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Valuation of Equity-Linked Death Benefits on Two Lives with Dependence

Kokou Essiomle and Franck Adékambi *

School of Economics, University of Johannesburg, Johannesburg 2006, South Africa
* Correspondence: fadekambi@uj.ac.za

Abstract: The purpose of this paper is to investigate equity-linked death benefits for joint alive and last survivor individuals. Utilizing Farlie–Gumbel–Morgenstern (FGM) type dependency modeling framework, we first analyze the joint distribution of the couple (joint alive and last survival density) when marginal distributions follow mixed exponentials and weighted exponentials distributions. Then, we derive the price of the guaranteed minimum death benefit (GMDB) product. In addition, we provide closed analytical expressions of the price of some financial contingent claim contracts (classical and exotic options). Furthermore, we present some numerical results to support our theoretical results. We show in our numerical example that it is important to model the dependency between two lives (couple) since the price changes as the copula parameter changes.

Keywords: equity-linked death benefits; lookback option; multi-life; Farlie–Gumbel–Morgenstern copula; weighted exponentials distributions

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Citation: Essiomle, Kokou, and Franck Adékambi. 2023. Valuation of Equity-Linked Death Benefits on Two Lives with Dependence. *Risks* 11: 21. https://doi.org/10.3390/risks11010021

Academic Editors: Andrea Bencsik, Janos Fustos and Maria Jakubik

Received: 8 December 2022
Revised: 3 January 2023
Accepted: 10 January 2023
Published: 12 January 2023

1. Introduction

Consider the problem of a Guaranteed Minimum Death Benefit (GMDB) rider that guarantees the following payment to the customer's estate when the customer dies, $\max(S(T_x), K)$, where T_x is the time-until-death random variable for a life aged x, and K is the guaranteed amount. Because

$$\max(S(T_x),\ K) \quad = \quad S(T_x) + (K - S(T_x))_+,$$

the problem is equivalent to determining the price for an exotic put option when valuing such a product. Several studies have been conducted on classical European and American options with fixed maturities.

The transformation of the market and the risk appetite of investors and policyholders have made classical life contingency products less attractive to policyholders, requiring providers of insurance and financial contracts to develop more complex products than classical (traditional) products, such as variable annuities and minimum guaranteed benefit. These products combine actuarial and financial principles.

This makes the valuation of these products a complex problem that requires deep knowledge of both actuarial and financial pricing techniques (fair valuation). Examples of a new approach of fair valuation that combines both market-consistent and actuarial methods can be found in Dhaene et al. (2017).

In general, scholars have been working on developing accurate pricing models for these products over the past years. In Gerber et al. (2012), the authors addressed the Guaranteed Minimum Death Benefit pricing issue based on the assumption that the death time is exponential (mixed exponentials) and the expected discounted value of the payment was calculated. Extending further in Gerber et al. (2013), they derived the call, put, lookback, and barrier options prices based on the upward and downward stock's movement.

The lookback option pricing problem was previously studied by Gerber and Shiu (2003). By using Laplace transform techniques, they derived the expected discounted value of dividend payments based on a standard Brownian motion model of the company's aggregate net income. As a result, they were able to derive the price of some European lookback call and put options in both fixed and floating strike prices. The authors concluded their work by analysing the possibility of stochastic guaranteed levels. Similarly, Buchen and Konstandatos (2005) presented a new way to price lookback options using the Black and Scholes framework. They showed in their analysis that the lookback option is an integrated version of the barrier option. As a result of the work of these researchers, the portfolio present value can be used to price exotic options using an arbitrage-free framework.

Scholars find it challenging to choose a process that can adequately explain the time evolution of the underlying stock process and provide a closed tractable expression of any financial contingent contract. Stochastic process models used by researchers tend to be unrealistic (the Black–Scholes framework). According to Linetsky (2004), the price of a lookback option can be derived in terms of spectral expansion by linking the diffusion maximum and minimum to the hitting times and the spectral decomposition of diffusion hitting times. In addition, he showed that a closed analytical form can be obtained under the constant elasticity variance diffusion framework. Zhang et al. (2020) utilized the exponential Lévy process for modeling the stock price process to analyze the GMDB pricing problem. By using the Fast Fourier Transform, they derived the price of the GMDB and obtained the price for various payoffs. Their numerical results were compared to those computed using B-spline functions of different orders to demonstrate the efficiency and accuracy of their proposed algorithm.

With GMDB, Wang et al. (2021) analyzed the valuation problem of equity-linked annuities with regime-switching jump diffusion models. Their method of Fourier series expansion and Fourier transform has been used to derive closed expressions for some GMDB contracts. Their method's effectiveness was demonstrated by numerical values that confirmed its efficiency. Accordingly, Ai and Zhang (2022) adopted an exponential regime switching Lévy process for the stock and, by using Fourier constant series expansion techniques, they derived explicit expressions for the price of life contingent lookback options embedded in variables GMDB. Unlike Wang et al. (2021), Kirkby and Nguyen (2021) focused their work on determining the payoff of equity-linked GMDBs and they were able to derive a closed form of the price of such products when the risky index process follows the exponential Lévy process.

The valuation of such an option requires a distribution that reflects the policyholder's life expectancy or mortality well. Thus, Shevchenko and Luo (2016) assumed stochastic mortality behavior among policyholders. The authors examined the existence of a numerical pricing method via option stochastic control. They developed a method for valuing variable annuities based on Gauss–Hermite quadrature. In their analysis, both incomplete and complete markets were considered. The problem of finding the optimal fund to finance a pensioner of age (x) was previously analyzed by Dufresne (2007). By assuming a mixed exponential distribution of the lifetime of an individual aged (x), he derived analytic expressions for the stochastic life annuity distribution.

Most of the insurance and finance research has focused on the death or survival of only one family member. Buying insurance or financial contracts is generally done to protect savings, so it is important to look at the life statuses of both the wife and the husband. As a result, "joint life or last survival" insurance contracts are used in pensions. For a comprehensive overview of annuity for a married couple, we refer the readers to Brown and Poterba (1999) and Matvejevs and Matvejevs (2001).

In this paper, we investigate the valuation problem of GMDBs as in Gerber et al. (2012). For the time to maturity of the option, we take into account both the joint life and the last survival lifetime, unlike Gerber et al. (2012). Additionally, we assume that the lifetime random variables in the married couple are interdependent. Gerber et al. (2012) results are reviewed under a specific dependency structure and a hyper and weighted exponential distributions assumption. To the best of the authors' knowledge, this is the first study to consider married couples when pricing GMDB contracts.

The paper is structured as follows. In Section 2, we set up the model and derive the distributions of both joint life and last survival status. Then, we derive the discounted density of Brownian motion in Section 3. We go on to discuss some classical options valuations problems in Sections 4 and 5 and analyze the impact of the dependency as well as the type of life status under consideration on the price in Section 6. Finally, in Section 7, we conclude the paper with an explanation of the limitations and possible extensions of the work.

2. Model

2.1. Multiple-Life Insurance Model

In this section, we apply the mixed exponential distributions in the context of joint-life insurance modelling. This family of distributions allows us to derive some closed-form expression for many useful actuarial quantities. The survival of the two lives is referred to as the status of interest or simply the status. There are two common types of status: the joint life status and the last survival status.

2.2. Joint Life Status

The joint-life status is one that requires the survival of both lives. Accordingly, the status terminates upon the first death of one of the two lives. The joint-life status of two lives (x) and (y) will be denoted by (x, y), and the moment of death random variable is given by $T_{(x,y)} = \min\left(T_x,\ T_y\right)$. If the random variables T_x and T_y are dependent and model this dependency via the Farlie–Gumbel–Morgenstern (FGM) copula then, the joint distribution of $\left(T_x, T_y\right)$ is defined as follows:

$$f_{\left(T_x,T_y\right)}(u,\ v) = (f_{T_x}(u) - \theta h_{T_x}(u))f_{T_y}(v) + 2\theta h_{T_x}(u)f_{T_y}(v)\bar{F}_{T_y}(v),$$

where $h_{T_x}(u) \;=\; f_{T_x}(u)(1 - 2F_{T_x}(u))$ and $-1 \leq \theta \leq 1$ is the FGM copula's parameter.

Hereafter, we denote by

$$\begin{aligned}
\eta^{\alpha,\beta}_{\lambda_i,\lambda_j\gamma_k,\gamma_l} &= \alpha_i\alpha_j\beta_k\beta_l(\lambda_i + \lambda_j + \gamma_k + \gamma_l); \quad \eta_{\lambda_i,\lambda_j,\gamma_k,\gamma_l} \;=\; \lambda_i + \lambda_j + \gamma_k + \gamma_l,\\
\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(i_1,k_1) &= \alpha_i^{i_1}\lambda_i\beta_k^{k_1}\gamma_k(i_1\lambda_i + k_1\gamma_k); \quad \eta_{\lambda_i,\gamma_k}(i_1,k_1) \;=\; i_1\lambda_i + k_1\gamma_k,\\
\eta^{\alpha,\beta}_{\lambda_i,\lambda_j,\gamma_k}(i_1,j_1,k_1) &= \alpha_i^{i_1}\lambda_i\alpha_j^{j_1}\lambda_j\beta_k^{k_1}\gamma_k(i_1\lambda_i + j_1\lambda_j + k_1\gamma_k),\\
\eta^{\alpha,\beta}_{\lambda_i,\gamma_k,\gamma_l}(i_1,k_1,l_1) &= \alpha_i^{i_1}\lambda_i\beta_k^{k_1}\beta_k\beta_l^{l_1}\gamma_k(i_1\lambda_i + k_1\gamma_k + l_1\gamma_l),\\
\eta_{\lambda_i,\lambda_j,\gamma_k}(i_1,j_1,k_1) &= i_1\lambda_i + j_1\lambda_j + k_1\gamma_k; \quad \eta_{\lambda_i,\gamma_k,\gamma_l}(i_1,k_1,l_1) \;=\; i_1\lambda_i + k_1\gamma_k + l_1\gamma_l.
\end{aligned}$$

Proposition 1. *If T_x and T_y follow hyper-exponential distributions with density functions, $f_{T_x}(t) \;=\; \sum_{i=1}^{n}\alpha_i\lambda_i e^{-\lambda_i t}$, $f_{T_y}(t) \;=\; \sum_{i=1}^{m}\beta_i\gamma_i e^{-\gamma_i t}$, for $t \geq 0$. Then,*

$$\begin{aligned}
f_{T_{(x,y)}}(w) \;=\;& (1+\theta)\sum_{k=1}^{n}\sum_{i=1}^{m}\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(1,1)e^{-\eta_{\lambda_k,\gamma_i}(1,1)w} - \theta\Bigg(\sum_{k=1}^{m}\sum_{i=1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,1)e^{-\eta_{\lambda_i,\gamma_k}(2,1)w}\\
&+\; 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\eta^{\alpha,\beta}_{\lambda_k,\gamma_i,\gamma_j}(1,1,1)e^{-\eta_{\lambda_k,\gamma_i,\gamma_j}(1,1,1)w}\Bigg)\\
&+\; 2\theta\Bigg(\sum_{k=1}^{m}\sum_{i=1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(1,2)e^{-\eta_{\lambda_i,\gamma_k}(1,2)w} + 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\lambda_j,\gamma_k}(1,1,1)e^{-\eta_{\lambda_i,\lambda_j,\gamma_k}(1,1,1)w}\Bigg)\\
&-\; 2\theta\Bigg(\sum_{k=1}^{m}\sum_{i=1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,2)e^{-\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,2)w} + 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\lambda_j,\gamma_k}(1,1,2)e^{-\eta_{\lambda_i,\lambda_j,\gamma_k}(1,1,2)w}\\
&+\; 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\eta^{\alpha,\beta}_{\lambda_k,\gamma_i,\gamma_j}(2,1,1)e^{-\eta^{\alpha,\beta}_{\lambda_k,\gamma_i,\gamma_j}(2,1,1)w}\\
&+\; 4\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\sum_{i=+1}^{n-1}\sum_{j=i+1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\lambda_j\gamma_k,\gamma_l}e^{-\eta^{\alpha,\beta}_{\lambda_i,\lambda_j\gamma_k,\gamma_l}w}\Bigg),
\end{aligned}$$

where n and m represent the number of exponential distributions used to construct the hyper-exponential distribution, α_i and β_i represent the probability weight of the distribution and satisfy the following constraint $\sum_{i=1}^{n}\alpha_i = \sum_{j=1}^{m}\beta_j = 1$.

Proof. See Appendix A. □

2.3. The Last Survivor Status

The other common status is the last-survivor status. The last-survivor status is one that ends upon the death of both lives. That is, the status survives as long as at least one of the component members remains alive. The last-survivor status of two lives (x) and (y) will be denoted by $(\overline{xy})$, and the moment of death random variable is given by $T_{\overline{(xy)}} = \max(T_x,\ T_y)$.

Proposition 2. *If T_x and T_y follow hyper-exponential distributions with density functions,*

$$f_{T_x}(t) = \sum_{i=1}^{n}\alpha_i\lambda_i e^{-\lambda_i t},\ f_{T_y}(t) = \sum_{i=1}^{m}\beta_i\gamma_i e^{-\gamma_i t},\ \textit{for}\ \ t \geq 0.$$

Then, the density of $T_{\overline{(xy)}}$ is given by

$$\begin{aligned}
f_{\left(T_{\overline{(xy)}}\right)}(w) \;=\;& \sum_{i=1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(1,0)e^{-\eta_{\lambda_i,\gamma_k}(1,0)w} + (1-\theta)\sum_{i=1}^{m}\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(0,1)e^{-\eta_{\lambda_k,\gamma_i}(0,1)w} \\
&+\ \theta\left(\sum_{i=1}^{m}\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(0,2)e^{-\eta_{\lambda_k,\gamma_i}(0,2)w} + 2\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\eta^{\alpha,\beta}_{\lambda_i,\gamma_k,\gamma_l}(0,1,1)e^{-\eta_{\lambda_i,\gamma_k,\gamma_l}(0,1,1)w}\right) \\
&+\ \theta\left(\sum_{k=1}^{m}\sum_{i=1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,1)e^{-\eta_{\lambda_i,\gamma_k}(2,1)w} + 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\lambda_j,\gamma_k}(1,1,1)e^{-\eta_{\lambda_i,\lambda_j,\gamma_k}(1,1,1)w}\right) \\
&+\ \theta\left(\sum_{k=1}^{n}\sum_{i=1}^{m}\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(1,2)e^{-\eta_{\lambda_k,\gamma_i}(1,2)w} + 2\sum_{i=1}^{n}\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\eta^{\alpha,\beta}_{\lambda_i,\gamma_k,\gamma_l}(1,1,1)e^{-\eta_{\lambda_i,\gamma_k,\gamma_l}(1,1,1)w}\right) \\
&-\ \theta\left(\sum_{k=1}^{m}\sum_{i=1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,2)e^{-\eta_{\lambda_i,\gamma_k}(2,2)w} + 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\lambda_j,\gamma_k}(1,1,2)e^{-\eta_{\lambda_i,\lambda_j,\gamma_k}(1,1,2)w}\right. \\
&+\ 2\sum_{i=1}^{n}\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\eta^{\alpha,\beta}_{\lambda_i,\gamma_k,\gamma_l}(2,1,1)e^{-\eta_{\lambda_i,\gamma_k,\gamma_l}(2,1,1)w} \\
&+\ \left.4\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\lambda_j\gamma_k,\gamma_l}e^{-\eta_{\lambda_i,\lambda_j\gamma_k,\gamma_l}w}\right).
\end{aligned}$$

Proof. See Appendix A. □

2.4. Special Case of the Weighted Exponential Distribution

The pdf of the weighted exponential (WE) distribution is unimodal (contrary to the pdf of the exponential distribution) and the corresponding hazard rate function (hrf) is increasing for all values of t. It also possesses various likelihood ratio properties. Additionally, all of its moments can be calculated explicitly—it follows that the related mean, variance, skewness, kurtosis, coefficient of variation, etc. can be computed easily. The technical details can be found in Gupta and Kundu (2001) and Das and Kundu (2016) . On the practical side, the WE distribution is suitable for modelling lifetime data when wear-out or ageing is present, providing a real alternative to the exponential distribution for this aim. The success of this weighted version of the exponential distribution has inspired a generation of researchers and practitioners for more in this direction.

The following definitions can be found in Chesneau et al. (2022).

Definition 1 (Weighted Exponential distribution)**.**
The random variable T_x (respectively, T_y) is said to have WE distribution, with the shape and scale parameters as $\alpha > 0$ and $\lambda > 0$, respectively, if the PDF of T_y is

$$f_{T_x}(x;\alpha,\lambda) = \frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left(1-e^{-\alpha\lambda x}\right);\ x > 0 \quad \text{and}\ \ 0\ \ \text{otherwise}.$$

Respectively,

$$f_{T_y}(y;\alpha,\lambda) = \frac{\alpha+1}{\alpha}\lambda e^{-\lambda y}\left(1-e^{-\alpha\lambda y}\right);\ y > 0 \quad \text{and}\ \ 0\ \ \text{otherwise}.$$

We will denote it as $WE(\alpha,\ \lambda)$.

Corollary 1. *If T_x and T_y follow hyper-exponential distributions with density functions,*

$$\begin{aligned} f_{T_x}(t) &= \alpha_1\lambda_1 e^{-\lambda_1 t} + \alpha_2\lambda_2 e^{-\lambda_2 t}(1+\alpha)\lambda e^{-(1+\alpha)\lambda t},\ \ t \geq 0 \\ f_{T_y}(t) &= \beta_1\gamma_1 e^{-\gamma_1 t} + \beta_2\gamma_2 e^{-\gamma_2 t}(1+\beta)\gamma e^{-(1+\beta)\gamma t},\ \ t \geq 0 \end{aligned}$$

where

$$\begin{aligned} \alpha_1 = \frac{\alpha+1}{\alpha},\ \ \lambda_1 &= \lambda,\ \ \alpha_2 = -\frac{1}{\alpha},\ \ \lambda_2 = (1+\alpha)\lambda, \\ \beta_1 = \frac{\beta+1}{\beta};\ \ \gamma_1 &= \gamma,\ \ \beta_2 = -\frac{1}{\beta}\ \ \text{and}\ \ \gamma_2 = (1+\beta)\gamma, \end{aligned}$$

then

$$\begin{aligned} f_{T_{(x,y)}}(w) &= (1+\theta)\sum_{k=1}^{2}\sum_{i=1}^{2}\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(1,1)e^{-\eta_{\lambda_k,\gamma_i}(1,1)w} \\ &- \theta\left(\sum_{k=1}^{2}\sum_{i=1}^{2}\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,1)e^{-\eta_{\lambda_i,\gamma_k}(2,1)w} + 2\sum_{k=1}^{2}\eta^{\alpha,\beta}_{\gamma_k,\lambda_1,\lambda_2}(1,1,1)e^{-\eta_{\gamma_k,\lambda_1,\lambda_2}(1,1,1)w}\right) \\ &+ 2\theta\left(\sum_{k=1}^{2}\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(1,2)e^{-\eta_{\lambda_i,\gamma_k}(1,2)w} + 2\sum_{i=1}^{2}\eta^{\alpha,\beta}_{\lambda_i,\gamma_1,\gamma_2}(1,1,1)e^{-\eta_{\lambda_i,\gamma_1,\gamma_2}(1,1,1)w}\right) \\ &- 2\theta\left(\sum_{k=1}^{2}\sum_{i=1}^{2}\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,2)e^{-\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,2)w} + 2\sum_{k=1}^{2}\eta^{\alpha,\beta}_{\lambda_1,\lambda_1,\gamma_k}(1,1,2)e^{-\eta_{\lambda_1,\lambda_1,\gamma_k}(1,1,2)w}\right) \\ &- 4\left(\sum_{k=1}^{2}\eta^{\alpha,\beta}_{\lambda_k,\gamma_1,\gamma_2}(2,1,1)e^{-\eta^{\alpha,\beta}_{\lambda_k,\gamma_1,\gamma_1}(2,1,1)w} + 2\eta^{\alpha,\beta}_{\lambda_1,\lambda_2\gamma_1,\gamma_2}e^{-\eta^{\alpha,\beta}_{\lambda_1,\lambda_2\gamma_1,\gamma_2}w}\right). \end{aligned}$$

Proof. Replace the distribution of T_x and T_y in Proposition 1 with a weighted exponential to complete the proof. □

Corollary 2. *If T_x and T_y follow hyper-exponential distributions with density functions,*

$$\begin{aligned} f_{T_x}(t) &= \alpha_1\lambda_1 e^{-\lambda_1 t} + \alpha_2\lambda_2 e^{-\lambda_2 t}(1+\alpha)\lambda e^{-(1+\alpha)\lambda t},\ \ t \geq 0 \\ f_{T_y}(t) &= \beta_1\gamma_1 e^{-\gamma_1 t} + \beta_2\gamma_2 e^{-\gamma_2 t}(1+\beta)\gamma e^{-(1+\beta)\gamma t},\ \ t \geq 0, \end{aligned}$$

where

$$\begin{aligned} \alpha_1 = \frac{\alpha+1}{\alpha},\ \ \lambda_1 &= \lambda,\ \ \alpha_2 = -\frac{1}{\alpha},\ \ \lambda_2 = (1+\alpha)\lambda, \\ \beta_1 = \frac{\beta+1}{\beta};\ \ \gamma_1 &= \gamma,\ \ \beta_2 = -\frac{1}{\beta}\ \ \text{and}\ \ \gamma_2 = (1+\beta)\gamma, \end{aligned}$$

then

$$
\begin{aligned}
f_{\left(T_{\overline{(xy)}}\right)}(w) &= \sum_{i=1}^{2} \eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(1,0)e^{-\eta_{\lambda_i,\gamma_k}(1,0)w} + (1-\theta)\sum_{i=1}^{2} \eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(0,1)e^{-\eta_{\lambda_k,\gamma_i}(0,1)w} \\
&+ \theta\left(\sum_{i=1}^{2} \eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(0,2)e^{-\eta_{\lambda_k,\gamma_i}(0,2)w} + 2\eta^{\alpha,\beta}_{\lambda_i,\gamma_1,\gamma_2}(0,1,1)e^{-\eta_{\lambda_i,\gamma_1,\gamma_2}(0,1,1)w}\right) \\
&+ \theta\left(\sum_{k=1}^{2}\sum_{i=1}^{2} \eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,1)e^{-\eta_{\lambda_i,\gamma_k}(2,1)w} + 2\sum_{k=1}^{2} \eta^{\alpha,\beta}_{\lambda_1,\lambda_2,\gamma_k}(1,1,1)e^{-\eta_{\lambda_1,\lambda_2,\gamma_k}(1,1,1)w}\right) \\
&+ \theta\left(\sum_{k=1}^{2}\sum_{i=1}^{2} \eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(1,2)e^{-\eta_{\lambda_k,\gamma_i}(1,2)w} + 2\sum_{i=1}^{2} \eta^{\alpha,\beta}_{\lambda_i,\gamma_1,\gamma_2}(1,1,1)e^{-\eta_{\lambda_i,\gamma_1,\gamma_2}(1,1,1)w}\right) \\
&- \theta\left(\sum_{k=1}^{2}\sum_{i=1}^{2} \eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,2)e^{-\eta_{\lambda_i,\gamma_k}(2,2)w} + 2\sum_{k=1}^{2} \eta^{\alpha,\beta}_{\lambda_1,\lambda_2,\gamma_k}(1,1,2)e^{-\eta_{\lambda_1,\lambda_2,\gamma_k}(1,1,2)w}\right) \\
&+ 2\left(\sum_{i=1}^{2} \eta^{\alpha,\beta}_{\lambda_i,\gamma_1,\gamma_2}(2,1,1)e^{-\eta_{\lambda_i,\gamma_1,\gamma_2}(2,1,1)w} + 2\eta^{\alpha,\beta}_{\lambda_1,\lambda_2\gamma_1,\gamma_2}e^{-\eta_{\lambda_1,\lambda_2\gamma_1,\gamma_2}w}\right).
\end{aligned}
$$

Proof. Replace the distribution of T_x and T_y in Proposition 2 with a weighted exponential to complete the proof. □

3. Exponential Stopping of Brownian Motion

As in Gerber et al. (2012), let us define

$$X(t) = \mu t + \sigma W(t),\ t \geq 0, \tag{1}$$

where $W(t)$ is a standard Brownian motion (Wiener process), and μ and $\sigma > 0$ are constants. Further, let $M(t) = \max\{X(s) :\ 0 \leq s \leq t\}$ denote the running maximum of the process. Let $f_{X(t),M(t)}(x,y)$, $y \geq \max(x,0)$, denote the joint probability density function of $X(t)$ and $M(t)$. Then, the process $X(t)$ is stopped at time $T_{(x,y)}$ or $T_{(\overline{xy})}$, an independent random variable with density defined in Propositions (1) and (2). Hereafter, for simplicity of notation we will denote $T_{(x,y)}$ by τ and $T_{(\overline{xy})}$ by $\overline{\tau}$.

For $-\delta < \min_{1\leq i\leq n, 1\leq j\leq m}\{\lambda_i, \gamma_j\}$, we define the following functions:

$$
\begin{aligned}
f^{\delta}_{X(\tau),M(\tau)}(x,y) &= \int_0^\infty e^{-\delta t} f_{X(t),M(t)}(x,y) f_{\tau}(t)\mathrm{d}t \ \ y \geq \max(x,\ 0), && (2) \\
f^{\delta}_{X(\overline{\tau}),M(\overline{\tau})}(x,y) &= \int_0^\infty e^{-\delta t} f_{S(t),M(t)}(x,y) f_{\overline{\tau}}(t)\mathrm{d}t \ \ y \geq \max(x,\ 0). && (3)
\end{aligned}
$$

Such functions are referred to as discounted density functions, in the case of negative δ, the adjective inflated might be more appropriate.

Under the conditions of Proposition 1, the joint discounted density of $X(\tau)$ and $M(\tau)$ is given by

$$\begin{aligned}
\frac{\sigma^2}{2} f^{\delta}_{X(\tau),M(\tau)}(x,y) &= (1+\theta)\sum_{k=1}^{n}\sum_{i=1}^{m}\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(1,1)\exp\Big[-b^{(1)}_{i,k}x-\Big(a^{(1)}_{i,k}-b^{(1)}_{i,k}\Big)y\Big] \\
&- \theta\Bigg(\sum_{k=1}^{m}\sum_{i=1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,1)\exp\Big[-b^{(2)}_{i,k}x-\Big(a^{(2)}_{i,k}-b^{(2)}_{i,k}\Big)y\Big] \\
&+ 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\eta^{\alpha,\beta}_{\lambda_k,\gamma_i,\gamma_j}(1,1,1)\exp\Big[-b^{(3))}_{k,i,j}x-\Big(a^{(3)}_{k,i,j}-b^{(3)}_{k,i,j}\Big)y\Big]\Bigg) \\
&+ 2\theta\Bigg(\sum_{i=1}^{n}\sum_{k=1}^{m}\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(1,2)\exp\Big[-b^{(4)}_{i,k}x-\Big(a^{(4)}_{i,k}-b^{(4)}_{i,k}\Big)y\Big] \\
&+ 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\lambda_j,\gamma_k}(1,1,1)\exp\Big[-b^{(5))}_{k,i,j}x-\Big(a^{(5)}_{k,i,j}-b^{(5)}_{k,i,j}\Big)y\Big]\Bigg) \\
&- 2\theta\Bigg(\sum_{k=1}^{m}\sum_{i=1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,2)\exp\Big[-b^{(6)}_{k,i}x-\Big(a^{(6)}_{k,i}-b^{(6)}_{k,i}\Big)y\Big] \\
&+ 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\lambda_j,\gamma_k}(1,1,2)\exp\Big[-b^{(7)}_{j,k,i}x-\Big(a^{(7)}_{j,k,i}-b^{(7)}_{j,k,i}\Big)y\Big] \\
&+ 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\eta^{\alpha,\beta}_{\lambda_k,\gamma_i,\gamma_j}(2,1,1)\exp\Big[-b^{(8)}_{k,i,j}x-\Big(a^{(8)}_{k,i,j}-b^{(8)}_{k,i,j}\Big)y\Big] \\
&+ 4\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\sum_{i=+1}^{n-1}\sum_{j=i+1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\lambda_j\gamma_k,\gamma_l}\exp\Big[-b^{(9)}_{i,j,k,l}x-\Big(a^{(9)}_{i,j,k,l}-b^{(9)}_{i,j,k,l}\Big)y\Big]\Bigg),
\end{aligned}$$

where $a^{(n)}_{k,l,i,j} > 0, \quad b^{(n)}_{k,l,i,j} < 0$ are solutions of quadratic equations in Formula (A2) and $y \geq \max(x,0)$.

Let us recall the following formulas that can be found in the books of Jeanblanc et al. (2009) and Borodin and Salminen (2002):

$$\begin{aligned}
f_{X(t),M(t)}(x,y) &= \frac{2y-x}{2\sqrt{\pi D^3 t^3}}\exp\left(\frac{\mu x-\frac{1}{2}\mu^2 t-\frac{(2y-x)^2}{2t}}{2D}\right), \\
\int_0^\infty e^{-\eta t}\frac{a e^{-\frac{a^2}{2t}}}{\sqrt{2\pi t^3}}\mathrm{d}t &= e^{-a\sqrt{2\eta}}, \quad \eta \geq 0. \qquad (4)
\end{aligned}$$

Let us prove the first term of the formula in the above result for the discounted joint density. Let $D = \frac{\sigma^2}{2}$; from Equations (2) and (4) we have

$$\begin{aligned}
&\sum_{k=1}^{n}\sum_{i=1}^{m}\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(1,1)\int_0^\infty e^{-\delta t}\frac{2y-x}{2\sqrt{\pi D^3 t^3}}e^{\frac{\mu x-\frac{1}{2}\mu^2 t-\frac{(2y-x)^2}{2t}}{2D}}e^{-\eta_{\lambda_k,\gamma_i}(1,1)t}\mathrm{d}t \\
=& \frac{1}{D}\sum_{k=1}^{n}\sum_{i=1}^{m}\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(1,1)e^{\frac{\mu x}{2D}}\int_0^\infty e^{-\epsilon s}\frac{a}{\sqrt{2\pi s^3}}e^{-\frac{a^2}{2s}}\mathrm{d}t \\
=& \frac{1}{D}\sum_{k=1}^{n}\sum_{i=1}^{m}\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(1,1)e^{\frac{\mu x}{2D}}e^{-a\sqrt{2\epsilon}},
\end{aligned}$$

where $\epsilon = \frac{\mu^2}{4D^2} + \frac{\eta_{\lambda_k,\gamma_i}(1,1)+\delta}{D}$, $a = \frac{2y-x}{\sqrt{2}}$.

Now consider the quadratic equation,

$$D\zeta^2 + \mu\zeta - \big(\eta_{\lambda_k,\gamma_i}(1,1)+\delta\big) = 0;$$

its solution is given by

$$\begin{aligned}\Delta &= \mu^2 + 4D\big(\eta_{\lambda_k,\gamma_i}(1,1) + \delta\big)\\ a^{(1)}_{i,k} &= \frac{-\mu+\sqrt{\Delta}}{2D}, \qquad b^{(1)}_{i,k} = -\frac{\mu+\sqrt{\Delta}}{2D}\\ a^{(1)}_{i,k} + b^{(1)}_{i,k} &= -\frac{\mu}{D}, \qquad a^{(1)}_{i,k} - b^{(1)}_{i,k} = 2\sqrt{\epsilon}.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{D}\sum_{k=1}^{n}\sum_{i=1}^{m}\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(1,1)e^{\frac{\mu x}{2D}}e^{-a\sqrt{2\epsilon}} &= \frac{1}{D}\sum_{k=1}^{n}\sum_{i=1}^{m}\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(1,1)e^{\frac{-\left(a^{(1)}_{i,k}+b^{(1)}_{i,k}\right)x}{2}}e^{-(2y-x)\left(\frac{a^{(1)}_{i,k}-b^{(1)}_{i,k}}{2}\right)}\\ &= \frac{1}{D}\sum_{k=1}^{n}\sum_{i=1}^{m}\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(1,1)\exp\Big[-b^{(1)}_{i,k}x - \Big(a^{(1)}_{i,k} - b^{(1)}_{i,k}\Big)y\Big].\end{aligned}$$

This completes the proof.

It can similarly be proven that the discounted joint density $X(\overline{\tau})$ is given by

$$\begin{aligned}\frac{\sigma^2}{2}f^{\delta}_{X(\overline{\tau}),M(\overline{\tau})}(x,y) &= \sum_{i=1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(1,0)e^{-d^{(1)}_i x-\left(c^{(1)}_i-d^{(1)}_i\right)y} + (1-\theta)\sum_{i=1}^{m}\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(0,1)e^{-d^{(2)}_i x-\left(c^{(2)}_i-d^{(2)}_i\right)y}\\ &+ \theta\Bigg(\sum_{i=1}^{m}\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(0,2)e^{-d^{(3)}_i x-\left(c^{(3)}_i-d^{(3)}_i\right)y}\\ &+ 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\eta^{\alpha,\beta}_{\lambda_i,\gamma_k,\gamma_l}(0,1,1)e^{-d^{(4)}_{i,j}x-\left(c^{(4)}_{i,j}-d^{(4)}_{i,j}\right)y}\Bigg)\\ &+ \theta\Bigg(\sum_{k=1}^{m}\sum_{i=1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,1)\exp\Big[-d^{(5)}_{k,i}x-\Big(c^{(5)}_{k,i}-d^{(5)}_{k,i}\Big)y\Big]\\ &+ 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\lambda_j,\gamma_k}(1,1,1)\exp\Big[-d^{(6)}_{k,i,j}x-\Big(c^{(6)}_{k,i,j}-d^{(6)}_{k,i,j}\Big)y\Big]\Bigg)\\ &+ \theta\Bigg(\sum_{k=1}^{n}\sum_{i=1}^{m}\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(1,2)\exp\Big[-d^{(5)}_{k,i}x-\Big(c^{(7)}_{k,i}-d^{(7)}_{k,i}\Big)y\Big]\\ &+2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\eta^{\alpha,\beta}_{\lambda_i,\gamma_k,\gamma_l}(1,1,1)\exp\Big[-d^{(8)}_{k,i,j}x-\Big(c^{(8)}_{k,i,j}-d^{(8)}_{k,i,j}\Big)y\Big]\Bigg)\\ &- \theta\Bigg(\sum_{i=1}^{m}\sum_{j=1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,2)\exp\Big[-d^{(9)}_{i,j}x-\Big(c^{(9)}_{i,j}-d^{(9)}_{i,j}\Big)y\Big]\\ &+ 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\lambda_j,\gamma_k}(1,1,2)\exp\Big[-d^{(10)}_{k,i,j}x-\Big(c^{(10)}_{k,i,j}-d^{(10)}_{k,i,j}\Big)y\Big]\\ &+ 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\eta^{\alpha,\beta}_{\lambda_i,\gamma_k,\gamma_l}(2,1,1)\exp\Big[-d^{(11)}_{k,i,j}x-\Big(c^{(11)}_{k,i,j}-d^{(11)}_{k,i,j}\Big)y\Big]\\ &+ 4\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\eta^{\alpha,\beta}_{\lambda_i,\lambda_j\gamma_k,\gamma_l}e^{-d^{(12)}_{k,l,i,j}x-\left(c^{(12)}_{k,l,i,j}-d^{(12)}_{k,l,i,j}\right)y}\Bigg),\end{aligned}$$

where $c^{(n)}_{k,l,i,j} > 0$, $d^{(n)}_{k,l,i,j} < 0$ are solutions of quadratic equations in Formula (A3) and $y \geq \max(x,0)$.

Integrating over y (respectively over x) yields the discounted densities. Below, are the formulas

$$
\begin{aligned}
f^{\delta}_{X(\tau)}(x) \quad = \quad & (1+\theta)\sum_{k=1}^{n}\sum_{i=1}^{m}\chi^{(1)}_{i,k}e^{-a^{(1)}_{i,k}x\mathbb{1}_{x\geq 0}-b^{(1)}_{i,k}x\mathbb{1}_{x<0}} - \theta\Bigg(\sum_{k=1}^{m}\sum_{i=1}^{n}\chi^{(2)}_{i,k}e^{-a^{(2)}_{i,k}x\mathbb{1}_{x\geq 0}-b^{(2)}_{i,k}x\mathbb{1}_{x<0}} \\
+ \quad & 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\chi^{(3)}_{k,i,j}e^{-a^{(3))}_{k,i,j}x\mathbb{1}_{x\geq 0}-b^{(3)}_{k,i,j}x\mathbb{1}_{x<0}}\Bigg) + 2\theta\Bigg(\sum_{k=1}^{n}\sum_{i=1}^{m}\chi^{(4)}_{i,k}e^{-a^{(4)}_{i,k}x\mathbb{1}_{x\geq 0}-b^{(4)}_{i,k}x\mathbb{1}_{x<0}} \\
+ \quad & 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\chi^{(5)}_{k,i,j}e^{-a^{(5))}_{k,i,j}x\mathbb{1}_{x\geq 0}-b^{(5)}_{k,i,j}x\mathbb{1}_{x<0}}\Bigg) - 2\theta\sum_{k=1}^{m}\sum_{i=1}^{n}\chi^{(6)}_{k,i}e^{-a^{(6)}_{i,j}x\mathbb{1}_{x\geq 0}-b^{(6)}_{k,i}x\mathbb{1}_{x<0}} \\
- \quad & 4\theta\Bigg(\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\chi^{(7)}_{j,k,i}e^{-a^{(7)}_{j,k,i}x\mathbb{1}_{x\geq 0}-b^{(7)}_{j,k,i}x\mathbb{1}_{x<0}} + \sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\chi^{(8)}_{k,i,j}e^{-a^{(8)}_{k,i,j}x\mathbb{1}_{x\geq 0}-b^{(8)}_{k,i,j}x\mathbb{1}_{x<0}}\Bigg) \\
- \quad & 8\theta\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\sum_{i=+1}^{n-1}\sum_{j=i+1}^{n}\chi^{(9)}_{i,j,k,l}\exp\Big[-a^{(9)}_{i,j,k,l}x\mathbb{1}_{x\geq 0}-b^{(9)}_{i,j,k,l}x\mathbb{1}_{x<0}\Big], \qquad (5)
\end{aligned}
$$

and

$$
\begin{aligned}
(-1)\times f^{\delta}_{M(\tau)}(y) \quad = \quad & (1+\theta)\sum_{k=1}^{n}\sum_{i=1}^{m}\frac{\chi^{(1)}_{i,k}}{b^{(1)}_{i,k}}e^{-a^{(1)}_{i,k}y}\mathbb{1}_{y\geq 0} - \theta\Bigg(\sum_{k=1}^{m}\sum_{i=1}^{n}\frac{\chi^{(2)}_{i,k}}{b^{(2)}_{i,k}}e^{-a^{(2)}_{i,k}y}\mathbb{1}_{y\geq 0} \\
+ \quad & 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\frac{\chi^{(3)}_{k,i,j}}{b^{(3)}_{k,i,j}}e^{-a^{(3))}_{k,i,j}y}\mathbb{1}_{y\geq 0}\Bigg) + 2\theta\Bigg(\sum_{k=1}^{n}\sum_{i=1}^{m}\frac{\chi^{(5)}_{i,k}}{b^{(4)}_{i,k}}e^{-a^{(4)}_{i,k}y}\mathbb{1}_{y\geq 0} \\
+ \quad & 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\chi^{(5)}_{k,i,j}}{b^{(5)}_{k,i,j}}e^{-a^{(5))}_{k,i,j}y}\mathbb{1}_{y\geq 0}\Bigg) - 2\theta\sum_{k=1}^{m}\sum_{i=1}^{n}\frac{\chi^{(6)}_{k,i}}{b^{(6)}_{k,i}}e^{-a^{(6)}_{k,i}y}\mathbb{1}_{y\geq 0} \\
- \quad & 4\theta\Bigg(\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\chi^{(7)}_{j,k,i}}{b^{(7)}_{j,k,i}}e^{-a^{(7)}_{j,k,i}y}\mathbb{1}_{y\geq 0} + \sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\frac{\chi^{(8)}_{k,i,j}}{b^{(8)}_{k,i,j}}e^{-a^{(8)}_{k,i,j}y}\mathbb{1}_{y\geq 0}\Bigg) \\
- \quad & 8\theta\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\sum_{i=+1}^{n-1}\sum_{j=i+1}^{n}\frac{\chi^{(9)}_{i,j,k,l}}{b^{(9)}_{i,j,k,l}}\exp\Big[-a^{(9)}_{i,j,k,l}y\Big]\mathbb{1}_{y\geq 0}, \qquad (6)
\end{aligned}
$$

where

$$
\begin{aligned}
\chi^{(1)}_{i,k} \quad &= \quad \frac{2\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(1,1)}{\sigma^2\Big(a^{(1)}_{i,k}-b^{(1)}_{i,k}\Big)}, \quad \chi^{(2)}_{i,k} = \frac{2\beta_k\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,1)}{\sigma^2\Big(a^{(2)}_{i,k}-b^{(2)}_{i,k}\Big)}, \quad \chi^{(3)}_{k,i,j} = \frac{2\eta^{\alpha,\beta}_{\lambda_k,\gamma_i,\gamma_j}(1,1,1)}{\sigma^2\Big(a^{(3)}_{k,i,j}-b^{(3)}_{k,i,j}\Big)}, \\
\chi^{(4)}_{i,k} \quad &= \quad \frac{2\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(1,2)}{\sigma^2\Big(a^{(4)}_{i,k}-b^{(4)}_{i,k}\Big)}, \quad \chi^{(5)}_{k,i,j} = \frac{2\eta^{\alpha,\beta}_{\lambda_i,\lambda_j,\gamma_k}(1,1,1)}{\sigma^2\Big(a^{(5)}_{k,i,j}-b^{(5)}_{k,i,j}\Big)}, \quad \chi^{(6)}_{k,i} = \frac{2\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,2)}{\sigma^2\Big(a^{(6)}_{k,i}-b^{(6)}_{k,i}\Big)}, \\
\chi^{(7)}_{j,k,i} \quad &= \quad \frac{2\eta^{\alpha,\beta}_{\lambda_i,\lambda_j,\gamma_k}(1,1,2)}{\sigma^2\Big(a^{(7)}_{k,i,j}-b^{(7)}_{k,i,j}\Big)}, \ \chi^{(8)}_{k,i,j} \ = \ \frac{2\eta^{\alpha,\beta}_{\lambda_k,\gamma_i,\gamma_j}(2,1,1)}{\sigma^2\Big(a^{(8)}_{k,i,j}-b^{(8)}_{k,i,j}\Big)}, \quad \chi^{(9)}_{i,j,k,l} \ = \ \frac{2\eta^{\alpha,\beta}_{\lambda_i,\lambda_j\gamma_k,\gamma_l}}{\sigma^2\Big(a^{(9)}_{i,j,k,l}-b^{(9)}_{i,j,k,l}\Big)}. \qquad (7)
\end{aligned}
$$

Similarly, when we consider the last survival distribution, we have

$$
\begin{aligned}
f^{\delta}_{X(\overline{\tau})}(x) &= \sum_{i=1}^{n} \kappa_i^{(1)} e^{-c_i^{(1)} x \mathbb{1}_{x\geq 0} - d_i^{(1)} x \mathbb{1}_{x<0}} + (1-\theta) \sum_{i=1}^{m} \kappa_i^{(2)} e^{-c_i^{(2)} x \mathbb{1}_{x\geq 0} - d_i^{(2)} x \mathbb{1}_{x<0}} \\
&+ \theta \left(\sum_{i=1}^{m} \kappa_i^{(3)} e^{-c_i^{(0)} x \mathbb{1}_{x\geq 0} - d_i^{(3)} x \mathbb{1}_{x<0}} + 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \kappa_{i,j}^{(4)} e^{-c_{i,j}^{(4)} x \mathbb{1}_{x\geq 0} - d_{i,j}^{(4)} x \mathbb{1}_{x<0}} \right) \\
&+ \theta \left(\sum_{k=1}^{m} \sum_{i=1}^{n} \kappa_{k,i}^{(5)} e^{-c_{k,i}^{(5)} x \mathbb{1}_{x\geq 0} - d_{k,i}^{(5)} x \mathbb{1}_{x<0}} + 2 \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \kappa_{k,i,j}^{(6)} e^{-c_{k,i,j}^{(6)} x \mathbb{1}_{x\geq 0} - d_{k,i,j}^{(6)} x \mathbb{1}_{x<0}} \right) \\
&+ \theta \left(\sum_{k=1}^{n} \sum_{i=1}^{m} \kappa_{k,i}^{(7)} e^{-c_{k,i}^{(7)} x \mathbb{1}_{x\geq 0} - d_{k,i}^{(7)} x \mathbb{1}_{x<0}} + 2 \sum_{k=1}^{n} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \kappa_{k,i,j}^{(8)} e^{-c_{k,i,j}^{(8)} x \mathbb{1}_{x\geq 0} - d_{k,i,j}^{(8)} x \mathbb{1}_{x<0}} \right) \\
&- \theta \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \kappa_{i,j}^{(9)} e^{-c_{i,j}^{(9)} x \mathbb{1}_{x\geq 0} - d_{i,j}^{(9)} x \mathbb{1}_{x\geq 0}} + 2 \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \kappa_{k,i,j}^{(10)} e^{-c_{k,i,j}^{(10)} x \mathbb{1}_{x\geq 0} - d_{k,i,j}^{(10)} x \mathbb{1}_{x<0}} \right. \\
&+ 2 \sum_{k=1}^{n} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \kappa_{k,i,j}^{(11)} \exp\left[-c_{k,i,j}^{(11)} x \mathbb{1}_{x\geq 0} - d_{k,i,j}^{(11)} x \mathbb{1}_{x<0} \right] \\
&+ \left. 4 \sum_{k=1}^{m-1} \sum_{l=k+1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \kappa_{k,l,i,j}^{(12)} \exp\left[-c_{k,l,i,j}^{(12)} x \mathbb{1}_{x\geq 0} - d_{k,l,i,j}^{(12)} x \mathbb{1}_{x<0} \right] \right)
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
(-1) \times f^{\delta}_{M(\overline{\tau})}(y) &= \sum_{i=1}^{n} \frac{\kappa_i^{(1)}}{d_i^{(1)}} e^{-c_i^{(1)} y} \mathbb{1}_{y\geq 0} + (1-\theta) \sum_{i=1}^{m} \frac{\kappa_i^{(2)}}{d_i^{(2)}} e^{-c_i^{(2)} y} \mathbb{1}_{x\geq 0} \\
&+ \theta \left(\sum_{i=1}^{m} \frac{\kappa_i^{(3)}}{d_i^{(3)}} e^{-c_i^{(0)} y} \mathbb{1}_{y\geq 0} + 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{\kappa_{i,j}^{(4)}}{d_{i,j}^{(4)}} e^{-c_{i,j}^{(4)} y} \mathbb{1}_{y\geq 0} \right) \\
&+ \theta \left(\sum_{k=1}^{m} \sum_{i=1}^{n} \frac{\kappa_{k,i}^{(5)}}{d_{k,i}^{(5)}} e^{-c_{k,i}^{(5)} y} \mathbb{1}_{y\geq 0} + 2 \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\kappa_{k,i,j}^{(6)}}{d_{k,i,j}^{(6)}} e^{-c_{k,i,j}^{(6)} y} \mathbb{1}_{y\geq 0} \right) \\
&+ \theta \left(\sum_{k=1}^{n} \sum_{i=1}^{m} \frac{\kappa_{k,i}^{(7)}}{d_{k,i}^{(7)}} e^{-c_{k,i}^{(7)} y} \mathbb{1}_{y\geq 0} + 2 \sum_{k=1}^{n} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{\kappa_{k,i,j}^{(8)}}{d_{k,i,j}^{(8)}} e^{-c_{k,i,j}^{(8)} y} \mathbb{1}_{y\geq 0} \right) \\
&- \theta \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\kappa_{i,j}^{(9)}}{d_{i,j}^{(9)}} e^{-c_{i,j}^{(9)} y} \mathbb{1}_{y\geq 0} + 2 \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\kappa_{k,i,j}^{(10)}}{d_{k,i,j}^{(10)}} e^{-c_{k,i,j}^{(10)} y} \mathbb{1}_{x\geq 0} \right. \\
&+ 2 \sum_{k=1}^{n} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{\kappa_{k,i,j}^{(11)}}{d_{k,i,j}^{(11)}} \exp\left[-c_{k,i,j}^{(11)} y \right] \mathbb{1}_{y\geq 0} \\
&+ \left. 4 \sum_{k=1}^{m-1} \sum_{l=k+1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\kappa_{k,l,i,j}^{(12)}}{d_{k,l,i,j}^{(12)}} \exp\left[-c_{k,l,i,j}^{(12)} y \right] \mathbb{1}_{y\geq 0} \right),
\end{aligned} \tag{9}
$$

where

$$
\begin{aligned}
\kappa_i^{(1)} &= \frac{2\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(1,0)}{\sigma^2\left(c_i^{(1)} - d_i^{(1)}\right)}, \quad \kappa_i^{(2)} = \frac{2\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(0,1)}{\sigma^2\left(c_i^{(2)} - d_i^{(2)}\right)}, \quad \kappa_i^{(3)} = \frac{2\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(0,2)}{\sigma^2\left(c_i^{(3)} - d_i^{(3)}\right)}, \\
\kappa_{i,j}^{(4)} &= \frac{2\eta^{\alpha,\beta}_{\lambda_i,\gamma_k,\gamma_l}(0,1,1)}{\sigma^2\left(c_{i,j}^{(4)} - d_{i,j}^{(4)}\right)}, \quad \kappa_{k,i}^{(5)} = \frac{2\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,1)}{\sigma^2\left(c_{k,i}^{(5)} - d_{k,i}^{(5)}\right)}, \quad \kappa_{k,i,j}^{(6)} = \frac{2\eta^{\alpha,\beta}_{\lambda_i,\lambda_j,\gamma_k}(1,1,1)}{\sigma^2\left(c_{k,i,j}^{(6)} - d_{k,i,j}^{(6)}\right)}, \\
\kappa_{k,i}^{(7)} &= \frac{2\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(1,2)}{\sigma^2\left(c_{k,i}^{(7)} - d_{k,i}^{(7)}\right)}, \quad \kappa_{k,i,j}^{(8)} = \frac{2\eta^{\alpha,\beta}_{\lambda_i,\gamma_k,\gamma_l}(1,1,1)}{\sigma^2\left(c_{k,i,j}^{(8)} - d_{k,i,j}^{(8)}\right)}, \quad \kappa_{i,j}^{(9)} = \frac{2\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,2)}{\sigma^2\left(c_{i,j}^{(9)} - d_{i,j}^{(9)}\right)}, \\
\kappa_{k,i,j}^{(10)} &= \frac{2\eta^{\alpha,\beta}_{\lambda_i,\lambda_j,\gamma_k}(1,1,2)}{\sigma^2\left(c_{k,i,j}^{(10)} - d_{k,i,j}^{(10)}\right)}, \quad \kappa_{k,i,j}^{(11)} = \frac{2\eta^{\alpha,\beta}_{\lambda_i,\gamma_k,\gamma_l}(2,1,1)}{\sigma^2\left(c_{k,i,j}^{(11)} - d_{k,i,j}^{(11)}\right)}, \quad \kappa_{k,l,i,j}^{(12)} = \frac{2\eta^{\alpha,\beta}_{\lambda_i,\lambda_j\gamma_k,\gamma_l}}{\sigma^2\left(c_{k,l,i,j}^{(12)} - d_{k,l,i,j}^{(12)}\right)}.
\end{aligned} \tag{10}
$$

(i) By letting $n = m = 2$ in Equations (5) and (6), we obtain the discounted densities for weighted exponential distribution in the case of joint alive status;

(ii) By letting $n = m = 2$ in Equations (8) and (9), we obtain the discounted densities for weighted exponential distribution in the case of last survival status.

As we have a closed expression of the discounted density, in both (joint life and last survival) scenarios, we are able to valuate some financial contingent claims and give a closed-analytical expressions. In the following section, we will analyse some basic options and derive some closed-expression of their prices.

4. Valuation of Basic Options

Hereafter, $S(t)$ represents the underlying stock price process defined by

$$S(t) \;=\; S(0)e^{X(t)}, \tag{11}$$

where $X(t)$ is the linear Brownian motion defined in Equation (1). As shown in Gerber et al. (2012), we have

$$\mathbf{E}[S(t)] \;=\; S(0)e^{\nu t},$$

where

$$\nu \;=\; \mu + \frac{\sigma^2}{2}. \tag{12}$$

The functions $b(S(\tau)), b(S(\overline{\tau}))$ represent the payoff of the financial contingent claim, then, valuing GMDB entails determining

$$\mathbf{E}\Big[e^{-\delta\tau} b(S(\tau))\Big] \quad \text{and} \quad \mathbf{E}\Big[e^{-\delta\overline{\tau}} b(S(\overline{\tau}))\Big]. \tag{13}$$

For a different type of option payoff, we can find the corresponding function $b()$ (see Gerber et al. 2012).

By assumption, $S(t)$ and T_x or T_y are independent. So $S(t)$ and τ or $\overline{\tau}$ are also independent. Thus, Equation (13) becomes

$$\begin{aligned} \mathbf{E}\Big[e^{-\delta\tau} b(S(\tau))\Big] &= \int_{-\infty}^{\infty} b(s(0)e^x) f^{\delta}_{X(\tau)}(x)\mathrm{d}x, \\ \mathbf{E}\Big[e^{-\delta\overline{\tau}} b(S(\overline{\tau}))\Big] &= \int_{-\infty}^{\infty} b(s(0)e^x) f^{\delta}_{X(\overline{\tau})}(x)\mathrm{d}x. \end{aligned} \tag{14}$$

For the special case where $b(x) = x$ we have

$$\begin{aligned} \mathbf{E}\Big[e^{-\delta\tau} b(S(\tau))\Big] \;=\; & \Bigg[(1+\theta)\sum_{k=1}^{n}\sum_{i=1}^{m} \frac{\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(1,1)}{\delta + \eta_{\lambda_k,\gamma_i}(1,1) \;-\nu} - \theta\Bigg(\sum_{k=1}^{m}\sum_{i=1}^{n} \frac{\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,1)}{\delta + \eta_{\lambda_i,\gamma_k}(2,1) - \nu} \\ &+ 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \frac{\eta^{\alpha,\beta}_{\lambda_k,\gamma_i,\gamma_j}(1,1,1)}{\delta + \eta_{\lambda_k,\gamma_i,\gamma_j}(1,1,1) - \nu}\Bigg) + 2\theta\Bigg(\sum_{i=1}^{n}\sum_{k=1}^{m} \frac{\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(1,2)}{\delta + \eta_{\lambda_i,\gamma_k}(1,2) - \nu} \\ &+ 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{\eta^{\alpha,\beta}_{\lambda_i,\lambda_j,\gamma_k}(1,1,1)}{\delta + \eta_{\lambda_i,\lambda_j,\gamma_k}(1,1,1) - \nu}\Bigg) - 2\theta\sum_{k=1}^{m}\sum_{i=1}^{n} \frac{\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,2)}{\delta + \eta_{\lambda_i,\gamma_k}(2,2) - \nu} \\ &- 4\theta\Bigg(\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{\eta^{\alpha,\beta}_{\lambda_i,\lambda_j,\gamma_k}(1,1,2)}{\delta + \eta_{\lambda_i,\lambda_j,\gamma_k}(1,1,2) - \nu} \\ &+ \sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \frac{\eta^{\alpha,\beta}_{\lambda_k,\gamma_i,\gamma_j}(2,1,1)}{\delta + \eta_{\lambda_k,\gamma_i,\gamma_j}(2,1,1) - \nu}\Bigg) \\ &- 8\theta\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\sum_{i=+1}^{n-1}\sum_{j=i+1}^{n} \frac{\eta^{\alpha,\beta}_{\lambda_i,\lambda_j\gamma_k,\gamma_l}}{\delta + \eta_{\lambda_i,\lambda_j\gamma_k,\gamma_l} - \nu}\Bigg] S(0). \end{aligned} \tag{15}$$

Proof. Without loss of generality, we will only prove the first part of Equation (15):

$$
\begin{aligned}
\mathbf{E}\Big[e^{-\delta\tau}b(S(\tau))\Big] &= \int_{-\infty}^{\infty} b(s(0)e^{x}) f^{\delta}_{X(\tau)}(x)\mathrm{d}x \\
&= S(0)\sum_{k=1}^{n}\sum_{i=1}^{m}\chi^{(1)}_{i,k}\left[\frac{1}{1-b^{1}_{i,k}}+\frac{1}{a^{(1)}_{i,k}-1}\right] \\
&= S(0)\sum_{k=1}^{n}\sum_{i=1}^{m}\frac{\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(1,1)}{D\left(1-b^{1}_{i,k}\right)\left(a^{(1)}_{i,k}-1\right)} \\
&= S(0)\sum_{k=1}^{n}\sum_{i=1}^{m}\frac{\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(1,1)}{\delta+\eta_{\lambda_k,\gamma_i}(1,1)}\times\frac{-a^{(1)}_{i,k}b^{(1)}_{i,k}}{\left(1-b^{1}_{i,k}\right)\left(a^{(1)}_{i,k}-1\right)} \\
&= S(0)\sum_{k=1}^{n}\sum_{i=1}^{m}\frac{\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(1,1)}{\delta+\eta_{\lambda_k,\gamma_i}(1,1)-\nu},
\end{aligned}
$$

where $D=\frac{\sigma^2}{2}$. □

If we consider the last survival, we get

$$
\begin{aligned}
\mathbf{E}\Big[e^{-\delta\overline{\tau}}b(S(\overline{\tau}))\Big] &= \Bigg[\sum_{i=1}^{n}\frac{\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(1,0)}{\delta+\eta_{\lambda_i,\gamma_k}(1,0)-\nu}+(1-\theta)\sum_{i=1}^{m}\frac{\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(0,1)}{\delta+\eta_{\lambda_k,\gamma_i}(0,1)-\nu} \\
&+ \theta\left(\sum_{i=1}^{m}\frac{2\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(0,2)}{\delta+\eta_{\lambda_k,\gamma_i}(0,2)-\nu}+2\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\frac{\eta^{\alpha,\beta}_{\lambda_i,\gamma_k,\gamma_l}(0,1,1)}{\delta+\eta_{\lambda_i,\gamma_k,\gamma_l}(0,1,1)-\nu}\right) \\
&+ \theta\left(\sum_{k=1}^{m}\sum_{i=1}^{n}\frac{\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,1)}{\delta+\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,1)-\nu}+2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\eta^{\alpha,\beta}_{\lambda_i,\lambda_j,\gamma_k}(1,1,1)}{\delta+\eta^{\alpha,\beta}_{\lambda_i,\lambda_j,\gamma_k}(1,1,1)-\nu}\right) \\
&+ \theta\left(\sum_{k=1}^{n}\sum_{i=1}^{m}\frac{\eta^{\alpha,\beta}_{\lambda_k,\gamma_i}(1,2)}{\delta+\eta_{\lambda_k,\gamma_i}(1,2)-\nu}+2\sum_{i=1}^{n}\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\frac{\eta^{\alpha,\beta}_{\lambda_i,\gamma_k,\gamma_l}(1,1,1)}{\delta+\eta_{\lambda_i,\gamma_k,\gamma_l}(1,1,1)-\nu}\right) \\
&- \theta\left(\sum_{k=1}^{m}\sum_{j=1}^{n}\frac{\eta^{\alpha,\beta}_{\lambda_i,\gamma_k}(2,2)}{\delta+\eta_{\lambda_i,\gamma_k}(2,2)-\nu}+2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\eta^{\alpha,\beta}_{\lambda_i,\lambda_j,\gamma_k}(1,1,2)}{\delta+\eta_{\lambda_i,\lambda_j,\gamma_k}(1,1,2)-\nu}\right. \\
&+ 2\sum_{i=1}^{n}\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\frac{\eta^{\alpha,\beta}_{\lambda_i,\gamma_k,\gamma_l}(2,1,1)}{\delta+\eta^{\alpha,\beta}_{\lambda_i,\gamma_k,\gamma_l}(2,1,1)-\nu} \\
&+ \left.4\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\eta^{\alpha,\beta}_{\lambda_i,\lambda_j\gamma_k,\gamma_l}}{\delta+\eta_{\lambda_i,\lambda_j\gamma_k,\gamma_l}-\nu}\right)\Bigg]S(0).
\end{aligned}
\tag{16}
$$

Remark 1.

(1) *An equivalent formula similar to Equations (15) and (16) can be obtained if T_x and T_y follow Weighted exponential distributions;*

(2) *If $\nu=\delta$, it is straightforward to show that $\mathbf{E}[e^{-\tau}S(\tau)]=\mathbf{E}\big[e^{-\overline{\tau}}S(\overline{\tau})\big]=S(0)$ which is the result in risk neutral pricing framework when δ represents the risk-free interest rate and the market is complete.*

Hereafter, we denote $\mathbf{E}\big[e^{-\delta\tau}b(S(\tau))\big]:=\pi_1$ and $\mathbf{E}\big[e^{-\delta\overline{\tau}}b(S(\overline{\tau}))\big]:=\pi_2$

4.1. Out-of-Money All-or-Nothing Call Option

The payoff of this type of contract is given by

$$
b(x)=x^{n_0}\mathbb{1}_{(x>K)},
\tag{17}
$$

where K is the strike price.

If we consider this type of financial contract, the price of the joint life contract is given by

$$
\begin{aligned}
\pi_1 \quad = \quad & (1+\theta)\sum_{k=1}^{n}\sum_{i=1}^{m}\frac{\chi^{(1)}_{i,k}}{a^{(1)}_{i,k}-n_0}K^{n_0}\left(\frac{S(0)}{K}\right)^{a^{(1)}_{i,k}} - \theta\left(\sum_{k=1}^{m}\sum_{i=1}^{n}\frac{\chi^{(2)}_{i,k}}{a^{(2)}_{i,k}-n_0}K^{n_0}\left(\frac{S(0)}{K}\right)^{a^{(2)}_{i,k}}\right.\\
+ \quad & \left. 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\frac{\chi^{(3)}_{k,i,j}}{a^{(3)}_{k,i,j}-n_0}K^{n_0}\left(\frac{S(0)}{K}\right)^{a^{(3)}_{k,i,j}}\right) + 2\theta\left(\sum_{k=1}^{n}\sum_{i=1}^{m}\frac{\chi^{(4)}_{i,k}}{a^{(4)}_{i,k}-n_0}K^{n_0}\left(\frac{S(0)}{K}\right)^{a^{(4)}_{i,k}}\right.\\
+ \quad & \left. 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\chi^{(5)}_{k,i,j}}{a^{(5)}_{k,i,j}-n_0}K^{n_0}\left(\frac{S(0)}{K}\right)^{a^{(5)}_{k,i,j}}\right) - 2\theta\sum_{k=1}^{m}\sum_{i=1}^{n}\frac{\chi^{(6)}_{k,i}}{a^{(6)}_{k,i}-n_0}K^{n_0}\left(\frac{S(0)}{K}\right)^{a^{(6)}_{k,i}}\\
- \quad & 4\theta\left(\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\chi^{(7)}_{j,k,i}}{a^{(7)}_{j,k,i}-n_0}K^{n_0}\left(\frac{S(0)}{K}\right)^{a^{(7)}_{j,k,i}} + \sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\frac{\chi^{(8)}_{k,i,j}}{a^{(8)}_{k,i,j}-n_0}K^{n_0}\left(\frac{S(0)}{K}\right)^{a^{(8)}_{k,i,j}}\right)\\
- \quad & 8\theta\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\sum_{i=+1}^{n-1}\sum_{j=i+1}^{n}\frac{\chi^{(9)}_{i,j,k,l}}{a^{(9)}_{i,j,k,l}-n_0}K^{n_0}\left(\frac{S(0)}{K}\right)^{a^{(9)}_{i,j,k,l}}. \qquad (18)
\end{aligned}
$$

This formula is valid if n_0 is less than the smallest non-negative root of Equation (A2).

Similarly, if we consider the last survival case and if n_0 does not exceed the smallest non-negative root of Equation (A3) then, the price of this financial contract becomes

$$
\begin{aligned}
\pi_2 \quad = \quad & \sum_{i=1}^{n}\frac{\kappa^{(1)}_{i}}{c^{(1)}_{i}-n_0}K^{n_0}\left(\frac{S(0)}{K}\right)^{c^{(1)}_{i}} + (1-\theta)\sum_{i=1}^{m}\frac{\kappa^{(2)}_{i}}{c^{(2)}_{i}-n_0}K^{n_0}\left(\frac{S(0)}{K}\right)^{c^{(2)}_{i}}\\
+ \quad & \theta\left(\sum_{i=1}^{m}\frac{\kappa^{(3)}_{i}}{c^{(3)}_{i}-n_0}K^{n_0}\left(\frac{S(0)}{K}\right)^{c^{(3)}_{i}} + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\frac{\kappa^{(4)}_{i,j}}{c^{(4)}_{i,j}-n_0}K^{n_0}\left(\frac{S(0)}{K}\right)^{c^{(4)}_{i,j}}\right)\\
+ \quad & \theta\left(\sum_{k=1}^{m}\sum_{i=1}^{n}\frac{\kappa^{(5)}_{k,i}}{c^{(5)}_{k,i}-n_0}K^{n_0}\left(\frac{S(0)}{K}\right)^{c^{(5)}_{k,i}} + 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\kappa^{(6)}_{k,i,j}}{c^{(6)}_{k,i,j}-n_0}K^{n_0}\left(\frac{S(0)}{K}\right)^{c^{(6)}_{k,i,j}}\right)\\
+ \quad & \theta\left(\sum_{k=1}^{n}\sum_{i=1}^{m}\frac{\kappa^{(7)}_{k,i}}{c^{(7)}_{k,i}-n_0}K^{n_0}\left(\frac{S(0)}{K}\right)^{c^{(7)}_{i}} + 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\frac{\kappa^{(8)}_{k,i,j}}{c^{(8)}_{k,i,j}-n_0}K^{n_0}\left(\frac{S(0)}{K}\right)^{c^{(8)}_{k,i,j}}\right)\\
- \quad & \theta\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\kappa^{(9)}_{i,j}}{c^{(9)}_{i,j}-n_0}K^{n_0}\left(\frac{S(0)}{K}\right)^{c^{(9)}_{i,j}} + 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\kappa^{(10)}_{k,i,j}}{c^{(10)}_{k,i,j}-n_0}K^{n_0}\left(\frac{S(0)}{K}\right)^{c^{(10)}_{k,i,j}}\right.\\
+ \quad & 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\frac{\kappa^{(11)}_{k,i,j}}{c^{(11)}_{k,i,j}-n_0}K^{n_0}\left(\frac{S(0)}{K}\right)^{c^{(11)}_{k,i,j}}\\
+ \quad & \left. 4\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\kappa^{(12)}_{k,l,i,j}}{c^{(12)}_{k,l,i,j}-n_0}K^{n_0}\left(\frac{S(0)}{K}\right)^{c^{(12)}_{k,l,i,j}}\right). \qquad (19)
\end{aligned}
$$

4.2. At-the-Money All-or-Nothing Call Option

At-the-money option, the stock price at time 0 equal the strike price. Under this condition, Equations (18) and (19) become

$$
\begin{aligned}
\pi_1 \quad = \quad & (1+\theta)\sum_{k=1}^{n}\sum_{i=1}^{m}\frac{\chi^{(1)}_{i,k}K^{n_0}}{a^{(1)}_{i,k}-n_0} - \theta\left(\sum_{k=1}^{m}\sum_{i=1}^{n}\frac{\chi^{(2)}_{i,k}K^{n_0}}{a^{(2)}_{i,k}-n_0} + 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\frac{\chi^{(3)}_{k,i,j}}{a^{(3)}_{k,i,j}-n_0}\right)\\
+ \quad & 2\theta\left(\sum_{k=1}^{n}\sum_{i=1}^{m}\frac{\chi^{(4)}_{i,k}K^{n_0}}{a^{(4)}_{i,k}-n_0} + 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\chi^{(5)}_{k,i,j}}{a^{(5)}_{k,i,j}-n_0}\right) - 2\theta\sum_{k=1}^{m}\sum_{i=1}^{n}\frac{\chi^{(6)}_{k,i}K^{n_0}}{a^{(6)}_{k,i}-n_0}\\
- \quad & 4\theta\left(\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\chi^{(7)}_{j,k,i}K^{n_0}}{a^{(7)}_{j,k,i}-n_0} + \sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\frac{\chi^{(8)}_{k,i,j}K^{n_0}}{a^{(8)}_{k,i,j}-n_0}2\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\chi^{(9)}_{i,j,k,l}K^{n_0}}{a^{(9)}_{i,j,k,l}-n_0}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\pi_2 \quad = \quad & \sum_{i=1}^{n} \frac{\kappa_i^{(1)} K^{n_0}}{c_i^{(1)} - n_0} + (1-\theta) \sum_{i=1}^{n} \frac{\kappa_i^{(2)} K^{n_0}}{c_i^{(2)} - n_0} + \theta \left(\sum_{i=1}^{m} \frac{\kappa_i^{(3)} K^{n_0}}{c_i^{(3)} - n_0} + 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{\kappa_{i,j}^{(4)} K^{n_0}}{c_{i,j}^{(4)} - n_0} \right) \\
+ \quad & \theta \left(\sum_{k=1}^{m} \sum_{i=1}^{n} \frac{\kappa_{k,i}^{(5)} K^{n_0}}{c_{k,i}^{(5)} - n_0} + 2 \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\kappa_{k,i,j}^{(6)} K^{n_0}}{c_{k,i,j}^{(6)} - n_0} \right) \\
+ \quad & \theta \left(\sum_{k=1}^{n} \sum_{i=1}^{m} \frac{\kappa_{k,i}^{(7)} K^{n_0}}{c_{k,i}^{(7)} - n_0} + 2 \sum_{k=1}^{n} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{\kappa_{k,i,j}^{(8)} K^{n_0}}{c_{k,i,j}^{(8)} - n_0} \right) \\
- \quad & \theta \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\kappa_{i,j}^{(9)} K^{n_0}}{c_{i,j}^{(9)} - n_0} + 2 \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\kappa_{k,i,j}^{(10)} K^{n_0}}{c_{k,i,j}^{(10)} - n_0} + 2 \sum_{k=1}^{n} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{\kappa_{k,i,j}^{(11)} K^{n_0}}{c_{k,i,j}^{(11)} - n_0} \right. \\
+ \quad & \left. 4 \sum_{k=1}^{m-1} \sum_{l=k+1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\kappa_{k,l,i,j}^{(12)} K^{n_0}}{c_{k,l,i,j}^{(12)} - n_0} \right).
\end{aligned}
$$

4.3. Out-of-Money Call Option

The payoff of this type of contract is given by

$$
b(x) = (x - K)_+ \; = \; x \mathbb{1}_{(x>K)} - K \mathbb{1}_{(x>K)}. \tag{20}
$$

As the option is out-of money, $S(0) < K$. Then applying, Equation (18) yields the following when we consider the joint life contract.

$$
\begin{aligned}
\pi_1 \quad = \quad & (1+\theta) \sum_{k=1}^{n} \sum_{i=1}^{m} \frac{K \chi_{i,k}^{(1)}}{a_{i,k}^{(1)} \left(a_{i,k}^{(1)} - 1 \right)} \left(\frac{S(0)}{K} \right)^{a_{i,k}^{(1)}} - \theta \left(\sum_{k=1}^{m} \sum_{i=1}^{n} \frac{K \chi_{i,k}^{(2)}}{a_{i,k}^{(2)} \left(a_{i,k}^{(2)} - 1 \right)} \left(\frac{S(0)}{K} \right)^{a_{i,k}^{(2)}} \right. \\
+ \quad & \left. 2 \sum_{k=1}^{n} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{K \chi_{k,i,j}^{(3)}}{a_{k,i,j}^{(3)} \left(a_{k,i,j}^{(3)} - 1 \right)} \left(\frac{S(0)}{K} \right)^{a_{k,i,j}^{(3)}} \right) + 2\theta \left(\sum_{k=1}^{n} \sum_{i=1}^{m} \frac{K \chi_{i,k}^{(4)}}{a_{i,k}^{(4)} \left(a_{i,k}^{(4)} - 1 \right)} \left(\frac{S(0)}{K} \right)^{a_{i,k}^{(4)}} \right. \\
+ \quad & \left. 2 \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{K \chi_{k,i,j}^{(5)}}{a_{k,i,j}^{(5)} \left(a_{k,i,j}^{(5)} - 1 \right)} \left(\frac{S(0)}{K} \right)^{a_{k,i,j}^{(3)}} \right) - 2\theta \sum_{k=1}^{m} \sum_{i=1}^{n} \frac{K \chi_{k,i}^{(6)}}{a_{k,i}^{(6)} \left(a_{k,i}^{(6)} - 1 \right)} \left(\frac{S(0)}{K} \right)^{a_{k,i}^{(6)}} \\
- \quad & 4\theta \left(\sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{K \chi_{j,k,i}^{(7)}}{a_{j,k,i}^{(7)} \left(a_{j,k,i}^{(7)} - 1 \right)} \left(\frac{S(0)}{K} \right)^{a_{j,k,i}^{(7)}} + \sum_{k=1}^{n} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{K \chi_{k,i,j}^{(8)}}{a_{k,i,j}^{(8)} \left(a_{k,i,j}^{(8)} - 1 \right)} \left(\frac{S(0)}{K} \right)^{a_{k,i,j}^{(8)}} \right) \\
- \quad & 8\theta \sum_{k=1}^{m-1} \sum_{l=k+1}^{m} \sum_{i=+1}^{n-1} \sum_{j=i+1}^{n} \frac{K \chi_{i,j,k,l}^{(9)}}{a_{i,j,k,l}^{(9)} \left(a_{i,j,k,l}^{(9)} - 1 \right)} \left(\frac{S(0)}{K} \right)^{a_{i,j,k,l}^{(9)}}.
\end{aligned} \tag{21}
$$

Similarly, we consider the last survival contract, the price of out-of-money call option is

$$
\begin{aligned}
\pi_2 \quad = \quad & \sum_{i=1}^{n} \frac{K\kappa_i^{(1)}}{c_i^{(1)}\left(c_i^{(1)}-1\right)}\left(\frac{S(0)}{K}\right)^{c_i^{(1)}} + (1-\theta)\sum_{i=1}^{m} \frac{K\kappa_i^{(2)}}{c_i^{(2)}\left(c_i^{(2)}-1\right)}\left(\frac{S(0)}{K}\right)^{c_i^{(2)}} \\
+ \quad & \theta\left(\sum_{i=1}^{m} \frac{K\kappa_i^{(3)}}{c_i^{(3)}\left(c_i^{(3)}-1\right)}\left(\frac{S(0)}{K}\right)^{c_i^{(3)}} + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \frac{K\kappa_{i,j}^{(4)}}{c_{i,j}^{(4)}\left(c_{i,j}^{(4)}-1\right)}\left(\frac{S(0)}{K}\right)^{c_{i,j}^{(4)}}\right) \\
+ \quad & \theta\left(\sum_{k=1}^{m}\sum_{i=1}^{n} \frac{K\kappa_{k,i}^{(5)}}{c_{k,i}^{(5)}\left(c_{k,i}^{(5)}-1\right)}\left(\frac{S(0)}{K}\right)^{c_{k,i}^{(5)}} + 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{K\kappa_{k,i,j}^{(6)}}{c_{k,i,j}^{(6)}\left(c_{k,i,j}^{(6)}-1\right)}\left(\frac{S(0)}{K}\right)^{c_{k,i,j}^{(6)}}\right) \\
+ \quad & \theta\left(\sum_{k=1}^{n}\sum_{i=1}^{m} \frac{K\kappa_{k,i}^{(7)}}{c_{k,i}^{(7)}\left(c_{k,i}^{(7)}-1\right)}\left(\frac{S(0)}{K}\right)^{c_i^{(7)}} + 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \frac{K\kappa_{k,i,j}^{(8)}}{c_{k,i,j}^{(8)}\left(c_{k,i,j}^{(8)}-1\right)}\left(\frac{S(0)}{K}\right)^{c_{k,i,j}^{(8)}}\right) \\
- \quad & \theta\left(\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{K\kappa_{i,j}^{(9)}}{c_{i,j}^{(9)}\left(c_{i,j}^{(9)}-1\right)}\left(\frac{S(0)}{K}\right)^{c_{i,j}^{(9)}} + 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{K\kappa_{k,i,j}^{(10)}}{c_{k,i,j}^{(10)}\left(c_{k,i,j}^{(10)}-1\right)}\left(\frac{S(0)}{K}\right)^{c_{k,i,j}^{(10)}}\right. \\
+ \quad & 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \frac{K\kappa_{k,i,j}^{(11)}}{c_{k,i,j}^{(11)}\left(c_{k,i,j}^{(11)}-1\right)}\left(\frac{S(0)}{K}\right)^{c_{k,i,j}^{(11)}} \\
+ \quad & \left.4\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{K\kappa_{k,l,i,j}^{(12)}}{c_{k,l,i,j}^{(12)}\left(c_{k,l,i,j}^{(12)}-1\right)}\left(\frac{S(0)}{K}\right)^{c_{k,l,i,j}^{(12)}}\right).
\end{aligned} \tag{22}
$$

4.4. At-the Money Call Option

At-the-money option, the stock price at time 0 equal the strike price. Under this condition, Equations (21) and (22) become

$$
\begin{aligned}
\pi_1 \quad = \quad & (1+\theta)\sum_{k=1}^{n}\sum_{i=1}^{m} \frac{S(0)\chi_{i,k}^{(1)}}{a_{i,k}^{(1)}\left(a_{i,k}^{(1)}-1\right)} - \theta\left(\sum_{k=1}^{m}\sum_{i=1}^{n} \frac{S(0)\chi_{i,k}^{(2)}}{a_{i,k}^{(2)}\left(a_{i,k}^{(2)}-1\right)}\right. \\
+ \quad & \left.2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \frac{S(0)\chi_{k,i,j}^{(3)}}{a_{k,i,j}^{(3)}\left(a_{k,i,j}^{(3)}-1\right)}\right) + 2\theta\left(\sum_{k=1}^{n}\sum_{i=1}^{m} \frac{S(0)\chi_{i,k}^{(4)}}{a_{i,k}^{(4)}\left(a_{i,k}^{(4)}-1\right)}\right. \\
+ \quad & \left.2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{S(0)\chi_{k,i,j}^{(5)}}{a_{k,i,j}^{(5)}\left(a_{k,i,j}^{(5)}-1\right)}\right) - 2\theta\sum_{k=1}^{m}\sum_{i=1}^{n} \frac{S(0)\chi_{k,i}^{(6)}}{a_{k,i}^{(6)}\left(a_{k,i}^{(6)}-1\right)} \\
- \quad & 4\theta\left(\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{S(0)\chi_{j,k,i}^{(7)}}{a_{j,k,i}^{(7)}\left(a_{j,k,i}^{(7)}-1\right)} + \sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \frac{S(0)\chi_{k,i,j}^{(8)}}{a_{k,i,j}^{(8)}\left(a_{k,i,j}^{(8)}-1\right)}\right) \\
- \quad & 8\theta\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{S(0)\chi_{i,j,k,l}^{(9)}}{a_{i,j,k,l}^{(9)}\left(a_{i,j,k,l}^{(9)}-1\right)}.
\end{aligned}
$$

$$
\begin{aligned}
\pi_2 \quad = \quad & \sum_{i=1}^{n} \frac{S(0)\kappa_i^{(1)}}{c_i^{(1)}\left(c_i^{(1)}-1\right)} + (1-\theta)\sum_{i=1}^{m} \frac{S(0)\kappa_i^{(2)}}{c_i^{(2)}\left(c_i^{(2)}-1\right)} + \theta\left(\sum_{i=1}^{m} \frac{S(0)\kappa_i^{(3)}}{c_i^{(3)}\left(c_i^{(3)}-1\right)}\right. \\
+ \quad & 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \left.\frac{S(0)\kappa_{i,j}^{(4)}}{c_{i,j}^{(4)}\left(c_{i,j}^{(4)}-1\right)}\right) + \theta\left(\sum_{k=1}^{m}\sum_{i=1}^{n} \frac{S(0)\kappa_{k,i}^{(5)}}{c_{k,i}^{(5)}\left(c_{k,i}^{(5)}-1\right)} + 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{S(0)\kappa_{k,i,j}^{(6)}}{c_{k,i,j}^{(6)}\left(c_{k,i,j}^{(6)}-1\right)}\right) \\
+ \quad & \theta\left(\sum_{k=1}^{n}\sum_{i=1}^{m} \frac{S(0)\kappa_{k,i}^{(7)}}{c_{k,i}^{(7)}\left(c_{k,i}^{(7)}-1\right)} + 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \frac{S(0)\kappa_{k,i,j}^{(8)}}{c_{k,i,j}^{(8)}\left(c_{k,i,j}^{(8)}-1\right)}\right) \\
- \quad & \theta\left(\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{S(0)\kappa_{i,j}^{(9)}}{c_{i,j}^{(9)}\left(c_{i,j}^{(9)}-1\right)} + 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{S(0)\kappa_{k,i,j}^{(10)}}{c_{k,i,j}^{(10)}\left(c_{k,i,j}^{(10)}-1\right)}\right. \\
+ \quad & \left. 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \frac{S(0)\kappa_{k,i,j}^{(11)}}{c_{k,i,j}^{(11)}\left(c_{k,i,j}^{(11)}-1\right)} + 4\sum_{k=1}^{m}\sum_{l=k+1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{S(0)\kappa_{k,l,i,j}^{(12)}}{c_{k,l,i,j}^{(12)}\left(c_{k,l,i,j}^{(12)}-1\right)}\right).
\end{aligned}
$$

4.5. Out-of-Money All-or-Nothing Put Option

For such a financial contingent, the function $b()$ is defined by

$$
b(x) = x^{n_0}\mathbb{1}_{(x<K)}, \tag{23}
$$

where $n_0 \in \mathbb{N}$ and $K < S(0)$.

By considering the joint live status, one can show that the price of such a contingent claim can be expressed as follows:

$$
\begin{aligned}
\pi_1 \quad = \quad & (1+\theta)\sum_{k=1}^{n}\sum_{i=1}^{m} \frac{\chi_{i,k}^{(1)}K^{n_0}}{n_0-b_{i,k}^{(1)}}\left(\frac{K}{S(0)}\right)^{-b_{i,k}^{(1)}} - \theta\left(\sum_{k=1}^{m}\sum_{i=1}^{n} \frac{\chi_{i,k}^{(2)}K^{n_0}}{n_0-b_{i,k}^{(2)}}\left(\frac{K}{S(0)}\right)^{-b_{i,k}^{(2)}}\right. \\
+ \quad & \left. 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \frac{\chi_{k,i,j}^{(3)}K^{n_0}}{n_0-b_{k,i,j}^{(3)}}\left(\frac{K}{S(0)}\right)^{-b_{k,i,j}^{(3)}}\right) + 2\theta\left(\sum_{k=1}^{n}\sum_{i=1}^{m} \frac{\chi_{i,k}^{(4)}K^{n_0}}{n_0-b_{i,k}^{(4)}}\left(\frac{K}{S(0)}\right)^{-b_{i,k}^{(4)}}\right. \\
+ \quad & \left. 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{\chi_{k,i,j}^{(5)}K^{n_0}}{n_0-b_{k,i,j}^{(5)}}\left(\frac{K}{S(0)}\right)^{-b_{k,i,j}^{(5)}}\right) - 2\theta\sum_{k=1}^{m}\sum_{i=1}^{n} \frac{\chi_{k,i}^{(6)}K^{n_0}}{n_0-b_{k,i}^{(6)}}\left(\frac{K}{S(0)}\right)^{-b_{k,i}^{(6)}} \\
- \quad & 4\theta\left(\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{\chi_{j,k,i}^{(7)}K^{n_0}}{n_0-b_{j,k,i}^{(7)}}\left(\frac{K}{S(0)}\right)^{-b_{j,k,i}^{(7)}} + \sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \frac{\chi_{k,i,j}^{(8)}K^{n_0}}{n_0-b_{k,i,j}^{(8)}}\left(\frac{K}{S(0)}\right)^{-b_{k,i,j}^{(8)}}\right) \\
- \quad & 8\theta\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\sum_{i=+1}^{n-1}\sum_{j=i+1}^{n} \frac{\chi_{i,j,k,l}^{(9)}K^{n_0}}{n_0-b_{i,j,k,l}^{(9)}}\left(\frac{K}{S(0)}\right)^{-b_{i,j,k,l}^{(9)}}.
\end{aligned} \tag{24}
$$

Proof. Without loss of generality, let us show the first part of the formula.

Let $k = \ln\left[\frac{K}{S(0)}\right]$, since the contract is a put option, $k < 0$. Hence we have:

$$
\begin{aligned}
\chi_{i,k}^{(1)}\int_{-\infty}^{k}(S(0)e^x)^{n_0}e^{b_{i,k}^{(1)}x}\mathrm{d}x \quad &= \quad \chi_{i,k}^{(1)}(S(0))^{n_0}\frac{e^{-\left(b_{i,k}^{(1)}-n_0\right)k}}{-\left(b_{i,k}^{(1)}-n_0\right)} \\
&= \quad \frac{\chi_{i,k}^{(1)}K^{n_0}}{n_0-b_{i,k}^{(1)}}\left[\frac{K}{S(0)}\right]^{-b_{i,k}^{(1)}}.
\end{aligned}
$$

We complete the proof by using the linearity of the integral. □

Similarly, when the contract is written on the last survival status, we have

$$
\begin{aligned}
\pi_2 \;=\;& \sum_{i=1}^{n} \frac{\kappa_i^{(1)} K^{n_0}}{n_0 - d_i^{(1)}} \left(\frac{K}{S(0)}\right)^{-d_i^{(1)}} + (1-\theta) \sum_{i=1}^{m} \frac{\kappa_i^{(2)} K^{n_0}}{n_0 - d_i^{(2)}} \left(\frac{K}{S(0)}\right)^{-d_i^{(2)}} \\
+\;& \theta \left(\sum_{i=1}^{m} \frac{\kappa_i^{(3)} K^{n_0}}{n_0 - d_i^{(3)}} \left(\frac{K}{S(0)}\right)^{-d_i^{(3)}} + 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{\kappa_{i,j}^{(4)} K^{n_0}}{n_0 - d_{i,j}^{(4)}} \left(\frac{K}{S(0)}\right)^{-d_{i,j}^{(4)}} \right) \\
+\;& \theta \left(\sum_{k=1}^{m} \sum_{i=1}^{n} \frac{\kappa_{k,i}^{(5)} K^{n_0}}{n_0 - d_{k,i}^{(5)}} \left(\frac{K}{S(0)}\right)^{-d_{k,i}^{(5)}} + 2 \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\kappa_{k,i,j}^{(6)} K^{n_0}}{n_0 - d_{k,i,j}^{(6)}} \left(\frac{K}{S(0)}\right)^{-d_{k,i,j}^{(6)}} \right) \\
+\;& \theta \left(\sum_{k=1}^{n} \sum_{i=1}^{m} \frac{\kappa_{k,i}^{(7)} K^{n_0}}{n_0 - d_{k,i}^{(7)}} \left(\frac{K}{S(0)}\right)^{c_i^{(7)}} + 2 \sum_{k=1}^{n} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{\kappa_{k,i,j}^{(8)} K^{n_0}}{n_0 - d_{k,i,j}^{(8)}} \left(\frac{K}{S(0)}\right)^{-d_{k,i,j}^{(8)}} \right) \\
-\;& \theta \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\kappa_{i,j}^{(9)} K^{n_0}}{n_0 - d_{i,j}^{(9)}} \left(\frac{K}{S(0)}\right)^{-d_{i,j}^{(9)}} + 2 \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\kappa_{k,i,j}^{(10)} K^{n_0}}{n_0 - d_{k,i,j}^{(10)}} \left(\frac{K}{S(0)}\right)^{-d_{k,i,j}^{(10)}} \right. \\
+\;& 2 \sum_{k=1}^{n} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{\kappa_{k,i,j}^{(11)} K^{n_0}}{n_0 - d_{k,i,j}^{(11)}} \left(\frac{K}{S(0)}\right)^{-d_{k,i,j}^{(11)}} \\
+\;& \left. 4 \sum_{k=1}^{m-1} \sum_{l=k+1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\kappa_{k,l,i,j}^{(12)} K^{n_0}}{n_0 - d_{k,l,i,j}^{(12)}} \left(\frac{K}{S(0)}\right)^{-d_{k,l,i,j}^{(12)}} \right).
\end{aligned}
\tag{25}
$$

Remark 2.

(1) *When we let $K = S(0)$ in Equations (24) and (25), we obtain the price when the option is at-the money;*

(2) *When $n = 2$ and $m = 2$, we obtain the formula for the option price when individual lifetime follows weighted exponential distribution .*

4.6. Out-of-Money Put Option

For this specific financial contingent contract we have

$$b(x) = (K - x)_+ \;=\; K\mathbb{1}_{(x<K)} - x\mathbb{1}_{(x<K)}.$$

Applying Equation (24) for $n_0 = 0$ and $n_0 = 1$ twice, we have for joint life status,

$$
\begin{aligned}
\pi_1 \;=\;& (1+\theta) \sum_{k=1}^{n} \sum_{i=1}^{m} \frac{\chi_{i,k}^{(1)} K}{b_{i,k}^{(1)} \left(b_{i,k}^{(1)} - 1\right)} \left(\frac{K}{S(0)}\right)^{-b_{i,k}^{(1)}} - \theta \left(\sum_{k=1}^{m} \sum_{i=1}^{n} \frac{\chi_{i,k}^{(2)} K}{b_{i,k}^{(2)} \left(b_{i,k}^{(2)} - 1\right)} \left(\frac{K}{S(0)}\right)^{-b_{i,k}^{(2)}} \right. \\
+\;& \left. 2 \sum_{k=1}^{n} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{\chi_{k,i,j}^{(3)} K}{b_{k,i,j}^{(3)} \left(b_{k,i,j}^{(3)} - 1\right)} \left(\frac{K}{S(0)}\right)^{-b_{k,i,j}^{(3)}} \right) + 2\theta \left(\sum_{k=1}^{n} \sum_{i=1}^{m} \frac{\chi_{i,k}^{(4)} K}{b_{i,k}^{(4)} \left(b_{i,k}^{(4)} - 1\right)} \left(\frac{K}{S(0)}\right)^{-b_{i,k}^{(4)}} \right. \\
+\;& \left. 2 \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\chi_{k,i,j}^{(5)} K}{b_{k,i,j}^{(5)} \left(b_{k,i,j}^{(5)} - 1\right)} \left(\frac{K}{S(0)}\right)^{-b_{k,i,j}^{(5)}} \right) - 2\theta \sum_{k=1}^{m} \sum_{i=1}^{n} \frac{\chi_{k,i}^{(6)} K}{b_{k,i}^{(4)} \left(b_{k,i}^{(6)} - 1\right)} \left(\frac{K}{S(0)}\right)^{-b_{k,i}^{(6)}} \\
-\;& 4\theta \left(\sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\chi_{j,k,i}^{(7)} K}{b_{j,k,i}^{(7)} \left(b_{j,k,i}^{(7)} - 1\right)} \left(\frac{K}{S(0)}\right)^{-b_{j,k,i}^{(7)}} \right. \\
+\;& \left. \sum_{k=1}^{n} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{\chi_{k,i,j}^{(8)} K}{b_{k,i,j}^{(8)} \left(b_{k,i,j}^{(8)} - 1\right)} \left(\frac{K}{S(0)}\right)^{-b_{k,i,j}^{(8)}} \right) \\
-\;& 8\theta \sum_{k=1}^{m-1} \sum_{l=k+1}^{m} \sum_{i=+1}^{n-1} \sum_{j=i+1}^{n} \frac{\chi_{i,j,k,l}^{(9)} K}{b_{i,j,k,l}^{(9)} \left(b_{i,j,k,l}^{(9)} - 1\right)} \left(\frac{K}{S(0)}\right)^{-b_{i,j,k,l}^{(9)}}.
\end{aligned}
\tag{26}
$$

Similarly, for the last survival status, we have

$$
\begin{aligned}
\pi_2 \;=\;& \sum_{i=1}^{n} \frac{K\kappa_i^{(1)}}{d_i^{(1)}\left(d_i^{(1)}-1\right)}\left(\frac{K}{S(0)}\right)^{-d_i^{(1)}} + (1-\theta)\sum_{i=1}^{m} \frac{K\kappa_i^{(2)}}{d_i^{(2)}\left(d_i^{(2)}-1\right)}\left(\frac{K}{S(0)}\right)^{-d_i^{(2)}} \\
&+ \theta\left(\sum_{i=1}^{m} \frac{K\kappa_i^{(3)}}{d_i^{(3)}\left(d_i^{(3)}-1\right)}\left(\frac{K}{S(0)}\right)^{-d_i^{(3)}} + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \frac{K\kappa_{i,j}^{(4)}}{d_{i,j}^{(4)}\left(d_{i,j}^{(4)}-1\right)}\left(\frac{K}{S(0)}\right)^{-d_{i,j}^{(4)}}\right) \\
&+ \theta\left(\sum_{k=1}^{m}\sum_{i=1}^{n} \frac{K\kappa_{k,i}^{(5)}}{d_{k,i}^{(5)}\left(d_{k,i}^{(5)}-1\right)}\left(\frac{K}{S(0)}\right)^{-d_{k,i}^{(5)}} + 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{K\kappa_{k,i,j}^{(6)}}{d_{k,i,j}^{(6)}\left(d_{k,i,j}^{(6)}-1\right)}\left(\frac{K}{S(0)}\right)^{-d_{k,i,j}^{(6)}}\right) \\
&+ \theta\left(\sum_{k=1}^{n}\sum_{i=1}^{m} \frac{K\kappa_{k,i}^{(7)}}{d_{k,i}^{(7)}\left(d_{k,i}^{(7)}-1\right)}\left(\frac{K}{S(0)}\right)^{-d_i^{(7)}} + 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \frac{K\kappa_{k,i,j}^{(8)}}{d_{k,i,j}^{(8)}\left(d_{k,i,j}^{(8)}-1\right)}\left(\frac{K}{S(0)}\right)^{-d_{k,i,j}^{(8)}}\right) \\
&- \theta\left(\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{K\kappa_{i,j}^{(9)}}{d_{i,j}^{(9)}\left(d_{i,j}^{(9)}-1\right)}\left(\frac{K}{S(0)}\right)^{-d_{i,j}^{(9)}} + 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{K\kappa_{k,i,j}^{(10)}}{d_{k,i,j}^{(10)}\left(d_{k,i,j}^{(10)}-1\right)}\left(\frac{K}{S(0)}\right)^{-d_{k,i,j}^{(10)}}\right. \\
&+ 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \frac{K\kappa_{k,i,j}^{(11)}}{d_{k,i,j}^{(11)}\left(d_{k,i,j}^{(11)}-1\right)}\left(\frac{K}{S(0)}\right)^{-d_{k,i,j}^{(11)}} \\
&+ \left. 4\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{K\kappa_{k,l,i,j}^{(12)}}{d_{k,l,i,j}^{(12)}\left(d_{k,l,i,j}^{(12)}-1\right)}\left(\frac{K}{S(0)}\right)^{-d_{k,l,i,j}^{(12)}}\right).
\end{aligned} \tag{27}
$$

Remark 3.

(1) *When we let $K = S(0)$ in Equations (26) and (27), we obtain the price when the put option is at-the money;*

(2) *The price when the option is in-the-money can be computed by the put-call parity;*

(3) *When $n = 2$ and $m = 2$, we obtain the formula for the option price when the individual lifetime follows weighted exponential distribution.*

5. Valuation of Lookback Options

This section deals with lookback options when the exercise time is the time when the first person in the couple dies or when the last person dies. This exotic option allows the holder to review the stock price over the lifespan. In the following, we derive the price of this exotic call option with a fixed strike price and a put option with a floating strike price.

5.1. Fixed Strike Lookback Call Option

The payoff at time t is given by

$$
b(t) = \left[\max\left(H,\ \max_{0\,\le\,s\,\le\,t} S(s)\right) - K\right]_+ = \left[\max\left(H,\ S(0)e^{M(t)}\right) - K\right]_+, \tag{28}
$$

where $H \ge S(0)$ is a positive constant and can be interpreted as the minimum stock price guarantee. In order to valuate this type of option we need to distinguish whether the option is out-of-money or in-the-money.

5.2. Out-of-the-Money Fixed Strike Lookback Call Option

This condition is equivalent to $H < K$. Under this assumption, the payoff given in Equation (28) becomes

$$
b(t) = \left[S(0)e^{M(t)} - K\right]_+. \tag{29}
$$

The price π_1 and π_2 are respectively given by

$$
\begin{aligned}
\mathbf{E}\left[e^{\delta\tau}b(\tau)\right] &= \int_k^{\infty}(S(0)e^y-K)f^{\delta}_{M(\tau)}(y)\mathrm{d}y,\\
\mathbf{E}\left[e^{\delta\overline{\tau}}b(\overline{\tau})\right] &= \int_k^{\infty}(S(0)e^y-K)f^{\delta}_{M(\overline{\tau})}(y)\mathrm{d}y.
\end{aligned}
$$

If we consider the joint life status as the lifespan for the option and by Equation (29), the price is given by

$$
\begin{aligned}
\pi_1 &= (1+\theta)\sum_{k=1}^{n}\sum_{i=1}^{m}\frac{K\chi^{(1)}_{i,k}}{b^{(1)}_{i,k}a^{(1)}_{i,k}\left(1-a^{(1)}_{i,k}\right)}\left(\frac{S(0)}{K}\right)^{a^{(1)}_{i,k}}-\theta\left(\sum_{k=1}^{m}\sum_{i=1}^{n}\frac{K\chi^{(2)}_{i,k}}{b^{(2)}_{i,k}a^{(2)}_{i,k}\left(1-a^{(2)}_{i,k}\right)}\left(\frac{S(0)}{K}\right)^{a^{(2)}_{i,k}}\right.\\
&+ \left.2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\frac{K\chi^{(3)}_{k,i,j}}{b^{(3)}_{k,i,j}a^{(3)}_{k,i,j}\left(1-a^{(3)}_{k,i,j}\right)}\left(\frac{S(0)}{K}\right)^{a^{(3)}_{k,i,j}}\right)+2\theta\left(\sum_{k=1}^{n}\sum_{i=1}^{m}\frac{K\chi^{(4)}_{i,k}}{b^{(4)}_{i,k}a^{(4)}_{i,k}\left(1-a^{(4)}_{i,k}\right)}\left(\frac{S(0)}{K}\right)^{a^{(4)}_{i,k}}\right.\\
&+ \left.2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{K\chi^{(5)}_{k,i,j}}{b^{(5)}_{k,i,j}a^{(5)}_{k,i,j}\left(1-a^{(5)}_{k,i,j}\right)}\left(\frac{S(0)}{K}\right)^{a^{(5)}_{k,i,j}}\right)-2\theta\sum_{k=1}^{m}\sum_{i=1}^{n}\frac{K\chi^{(6)}_{k,i}}{b^{(6)}_{k,i}a^{(4)}_{k,i}\left(1-a^{(6)}_{k,i}\right)}\left(\frac{S(0)}{K}\right)^{a^{(6)}_{k,i}}\\
&- 4\theta\left(\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{K\chi^{(7)}_{j,k,i}}{b^{(5)}_{j,k,i}a^{(7)}_{j,k,i}\left(1-a^{(7)}_{j,k,i}\right)}\left(\frac{S(0)}{K}\right)^{a^{(7)}_{j,k,i}}+\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\frac{K\chi^{(8)}_{k,i,j}}{b^{(8)}_{k,i,j}a^{(8)}_{k,i,j}\left(1-a^{(8)}_{k,i,j}\right)}\right.\\
&\times \left.\left(\frac{S(0)}{K}\right)^{a^{(8)}_{k,i,j}}+2\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\sum_{i=+1}^{n-1}\sum_{j=i+1}^{n}\frac{K\chi^{(9)}_{i,j,k,l}}{b^{(9)}_{i,j,k,l}a^{(9)}_{i,j,k,l}\left(1-a^{(9)}_{i,j,k,l}\right)}\left(\frac{S(0)}{K}\right)^{a^{(7)}_{i,j,k,l}}\right).
\end{aligned}
\tag{30}
$$

Proof. Without loss of generality, we show the first part of the formula.

Let $k=\ln\left[\frac{K}{S(0)}\right]$, since the contract is an out-of-money lookback call option, $k>0$. Hence we have:

$$
\begin{aligned}
-\frac{\chi^{(1)}_{i,k}}{b^{(1)}_{i,k}}\int_k^{\infty}(S(0)e^y-K)e^{-a^{(1)}_{i,k}y}\mathrm{d}y &= -\frac{\chi^{(1)}_{i,k}}{b^{(1)}_{i,k}}\left[S(0)\int_k^{\infty}e^{-\left(a^{(1)}_{i,k}-1\right)y}\mathrm{d}y-K\int_k^{\infty}e^{-a^{(1)}_{i,k}y}\mathrm{d}y\right]\\
&= -\frac{\chi^{(1)}_{i,k}}{b^{(1)}_{i,k}}\left[\frac{K}{a^{(1)}_{i,k}-1}\left(\frac{S(0)}{K}\right)^{a^{(1)}_{i,k}}-\frac{K}{a^{(1)}_{i,k}}\left(\frac{S(0)}{K}\right)^{a^{(1)}_{i,k}}\right]\\
&= \frac{\chi^{(1)}_{i,k}}{b^{(1)}_{i,k}}\frac{K}{a^{(1)}_{i,k}\left(1-a^{(1)}_{i,k}\right)}\left[\frac{S(0)}{K}\right]^{a^{(1)}_{i,k}}.
\end{aligned}
$$

We complete the proof by using the linearity of the integral. □

Similarly, if we consider the last survival status as the lifespan for the option and by Equation (29), the price's formula becomes:

$$\begin{aligned}
\pi_2 \;=\;& \sum_{i=1}^{n} \frac{K\kappa_i^{(1)}}{d_i^{(1)} c_i^{(1)}\left(1-c_i^{(1)}\right)} \left(\frac{S(0)}{K}\right)^{c_i^{(1)}} + (1-\theta) \sum_{i=1}^{m} \frac{K\kappa_i^{(2)}}{d_i^{(2)} c_i^{(2)}\left(1-c_i^{(2)}\right)} \left(\frac{S(0)}{K}\right)^{c_i^{(2)}} \\
+\;& \theta\left(\sum_{i=1}^{m} \frac{K\kappa_i^{(3)}}{d_i^{(3)} c_i^{(3)}\left(1-c_i^{(3)}\right)} \left(\frac{S(0)}{K}\right)^{c_i^{(3)}} + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \frac{K\kappa_{i,j}^{(4)}}{d_{i,j}^{(4)} c_{i,j}^{(4)}\left(1-c_{i,j}^{(4)}\right)} \left(\frac{S(0)}{K}\right)^{c_{i,j}^{(4)}} \right) \\
+\;& \theta\left(\sum_{k=1}^{m}\sum_{i=1}^{n} \frac{K\kappa_{k,i}^{(5)}}{d_{k,i}^{(5)} c_{k,i}^{(5)}\left(1-c_{k,i}^{(5)}\right)} \left(\frac{S(0)}{K}\right)^{c_{k,i}^{(5)}} + 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{K\kappa_{k,i,j}^{(6)}}{d_{k,i,j}^{(6)} c_{k,i,j}^{(6)}\left(1-c_{k,i,j}^{(6)}\right)} \left(\frac{S(0)}{K}\right)^{c_{k,i,j}^{(6)}} \right) \\
+\;& \theta\left(\sum_{k=1}^{n}\sum_{i=1}^{m} \frac{K\kappa_{k,i}^{(7)}}{d_{k,i}^{(7)} c_{k,i}^{(7)}\left(1-c_{k,i}^{(7)}\right)} \left(\frac{S(0)}{K}\right)^{c_i^{(7)}} + 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \frac{K\kappa_{k,i,j}^{(8)}}{d_{k,i,j}^{(8)} c_{k,i,j}^{(8)}\left(1-c_{k,i,j}^{(8)}\right)} \left(\frac{S(0)}{K}\right)^{c_{k,i,j}^{(8)}} \right) \\
-\;& \theta\left(\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{K\kappa_{i,j}^{(9)}}{d_{i,j}^{(9)} c_{i,j}^{(9)}\left(1-c_{i,j}^{(9)}\right)} \left(\frac{S(0)}{K}\right)^{c_{i,j}^{(9)}} + 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{K\kappa_{k,i,j}^{(10)}}{d_{k,i,j}^{(10)} c_{k,i,j}^{(10)}\left(1-c_{k,i,j}^{(10)}\right)} \left(\frac{S(0)}{K}\right)^{c_{k,i,j}^{(10)}} \right. \\
+\;& 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \frac{K\kappa_{k,i,j}^{(11)}}{d_{k,i,j}^{(11)} c_{k,i,j}^{(11)}\left(1-c_{k,i,j}^{(11)}\right)} \left(\frac{S(0)}{K}\right)^{c_{k,i,j}^{(11)}} \\
+\;& \left. 4\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{K\kappa_{k,l,i,j}^{(12)}}{d_{k,l,i,j}^{(12)} c_{k,l,i,j}^{(12)}\left(1-c_{k,l,i,j}^{(12)}\right)} \left(\frac{S(0)}{K}\right)^{c_{k,l,i,j}^{(12)}} \right). \qquad (31)
\end{aligned}$$

Remark 4.

(1) *If the lookback call option is in-the-money ($K < H$), then the payoff becomes*

$$\max\left(H,\ S(0)e^{M(t)}\right) - K \;=\; H - K + \left[S(0)e^{M(t)} - H\right]_+.$$

The price is straightforward by Equations (30) and (31);

(2) *If $n = 2$ and $m = 2$ in Equations (30) and (31), we obtain the price for weighted exponential distribution.*

5.3. Floating Strike Lookback Put Option

The payoff of this exotic option is given by

$$\begin{aligned}
b(t) \;&=\; \left[\max\left(H,\ \max_{0\le s\le t} S(s)\right) - S(t)\right]_+ \\
&=\; \max\left(H,\ \max_{0\le s\le t} S(s)\right) - S(t) \\
&=\; \left[S(0)e^{M(t)} - H\right]_+ + H - S(t). \qquad (32)
\end{aligned}$$

Remark 5. *From Equation (32), we see that the payoff of this particular floating lookback option can be split into the classical fixed lookback call option and the forward option.*

From the remark above, we can obtain the price of such an option for a joint lifespan scenario by combining Equations (15) and (30) and replace K with H.

Similarly, for the last survival lifespan scenario we can derive the price for this option by combining Equations (16) and (31) and replace K with H.

6. Numerical Simulation

To illustrate the impact of dependency structure on joint lifespan status and last survival status, we present numerical examples in this section. Gerber's results are consistent with ours when the copula's parameters $\theta = 0$, $n = m = 1$ for the joint life status.

For numerical simulation, we parameterized the lifespan distribution as follows: $m = n = 2$, $\lambda_1 = 0.016$; $\lambda_2 = 0.014$; $\alpha_1 = 0.35$; $\alpha_2 = 0.65$ and $\gamma_1 = 0.019$; $\gamma_2 = 0.017$; $\beta_1 = 0.40$; $\beta_2 = 0.60$. This means that the life expectancy of the first person in the couple is approximately 56.34675 years and the second person is 68.30357.

From Tables 1 and 2, one can clearly see the impact of the dependency parameter, as for negative values of θ, the joint life status has a higher price than last survival, and the reverse is true for positive values of θ. Moreover, it can be seen from Table 3 that, regardless of the value of θ, the price of out-of-money lookback options is cheaper for joint life status than last survival. This is because the joint life option is expected to be exercised sooner than the last survival option, and also because the option is out-of-money. Additionally, the price increases as the copula's parameter increases.

Table 1. Out-of-money call option for $\sigma = 0.25, \delta = 8\%,\ \mu = \delta - \frac{\sigma^2}{2}$.

$\boldsymbol{\theta}$		$\mathbf{-0.33}$		$\mathbf{0}$		$\mathbf{0.33}$	
$\boldsymbol{K}$	$\boldsymbol{S(0)}$	$\boldsymbol{(x,y)}$	$\boldsymbol{(\overline{xy})}$	$\boldsymbol{(x,y)}$	$\boldsymbol{(\overline{xy})}$	$\boldsymbol{(x,y)}$	$\boldsymbol{(\overline{xy})}$
200	180	8.880042	6.121062	7.251288	8.852058	5.622535	11.583054
150	130	5.952699	4.160420	4.887243	5.995451	3.821787	7.830483
100	90	4.440021	3.060531	3.625644	4.426029	2.811267	5.791527

Table 2. Out-of-money put option for $\sigma = 0.25, \delta = 8\%,\ \mu = \delta - \frac{\sigma^2}{2}$.

θ		$\mathbf{-0.33}$		$\mathbf{0}$		$\mathbf{0.33}$	
$\boldsymbol{S(0)}$	$\boldsymbol{K}$	$\boldsymbol{(x,y)}$	$\boldsymbol{(\overline{xy})}$	$\boldsymbol{(x,y)}$	$\boldsymbol{(\overline{xy})}$	$\boldsymbol{(x,y)}$	$\boldsymbol{(\overline{xy})}$
200	180	3.966542	2.427200	3.119718	3.662473	2.272894	4.897746
150	130	2.560014	1.588089	2.024758	2.388694	1.489502	3.189298
100	90	1.983271	1.213600	1.559859	1.831236	1.136447	2.448873

Table 3. Out-of-the-money fixed strike lookback call option for $\sigma = 0.25, \delta = 8\%,\ \mu = \delta - \frac{\sigma^2}{2}$.

$\boldsymbol{\theta}$		$\mathbf{-0.33}$		$\mathbf{0}$		$\mathbf{0.33}$	
$\boldsymbol{K}$	$\boldsymbol{S(0)}$	$\boldsymbol{(x,y)}$	$\boldsymbol{(\overline{xy})}$	$\boldsymbol{(x,y)}$	$\boldsymbol{(\overline{xy})}$	$\boldsymbol{(x,y)}$	$\boldsymbol{(\overline{xy})}$
200	180	4.796560	5.731933	5.702499	7.754062	6.608438	9.776192
150	130	3.217272	3.894777	3.843399	5.251825	4.469526	6.608873
100	90	2.398280	2.865966	2.851250	3.877031	3.304219	4.888096

7. Conclusions and Discussion

This paper analyses the Guaranteed Minimum Death Benefit contract when the exercise time is the lifespan for the joint life and last survival status. Under a certain dependency structure, and considering mixed exponential distributions, we derived a closed-analytical expression of the price of exotic options. Furthermore, the impact of the dependency in the married couple on the price as well as the effect of incorporating the joint life and last survival status on the model are shown via numerical illustration.

Our result is consistent with Gerber et al. (2012, 2013)'s work where the dependency parameter $\theta = 0$ for the joint life status and where $n = m = 1$. Hence, this work not only generalises previous works, but incorporates the couple's lifespan in the analysis of the GMDB contract pricing.

Because the phase-type distribution family is weakly dense in the non-negative real axis of random variable and mixed exponential distributions belonging to that family, our results are therefore more general and extend the existing results in the literature.

Despite this result adding to the literature, it has a downside in that we did not use any real data to check its goodness-of-fit. In addition, the underlying stock price process under consideration does not account for the jump in the stock price; however, there is evidence that, in some scenarios, one can observe jumps in the stock price.

Our results could be reviewed by considering an adequate diffusion process (Lévy process, regime switching model) and by using a copula that can model a more complex dependency.

Author Contributions: The authors have contributed equally to this work from the Introduction to the Conclusion. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Global Excellence Stature (GES) 4.0 scholarship program and the NRF INCENTIVE GRANT from the University of Johannesburg.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

This section provides the proof of Propositions 1 and 2 and the quadratic equations.

Proof of Proposition 1.

$$
\begin{aligned}
\mathbf{P}\Big(T_{(x,y)} \geq w\Big) &= \mathbf{P}\big(\min\big(T_x, T_y\big) \geq w\big) \\
&= \mathbf{P}(T_x \geq w,\ T_x \geq w) \\
&= \int_w^\infty \int_w^\infty f_{\left(T_x, T_y\right)}(u, v)\mathrm{d}u\mathrm{d}v \\
&= \int_w^\infty \int_w^\infty \Big\{(f_{T_x}(u) - \theta h_{T_x}(u))f_{T_y}(v) + 2\theta h_{T_y} f_{T_y}(v)\bar{F}_{T_y}(w)\Big\}\mathrm{d}u\mathrm{d}u \\
&= \bar{F}_{T_y}(w)\int_w^\infty (f_{T_x}(u) - \theta h_{T_x}(u))\mathrm{d}u - 2\theta\Big(\bar{F}_{T_y}(w)\Big)^2 \int_w^\infty h_{T_x}(u)\mathrm{d}u \\
&= \bar{F}_{T_y}(w)\int_w^\infty f_{T_x}(u)\mathrm{d}u - \theta\bar{F}_{T_y}(w)\int_w^\infty f_{T_x}(u)\Big(1 - 2F_{T_y}(u)\Big)\mathrm{d}u \\
&- 2\theta\Big(\bar{F}_{T_y}(w)\Big)^2 \int_w^\infty f_{T_x}(u)(1 - 2F_{T_x}(u))\mathrm{d}u \\
&= \bar{F}_{T_y}(w)\bar{F}_{T_x}(w) - \theta\bar{F}_{T_y}(w)\bar{F}_{T_x}(w) + 2\theta\bar{F}_{T_y}(w)\int_w^\infty f_{T_x}(u)F_{T_x}(u)\mathrm{d}u \\
&- \theta\Big(\bar{F}_{T_y}(w)\Big)^2\bar{F}_{T_y}(w) + 4\theta\Big(\bar{F}_{T_y}(w)\Big)^2 \int_w^\infty f_{T_x}(u)F_{T_x}(u)\mathrm{d}u \\
&= \bar{F}_{T_y}(w)\bar{F}_{T_x}(w) - \theta\bar{F}_{T_y}(w)\bar{F}_{T_x}(w) + \theta\bar{F}_{T_y}(w)\Big(1 - F^2_{T_w}\Big) \\
&- 2\theta\Big(\bar{F}_{T_y}(w)\Big)^2\bar{F}_{T_x}(w) + 2\theta\Big(\bar{F}_{T_y}(w)\Big)^2\Big(1 - F^2_{T_x}\Big).
\end{aligned}
$$

$$\begin{aligned}
\mathbf{P}\Big(T_{(x,y)} \geq w\Big) &= \bar{F}_{T_y}(w)\bar{F}_{T_x}(w) - \theta\bar{F}_{T_y}(w)\bar{F}_{T_x}(w) + \theta\bar{F}_{T_y}(w)\Big(1 - F^2_{T_x}(w)\Big) \\
&- 2\theta\bar{F}^2_{T_y}(w)\bar{F}_{T_x}(w) + 2\theta\bar{F}^2_{T_y}(w)\Big(1 - F^2_{T_x}(w)\Big) \\
&= (1-\theta)\bar{F}_{T_y}(w)\bar{F}_{T_x}(w) + \theta\bar{F}_{T_y}(w) - \theta\bar{F}_{T_y}(w)\Big(1 - 2\bar{F}_{T_x}(w) + \bar{F}^2_{T_x}(w)\Big) \\
&- 2\theta\bar{F}^2_{T_y}(w)\bar{F}_{T_x}(w) + 2\theta\bar{F}^2_{T_y}(w) - 2\theta\bar{F}^2_{T_y}(w)\Big(1 - 2\bar{F}_{T_x}(w) + \bar{F}^2_{T_x}(w)\Big) \\
&= (1-\theta)\bar{F}_{T_y}(w)\bar{F}_{T_x}(w) + \theta\bar{F}_{T_y}(w) - \theta\bar{F}_{T_y}(w) + 2\theta\bar{F}_{T_y}(w)\bar{F}_{T_x}(w) \\
&- \theta\bar{F}_{T_y}(w)\bar{F}^2_{T_x}(w) - 2\theta\bar{F}^2_{T_y}(w)\bar{F}_{T_x}(w) + 2\theta\bar{F}^2_{T_y}(w) - 2\theta\bar{F}^2_{T_y}(w) \\
&+ 4\theta\bar{F}^2_{T_y}(w)\bar{F}_{T_x}(w) - 2\theta\bar{F}^2_{T_y}(w)\bar{F}^2_{T_x}(w) \\
&= (1+\theta)\bar{F}_{T_y}(w)\bar{F}_{T_x}(w) - \theta\bar{F}_{T_y}(w)\bar{F}^2_{T_x}(w) + 2\theta\bar{F}^2_{T_y}(w)\bar{F}_{T_x}(w) \\
&- 2\theta\bar{F}^2_{T_y}(w)\bar{F}^2_{T_x}(w).
\end{aligned}$$

$$\begin{aligned}
\mathbf{P}\Big(T_{(x,y)} \geq w\Big) &= (1+\theta)\bar{F}_{T_y}(w)\bar{F}_{T_x}(w) - \theta\bar{F}_{T_y}(w)\bar{F}^2_{T_x}(w) + 2\theta\bar{F}^2_{T_y}(w)\bar{F}_{T_x}(w) \\
&+ 2\theta\bar{F}^2_{T_y}(w)\bar{F}^2_{T_x}(w).
\end{aligned} \tag{A1}$$

$$\begin{aligned}
\bar{F}_{T_x}(w)\bar{F}_{T_y}(w) &= \sum_{i=1}^{n}\alpha_i e^{-\lambda_i w}\sum_{k=1}^{m}\beta_k e^{-\gamma_k w} = \sum_{i=1}^{n}\sum_{k=1}^{m}\alpha_i\beta_k e^{-(\lambda_i+\gamma_k)w}, \\
\bar{F}^2_{T_x}(w) &= \sum_{i=1}^{n}\alpha_i^2 e^{-2\lambda_i w} + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\alpha_i\alpha_j e^{-\left(\lambda_i+\lambda_j\right)w}, \\
\bar{F}^2_{T_y}(w) &= \sum_{i=1}^{m}\beta_i^2 e^{-2\gamma_i w} + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\beta_i\beta_j e^{-\left(\gamma_i+\gamma_j\right)w}.
\end{aligned}$$

$$\begin{aligned}
\bar{F}_{T_y}(w)\bar{F}^2_{T_x}(w) &= \sum_{k=1}^{m}\beta_k e^{-\gamma_k t}\left[\sum_{i=1}^{n}\alpha_i^2 e^{-2\lambda_i w} + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\alpha_i\alpha_j e^{-\left(\lambda_i+\lambda_j\right)w}\right] \\
&= \sum_{k=1}^{m}\sum_{i=1}^{n}\beta_k\alpha_i^2 e^{-(\gamma_k+2\lambda_i)t} + 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\beta_k\alpha_i\alpha_j e^{-\left(\gamma_k+\lambda_i+\lambda_j\right)w}
\end{aligned}$$

$$\begin{aligned}
\bar{F}^2_{T_y}(w)\bar{F}_{T_x}(w) &= \sum_{k=1}^{n}\alpha_k e^{-\lambda_k t}\left[\sum_{i=1}^{m}\beta_i^2 e^{-2\gamma_i w} + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\beta_i\beta_j e^{-\left(\gamma_i+\gamma_j\right)w}\right] \\
&= \sum_{k=1}^{n}\sum_{i=1}^{m}\alpha_k\beta_i^2 e^{-(\lambda_k+2\gamma_i)t} + 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\alpha_k\beta_i\beta_j e^{-\left(\lambda_k+\gamma_i+\gamma_j\right)w}
\end{aligned}$$

$$
\begin{aligned}
\bar{F}^2_{T_y}(w)\bar{F}^2_{T_x}(w) &= \left(\sum_{i=1}^{m}\beta_i^2 e^{-2\gamma_i w} + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\beta_i\beta_j e^{-\left(\gamma_i+\gamma_j\right)w}\right) \\
&\times \left(\sum_{i=1}^{n}\alpha_i^2 e^{-2\lambda_i w} + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\alpha_i\alpha_j e^{-\left(\lambda_i+\lambda_j\right)w}\right) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}\beta_i^2\alpha_j^2 e^{-\left(2\gamma_i+2\lambda_j\right)w} + 2\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\beta_k^2\alpha_i\alpha_j e^{-\left(2\gamma_k+\lambda_i+\lambda_j\right)w} \\
&+ 2\sum_{k=1}^{n}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\alpha_k^2\beta_i\beta_j e^{-\left(2\lambda_k+\gamma_i+\gamma_j\right)w} \\
&+ 4\sum_{k=1}^{m-1}\sum_{l=k+1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\beta_k\beta_l\alpha_i\alpha_j e^{-\left(\gamma_k+\gamma_l+\lambda_i+\lambda_j\right)w}.
\end{aligned}
$$

This completes the proof by plugging the above results into Equation (A1). □

Proof of Proposition 2.

$$
\begin{aligned}
\left(T_{\overline{(xy)}} \le w\right) &= \mathbf{P}\left(\max\left(T_x,\ T_y\right) \le w\right) \\
&= \mathbf{P}\left(T_x \le w,\ T_y \le w\right) \\
&= \int_0^w\int_0^w f_{\left(T_x,T_y\right)}(u,v)\,\mathrm{d}u\mathrm{d}v \\
&= \int_0^w\int_0^w \left\{\left(f_{T_x}(u)-\theta h_{T_x}(u)\right)f_{T_y}(v) + 2\theta h_{T_x}(u)f_{T_y}(v)\overline{F}_{T_y}(v)\right\}\mathrm{d}u\mathrm{d}v \\
&= F_{T_y}(w)\int_0^w\left(f_{T_x}(u)-\theta h_{T_x}(u)\right)\mathrm{d}u - \theta\left(\overline{F}^2_{T_y}(w)-1\right)\int_0^w h_{T_x}(u)\mathrm{d}u \\
&= F_{T_y}(w)\int_0^w f_{T_x}(u)\mathrm{d}u - \theta F_{T_y}(w)\int_0^w h_{T_x}(u)\mathrm{d}u + \theta\left(1-\overline{F}^2_{T_y}(w)\right)\int_0^w h_{T_x}(u)\mathrm{d}u.
\end{aligned}
$$

But,

$$
\begin{aligned}
\int_0^w h_{T_x}(u)\mathrm{d}u &= \int_0^w f_{T_x}(u)\left(1-2F_{T_x}(u)\right)\mathrm{d}u \\
&= F_{T_x}(w) - F^2_{T_x}(w).
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{P}\left(T_{\overline{(xy)}} \le w\right) &= F_{T_y}(w)\int_0^w f_{T_x}(u)\mathrm{d}u - \theta F_{T_y}(w)\int_0^w h_{T_x}(u)\mathrm{d}u + \theta\left(1-\bar{F}^2_{T_y}(w)\right)\int_0^w h_{T_x}(u)\mathrm{d}u \\
&= F_{T_y}(w)F_{T_x}(w) - \theta F_{T_y}(w)\left(F_{T_x} - F^2_{T_x}(w)\right) + \theta\left(1-\overline{F}(w)\right)\left(F_{T_x}(w) - F^2_{T_x}(w)\right) \\
&= (1-\theta)F_{T_y}(w)F_{T_x}(w) + \theta F_{T_x}(w) - \theta F^2_{T_x}(w) + \theta F_{T_y}(w)F^2_{T_x}(w) \\
&- \theta F^2_{T_y}(w)F_{T_x}(w) + \theta F^2_{T_y}(w)F^2_{T_x}(w) \\
&= \left(1-\overline{F}_{T_x}(w)\right) + (\theta-1)\overline{F}_{T_y}(w) - \theta\overline{F}^2_{T_y}(w) - \theta\overline{F}_{T_x}(w)\overline{F}^2_{T_y}(w) \\
&- \theta\overline{F}_{T_y}(w)\overline{F}^2_{T_x}(w) + \theta\overline{F}^2_{T_x}(w)\overline{F}^2_{T_y}(w) \\
\mathbf{P}\left(T_{\overline{(xy)}} \le w\right) &= \left(1-\overline{F}_{T_x}(w)\right) + (\theta-1)\overline{F}_{T_y}(w) - \theta\overline{F}^2_{T_y}(w) - \theta\bar{F}_{T_x}(w)\overline{F}^2_{T_y}(w) \\
&\quad -\theta\overline{F}_{T_y}(w)\overline{F}^2_{T_x}(w) + \theta\overline{F}^2_{T_x}(w)\bar{F}^2_{T_y}(w).
\end{aligned}
$$

Plugging, the analytical expression of $\overline{F}_{T_y}$, $\overline{F}_{T_x}$ in the above relation completes the proof. □

For a given $i,\ j,\ k,\ l$, let defined the following quadratic equations.

$$
\begin{aligned}
\frac{\sigma^2}{2}\zeta^2+\mu\zeta-(\lambda_i+\gamma_k+\delta) &= 0, & \frac{\sigma^2}{2}\zeta^2+\mu\zeta-(2\lambda_i+\gamma_k+\delta) &= 0,\\
\frac{\sigma^2}{2}\zeta^2+\mu\zeta-(\lambda_k+\gamma_i+\gamma_j+\delta) &= 0, & \frac{\sigma^2}{2}\zeta^2+\mu\zeta-(\lambda_i+2\gamma_k+\delta) &= 0,\\
\frac{\sigma^2}{2}\zeta^2+\mu\zeta-(\gamma_i+\lambda_j+\lambda_k+\delta) &= 0, & \frac{\sigma^2}{2}\zeta^2+\mu\zeta-(2\lambda_i+2\gamma_j+\delta) &= 0,\\
\frac{\sigma^2}{2}\zeta^2+\mu\zeta-(\lambda_j+\lambda_k+2\gamma_i+\delta) &= 0, & \frac{\sigma^2}{2}\zeta^2+\mu\zeta-(2\lambda_k+\gamma_i+\gamma_j+\delta) &= 0,\\
\frac{\sigma^2}{2}\zeta^2+\mu\zeta-(\lambda_i+\lambda_j+\gamma_k+\gamma_l+\delta) &= 0. & & \qquad\qquad \text{(A2)}
\end{aligned}
$$

where the superscript $(i),\ \ i=1,\cdots,9$ indicates that the root correspond to the first up to the ninth equation.

For the last survival status, we consider the following quadratic equations:

$$
\begin{aligned}
\frac{\sigma^2}{2}\zeta^2+\mu\zeta-(\delta+\lambda_i) &= 0, & \frac{\sigma^2}{2}\zeta^2+\mu\zeta-(\delta+\gamma_i) &= 0,\\
\frac{\sigma^2}{2}\zeta^2+\mu\zeta-(\delta+2\gamma_i) &= 0, & \frac{\sigma^2}{2}\zeta^2+\mu\zeta-(\delta+\gamma_i+\gamma_j) &= 0,\\
\frac{\sigma^2}{2}\zeta^2+\mu\zeta-(\delta+\gamma_k+2\lambda_i) &= 0, & \frac{\sigma^2}{2}\zeta^2+\mu\zeta-(\delta+\gamma_k+\lambda_i+\lambda_j) &= 0,\\
\frac{\sigma^2}{2}\zeta^2+\mu\zeta-(\delta+2\gamma_i+\lambda_k) &= 0, & \frac{\sigma^2}{2}\zeta^2+\mu\zeta-(\delta+\lambda_k+\gamma_i+\gamma_j) &= 0,\\
\frac{\sigma^2}{2}\zeta^2+\mu\zeta-(\delta+2\gamma_i+2\lambda_j) &= 0, & \frac{\sigma^2}{2}\zeta^2+\mu\zeta-(\delta+2\gamma_k+\lambda_i+\lambda_j) &= 0,\\
\frac{\sigma^2}{2}\zeta^2+\mu\zeta-(\delta+2\lambda_k+\gamma_i+\gamma_j) &= 0, & \frac{\sigma^2}{2}\zeta^2+\mu\zeta-(\delta+\gamma_k+\gamma_l+\lambda_i+\lambda_j) &= 0. \qquad \text{(A3)}
\end{aligned}
$$

References

Ai, Meiqiao, and Zhimin Zhang. 2022. Pricing some life-contingent lookback options under regime-switching Lévy models. *Journal of Computational and Applied Mathematics* 404: 114082. [CrossRef]

Borodin, Andrei N., and Paavo Salminen. 2002. *Handbook of Brownian Motion-Facts and Formulae*. Basel, Boston and Berlin: Birkhäuser.

Brown, Jeffrey R., and James M. Poterba. 1999. *Joint Life Annuities and Annuity Demand by Married Couples*. Cambridge: National Bureau of Economic Research.

Buchen, Peter, and Otto Konstandatos. 2005. A new method of pricing lookback options. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics* 15: 245–59. [CrossRef]

Chesneau, Christophe, Vijay Kumar, Mukti Khetan, and Mohd Arshad. 2022. On a Modified Weighted Exponential Distribution with Applications. *Mathematical and Computational Applications* 27: 17. [CrossRef]

Das, Suchismita, and Debasis Kundu. 2016. On weighted exponential distribution and its length biased version. *Journal of the Indian Society for Probability and Statistics* 17: 57–77. [CrossRef]

Dhaene, Jan, Ben Stassen, Karim Barigou, Daniël Linders, and Ze Chen. 2017. Fair valuation of insurance liabilities: Merging actuarial judgement and market-consistency. *Insurance: Mathematics and Economics* 76: 14–27. [CrossRef]

Dufresne, Daniel. 2007. Stochastic life annuities. *North American Actuarial Journal* 11: 136–57. [CrossRef]

Gerber, Hans U., and Elias S. W. Shiu. 2003. Pricing lookback options and dynamic guarantees. *North American Actuarial Journal* 7: 48–66. [CrossRef]

Gerber, Hans U., Elias S. W. Shiu, and Hailiang Yang. 2012 Valuing equity-linked death benefits and other contingent options: A discounted density approach. *Insurance: Mathematics and Economics* 53: 615–23. [CrossRef]

Gerber, Hans U., Elias S. W. Shiu, and Hailiang Yang. 2013. Valuing equity-linked death benefits in jump diffusion models. *Insurance: Mathematics and Economics* 51: 73–92. [CrossRef]

Gupta, Rameshwar D., and Debasis Kundu. 2001. Generalized exponential distribution: Different method of estimations. *Journal of Statistical Computation and Simulation* 69: 315–37. [CrossRef]

Jeanblanc, Monique, Marc Yor, and Marc Chesney. 2009. *Mathematical Methods for Financial Markets*. New York: Springer Science & Business Media.

Kirkby, J. Lars, and Duy Nguyen. 2021. Equity-linked guaranteed minimum death benefits with dollar cost averaging. *Insurance: Mathematics and Economics* 100: 408–28. [CrossRef]

Linetsky, Vadim. 2004. Lookback options and diffusion hitting times: A spectral expansion approach. *Finance and Stochastics* 8: 373–98. [CrossRef]

Matvejevs, Andrejs, and Aleksandrs Matvejevs. 2001. Insurance models for joint life and last survivor benefits. *Informatica* 12: 547–58.

Shevchenko, Pavel V., and Xiaolin Luo. 2016. A unified pricing of variable annuity guarantees under the optimal stochastic control framework. *Risks* 4: 22. [CrossRef]

Wang, Yayun, Zhimin Zhang, and Wenguang Yu. 2021. Pricing equity-linked death benefits by complex Fourier series expansion in a regime-switching jump diffusion model. *Applied Mathematics and Computations* 399: 126031. [CrossRef]

Zhang, Zhimin, Yaodi Yong, and Wenguang Yu. 2020. Valuing equity-linked death benefits in general exponential Lévy models. *Journal of Computational and Applied Mathematics* 365: 112377. [CrossRef]