} , such as \mathcal{C}^1 or \mathcal{C}^2 . Let the corresponding cedent’s and reinsurer’s total losses under the ceded function f^* be denoted by C_{f^*} and R_{f^*} , respectively. Then, f^* is a Pareto-optimal reinsurance policy if there is no ceded function $f \neq f^*$ belonging to the admissible set \mathcal{C} , such that:

$$\text{VaR}_{\alpha_c}(C_f) \leq \text{VaR}_{\alpha_c}(C_{f^*})$$

and:

$$\text{VaR}_{\alpha_r}(R_f) \leq \text{VaR}_{\alpha_r}(R_{f^*}),$$

with at least one of the inequalities being strict. To find the Pareto-optimal reinsurance policies, we utilize the following proposition.

Proposition 1. All Pareto-optimal reinsurance policies f in \mathcal{C}^i , $i \in \{1, 2\}$, can be determined by solving the problem:

$$\min_{f \in \mathcal{C}^i} \left\{ \beta \text{VaR}_{\alpha_c}(C_f) + (1 - \beta) \text{VaR}_{\alpha_r}(R_f) \right\}, \tag{2}$$

where $0 \leq \beta \leq 1$.

Proof. Similar to the discussion on page 90 of [30], one method to find Pareto-optimal decisions is to choose two positive constants k_1, k_2 and find:

$$\min_{f \in \mathcal{C}^i} \left\{ k_1 \text{VaR}_{\alpha_c}(C_f) + k_2 \text{VaR}_{\alpha_r}(R_f) \right\}.$$

Without loss of generality, we set $k_1 = \beta$ and $k_2 = 1 - \beta$ with $0 \leq \beta \leq 1$. In more detail, let g be a function belonging to \mathcal{C}^i and minimizing (2), then there cannot exist in \mathcal{C}^i any function $f \neq g$ such that $\text{VaR}_{\alpha_c}(C_f) \leq \text{VaR}_{\alpha_c}(C_g)$ and $\text{VaR}_{\alpha_r}(R_f) \leq \text{VaR}_{\alpha_r}(R_g)$ with at least one of the inequalities being strict, because otherwise, we would have:

$$\beta \text{VaR}_{\alpha_c}(C_f) + (1 - \beta) \text{VaR}_{\alpha_r}(R_f) < \beta \text{VaR}_{\alpha_c}(C_g) + (1 - \beta) \text{VaR}_{\alpha_r}(R_g).$$

This is a contradiction to the assumed property of function g .

Furthermore, for any two ceded functions $f_1, f_2 \in \mathcal{C}^i$, the family $\{f_{\gamma}, 0 \leq \gamma \leq 1\}$ of ceded functions defined by $f_{\gamma}(x) = \gamma f_1(x) + (1 - \gamma) f_2(x)$, is a subset of \mathcal{C}^i and satisfies:

$$\text{VaR}_{\alpha_c}(C_{f_{\gamma}}) = \gamma \text{VaR}_{\alpha_c}(C_{f_1}) + (1 - \gamma) \text{VaR}_{\alpha_c}(C_{f_2}) \tag{3}$$

and:

$$\text{VaR}_{\alpha_r}(R_{f_{\gamma}}) = \gamma \text{VaR}_{\alpha_r}(R_{f_1}) + (1 - \gamma) \text{VaR}_{\alpha_r}(R_{f_2}). \tag{4}$$

Equation (3) is satisfied because:

$$\begin{aligned}\text{VaR}_{\alpha_c}(C_{f_\gamma}) &= \text{VaR}_{\alpha_c}(I_{f_\gamma}(X) + P_{f_\gamma}) \\ &= \text{VaR}_{\alpha_c}(\gamma C_{f_1} + (1 - \gamma)C_{f_2}) \\ &= \gamma \text{VaR}_{\alpha_c}(C_{f_1}) + (1 - \gamma)\text{VaR}_{\alpha_c}(C_{f_2}),\end{aligned}$$

where the last equality is due to the fact that C_{f_1} and C_{f_2} are non-decreasing functions of the same random variable X and therefore comonotonic. Similarly, Equation (4) is satisfied. Therefore, Condition C on page 90 of [30] is satisfied, and we conclude that all Pareto-optimal reinsurance policies in \mathcal{C}^i can be found by solving Problem (2). \square

In view of Proposition 1, throughout the rest of this paper, we seek optimal reinsurance policies by solving the optimization problem:

$$\min_{f \in \mathcal{C}^i} \left\{ \beta \text{VaR}_{\alpha_c}(C_f) + (1 - \beta) \text{VaR}_{\alpha_r}(R_f) \right\}$$

for $i \in \{1, 2\}$, which is equivalent to minimizing:

$$\mathcal{H}(f) = \beta \text{VaR}_{\alpha_c}(-f(X) + P_f) + (1 - \beta) \text{VaR}_{\alpha_r}(f(X) - P_f). \quad (5)$$

As shown by [13], we have $\mathcal{C}^1 \subset \mathcal{C}^2$, and every function $f \in \mathcal{C}^2$ is Lipschitz-continuous and, hence, continuous. Consequently (e.g., [27]), for every $f \in \mathcal{C}^2$, we have $\text{VaR}_{\alpha}(f(X)) = f(\text{VaR}_{\alpha}(X))$, and thus, with $a_c = \text{VaR}_{\alpha_c}(X)$ and $a_r = \text{VaR}_{\alpha_r}(X)$, the optimization problem becomes:

$$\min_{f \in \mathcal{C}^i} \mathcal{H}(f) = \min_{f \in \mathcal{C}^i} \left\{ -\beta \cdot f(a_c) + (1 - \beta) \cdot f(a_r) + (2\beta - 1)(1 + \theta) \mathbf{E}[f(X)] \right\}, \quad i = 1, 2. \quad (6)$$

Since we allow $S_X(0) < 1$, the relationships between the probability levels α_c and α_r , as well as $S_X(0)$ need to be discussed. Namely, we have the following observations:

1. If $1 - \alpha_c \geq S_X(0)$ and $1 - \alpha_r \geq S_X(0)$, then $a_c = a_r = 0$. Thus,
 - when $\beta > 1/2$, the solution to Problem (6) is $f^*(x) = 0$ for all x ;
 - when $\beta < 1/2$, the solution is $f^*(x) = x$;
 - when $\beta = 1/2$, the objective function is always zero.
2. If $1 - \alpha_c < S_X(0)$ and $1 - \alpha_r \geq S_X(0)$, then $a_c > 0$ and $a_r = 0$. Thus,
 - when $\beta < 1/2$, the optimal ceded function is $f^*(x) = x$;
 - when $\beta > 1/2$, the form of the optimal ceded function is similar to the case when $\beta = 1$, with only the risk and the profit of the cedent considered (the solution for the latter case can be found in Case 2 of Sections 3.2 and 4.2 below);
 - when $\beta = 1/2$, the optimal ceded function is $f^*(x) = x$.
3. If $1 - \alpha_c \geq S_X(0)$ and $1 - \alpha_r < S_X(0)$, then $a_c = 0$ and $a_r > 0$. Thus,
 - when $\beta > 1/2$, the solution to Problem (6) is $f^*(x) = 0$ for all x ;
 - when $\beta < 1/2$, the form of the optimal ceded function is similar to the case when $\beta = 0$, with only the risk and the profit of the reinsurer being considered (the solution for the latter case can be found in Case 3 of Sections 3.2 and 4.2 below).
 - when $\beta = 1/2$, the optimal ceded function is $f^*(x) = 0$ for all x .

Throughout the rest of this paper, we only consider the optimal forms of reinsurance policies under the conditions $1 - \alpha_c < S_X(0)$ and $1 - \alpha_r < S_X(0)$.

Now, we are ready to determine the optimal forms of f , the task that makes up the contents of the following two sections. Namely, in Section 3, we consider the case when the admissible set of ceded

functions is \mathcal{C}^1 and in Section 4 when the admissible set is \mathcal{C}^2 . As noted earlier, both classes are of interest in the broad context of economic theory, with the class \mathcal{C}^2 being more relevant to reinsurance policies. Nevertheless, the class \mathcal{C}^1 includes the important quota share and excess-of-loss reinsurance policies that provide natural reference points for analysing the optimal reinsurance policies in \mathcal{C}^2 .

3. Optimal Reinsurance Policies When $f \in \mathcal{C}^1$

In this section, we determine optimal insurance policies under the condition that $f \in \mathcal{C}^1$, which means that f is convex and non-decreasing and the retained loss function $I_f(x) = x - f(x)$ is non-decreasing. These conditions are also assumed by [11,12], where they in fact require that f is Lipschitz-continuous (cf., e.g., Section 2 of [13]) and that $f(x)$ linearly tends to infinity when $x \rightarrow \infty$.

3.1. Functional Form of the Ceded Function

Here, we determine the functional form of the solution to the minimization problem:

$$\min_{f \in \mathcal{C}^1} \mathcal{H}(f) = \min_{f \in \mathcal{C}^1} \left\{ -\beta \cdot f(a_c) + (1 - \beta) \cdot f(a_r) + (2\beta - 1)(1 + \theta) \mathbf{E}[f(X)] \right\}. \tag{7}$$

We subdivide our following analysis into three cases.

3.1.1. Case 1: $\beta > 1/2$

In this case, the coefficients in front of $f(a_r)$ and $\mathbf{E}[f(X)]$ on the right-hand side of Equation (7) are positive, and the coefficient in front of $f(a_c)$ is negative. Thus, for any ceded function f , we determine the functional form of an optimal ceded function f^* , such that $f^*(a_c) = f(a_c)$ and:

$$(1 - \beta)f^*(a_r) + (2\beta - 1)P_{f^*} \leq (1 - \beta)f(a_r) + (2\beta - 1)P_f.$$

This requires $f^*(a_r)$ and also the entire function f^* to be as small as possible.

As we see from Figure 1, the convexity of f implies that the above requirements are satisfied by the ceded function:

$$f^*(x) = c(x - d)_+, \tag{8}$$

where $c \in (f'(a_c-), f'(a_c+))$ and $d \in [0, a_c]$ are any constants. Since the slope of f should not exceed one, we must have $c \in [0, 1]$.

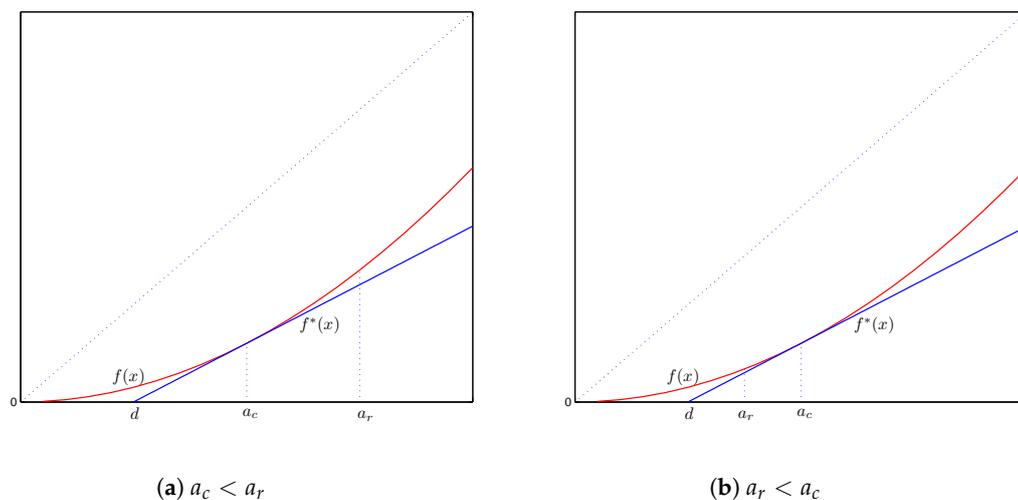


Figure 1. Optimal ceded functions in \mathcal{C}^1 : Case 1.

Remark 1. It is clear from the above proof that the result for the optimal form of reinsurance policy is valid as long as $P_{f_1} \leq P_{f_2}$ whenever $f_1(x) \leq f_2(x)$ for all x . Obviously, this condition is satisfied by the distortion premium calculation principle (e.g., [2]), which has been assumed in, for example [8,25], among others. For a discussion of the validity of this condition in the case of the weighted premium calculation principle, we refer to [3]. In the current paper, we adopt the simplest expectation premium principle (Equation (1)) for the simplicity of presentation.

3.1.2. Case 2: $\beta < 1/2$

In this case, the coefficient in front of $f(a_r)$ on the right-hand side of Equation (7) is positive, and those in front of $f(a_c)$ and $E[f(X)]$ are negative. Therefore, to solve Problem (7), for any ceded function f , we search for a function f^* , such that $f^*(a_r) = f(a_r)$ and:

$$\beta f^*(a_c) + (1 - 2\beta)P_{f^*} \geq \beta f(a_c) + (1 - 2\beta)P_f,$$

which requires $f^*(a_c)$ and also the entire function f^* to be as large as possible.

As we see from Figure 2, the convexity of f implies that the above requirements are satisfied by the ceded function:

$$f^*(x) = \begin{cases} \eta x & \text{when } 0 \leq x < a_r, \\ x - (1 - \eta)a_r & \text{when } x \geq a_r, \end{cases} \tag{9}$$

where $\eta \in [0, 1]$ can be any constant.

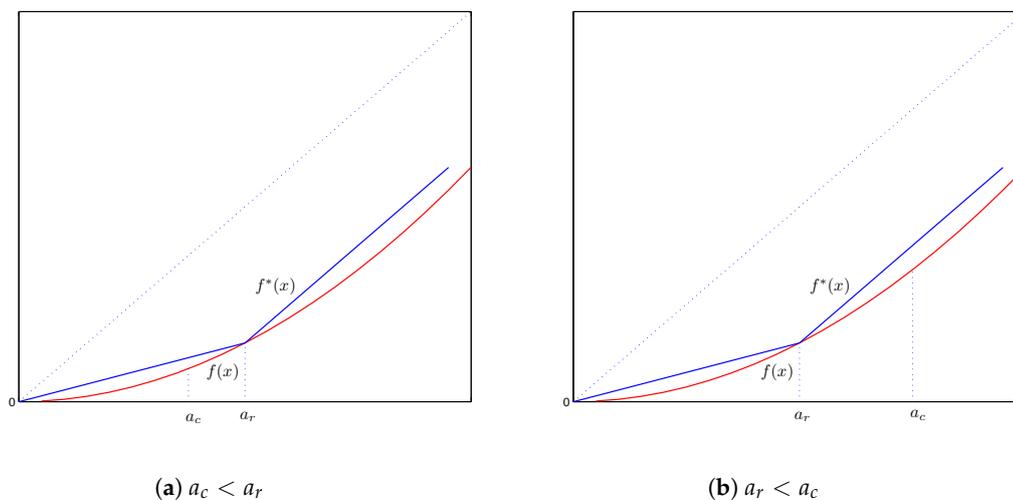


Figure 2. Optimal ceded functions in C^1 : Case 2.

3.1.3. Case 3: $\beta = 1/2$

In this case, Problem (7) simplifies to:

$$\min_{f \in C^1} \{f(a_r) - f(a_c)\}. \tag{10}$$

Since f is non-decreasing, we have that when $a_c < a_r$, then Problem (10) is solved by any ceded function f^* , which is constant on the interval $[a_c, a_r]$. Since f^* has to be convex, this in turn requires f^* to be constant on $[0, a_c]$. Since $f^*(0) = 0$, we conclude that any function f^* in C^1 with $f^*(x) = 0$ on $[0, a_r]$ is Pareto-optimal.

When $a_c > a_r$, then because the slope of the ceded function is no more than one, Problem (10) is solved by f^* , which increases at the rate of one in the interval $[a_r, a_c]$, which in turn requires f^*

to increase at the rate of one for all $x > a_c$ because of the convexity assumption. In summary, any function f^* in C^1 with $f^{*'}(x) = 1$ on $[a_r, \infty)$ is Pareto-optimal.

When $\alpha_c = \alpha_r$, the objective function is constant.

3.2. Parameter Values of the Optimal Ceded Function

When $\beta > 1/2$, then the optimal ceded function f^* is given by Equation (8) for which the parameters c and d need to be determined. When $\beta < 1/2$, then the optimal ceded function is given by Equation (9) for which the parameter η needs to be determined. We accomplish these tasks below by subdividing our considerations into four cases.

3.2.1. Case 1: $\beta > 1/2$ and $\alpha_c < \alpha_r$

In this case, the optimal ceded function is given by Equation (8) with $d < \alpha_c < \alpha_r$, and optimization Problem (7) becomes:

$$\min_{(c,d) \in [0,1] \times [0,a_c]} g_1(c,d),$$

where:

$$g_1(c,d) = c \left(-\beta(a_c - d) + (1 - \beta)(a_r - d) + (2\beta - 1)(1 + \theta) \int_d^\infty S_X(t)dt \right).$$

Following [11,12], we use the notations:

$$\begin{aligned} \theta^* &= \frac{1}{1 + \theta'} \\ d^* &= S_X^{-1}(\theta^*), \\ Q(\beta, a_c, a_r) &= \frac{\beta a_c - (1 - \beta)a_r}{2\beta - 1}, \\ U(x) &= S_X^{-1}(x) + (1 + \theta) \int_{S_X^{-1}(x)}^\infty S_X(t)dt. \end{aligned}$$

Theorem 1. Under the conditions $\beta > 1/2$ and $\alpha_c < \alpha_r$, the optimal ceded function is $f^*(x) = c(x - d)_+$ with the following parameters:

1. $c = 1$ and $d = d^*$ when $\theta^* < S_X(0)$ and $U(\theta^*) < Q(\beta, a_c, a_r)$;
2. $c \in [0, 1]$ is any constant and $d = d^*$ when $\theta^* < S_X(0)$ and $U(\theta^*) = Q(\beta, a_c, a_r)$;
3. $c = 1$ and $d = 0$ when $\theta^* \geq S_X(0)$ and $(1 + \theta)\mathbf{E}[X] < Q(\beta, a_c, a_r)$;
4. $c \in [0, 1]$ is any constant and $d = 0$ when $\theta^* \geq S_X(0)$ and $(1 + \theta)\mathbf{E}[X] = Q(\beta, a_c, a_r)$.

If none of the above conditions are satisfied, then $f^*(x) = 0$ for all x .

Proof. We only prove Part (1) because the proofs of the other parts are similar. To minimize function $g_1(c, d)$ over $(c, d) \in [0, 1] \times [0, a_c]$, we first take the derivative of $g_1(c, d)$ with respect to d and have:

$$\frac{\partial g_1(c,d)}{\partial d} = c(2\beta - 1) \left(1 - (1 + \theta)S_X(d) \right),$$

which is an increasing function in d . Consequently, the function $g_1(c, d)$ is convex in d . Since $\theta^* < S_X(0)$, the derivative $\partial g_1(c, d)/\partial d$ is negative at $d = 0$ and is equal to zero at d^* . It is easy to show that $a_c < a_r$ if and only if $a_c > Q(\beta, a_c, a_r)$. Then, the condition $U(\theta^*) < Q(\beta, a_c, a_r)$ indicates that $d^* < U(\theta^*) < a_c$, and so, the deductible level d^* minimizes the function $g_1(c, d)$ when $c > 0$.

Next, setting $d = d^*$, we have:

$$g_1(c, d^*) = c(2\beta - 1) \left(U(\theta^*) - Q(\beta, a_c, a_r) \right) < 0. \tag{11}$$

Because $U(\theta^*) < Q(\beta, a_c, a_r)$ by assumption, $g_1(c, d^*)$ is minimized at $c = 1$. Overall, assuming $c > 0$, function $g_1(c, d)$ is minimized at $(c, d) = (1, d^*)$. Noting that $g_1(0, d) = 0 > g_1(1, d^*)$, we obtain the desired result. \square

Remark 2. We have the following observations:

- When $\beta = 1$, then only the cedent is considered. In this case, $Q(\beta, a_c, a_r) = a_c$ and $f^*(x) = (x - d^*)_+$ when $U(\theta^*) < a_c$. Therefore, when $U(\theta^*) > a_c$, then $f^*(x) = 0$ for all x , and the primary insurance company will not purchase any reinsurance policy. This result agrees with those derived by [11,12].
- When $\beta \searrow 1/2$, then $Q(\beta, a_c, a_r) \sim (a_c - a_r)/(4\beta - 2) < 0$, and the optimal value of c is zero. Therefore, $f^*(x) = 0$ for all x .
- The value of d^* in the excess-of-loss reinsurance policy does not depend on the choice of β whenever $U(\theta^*) \leq Q(\beta, a_c, a_r)$.

3.2.2. Case 2: $\beta > 1/2$ and $\alpha_c > \alpha_r$

In this case, the optimal ceded function is given by Equation (8) with $d < a_c$. The order between d and a_r is not, however, determined. Therefore, the optimization problem is:

$$\min_{(c,d) \in [0,1] \times [0,a_c]} g_2(c, d),$$

where:

$$g_2(c, d) = c \left(-\beta(a_c - d) + (1 - \beta)(a_r - d)_+ + (2\beta - 1)(1 + \theta) \int_d^\infty S_X(t) dt \right),$$

which is a continuous function in c and d . Note, however, that the left-hand derivative $\partial g_2(c, d) / \partial d|_{d=a_r-}$ is not equal to the right-hand derivative $\partial g_2(c, d) / \partial d|_{d=a_r+}$. With the additional notations:

$$\begin{aligned} \theta_\beta^* &= \frac{\beta}{(2\beta - 1)(1 + \theta)}, \\ d_\beta^* &= S_X^{-1}(\theta_\beta^*), \\ U_\beta(x) &= S_X^{-1}(x) + \frac{1}{\theta_\beta^*} \int_{S_X^{-1}(x)}^\infty S_X(t) dt, \end{aligned}$$

we have the following theorem.

Theorem 2. Under the conditions $\beta > 1/2$ and $\alpha_c > \alpha_r$, the optimal ceded function is $f^*(x) = c(x - d)_+$ with the following parameters:

1. $c = 1$ and $d = d^*$ when $1 - \alpha_r < \theta^* < S_X(0)$ and $Q(\beta, a_c, a_r) > U(\theta^*)$;
2. $c \in [0, 1]$ is any constant and $d = d^*$ when $1 - \alpha_r < \theta^* < S_X(0)$ and $Q(\beta, a_c, a_r) = U(\theta^*)$;
3. $c = 1$ and $d = a_r$ when $\theta^* < 1 - \alpha_r < \theta_\beta^*$, and $a_c > U_\beta(1 - \alpha_r)$;
4. $c \in [0, 1]$ is any constant and $d = a_r$ when $\theta^* < 1 - \alpha_r < \theta_\beta^*$, and $a_c = U_\beta(1 - \alpha_r)$;
5. $c = 1$ and $d = d_\beta^*$ when $1 - \alpha_c < \theta_\beta^* < 1 - \alpha_r$ and $a_c > U_\beta(\theta_\beta^*)$;
6. $c \in [0, 1]$ is any constant and $d = d_\beta^*$ when $1 - \alpha_c < \theta_\beta^* < 1 - \alpha_r$ and $a_c = U_\beta(\theta_\beta^*)$;
7. $c = 1$ and $d = 0$ when $\theta^* \geq S_X(0)$ and $Q(\beta, a_c, a_r) > (1 + \theta)\mathbf{E}[X]$;
8. $c \in [0, 1]$ is any constant and $d = 0$ when $\theta^* \geq S_X(0)$ and $Q(\beta, a_c, a_r) = (1 + \theta)\mathbf{E}[X]$.

If none of the conditions above are satisfied, then $f^*(x) = 0$ for all x .

Proof. We prove Parts (1), (3) and (5) only, because the proofs of the other parts are similar.

Part (1):

The derivative of $g_2(c, d)$ with respect to d is given by:

$$\frac{\partial g_2(c, d)}{\partial d} = \begin{cases} c(2\beta - 1)(1 - (1 + \theta)S_X(d)) & \text{when } d < a_r, \\ c(2\beta - 1)\left(\frac{\beta}{2\beta - 1} - (1 + \theta)S_X(d)\right) & \text{when } d > a_r. \end{cases}$$

Assuming $c > 0$, we have that $\partial g_2(c, d)/\partial d$ is increasing in $d > 0$. The condition $1 - \alpha_r < \theta^* < S_X(0)$ ensures that $\partial g_2(c, d)/\partial d$ is negative at $d = 0$, increases to zero at $d^* = S_X^{-1}(\theta^*) < a_r$ and becomes positive for $d > d^*$. Therefore, the objective function is minimized at $d = d^*$. At $d = d^*$, the derivative $\partial g_2(c, d)/\partial c$ is given by Formula (11). Therefore, as in the proof of Theorem 1, the condition $U(\theta^*) < Q(\beta, a_c, a_r)$ ensures that $g_2(c, d)$ is minimized at $(c, d) = (1, d^*)$.

Part (3):

When $\theta^* < 1 - \alpha_r < \theta_\beta^*$, the derivative $\partial g_2(c, d)/\partial d$ is negative for $d < a_r$ and positive for $d > a_r$. Therefore, the function $g_2(c, d)$ is minimized at $d = a_r$, assuming $c > 0$. Next, since:

$$g_2(c, a_r) = c\beta (U_\beta(1 - \alpha_r) - a_c)$$

and $U_\beta(1 - \alpha_r) < a_c$ by assumption, the function $g_2(c, d)$ is minimized at $(c, d) = (1, a_r)$ with $g_2(1, a_r) < 0$. Noting that $g_2(0, a_r) = 0 > g_2(1, a_r)$, the desired result follows.

Part (5):

Since $\theta^* < \theta_\beta^*$, the assumption $\theta_\beta^* < 1 - \alpha_r$ implies $\theta^* < 1 - \alpha_r$. Therefore, the derivative $\partial g_2(c, d)/\partial d$ is negative for $d < a_r$, equal to zero at $d = d_\beta^* \in (a_r, a_c)$ and positive afterwards. Therefore, the objective function is minimized at $d = d_\beta^*$. Note that the condition $a_c > U_\beta(\theta_\beta^*)$ implies $d_\beta^* < a_c$. Furthermore, since:

$$\begin{aligned} g_2(c, d_\beta^*) &= c \left(-\beta(a_c - d_\beta^*) + (2\beta - 1)(1 + \theta) \int_{d_\beta^*}^\infty S_X(t)dt \right) \\ &= c\beta (U_\beta(\theta_\beta^*) - a_c) \end{aligned}$$

and $U_\beta(\theta_\beta^*) < a_c$ by assumption, the objective function $g_2(c, d)$ is minimized at $(c, d) = (1, d_\beta^*)$ when $c > 0$. Noting that $g_2(0, d) = 0 > g_2(1, d_\beta^*)$, the desired result follows. \square

Remark 3. We have the following observations:

- When $\beta = 1$, then $\theta_\beta^* = \theta^*$ and $U_\beta(x) = U(x)$. Thus, the result is exactly the same as in the first bullet at the end of Case 1 above. The value of α_r makes no difference here because only the cedent’s risk is considered when $\beta = 1$.
- When $\beta \searrow 1/2$, then $Q(\beta, a_c, a_r) \sim (a_c - a_r)/(4\beta - 2) \nearrow \infty$, $\theta_\beta^* \nearrow \infty$ and $U_\beta(\theta_\beta^*) = 0$. Therefore, Parts (1) and (3) of Theorem 2 apply. We have:

$$f^*(x) = \begin{cases} (x - d^*)_+ & \text{when } \theta^* > 1 - \alpha_r, \\ (x - a_r)_+ & \text{when } \theta^* < 1 - \alpha_r. \end{cases}$$

3.2.3. Case 3: $\beta < 1/2$ and $\alpha_c < \alpha_r$

With the optimal ceded function f^* given by Equation (9), Problem (7) reduces to:

$$\min_{\eta \in [0,1]} g_3(\eta),$$

where:

$$g_3(\eta) = -\beta\eta a_c + (1 - \beta)\eta a_r + (2\beta - 1)(1 + \theta) \left(\eta \int_0^{a_r} x dF_X(x) + \int_{a_r}^{\infty} (x - a_r + \eta a_r) dF_X(x) \right).$$

Taking the derivative of $g_3(\eta)$ with respect to η , we have:

$$\begin{aligned} g'_3(\eta) &= -\beta a_c + (1 - \beta)a_r + (2\beta - 1)(1 + \theta) \left(\int_0^{a_r} x dF_X(x) + a_r S_X(a_r) \right) \\ &= (1 - 2\beta) (Q(\beta, a_c, a_r) - (1 + \theta)E[X \wedge a_r]), \end{aligned} \tag{12}$$

where $X \wedge a_r = \min\{X, a_r\}$. Therefore, $g_3(\eta)$ achieves its minimum at $\eta = 1$ when the quantity on the right-hand side of Equation (12) is negative. Otherwise, the minimum is at $\eta = 0$. Consequently, we have the following theorem.

Theorem 3. Under the conditions $\beta < 1/2$ and $\alpha_c < \alpha_r$, the optimal ceded function is:

$$f^*(x) = \begin{cases} \eta x & \text{when } 0 \leq x < a_r, \\ x - (1 - \eta)a_r & \text{when } x \geq a_r, \end{cases}$$

with the parameter:

$$\eta = \begin{cases} 1 & \text{when } (1 + \theta)E[X \wedge a_r] > Q(\beta, a_c, a_r), \\ 0 & \text{when } (1 + \theta)E[X \wedge a_r] < Q(\beta, a_c, a_r), \\ \text{any constant} \in [0, 1] & \text{when } (1 + \theta)E[X \wedge a_r] = Q(\beta, a_c, a_r). \end{cases}$$

Remark 4. A few observations follow:

- When $\beta \nearrow 1/2$, then $g'_3(\eta) \rightarrow (a_r - a_c)/2 > 0$. In this case, $\eta^* = 0$ and the optimal reinsurance policy is $f^*(x) = (x - a_r)_+$.
- When $\beta = 0$ and only the reinsurer’s risk is considered, Theorem 3 holds with $Q(\beta, a_c, a_r) = a_r$.

3.2.4. Case 4: $\beta < 1/2$ and $\alpha_c > \alpha_r$

With the optimal reinsurance function f^* given by Equation (9), Problem (7) becomes:

$$\min_{\eta \in [0,1]} g_4(\eta),$$

where

$$\begin{aligned} g_4(\eta) &= -\beta(a_c - a_r + \eta a_r) + (1 - \beta)\eta a_r \\ &\quad + (2\beta - 1)(1 + \theta) \left(\eta \int_0^{a_r} x dF_X(x) + \int_{a_r}^{\infty} (x - a_r + \eta a_r) dF_X(x) \right). \end{aligned}$$

Taking the derivative of $g_4(\eta)$ with respect to η , we get:

$$g'_4(\eta) = (1 - 2\beta) \left(a_r - (1 + \theta)E[X \wedge a_r] \right),$$

which yields the following theorem.

Theorem 4. Under the conditions $\beta < 1/2$ and $\alpha_c > \alpha_r$, the optimal ceded function is:

$$f^*(x) = \begin{cases} \eta x & \text{when } 0 \leq x < a_r, \\ x - (1 - \eta)a_r & \text{when } x \geq a_r, \end{cases}$$

with the parameter:

$$\eta = \begin{cases} 1 & \text{when } (1 + \theta)\mathbf{E}[X \wedge a_r] > a_r, \\ 0 & \text{when } (1 + \theta)\mathbf{E}[X \wedge a_r] < a_r, \\ \text{any constant } \in [0, 1] & \text{when } (1 + \theta)\mathbf{E}[X \wedge a_r] = a_r. \end{cases}$$

Note that Theorems 3 and 4 are quite similar, with the role of $Q(\beta, a_c, a_r)$ in the former theorem played by a_r in the latter one.

3.3. An Illustrative Example

In this section, we construct a numerical example to illustrate the Pareto optimality of the reinsurance policies that we derived above. Specifically, we assume that the loss variable X follows the exponential distribution with the survival function $S_X(x) = e^{-0.001x}$ for $x \geq 0$. Let the safety loading parameter be $\theta = 0.2$. Then, $\theta^* = 1/(1 + \theta) = 0.833$, $d^* = S_X^{-1}(\theta^*) = 182.3$ and $U(\theta^*) = 1182.3$. We discuss two scenarios.

3.3.1. Scenario A: $\alpha_c = 0.95$ and $\alpha_r = 0.99$

In this case, $a_c = 2995.7$ and $a_r = 4605.2$. Applying Theorems 1 and 3, we have:

$$f_1^*(x) = \begin{cases} (x - 4605.2)_+ & \text{when } \beta \in [0, 0.5), \\ 0 & \text{when } \beta \in (0.5, 0.654), \\ (x - 182.3)_+ & \text{when } \beta \in (0.654, 1]. \end{cases}$$

When $\beta = 0.5$, then:

$$f_{1A}^*(x) = \begin{cases} 0 & \text{when } x \leq 4605.2, \\ \text{unspecified} & \text{when } x > 4605.2. \end{cases}$$

When $\beta = 0.654$, then:

$$f_{1A}^*(x) = c(x - 182.3)_+$$

for any constant $c \in [0, 1]$. The values of $\text{VaR}(C_{f_{1A}^*})$ versus $\text{VaR}(R_{f_{1A}^*})$ are reported in Table 1.

Table 1. VaRs of the cedent and the reinsurer when $f \in \mathcal{C}^1$.

	$\text{VaR}_{\alpha_c}(C_{f_{1A}^*})$	$\text{VaR}_{\alpha_r}(R_{f_{1A}^*})$
$\beta \in [0, 0.5)$	3005.73	-10
$\beta = 0.5$	between 2995.73 and 3005.73	between -10 and 0
$\beta \in (0.5, 0.654)$	2995.73	0
$\beta = 0.654$	between 1182.32 and 2995.73	between 0 and 3422.85
$\beta \in (0.654, 1]$	1182.32	3422.85

We have the following observations:

- For $\beta \in (0.654, 1]$, the insurer is “more important”. As a result, it retains the “good” risk in the layer of losses $(0, S_X^{-1}(\theta^*))$ and cedes the rest. For $\beta \in [0, 0.5)$, the reinsurer is “more important”, and it assumes the risk above a_r . As a result, the chance of a payment is so small that its VaR does not increase; it actually reduces to -10 because of the collected premium. For $\beta \in (0.5, 0.654)$, no agreement is reached between the two parties.
- From Table 1, we see that when β gets larger and the cedent becomes increasingly important, then $\text{VaR}_{\alpha_c}(C_{f_{1A}^*})$ decreases, whereas $\text{VaR}_{\alpha_r}(R_{f_{1A}^*})$ increases.
- When $\beta = 0.5$ and $\beta = 0.654$, the optimal ceded functions are only partially specified, and the risk of the two parties varies in some range. For example, when $\beta = 0.5$, then $\text{VaR}_{\alpha_c}(C_{f_{1A}^*})$ is maximized by choosing $f_{1A}^*(x) = (x - 4605.2)_+$ because the cedent is choosing a maximal ceded function and paying a maximal reinsurance premium (within the partially-specified optimal ceded functions). However, its VaR does not reduce with such a high deductible value. On the other hand, $\text{VaR}_{\alpha_c}(C_{f_{1A}^*})$ is minimized with $f_{1A}^*(x) = 0$, within the partially-specified optimal ceded functions.

3.3.2. Scenario B: $\alpha_c = 0.99$ and $\alpha_r = 0.95$

In this case, we have $a_c = 4605.2$ and $a_r = 2995.7$. Applying Theorems 2 and 4, we have:

$$f_{1B}^*(x) = \begin{cases} (x - 2995.7)_+ & \text{when } \beta \in [0, 0.5), \\ (x - 182.3)_+ & \text{when } \beta \in (0.5, 1]. \end{cases} \tag{13}$$

When $\beta = 0.5$,

$$f_{1B}^*(x) = \begin{cases} x - d & \text{when } x \geq 2995.7, \\ \text{unspecified} & \text{when } x < 2995.7, \end{cases} \tag{14}$$

where $d \in [0, 2995.7]$ can be any constant. The values of $\text{VaR}(C_{f_{1B}^*})$ versus $\text{VaR}(R_{f_{1B}^*})$ are reported in Table 2.

Table 2. VaRs of the cedent and the reinsurer when $f \in \mathcal{C}^1$.

	$\text{VaR}_{\alpha_c}(C_{f_{1B}^*})$	$\text{VaR}_{\alpha_r}(R_{f_{1B}^*})$
$\beta \in [0, 0.5)$	3055.73	-60
$\beta = 0.5$	between 1182.32 and 3055.73	between -60 and 1813.41
$\beta \in (0.5, 1]$	1182.32	1813.41

4. Optimal Reinsurance Policy When $f \in \mathcal{C}^2$

In this section, we determine optimal reinsurance policies when $f \in \mathcal{C}^2$, that is when both f and the retained loss function I_f are non-decreasing. Comparing this situation with the earlier $f \in \mathcal{C}^1$, we can now deal with non-convex ceded functions, such as $f(x) = \min\{x, l\}$ for any retention level $l > 0$. Mathematically, the problem becomes:

$$\min_{f \in \mathcal{C}^2} \mathcal{H}(f) = \min_{f \in \mathcal{C}^2} \left\{ -\beta \cdot f(a_c) + (1 - \beta) \cdot f(a_r) + (2\beta - 1)(1 + \theta)E[f(X)] \right\}. \tag{15}$$

As pointed out in the Introduction, solutions to similar problems exist in the literature, and we refer to [7,8,25] for details and further references. Our contribution in this paper is to generalize the geometric arguments of [12] to the situation when the interests of both the cedent and the reinsurer are taken into account, and we do so in such a way that allows us to avoid lengthy mathematical arguments and consequently helps us to gain useful intuition. In addition, for all scenarios considered, we are able to provide explicit recipes for determining optimal reinsurance policies.

In Section 4.1 below, we derive optimal forms of ceded functions, and in Section 4.2, we determine parameter values of the optimal functions. Section 4.3 contains an illustrative numerical example,

which is a continuation of that of Section 3.3. Throughout the rest of this section, we assume $1 - \alpha_c < S_X(0)$ and $1 - \alpha_r < S_X(0)$.

4.1. Functional Form of the Ceded Function

We have subdivided our considerations into three cases.

4.1.1. Case 1: $\beta > 1/2$

Similarly to Case 1 of Section 3.1.1, we determine the functional form of the ceded function f^* in the following manner. For any $f \in \mathcal{C}^2$, we seek f^* , such that $f^*(a_c) = f(a_c)$ and:

$$(1 - \beta)f^*(a_r) + (2\beta - 1)P_{f^*} \leq (1 - \beta)f(a_r) + (2\beta - 1)P_f.$$

This requires $f^*(a_r)$, as well as the entire function f^* to be as small as possible for a fixed value of $f^*(a_c)$.

As we see from Figure 3, because f is non-decreasing with a slope not exceeding one, the aforementioned requirements are satisfied by the function:

$$f^*(x) = \min\{(x - d)_+, a_c - d\} = \begin{cases} 0 & \text{when } x \leq d, \\ x - d & \text{when } d < x < a_c, \\ a_c - d & \text{when } x \geq a_c, \end{cases} \tag{16}$$

where $d \in [0, a_c]$ can be any constant. The optimal value of d will be determined in Section 4.2 below. In reinsurance jargon, the above specified optimal form of the reinsurance policy is for the reinsurer to provide coverage over the layer (d, a_c) .

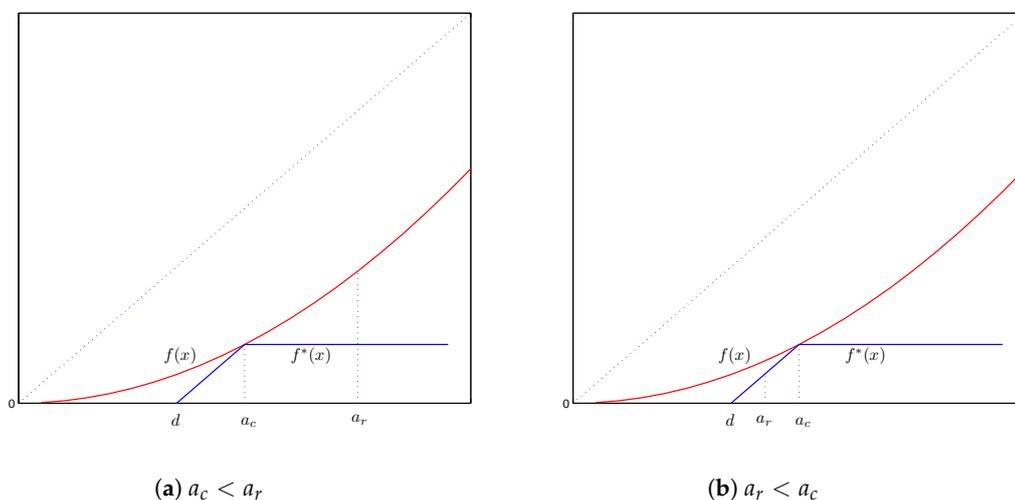


Figure 3. Optimal ceded functions in \mathcal{C}^1 : Case 1.

4.1.2. Case 2: $\beta < 1/2$

Similarly to Case 2 of Section 3.1.1, since the coefficients in front of $f(a_c)$ and P_f in objective Function (15) are negative, the optimal reinsurance policy is found by seeking f^* , such that $f^*(a_r) = f(a_r)$ and:

$$\beta f^*(a_c) + (1 - 2\beta)P_{f^*} \geq \beta f(a_c) + (1 - 2\beta)P_f.$$

As we see from Figure 4, these requirements are satisfied by the function:

$$\begin{aligned}
 f^*(x) &= \min\{x, d\} \mathbf{1}_{\{x \leq a_r\}} + (x - a_r + d) \mathbf{1}_{\{x > a_r\}} \\
 &= \begin{cases} x & \text{when } 0 \leq x \leq d, \\ d & \text{when } d < x \leq a_r, \\ x - a_r + d & \text{when } x > a_r, \end{cases} \tag{17}
 \end{aligned}$$

where $d \in [0, a_r]$ can be any constant. Hence, the optimal form of the reinsurance policy is for the reinsurer to provide a coverage except for the layer (d, a_r) . In other words, the insurer retains losses in the layer (d, a_r) .

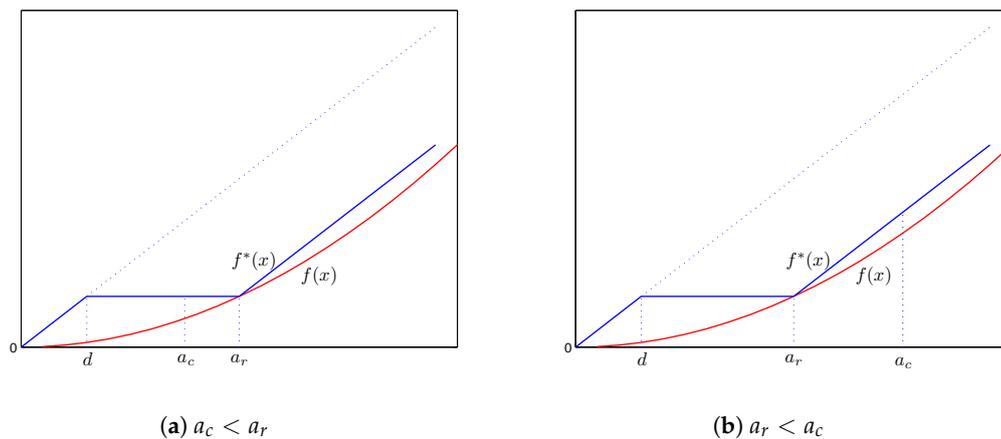


Figure 4. Optimal ceded functions in C^1 : Case 2.

4.1.3. Case 3: $\beta = 1/2$

In this case, the minimization problem (15) simplifies to:

$$\min_{f \in C^2} \{f(a_r) - f(a_c)\}.$$

When $\alpha_c < \alpha_r$, because the ceded function is non-decreasing, this requires f^* to be constant on the interval (a_c, a_r) . Therefore, any function f^* in C^2 with $f^*(x) = c$ on (a_c, a_r) , where $c \in [0, a_c]$ is a constant, is Pareto-optimal.

When $\alpha_c > \alpha_r$, because the slope of the ceded function cannot exceed one, the function f^* increases at the rate of one on the interval (a_r, a_c) . Therefore, any function f^* in C^2 with $f^{*'}(x) = 1$ on (a_r, a_c) is Pareto-optimal.

Finally, when $\alpha_c = \alpha_r$, then the objective function is always constant.

4.2. Parameter Values of the Optimal Ceded Function

In this section, we obtain parameter values of the optimal ceded functions that we derived in Section 4.1. Four cases are considered separately.

4.2.1. Case 1: $\beta > 1/2$ and $\alpha_c < \alpha_r$

Let:

$$\theta^* = \frac{1}{1 + \theta}$$

and:

$$d^* = S_X^{-1}(\theta^*).$$

Theorem 5. Under the conditions $\beta > 1/2$ and $\alpha_c < \alpha_r$, the optimal ceded function is $f^*(x) = \min\{(x - d)_+, a_c - d\}$ with the parameter:

1. $d = d^*$ when $1 - \alpha_c < \theta^* < S_X(0)$;
2. $d = 0$ when $\theta^* \geq S_X(0)$.

In addition, when $\theta^* \leq 1 - \alpha_c$, then $f^*(x) = 0$ for all x .

Proof. With the function f^* given by Equation (16), optimization Problem (15) becomes:

$$\min_{d \in [0, a_c]} g_5(d),$$

where:

$$g_5(d) = (2\beta - 1) \left((1 + \theta) \int_d^{a_c} S_X(x) dx - a_c + d \right).$$

The derivative:

$$g'_5(d) = (2\beta - 1)(1 - (1 + \theta)S_X(d))$$

is increasing in d . Therefore, when $1 - \alpha_c < \theta^* < S_X(0)$, then $g_5(d)$ is minimized at $0 < d^* < a_c$. When $\theta^* > S_X(0)$, then $g_5(d)$ is minimized at $d = 0$. Finally, when $\theta^* < 1 - \alpha_c$, then $g_5(d)$ is minimized at $d = a_c$, and so, $f^*(x) = 0$. \square

4.2.2. Case 2: $\beta > 1/2$ and $\alpha_c > \alpha_r$

With the function f^* given by Equation (16), optimization problem (15) reduces to:

$$\min_{d \in [0, a_c]} g_6(d),$$

where:

$$g_6(d) = -\beta(a_c - d) + (1 - \beta)(a_r - d)_+ + (2\beta - 1)(1 + \theta) \int_d^{a_c} S_X(x) dx.$$

Let:

$$\theta^*_\beta = \frac{\beta}{(2\beta - 1)(1 + \theta)}$$

and

$$d^*_\beta = S_X^{-1}(\theta^*_\beta).$$

We calculate the derivative:

$$g'_6(d) = \begin{cases} (2\beta - 1)(1 + \theta)(\theta^* - S_X(d)) & \text{when } d < a_r, \\ (2\beta - 1)(1 + \theta)(\theta^*_\beta - S_X(d)) & \text{when } d > a_r, \end{cases}$$

which is an increasing function in d , and so, we have the following theorem.

Theorem 6. Under the conditions $\beta > 1/2$ and $\alpha_c > \alpha_r$, the optimal ceded function is $f^*(x) = \min\{(x - d)_+, a_c - d\}$ with the parameter:

1. $d = d^*$ when $1 - \alpha_r < \theta^* < S_X(0)$;
2. $d = a_r$ when $\theta^* < 1 - \alpha_r < \theta^*_\beta$;
3. $d = d^*_\beta$ when $\theta^* < 1 - \alpha_r$ and $1 - \alpha_c < \theta^*_\beta < 1 - \alpha_r$;
4. $d = 0$ when $\theta^* \geq S_X(0)$.

If none of the above conditions are satisfied, then $f^*(x) = 0$ for all x .

Proof. We use similar arguments to those in Theorem 2. We illustrate them here by proving Part (1) only. When $1 - \alpha_r < \theta^* < S_X(0)$, the derivative $g'_6(d)$ reaches zero at $d^* = S_X^{-1}(\theta^*) \in (0, a_r)$ and then remains positive for $d > d^*$. Therefore, $g_6(d)$ reaches its minimum at $d^* = S_X^{-1}(\theta^*)$. With this, we conclude the proof of Theorem 6. \square

4.2.3. Case 3: $\beta < 1/2$ and $\alpha_c < \alpha_r$

With the function f^* given by Equation (17), optimization Problem (15) reduces to:

$$\min_{d \in [0, a_r]} g_7(d),$$

where the objective function is:

$$g_7(d) = \begin{cases} -\beta d + (1 - \beta)d + (2\beta - 1)(1 + \theta) \left(\int_0^d S_X(x)dx + \int_{a_r}^\infty S_X(x)dx \right) & \text{when } d < a_c, \\ -\beta a_c + (1 - \beta)d + (2\beta - 1)(1 + \theta) \left(\int_0^d S_X(x)dx + \int_{a_r}^\infty S_X(x)dx \right) & \text{when } d > a_c. \end{cases}$$

Thus:

$$g'_7(d) = \begin{cases} (1 - 2\beta)(1 + \theta)(\theta^* - S_X(d)) & \text{when } d < a_c, \\ (1 - 2\beta)(1 + \theta)(\theta^*_\beta - S_X(d)) & \text{when } d > a_c, \end{cases}$$

which leads us to the following theorem, whose proof is similar to that of Theorem 3 and thus omitted.

Theorem 7. Under the conditions $\beta < 1/2$ and $\alpha_c < \alpha_r$, the optimal ceded function is:

$$f^*(x) = \min\{x, d\} \mathbf{1}_{\{x \leq a_r\}} + (x - a_r + d) \mathbf{1}_{\{x > a_r\}}$$

with the parameter:

1. $d = d^*$ when $1 - \alpha_c < \theta^* < S_X(0)$;
2. $d = a_c$ when $\theta^* < 1 - \alpha_c < \theta^*_\beta$;
3. $d = d^*_\beta$ when $\theta^* < 1 - \alpha_c$ and $1 - \alpha_r < \theta^*_\beta < 1 - \alpha_c$;
4. $d = a_r$ when $\theta^* < 1 - \alpha_c$ and $\theta^*_\beta < 1 - \alpha_r$;
5. $d = 0$ when $\theta^* \geq S_X(0)$.

If none of the above conditions are satisfied, then $f^*(x) = 0$ for all x .

4.2.4. Case 4: $\beta < 1/2$ and $\alpha_c > \alpha_r$

With the function f^* given by Equation (17), optimization Problem (15) reduces to:

$$\min_{d \in [0, a_r]} g_8(d),$$

where:

$$g_8(d) = -\beta(a_c - a_r + d) + (1 - \beta)d + (2\beta - 1)(1 + \theta) \left(\int_0^d S_X(x)dx + \int_{a_r}^\infty S_X(x)dx \right).$$

Thus,

$$g'_8(d) = (1 - 2\beta)(1 + \theta)(\theta^* - S_X(d)),$$

which gives us the following theorem.

Theorem 8. Under the conditions $\beta < 1/2$ and $\alpha_c > \alpha_r$, the optimal ceded function is:

$$f^*(x) = \min\{x, d\} \mathbf{1}_{\{x \leq a_r\}} + (x - a_r + d) \mathbf{1}_{\{x > a_r\}}$$

with the parameter:

1. $d = d^*$ when $1 - \alpha_r < \theta^* < S_X(0)$;
2. $d = a_r$ when $\theta^* < 1 - \alpha_r < S_X(0)$;
3. $d = 0$ when $\theta^* \geq S_X(0)$.

If none of the above conditions are satisfied, then $f^*(x) = 0$ for all x .

4.3. The Illustrative Example Continued

In this subsection, we continue the illustrative example of Section 3.3, but now assume that the admissible class of ceded functions is \mathcal{C}^2 .

4.3.1. Scenario A: $\alpha_c = 0.95$ and $\alpha_r = 0.99$

Applying Theorems 5 and 7, we have:

$$f_{2A}^*(x) = \begin{cases} \min \{x, 182.3\} \mathbf{1}_{\{x \leq 4605.2\}} + (x - 4422.9) \mathbf{1}_{\{x > 4605.2\}} & \text{when } \beta \in [0, 0.5), \\ \min \{(x - 182.3)_+, 2813.4\} & \text{when } \beta \in (0.5, 1]. \end{cases}$$

When $\beta = 0.5$, then:

$$f_{2A}^*(x) = \begin{cases} d & \text{when } 2995.7 \leq x \leq 4605.2, \\ \text{unspecified} & \text{otherwise,} \end{cases}$$

where $d \in [0, 2995.7]$ can be any constant.

The values of $\text{VaR}(C_{f_{2A}^*})$ versus $\text{VaR}(R_{f_{2A}^*})$ are reported in Table 3.

Table 3. VaRs of the cedent and the reinsurer when $f \in \mathcal{C}^2$.

	$\text{VaR}_{\alpha_c}(C_{f_{2A}^*})$	$\text{VaR}_{\alpha_r}(R_{f_{2A}^*})$
$\beta \in [0, 0.5)$	3025.41	-29.68
$\beta = 0.5$	between 1122.32 and 3025.41	between -29.68 and 1873.41
$\beta \in (0.5, 1]$	1122.32	1873.41

We have the following observations:

- Since the cedent and the reinsurer have more choices when $f \in \mathcal{C}^2$, their VaRs under the optimal reinsurance policy f_{2A}^* are lower than the corresponding ones under f_{1A}^* . In particular, the reinsurer’s risk is reduced significantly even when $\beta = 1$.
- For $\beta \in [0, 0.5)$, the reinsurer assumes the “good” risk in the layer $(0, S_X^{-1}(\theta^*))$, as well as losses greater than 4422.9. The former layer creates profit, and the latter layer does not contribute to its VaR because the chance of penetration is too small compared with the probability level α_r used in its VaR.
- For $\beta \in (0.5, 1)$, the insurer retains the “good” risk in the layer $(0, S_X^{-1}(\theta^*))$, as well as the losses greater than 2813.4. The former layer creates profit, and the latter layer does not contribute to its VaR because the chance of penetration is too small compared with the probability level α_c used in its VaR.

4.3.2. Scenario B: $\alpha_c = 0.99$ and $\alpha_r = 0.95$

Applying Theorems 6 and 8, we have:

$$f_{2B}^*(x) = \begin{cases} \min \{x, 182.3\} \mathbf{1}_{\{x \leq 2995.7\}} + (x - 2813.4) \mathbf{1}_{\{x > 2995.7\}} & \text{when } \beta \in [0, 0.5), \\ \min \{(x - 182.3)_+, 4422.85\} & \text{when } \beta \in (0.5, 1]. \end{cases}$$

When $\beta = 0.5$,

$$f_{2B}^*(x) = \begin{cases} x - d & \text{when } x \in [2995.7, 4605.2], \\ \text{unspecified} & \text{when } x \in [0, 2995.7) \cup (4605.7, \infty), \end{cases} \tag{18}$$

where $d \in [0, 2995.7]$ can be any constant. The values of $\text{VaR}(C_{f_{2B}^*})$ versus $\text{VaR}(R_{f_{2B}^*})$ are reported in Table 4.

Table 4. VaRs of the cedent and the reinsurer when $f \in \mathcal{C}^2$.

	$\text{VaR}_{\alpha_c}(C_{f_{2B}^*})$	$\text{VaR}_{\alpha_r}(R_{f_{2B}^*})$
$\beta \in [0, 0.5)$	3073.43	-77.67
$\beta = 0.5$	between 1170.33 and 3073.43	between -77.67 and 1825.38
$\beta \in (0.5, 1]$	1170.33	1825.38

5. A Numerical Comparison of the Optimal Reinsurance Policies in \mathcal{C}^1 and \mathcal{C}^2

In Sections 3.3 and 4.3, we derived the Pareto-optimal reinsurance policies in \mathcal{C}^1 and \mathcal{C}^2 , respectively. In this section, we compare the two cases.

In Figure 5, we depict f_{1A}^* and f_{2A}^* obtained for Scenario A with the proportional reinsurance $f_1(x) = ax$ when a varies from zero to one and also with the excess-of-loss reinsurance $f_2(x) = (x - d)_+$ when the deductible level d varies from zero to $4605.2 = \max\{a_c, a_r\}$. The following can be concluded from the figure.

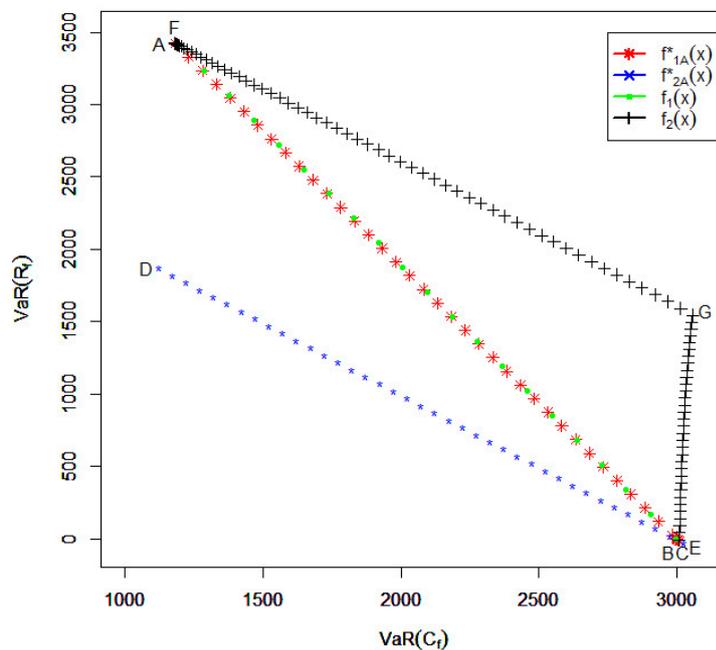


Figure 5. VaRs of the cedent and the reinsurer under different policies: Scenario A.

- The efficient frontier for the VaRs of the two parties with $f \in \mathcal{C}^1$ is represented by the path from $A = (1182.32, 3422.85)$ to $B = (2995.72, 0)$ and then to $C = (3005.73, -10)$. Note that the points between A and B represent the VaRs of the two parties resulting from the optimal policies obtained with $\beta = 0.5$. The points between B and C represent the VaRs of the two parties resulting from the optimal policies obtained with $\beta = 0.654$.

- The efficient frontier for the VaRs of the two parties when $f \in \mathcal{C}^2$ is represented by the path from $D = (1122.32, 1873.41)$ to $E = (3025.41, -29.68)$.
- For the quota-share reinsurance with $f_1(x) = ax$ where a ranges from zero to one, the VaRs of the two parties go from B to $F = (1200, 3405.2)$. When $f \in \mathcal{C}^1$, the quota-share reinsurance policy is quite close to the efficient frontier.
- For the excess-of-loss reinsurance $f_2(x) = (x - d)_+$ with d ranging from zero to $a_r = 4605.2$, the VaRs of the two parties go along the path $F \rightarrow A \rightarrow G \rightarrow C$ with $G = (3055.47, 1545.43)$.

From Figure 5, we conclude that if the reinsurer worries about the right-hand tail more than the primary insurer ($\alpha_c < \alpha_r$), then the difference between the efficient frontiers obtained for $f \in \mathcal{C}^1$ and $f \in \mathcal{C}^2$ is significant. This means that the convexity requirement in the definition of \mathcal{C}^1 is quite restrictive to the reinsurer, and the coverage with an upper limit (which is not allowed in \mathcal{C}^1) is valuable. In the case when the convexity of the ceded function must be required, quota-share policies are quite efficient.

In Figure 6, we compare f_{1B}^* and f_{2B}^* obtained for Scenario B with the quota-share reinsurance policies $f_1(x) = ax$ when a ranges from zero to one and the excess-of-loss reinsurance policies $f_2(x) = (x - d)_+$ when the deductible d ranges from zero to $4605.2 = \max\{a_c, a_r\}$.

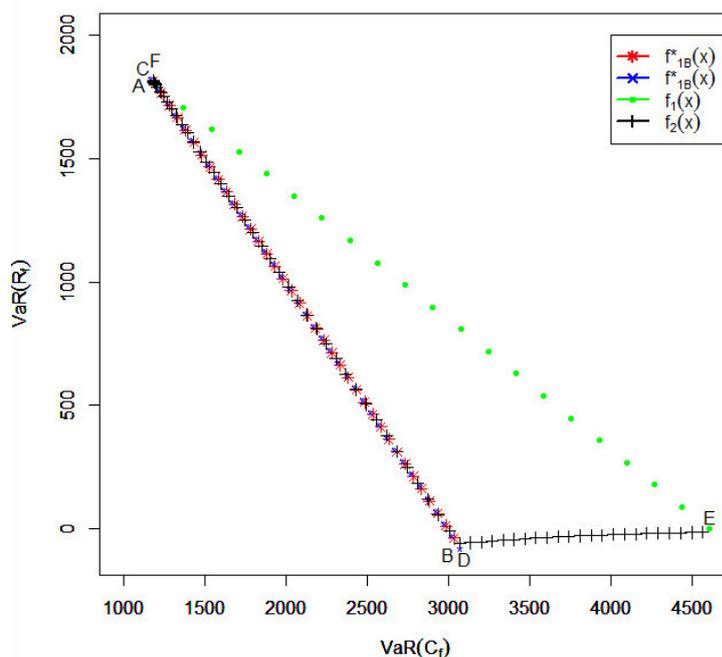


Figure 6. VaRs of the cedent and the reinsurer under different policies: Scenario B.

In particular, we observe the following:

- The efficient frontier for the VaRs of the two parties with $f \in \mathcal{C}^1$ is represented by the path from $A = (1182.32, 1813.41)$ to $B = (3055.73, -60)$.
- The efficient frontier for the VaRs of the two parties when $f \in \mathcal{C}^2$ is represented by the path from $C = (1170.33, 1825.38)$ to $D = (3073.43, -77.67)$. In fact, it can algebraically be shown that the path from B to A is actually a part of the path from D to C . That is, by allowing $f \in \mathcal{C}^2$, the efficient frontier is extended from the path $B \rightarrow A$ to the path $D \rightarrow C$.
- For the quota-share reinsurance with the parameter a ranging from zero to one, the VaRs of the two parties are represented by the path from $E = (4605.7, 0)$ to $F = (1200, 1795.7)$. We see that when $\alpha_c > \alpha_r$, the quota-share reinsurance policies are not efficient.

- For the excess-of-loss reinsurance with the parameter d ranging from zero to $a_c = 4605.2$, the VaRs of the two parties change along the path $F \rightarrow A \rightarrow B \rightarrow E$. We see that setting $d \in (0, a_r)$ is quite efficient, whereas setting $d \in (a_r, a_c)$ is not.

From Figure 6, we conclude that if the primary insurer worries about the right-hand tail more than the reinsurer ($\alpha_c > \alpha_r$), then the excess-of-loss policies with the deductible level ranging from $S_X^{-1}(\theta^*)$ to a_r provide a good part of the efficient frontier. The quota-share policies are in general inefficient.

6. Conclusions

In this paper, we have extended the geometric approach of [12] to obtain the optimal reinsurance policies accommodating both the cedent's and the reinsurer's interests. Specifically, we have derived the forms of optimal reinsurance functions and also specified their parameter values within two classical sets of admissible ceded functions. We have adopted the same value-at-risk measure for assessing risks of the two parties, but at possibly different probability levels. Illustrative numerical examples have been constructed to illuminate our theoretical findings and their practical implications.

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