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(ω, c)-Periodic Mild Solutions to Non-Autonomous Abstract Differential Equations

Luciano Abadias [1,†], Edgardo Alvarez [2,*] and Rogelio Grau [2,†]

[1] Departamento de Matemáticas, Instituto Universitario de Matemáticas y Aplicaciones, Universidad de Zaragoza, 50009 Zaragoza, Spain; labadias@unizar.es
[2] Departamento de Matemáticas y Estadística, Universidad del Norte, 080001 Barranquilla, Colombia; graur@uninorte.edu.co
* Correspondence: ealvareze@uninorte.edu.co
† These authors contributed equally to this work.

Abstract: We investigate the semi-linear, non-autonomous, first-order abstract differential equation $x'(t) = A(t)x(t) + f(t, x(t), \varphi[\alpha(t, x(t))]), t \in \mathbb{R}$. We obtain results on existence and uniqueness of (ω, c)-periodic (second-kind periodic) mild solutions, assuming that $A(t)$ satisfies the so-called Acquistapace–Terreni conditions and the homogeneous associated problem has an integrable dichotomy. A new composition theorem and further regularity theorems are given.

Keywords: Acquistapace–Terreni conditions; non-autonomous semi-linear equation; periodic; (ω, c)-periodic; delay

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1. Introduction

Of concern in the present paper is the existence and uniqueness of (ω, c)-periodic mild solutions for a class of semi-linear, non-autonomous equations. More precisely, our goal is to study the following problems:

$$x'(t) = A(t)x(t) + g(t, x(t)), \quad t \in \mathbb{R}, \tag{1}$$
$$x'(t) = A(t)x(t) + f(t, x(t), \varphi[\alpha(t, x(t))]), \quad t \in \mathbb{R}, \tag{2}$$
$$x'(t) = A(t)x(t) + f(t, x(t), x(t-h)), \quad h > 0, \ t \in \mathbb{R}. \tag{3}$$

In the above, $\{A(t)\}_{t\in\mathbb{R}}$ is a family of linear (usually unbounded) operators on a Banach space X, and $g : \mathbb{R} \times X \to X$, $f : \mathbb{R} \times X \times X \to X$, $\alpha : \mathbb{R} \times X \to \mathbb{R}$ and $\varphi : \mathbb{R} \to X$ are continuous functions satisfying suitable conditions.

The theory of non-autonomous differential equations has found applications in several areas of science and technology (see, e.g., recent developments [1–3]). The interest in this type of equations lies in the fact that a system subjected to external inputs can include periodic ones. Examples are included in the Floquet theory, which is used to study the stability of linear periodic systems in continuous time.

The concept of a vector-valued (ω, c)-periodic function was introduced by Alvarez et al. in [4]. In that work, the authors obtained several interesting properties of this type of function. After that, Li et al. in [5] studied the existence of (ω, c)-periodic solutions for a non-homogeneous problem which was impulsive. Then, Wang, Ren, and Zhou in [6] investigated the regularity of (ω, c)-periodic solutions of linear and semi-linear impulsive differential equations with boundary conditions. Additionally, Agaoglu et al. in [7] studied (ω, c)-periodic solutions for semi-linear equations in Banach spaces. Recently, the existence of (ω, c)-periodic solutions for a fractional differential equation has been studied by Mophou and Guérékata in [8].

The theory of mild solutions to the classical non-autonomous (linear and semi-linear) equations has been developed in the monographs [9–11]. A complete study of the first-order, non-autonomous Cauchy problem has already appeared in [12,13], in which the authors have established some precise and optimal conditions on the operator family $\{A(t)\}_{t\in\mathbb{R}}$ in order to obtain the existence and uniqueness of solutions.

Regularity of mild solutions (of a different kind, for example, almost periodic, almost automorphic, pseudo-almost periodic, pseudo-almost automorphic solutions, Stepanov almost periodic, etc.) for non-autonomous differential equations on $\mathbb{R}$

$$\mathcal{L}x = A(\cdot)x + f,$$

where $\mathcal{L}$ is a linear operator and the forcing term f is linear and/or nonlinear, have been widely developed in the literature under the assumption that the evolution family generated by $\{A(t)\}_{t\in\mathbb{R}}$ is exponentially stable (see, for example, refs. [14–19] and the references therein).

On the other hand, the concept of integrable dichotomy for periodic integro-differential equations was introduced in [20]. Recently, Pinto and Vidal [21] adapted this concept for the homogeneous system

$$x'(t) = A(t)x(t) \quad t \in \mathbb{R},$$

defined on a Banach space X. In that work, the authors investigated the existence of almost and pseudo-almost periodic mild solutions of the nonlinear system

$$x'(t) = A(t)x(t) + f(t, x(t), x(t-h)), \quad h > 0 \text{ (fixed)}, \ t \in \mathbb{R},$$

under the existence of an integrable dichotomy of the associated homogeneous linear system. It should be noted that the definition of integrable dichotomy generalizes the exponential stability. Thus, it is possible to obtain regularity results of mild solutions to non-autonomous problems under more general assumptions.

We remark that not much seems to be known about regularity results for (ω, c)-periodic mild solutions to (1.1), (1.2), and (1.3) (see Definitions 17, 20 and 26 below). Here, we are interested in showing that these problems have such kinds of mild solutions under appropriate conditions.

The main novelties of the present paper are the following.

- Our assumption on the evolution family generated by $\{A(t)\}_{t\in\mathbb{R}}$ is quite general. Indeed, we assume that the evolution family associated to the homogeneous linear problem has an integrable dichotomy, instead of employing the notion of exponential stability.
- Unlike other works, the computations involving c-norms are treated very carefully. This is very important when we are working in the space of (ω, c)-periodic functions because the norm changes with respect to the standard one.
- A new composition theorem (see Theorem 21) is given in order to obtain our second main result.

Our first main result (Theorem 15) ensures that linear (Definition 14) possesses a unique (ω, c)-periodic mild solution under the hypothesis that the homogeneous problem has an integrable dichotomy. The second main result (Theorem 18) shows that (1.1) has a unique (ω, c)-periodic mild solution under the hypothesis that the nonlinear term g satisfies the assumptions of composition Theorem 5 and a standard Lipschitz condition. The third main result (Theorem 23), which includes a more general nonlinearity, states that (1.2) has a unique (ω, c)-periodic mild solution. The composition theorem mentioned before is essential for the proof. The fourth main result (Theorem 28) gives a unique (ω, c)-periodic mild solution for the equation with constant delay (1.3). In this case, to achieve our goal, we use the translation invariant result of the space of (ω, c)-periodic functions. The Banach contraction principle is used to obtain these results.

This paper is structured as follows. In Section 2, firstly we introduce some notation and recall the definition of a (ω, c)-periodic function and its properties that will be used throughout the paper. We also recall some basic concepts of evolution families and integrable dichotomy. Section 3 is devoted to stating and showing the existence and uniqueness of (ω, c)-periodic mild solutions to the linear equation. In Section 4, we deal with regularity results for the semi-linear case. An example is also provided.

2. Preliminaries

Throughout this paper, we assume $c \in \mathbb{C} \setminus \{0\}$, $\omega > 0$, X will denote a complex Banach space with norm $\|\cdot\|$, and we will denote the space of continuous functions on $\mathbb{R}$ by

$$C(\mathbb{R}, X) := \{f : \mathbb{R} \to X : \ f \text{ is continuous}\}.$$

Additionally, we will denote the space of bounded and continuous functions on $\mathbb{R}$ as

$$BC(\mathbb{R}, X) := \{f : \mathbb{R} \to X : f \text{ is bounded and continuous}\},$$

the integrable functions on $\mathbb{R}$ as

$$L^1(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : f \text{ is integrable}\},$$

the space of continuous functions on $\mathbb{R} \times X$ by

$$C(\mathbb{R} \times X, X) := \{f : \mathbb{R} \times X \to X : f \text{ is continuous}\},$$

where $\mathbb{R} \times X$ is a Banach space with the norm $\|(t, x)\| = \max\{|t|, \|x\|\}$, and the space of bounded linear operators on X by

$$\mathcal{B}(X) := \{T : X \to X : \ T \text{ is bounded and linear}\},$$

and

$$C^1(I, \mathcal{B}(\mathrm{X})) := \{f : I \to \mathcal{B}(X) : \ f' \text{ is continuous}\},$$

with $I \subseteq \mathbb{R}$.

Definition 1 ([4])**.** *A function $g \in C(\mathbb{R}, X)$ is said to be (ω, c)-periodic if $g(t + \omega) = cg(t)$ for all $t \in \mathbb{R}$. ω is called the c-period of g. The collection of those functions with the same c-period ω will be denoted by $P_{\omega c}(\mathbb{R}, X)$. When $c = 1$ (ω-periodic case) we write $P_\omega(\mathbb{R}, X)$ instead of $P_{\omega 1}(\mathbb{R}, X)$. Using the principal branch of the complex Logarithm (i.e., the argument in $(-\pi, \pi]$) we define $c^{t/\omega} := \exp((t/\omega)Log(c))$. Additionally, we will use the notation $c^\wedge(t) := c^{t/\omega}$ and $|c|^\wedge(t) := |c^\wedge(t)| = |c|^{t/\omega}$.*

The following proposition gives a characterization of the (ω, c)-periodic functions. This result can be found in ([4], Proposition 2.2).

Proposition 2. *Let $f \in C(\mathbb{R}, X)$. Then, f is a (ω, c)-periodic if, and only if*

$$f(t) = c^\wedge(t)u(t), \quad c^\wedge(t) = c^{t/\omega}, \tag{4}$$

where $u(t)$ is a ω-periodic, X-valued function.

Remark 3. *It follows from the unique representation of the periodic functions that the decomposition in Proposition 2 is unique.*

The next remark describes the structure of the space $P_{\omega c}(\mathbb{R}, X)$ (see ([4], Remark 2.4)).

Remark 4. *From Definition 1, we can observe that $P_{\omega c}(\mathbb{R}, X)$ is a translation-invariant subspace over $\mathbb{C}$ of $C(\mathbb{R}, X)$, that is, if $h \geq 0$ (fixed) and $x \in P_{\omega c}(\mathbb{R}, X)$, then $x_h(\cdot) := x(\cdot - h) \in P_{\omega c}(\mathbb{R}, X)$. Furthermore, if $f \in P_{\omega c}(\mathbb{R}, X)$ is differentiable, then $f' \in P_{\omega c}(\mathbb{R}, X)$, and if $|c| = 1$, then $P_{\omega c}(\mathbb{R}, X)$ has only bounded functions if $|c| < 1$, then any element $f \in P_{\omega c}(\mathbb{R}, X)$ goes to* *zero as $t \to \infty$, and f is unbounded as $t \to -\infty$, and if $|c| > 1$, then f is unbounded as $t \to \infty$ and f goes to zero as $t \to -\infty$.*

From ([4], Theorem 2.10) we know that $P_{\omega c}(\mathbb{R}, X)$ is a Banach space with the norm

$$\|f\|_{\omega c} := \sup_{t \in [0,\omega]} \||c|^{\wedge}(-t)f(t)\|.$$

If $f \in P_{\omega c}(\mathbb{R}, X)$, then it is clear that $\|f\|_{\omega c} < \infty$. In this case, we say that f is c-bounded.

Let us denote the Nemytskii's operator associated with $F \in C(\mathbb{R} \times X, X)$ by $\mathcal{N}(\varphi)(\cdot) = F(\cdot, \varphi(\cdot))$. Then, we recall the following composition result (see ([4], Theorem 2.11)).

Theorem 5. *Let $F \in C(\mathbb{R} \times X, X)$ and $(\omega, c) \in \mathbb{R}^+ \times (\mathbb{C} \setminus \{0\})$ given. Then, the following statements are equivalent:*

(1) For every $\varphi \in P_{\omega c}(\mathbb{R}, X)$, we have $\mathcal{N}(\varphi) \in P_{\omega c}(\mathbb{R}, X)$;
(2) $F(t + \omega, cx) = cF(t, x)$ for all $(t, x) \in \mathbb{R} \times X$.

Now, let us consider the homogeneous system

$$x'(t) = A(t)x(t), \quad t \in \mathbb{R}. \tag{5}$$

In the following, we will assume that $\{U(t,s)\}_{t \geq s}$ is an evolution family of (5), that is, U is a classical solution of the system (5). For more details, see [10,11]. With this purpose, first we recall the definition of an evolution family, and some conditions which ensure the solvability of (5).

Definition 6. *A two-parameter family of bounded linear operators $\{U(t,s)\}_{t \geq s}$ on X is called an evolution family if*

1. *$U(t,r)U(r,s) = U(t,s)$ and $U(t,t) = I$ for all $t \geq r \geq s$ and $t, r, s \in \mathbb{R}$,*
2. *for each $x \in X$, the map $(t,s) \mapsto U(t,s)x$ is continuous on $t \geq s$.*

We denote by $R(\lambda, T) := (\lambda I - T)^{-1}$ for all λ on the resolvent set of the linear operator T ($\lambda \in \rho(T)$), and $\Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} \ : \ |\arg(\lambda)| \leq \theta\}$. We assume that $\{A(t)\}_{t \in \mathbb{R}}$ is a family of closed and densely defined linear operators on X, with domain $D(A(t))$, satisfying the so-called Acquistapace–Terreni conditions introduced in [12,13]:

(H1) There exist constants $\lambda_0 \geq 0$, $\theta \in (\frac{\pi}{2}, \pi)$, $K_1 \geq 0$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \quad \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{K_1}{1 + |\lambda|}, \text{ for all } \lambda \in \Sigma_\theta \cup \{0\}, \quad t \in \mathbb{R}.$$

(H2) There exist constants $K_2 \geq 0$, $\beta_1, \beta_2 \in (0, 1]$ with $\beta_1 + \beta_2 > 1$ such that

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \leq K_2|t - s|^{\beta_1}|\lambda|^{-\beta_2},$$

for all $t, s \in \mathbb{R}$, and $\lambda \in \Sigma_\theta$.

The following result follows from ([12], Theorem 2.3).

Theorem 7. *If the Acquistapace–Terreni conditions $(H1)$ and $(H2)$ are satisfied, then there exists a unique evolution family $\{U(t,s)\}_{t \geq s}$ on X such that:*

(a) $U(\cdot, s) \in C^1((s, \infty), \mathcal{B}(X))$, and $\frac{\partial U(t,s)}{\partial t} = A(t)U(t,s)$ for $t > s$.
(b) $\frac{\partial^+ U(t,s)x}{\partial s} = -U(t,s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$.

In that case, we say that the family $\{A(t)\}_{t\in\mathbb{R}}$ generates the evolution family $\{U(t,s)\}_{t\geq s}$. Sometimes, we shall write that $A(\cdot)$ generates the evolution family U.

Now we recall the notion of projections in the framework of evolution families and their interrelation.

Definition 8. *A strongly continuous function $P:\mathbb{R}\to\mathcal{B}(X)$ is called a projection-valued function if*

$$P^2(t)=P(t) \quad \text{for every} \quad t\in\mathbb{R}.$$

Given a projection-valued function $P:\mathbb{R}\to\mathcal{B}(X)$, we denote by Q the complementary projection-valued function, that is $Q(t)=I-P(t)$ for each $t\in\mathbb{R}$.

Definition 9. *We say that a projection-valued function $P:\mathbb{R}\to\mathcal{B}(X)$ is compatible with an evolution family U if*

(a) $U(t,s)P(s)=P(t)U(t,s)$ *for all* $t\geq s$.
(b) *The restriction* $U_Q(t,s):Q(s)X\to Q(t)X$ *is invertible for all* $t\geq s$ *(and we set* $U_Q(s,t)=U_Q(t,s)^{-1}$*).*

Remark 10. *If P is a projection-valued function compatible with an evolution family U, then*

1. $U(t,s)U_Q(s,t)Q(t)=Q(t)$ *for all* $t\geq s$.
2. $U_Q(s,t)U(t,s)Q(s)=Q(s)$ *for all* $t\geq s$.

Moreover, for all $t\geq r\geq s$, we have $U_Q(s,r)U_Q(r,t)=U_Q(s,t)$.

For a given evolution family U generated by $A(\cdot)$ and a projection-valued function P compatible with U, we denote the Green function associated to U and P and corresponding to the system (5) by

$$\Gamma(t,s):=\begin{cases} U(t,s)P(s), & t\geq s, t,s\in\mathbb{R}, \\ \\ -U_Q(t,s)Q(s), & t<s, t,s\in\mathbb{R}. \end{cases}$$

The next concept is due to Pinto and Vidal (see [21], Definition 4).

Definition 11. *Given a uniformly bounded projection P compatible with an evolution family U (generated by $A(\cdot)$), we say that the system (5) has an integrable dichotomy (λ,P) if there exists a function $\lambda:\mathbb{R}^2\to(0,\infty)$ such that*

$$\|\Gamma(t+\tau,s+\tau)\|\leq\lambda(t,s), \text{ for each } \tau\in\mathbb{R}, \tag{6}$$

and

$$I:=\sup_{t\in\mathbb{R}}\int_{\mathbb{R}}\lambda(t,s)\,ds<\infty. \tag{7}$$

In particular, if there is an integrable dichotomy, we denote

$$\mu:=\sup_{t,\tau\in\mathbb{R}}\int_{\mathbb{R}}\|\Gamma(t+\tau,s+\tau)\|\,ds<\infty.$$

Finally, we recall the following concept, which will be important to prove our main results.

Definition 12. *A continuous function $R:\mathbb{R}\times\mathbb{R}\to X$ is called bi-periodic if there exists $\omega>0$ such that*

$$R(t+\omega,s+\omega)=R(t,s) \quad \text{for all} \quad t,s\in\mathbb{R}.$$

For convenience, we say that R is ω-bi-periodic.

Example 13. *We have the following.*

- *If $R \in C(\mathbb{R} \times \mathbb{R}, X)$ and $R(t,s) = h(t-s)$ for some $h \in C(\mathbb{R}, X)$, then R is ω-bi-periodic, for any $\omega > 0$.*
- *If $g, h \in C(\mathbb{R}, X)$ are ω-periodic, then $R(t,s) = g(t)h(s)$ is ω-bi-periodic.*

3. The Linear Case

In this section, we deal with the existence and uniqueness of (ω, c)-periodic mild solutions to the linear problem

$$x'(t) = A(t)x(t) + h(t), \quad t \in \mathbb{R}, \tag{8}$$

where $A(\cdot)$ generates an evolution family U.

Definition 14. *A continuous function $x : \mathbb{R} \to X$ is called a (ω, c)-periodic mild solution to (8) on $\mathbb{R}$ if $x \in P_{\omega c}(\mathbb{R}, X)$ and satisfies the integral equation*

$$x(t) = U(t,s)x(s) + \int_s^t U(t,\sigma)h(\sigma)d\sigma \tag{9}$$

for all $t \geq s$, $t, s \in \mathbb{R}$.

The next theorem is the main result of this section.

Theorem 15. *Assume the following.*
(A1) $h \in P_{\omega c}(\mathbb{R}, X)$.
(A2) Γ is ω-bi-periodic, where ω is given in $(A1)$.
(A3) The evolution family generated by $A(t)$ has an integrable dichotomy (λ, P) satisfying

$$\lambda^{\sim}(t,\cdot) := |c|^{\wedge}(-(t-\cdot))\lambda(t,\cdot) \in L^1(\mathbb{R}), \quad \textit{for each } t \in \mathbb{R}, \tag{10}$$

and

$$I^{\sim} := \sup_{t\in\mathbb{R}} \int_{\mathbb{R}} \lambda^{\sim}(t,s)ds < \infty. \tag{11}$$

Then, (8) has a unique (ω, c)-periodic mild solution given by

$$x(t) = \int_{\mathbb{R}} \Gamma(t,s)h(s)\, ds, \quad t \in \mathbb{R}.$$

Proof. Let us define

$$x(t) = \int_{\mathbb{R}} \Gamma(t,s)h(s)\, ds, \quad t \in \mathbb{R}.$$

By $(A1)$ and $(A3)$ we have that the above integral is well-defined, since

$$\int_{\mathbb{R}} \|\Gamma(t,s)h(s)\|\, ds \leq \|h\|_{\omega c} \int_{\mathbb{R}} \lambda(t,s)|c|^{\wedge}(s)\, ds \leq \|h\|_{\omega c}|c|^{\wedge}(t)I^{\sim}.$$

Assertion 1. $x \in P_{\omega c}(\mathbb{R}, X)$.

Indeed, by Theorem 7 and Remark 10 we get the strong continuity of Γ in the first variable. Now, for $|\xi|$ small and $t \in \mathbb{R}$ fixed, we have

$$x(t+\xi) - x(t) = \int_{\mathbb{R}} [\Gamma(t+\xi, s+\xi)h(s+\xi) - \Gamma(t,s)h(s)]ds,$$

and the terms inside the integral can be bounded as follows:

$$\begin{aligned} \|\Gamma(t+\xi, s+\xi)h(s+\xi)\| &\leq \lambda(t,s)|c|^{\wedge}(s+\xi)\|h\|_{\omega c} \\ &\leq |c|^{\wedge}(t+\xi)|c|^{\wedge}(-(t-s))\lambda(t,s)\|h\|_{\omega c} \leq C\lambda^{\sim}(t,s), \end{aligned}$$

and $\|\Gamma(t,s)h(s)\| \le |c|^\wedge(t)|c|^\wedge(-(t-s))\lambda(t,s)\|h\|_{\omega c} \le C\lambda^\sim(t,s)$, where C is a constant (which does not depend on ξ). Since $\lambda^\sim(t,\cdot) \in L^1(\mathbb{R})$, it follows from the Dominated Convergence Theorem that $x \in C(\mathbb{R},X)$. Now, since $h \in P_{\omega c}(\mathbb{R},X)$, by definition of (ω,c)-periodicity and $(A2)$, we get

$$\begin{aligned} x(t+\omega) &= \int_{\mathbb{R}} \Gamma(t+\omega,s)h(s)\,ds = \int_{\mathbb{R}} \Gamma(t+\omega,\tau+\omega)h(\tau+\omega)\,d\tau \\ &= c\int_{\mathbb{R}} \Gamma(t,\tau)h(\tau)\,d\tau = c\,x(t), \quad t \in \mathbb{R}. \end{aligned}$$

So, we deduce that $x \in P_{\omega c}(\mathbb{R},X)$.

Assertion 2. *x is a (ω,c)-periodic mild solution of* (8).

By Assertion 1, we have $x \in P_{\omega c}(\mathbb{R},X)$. Next, for all $s \in \mathbb{R}$, we obtain

$$\begin{aligned} x(s) &= \int_{\mathbb{R}} \Gamma(s,\sigma)h(\sigma)\,d\sigma \\ &= \int_{-\infty}^{s} U(s,\sigma)P(\sigma)h(\sigma)d\sigma - \int_{s}^{\infty} U_Q(s,\sigma)Q(\sigma)h(\sigma)d\sigma. \end{aligned}$$

Now, for $t \ge s$, $t,s \in \mathbb{R}$, by Definitions 6, 8, 9, we get

$$\begin{aligned} U(t,s)x(s) &= \int_{-\infty}^{s} U(t,s)U(s,\sigma)P(\sigma)h(\sigma)d\sigma - \int_{s}^{\infty} U(t,s)U_Q(s,\sigma)Q(\sigma)h(\sigma)d\sigma \\ &= \int_{-\infty}^{s} U(t,\sigma)P(\sigma)h(\sigma)d\sigma - \int_{s}^{\infty} U_Q(t,\sigma)Q(\sigma)h(\sigma)d\sigma \\ &= \int_{-\infty}^{t} U(t,\sigma)P(\sigma)h(\sigma)d\sigma - \int_{s}^{t} U(t,\sigma)P(\sigma)h(\sigma)d\sigma \\ &- \int_{t}^{\infty} U_Q(t,\sigma)Q(\sigma)h(\sigma)d\sigma - \int_{s}^{t} U_Q(t,\sigma)Q(\sigma)h(\sigma)d\sigma \\ &= x(t) - \int_{s}^{t} U(t,\sigma)h(\sigma)d\sigma, \end{aligned}$$

and so x satisfies Equation (9), proving Assertion 2.

Assertion 3. *Uniqueness of x.*

Before, we proved that the unique (ω,c)-periodic solution of (5) is the trivial solution $x=0$. The proof is similar to ([20], Proposition 1). Indeed, let $B_0 \subset X$ be the set of initial conditions $\xi \in X$ pertaining to the (ω,c)-periodic solutions of (5). Let $\xi \in B_0$. First, assume that $(I-P)\xi \neq 0$. Define $\phi(t)^{-1} := \||c|^\wedge(-t)U(t,\cdot)(I-P(\cdot))\xi\|$. Since $(I-P)^2 = I-P$, we have

$$\begin{aligned} \int_{t}^{\infty} U(t,\cdot)(I-P(\cdot))\xi\;\phi(s)ds &= \int_{t}^{\infty} U(t,\cdot)(I-P(\cdot))\;(I-P(\cdot))\xi\phi(s)ds \\ &= \int_{t}^{\infty} U(t,\cdot)(I-P(\cdot))\;U(\cdot,s)U(s,\cdot)(I-P(\cdot))\xi\phi(s)ds. \end{aligned}$$

This implies that

$$\int_{t}^{\infty} \phi(t)\phi^{-1}(s)ds \le \int_{t}^{\infty} \||c|^\wedge(-(t-s))U(t,s)(I-P(s)\xi)\|ds \le I^\sim.$$

It follows from ([20], Lemma 1) that $\liminf_{s\in[t,\infty)} \phi(s) = 0$. This means that the function $t \mapsto \||c|^\wedge(-t)U(t,\cdot)(I-P(\cdot))\xi\|$ must be unbounded. Now, we suppose that $P\xi \neq 0$. Let $\phi(t)^{-1} := \||c|^\wedge(-t)U(t,\cdot)P(\cdot)\xi\|$. In a similar way as before but now with the integral on $(-\infty, t]$, we can conclude that $\liminf_{s\in(-\infty,t]} \phi(s) = 0$. Then, $\||c|^\wedge(-t)U(t,\cdot)P(\cdot)\xi\|$ is unbounded. Thus, the unique possibility is that $B_0 = \{0\}$, that is, $\xi = 0$.

Now, if there exists another mild solution $y \in P_{\omega c}(\mathbb{R}, X)$ of (8), then $u = x - y$ belongs to $P_{\omega c}(\mathbb{R}, X)$ and it is a (ω, c)-periodic mild solution to (5). It follows that $u = 0$, therefore, $x = y$. $\square$

Remark 16. *If U has an exponential dichotomy, that is, there exist $M > 0$ and $\varpi > 0$ such that*

$$\lambda(t,s) = \begin{cases} Me^{-\varpi(t-s)}, & t \geq s, \\ Me^{-\varpi(s-t)}, & s \geq t, \end{cases}$$

then (10) and (11) in $(A3)$ hold, provided that $e^{-\omega\varpi} < |c| \leq 1$. In fact,

$$\begin{aligned} I^\sim &= \sup_{t\in\mathbb{R}} \left(\int_{-\infty}^{t} |c|^\wedge(-(t-s))Me^{-\varpi(t-s)}ds + \int_t^\infty |c|^\wedge(-(t-s))Me^{-\varpi(s-t)}ds \right) \\ &= M \sup_{t\in\mathbb{R}} \left(\int_{-\infty}^{t} e^{-(\frac{\ln|c|}{\omega}+\varpi)(t-s)}ds + \int_t^\infty e^{(s-t)\frac{\ln|c|}{\omega}} e^{-\varpi(s-t)}ds \right) \\ &\leq M \sup_{t\in\mathbb{R}} \left(\int_0^\infty e^{-(\frac{\ln|c|+\omega\varpi}{\omega})s}ds + \int_0^\infty e^{-\varpi s}ds \right) \\ &= \frac{M(2\omega\varpi + \ln|c|)}{\varpi(\ln|c| + \omega\varpi)} < \infty. \end{aligned}$$

In the particular case that $P(t) = I$ for all $t \in \mathbb{R}$ (and hence $\lambda(t,s)$ can be taken as zero for $t < s$), then (10) and (11) hold whenever $e^{-\omega\varpi} < |c|$. Moreover, $I^\sim \leq \frac{M\omega}{\ln|c|+\omega\varpi}$.

4. The Semi-Linear Case

In this section, we present our main results. We study the existence and uniqueness of (ω, c)-periodic mild solutions for (1)–(3).

4.1. Semi-Linear Problem

In this subsection, we focus on the semi-linear problem (1), recall

$$x'(t) = A(t)x(t) + g(t, x(t)), \quad t \in \mathbb{R},$$

where $g : \mathbb{R} \times X \to X$ is a continuous function satisfying suitable conditions and $A(t)$ (usually unbounded) for each $t \in \mathbb{R}$ is a closed and densely defined linear operator with domain $D(A(t))$, satisfying $(H1)$ and $(H2)$.

Now, the linear case inspired us to introduce the following concept of the (ω, c)-periodic mild solution of a Cauchy-type problem.

Definition 17. *A continuous function $x : \mathbb{R} \to X$ is called a (ω, c)-periodic mild solution to (1) on $\mathbb{R}$ if $x \in P_{\omega c}(\mathbb{R}, X)$ and satisfies the integral equation*

$$x(t) = \int_{\mathbb{R}} \Gamma(t,\sigma)g(\sigma, x(\sigma))d\sigma \tag{12}$$

for all $t, s \in \mathbb{R}$.

In order to obtain our results, we assumed the following conditions.

(H3) $g \in C(\mathbb{R} \times X, X)$ and there exists $(\omega, c) \in \mathbb{R}^+ \times (\mathbb{C} \setminus \{0\})$ such that $g(t + \omega, cx) = cg(t, x)$ for all $t \in \mathbb{R}$ and for all $x \in X$.

(H4) There exists $L_g > 0$ such that $\|g(t,x) - g(t,y)\| \leq L_g\|x - y\|$ for all $t \in \mathbb{R}$ and for all $x, y \in X$.

(H5) Γ is ω-bi-periodic, where ω is given in $(H3)$.

Next, we have our first main result.

Theorem 18. *Suppose that* $(H1)$*–*$(H5)$ *and* $(A3)$ *hold, with* $L_g I^{\sim} < 1$*. Then,* (1) *has a unique* (ω, c)*-periodic mild solution.*

Proof. First, note that if $x \in P_{\omega c}(\mathbb{R}, X)$, according to $(H3)$ and Theorem 5, we have that $h(s) := g(s, x(s))$ is a (ω, c)-periodic function. Let us define the operator $\mathcal{G}$ on $P_{\omega c}(\mathbb{R}, X)$ by

$$\mathcal{G}x(t) = \int_{\mathbb{R}} \Gamma(t,s) g(s, x(s))\, ds = \int_{\mathbb{R}} \Gamma(t,s) h(s)\, ds, \quad t \in \mathbb{R}.$$

We can deduce that $\mathcal{G}x \in P_{\omega c}(\mathbb{R}, X)$ using $(H3)$ and proceeding as in the linear case.

Now, observe that x is the unique (ω, c)-periodic mild solution of (1) if, and only if x is the unique fixed point of $\mathcal{G}$, so we need to prove the existence and uniqueness of the fixed points of the operator $\mathcal{G}$ on $P_{\omega c}(\mathbb{R}, X)$.

To do that, we will use the Banach Fixed-Point Theorem. Indeed, let $x, y \in P_{\omega c}(\mathbb{R}, X)$. Then, $(H4)$, (6) and $(A3)$ imply

$$\begin{aligned}
\|\mathcal{G}(x) - \mathcal{G}(y)\|_{\omega c} &= \sup_{t \in [0,\omega]} \left\| |c|^\wedge(-t) \int_{\mathbb{R}} \Gamma(t,s)[g(s,x(s)) - g(s,y(s))]\, ds \right\| \\
&\le \sup_{t \in [0,\omega]} \int_{\mathbb{R}} \| |c|^\wedge(-(t-s)) \Gamma(t,s) \| \cdot L_g \cdot \| |c|^\wedge(-s)[x(s) - y(s)] \|\, ds \\
&\le L_g \|x - y\|_{\omega c} \sup_{t \in \mathbb{R}} \int_{\mathbb{R}} \lambda^{\sim}(t,s)\, ds \\
&= L_g I^{\sim} \|x - y\|_{\omega c}.
\end{aligned}$$

Since $L_g I^{\sim} < 1$, there exists a unique $z \in P_{\omega c}(\mathbb{R}, X)$ such that $\mathcal{G}z = z$, that is, $z(t) = \int_{\mathbb{R}} \Gamma(t,s) g(s, z(s))\, ds$. Hence z is the unique (ω, c)-periodic mild solution of (1). □

Corollary 19. *Assume that* $(H1)$*–*$(H5)$ *hold. Furthermore, suppose the evolution family generated by* $A(t)$ *has an exponential dichotomy, as in Remark 16. Then,* (1) *has a unique* (ω, c)*-periodic solution whenever*

$$\frac{L_g M(2\omega\varpi + \ln|c|)}{\varpi(\ln|c| + \omega\varpi)} < 1,$$

with $e^{-\omega\varpi} < |c| \le 1$*. In the particular case that* $P = I$*, the result holds, provided* $\frac{M L_g \omega}{\ln|c| + \omega\varpi} < 1$*, with* $e^{-\omega\varpi} < |c|$*.*

4.2. Semi-Linear Problem: A More General Case

In this subsection, we study (ω, c)-periodic mild solutions of (2), recall

$$x(t) = A(t)x(t) + f(t, x(t), \varphi[\alpha(t, x(t))]), \quad t \in \mathbb{R},$$

where $f : \mathbb{R} \times X \times X \to X$, $\varphi : \mathbb{R} \to X$, $\alpha : \mathbb{R} \times X \to \mathbb{R}$ are continuous functions and $A(t)$ for $t \in \mathbb{R}$ is as above.

In an analogous way to the preceding problem, we present the concept of the (ω, c)-periodic mild solution to (2) as follows.

Definition 20. *A continuous function* $x : \mathbb{R} \to X$ *is called a* (ω, c)*-periodic mild solution to* (2) *on* $\mathbb{R}$ *if* $x \in P_{\omega c}(\mathbb{R}, X)$ *and it satisfies the integral equation*

$$x(t) = \int_{\mathbb{R}} \Gamma(t,\sigma) f(\sigma, x(\sigma), \varphi[\alpha(\sigma, x(\sigma))]) d\sigma$$

for all $t, s \in \mathbb{R}$*.*

In order to investigate the regularity of (ω, c)-periodic solutions to (2), we have to show the following composition theorem.

Theorem 21. *Assume that the following conditions hold.*

(C1) $f \in C(\mathbb{R} \times X \times X, X)$ *and there exists* $(\omega, c) \in \mathbb{R}^+ \times (\mathbb{C} \setminus \{0\})$ *such that* $f(t+\omega, cx, cy) = cf(t, x, y)$ *for all* $t \in \mathbb{R}$ *and for all* $x, y \in X$.

(C2) $\alpha \in C(\mathbb{R} \times X, \mathbb{R})$ *is such that* $\alpha(t+\omega, cx) = c\alpha(t, x)$ *for all* $t \in \mathbb{R}$ *and for all* $x \in X$.

(C3) $\varphi \in C(\mathbb{R}, X)$ *and* $\varphi(ct) = c\,\varphi(t)$ *for all* $t \in \mathbb{R}$.

If $x \in P_{\omega c}(\mathbb{R}, X)$, *then* $f(\cdot, x(\cdot), \varphi[\alpha(\cdot, x(\cdot))]) \in P_{\omega c}(\mathbb{R}, X)$.

Proof. In view of $(C2)$ and Theorem 5, we have $\beta(\cdot) := \alpha(\cdot, x(\cdot)) \in P_{\omega c}(\mathbb{R}, \mathbb{R})$.

On the other hand, we claim that $y(\cdot) := \varphi(\beta(\cdot)) \in P_{\omega c}(\mathbb{R}, X)$. Indeed, since $y \in C(\mathbb{R}, X)$ and $\beta \in P_{\omega c}(\mathbb{R}, \mathbb{R})$, we obtain by $(C3)$ that

$$y(t+\omega) = \varphi(\beta(t+\omega)) = \varphi(c\,\beta(t)) = c\,\varphi(\beta(t)) = c\,y(t),$$

that is, $y \in P_{\omega c}(\mathbb{R}, X)$. Using this fact together with $(C1)$, we have

$$\begin{aligned} f(t+\omega, x(t+\omega), \varphi[\alpha(t+\omega, x(t+\omega))]) &= f(t+\omega, x(t+\omega), y(t+\omega)) \\ &= f(t+\omega, cx(t), cy(t)) \\ &= cf(t, x(t), y(t)) \\ &= cf(t, x(t), x[\alpha(t, x(t))]). \end{aligned}$$

This completes the proof of the theorem. □

Remark 22. *The function* $\varphi_1 : \mathbb{R} \to X$ *given by* $\varphi_1(t) = t \cdot x$ *for* $t \in \mathbb{R}$ *and* $x \in X$ *satisfies* $(C3)$. *Additionally, the function* $\varphi_2 : \mathbb{R} \to \mathbb{C}$ *given by* $\varphi_2(t) = t^{kn}$ *with* $c^{kn-1} = 1$ *satisfies this condition for* $k, n \in \mathbb{N}$.

The next theorem is our second main result.

Theorem 23. *Assume that* $(C1)$–$(C3)$, $(H1)$–$(H2)$, *and* $(A3)$ *hold. Let* Γ *be a* ω*-bi-periodic function, where* ω *is given in* $(C1)$. *Suppose the following conditions.*

1. *There exists* $C_1 > 0$ *such that*

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \le C_1(\|u_1 - u_2\| + \|v_1 - v_2\|)$$

for all $t \in \mathbb{R}$ *and for all* $u_1, u_2, v_1, v_2 \in X$.

2. *There exists* $C_2 > 0$ *such that*

$$\|\alpha(t, u) - \alpha(t, v)\| \le C_2\|u - v\|$$

for all $t \in \mathbb{R}$ *and for all* $u, v \in X$.

3. *There exists* $L_\varphi > 0$ *such that*

$$\|\varphi(t) - \varphi(s)\| \le L_\varphi |t - s|$$

for all $t, s \in \mathbb{R}$.

If $C_1(L_\varphi C_2 + 1)I^{\sim} < 1$, *then* (2) *has a unique* (ω, c)*-periodic mild solution.*

Proof. Let $x \in P_{\omega c}(\mathbb{R}, X)$. By conditions $(C1)$–$(C3)$,, we have by Theorem 21 that

$$h(\cdot) := f(\cdot, x(\cdot), \varphi[\alpha(\cdot, x(\cdot))]) \in P_{\omega c}(\mathbb{R}, X).$$

Let us define the operator $\mathcal{G}$ on $P_{\omega c}(\mathbb{R}, X)$ by

$$\mathcal{G}x(t) := \int_{\mathbb{R}} \Gamma(t,s) f(s, x(s), \varphi[\alpha(s, x(s))]) ds = \int_{\mathbb{R}} \Gamma(t,s) h(s) ds. \tag{13}$$

Note that, using the same arguments as in the proof of Theorem 18, we can prove that $\mathcal{G}$ maps $P_{\omega c}(\mathbb{R}, X)$ into $P_{\omega c}(\mathbb{R}, X)$ and each fixed point of $\mathcal{G}$ is a (ω, c)-periodic mild solution of (2).

We have only to prove that the operator $\mathcal{G}$ is a (strict) contraction in the Banach space $P_{\omega c}(\mathbb{R}, X)$. Indeed, given $x_1, x_2 \in P_{\omega c}(\mathbb{R}, X)$, by conditions (a)–(c), we have

$$\begin{aligned}
&\| |c|^\wedge(-t)(\mathcal{G}x_1(t) - \mathcal{G}x_2(t)) \| \\
&= \left\| |c|^\wedge(-t) \int_{\mathbb{R}} \Gamma(t,r)(f(r, x_1(r), \varphi[\alpha(r, x_1(r))]) - f(r, x_2(r), \varphi[\alpha(r, x_2(r))]))\, dr \right\| \\
&\leq \int_{\mathbb{R}} \| |c|^\wedge(-t)\Gamma(t,r)\| \cdot \| f(r, x_1(r), \varphi[\alpha(r, x_1(r))]) - f(r, x_2(r), \varphi[\alpha(r, x_2(r))]))\|\, dr \\
&\leq C_1 \int_{\mathbb{R}} \| |c|^\wedge(-t)\Gamma(t,r)\| \cdot (\|x_1(r) - x_2(r)\| + \|\varphi[\alpha(r, x_1(r))] - \varphi[\alpha(r, x_2(r))]\|)\, dr \\
&\leq C_1 \int_{\mathbb{R}} \| |c|^\wedge(-t)\Gamma(t,r)\| \cdot (\|x_1(r) - x_2(r)\| + L_\varphi \|\alpha(r, x_1(r)) - \alpha(r, x_2(r))\|)\, dr \\
&\leq C_1 \int_{\mathbb{R}} \| |c|^\wedge(-(t-r))\Gamma(t,r)\| \cdot (L_\varphi C_2 + 1) \| |c|^\wedge(-r)(x_1(r) - x_2(r))\|\, dr \\
&\leq C_1(L_\varphi C_2 + 1) \int_{\mathbb{R}} \lambda^\sim(t,r) \cdot \|x_1 - x_2\|_{\omega c}\, dr.
\end{aligned}$$

Then, using $(A3)$, we get

$$\begin{aligned}
\|\mathcal{G}x_1 - \mathcal{G}x_2\|_{\omega c} &= \sup_{t \in [0,\omega]} \| |c|^\wedge(-t)(\mathcal{G}x_1(t) - \mathcal{G}x_2(t)) \| \\
&\leq C_1(L_\varphi C_2 + 1) I^\sim \|x_1 - x_2\|_{\omega c}.
\end{aligned}$$

Since $C_1(L_\varphi C_2 + 1) I^\sim < 1$, we conclude that $\mathcal{G}$ has a unique fixed point $y \in P_{\omega c}(\mathbb{R}, X)$ by the contraction mapping principle, which is the unique (ω, c)-periodic mild solution of (2). $\square$

The next result is a direct consequence of Remark 16. The proof is similar to the one of Theorem 23.

Corollary 24. *Assume that $(C1)$–$(C3)$, $(H1)$–$(H2)$ and conditions (a)–(c) in Theorem 23 hold. Let Γ be a ω-bi-periodic function, where ω is given in $(C1)$. Furthermore, suppose the evolution family generated by $A(t)$ has an exponential dichotomy as in Remark 16. Then (2) has a unique (ω, c)-periodic solution whenever*

$$\frac{MC_1(L_\varphi C_2 + 1)(2\omega\varpi + \ln|c|)}{\varpi(\ln|c| + \omega\varpi)} < 1,$$

with $e^{-\omega\varpi} < |c| \leq 1$. In the particular case that $P = I$, the result holds provided $\frac{MC_1(L_\varphi C_2+1)\omega}{\ln|c|+\omega\varpi} < 1$, with $e^{-\omega\varpi} < |c|$.

Next, we will present an example that satisfies the conditions of Theorem 23.

Example 25. *Let $(X, \|\cdot\|) = (L^2(0,1), \|\cdot\|_2)$. Consider the operator A_B given by*

$$\begin{aligned}
&D(A_B) = \{x \in C^1[0,1] : \ x' \text{ is absolutely continuous on } [0,1],\ x'' \in X,\ x(0) = x(1) = 0\}, \\
&A_B x(r) = x''(r), \quad \forall r \in (0,1),\ \forall x \in D(A_B).
\end{aligned}$$

It is well-known that A_B is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ on X given by

$$T(t)x(r) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\langle x, e_n\rangle_{L^2} e_n(r),$$

where $e_n(r) = \sqrt{2}\sin n\pi r,\ n = 1, 2, \ldots$. Moreover, $\|T(t)\| \leq e^{-\pi^2 t}$ for $t \geq 0$ (see [22]). Let us define a family of linear operators $\{A(t)\}_{t\in\mathbb{R}}$ by

$$\begin{cases} D(A(t)) = D(A_B), & t \in \mathbb{R}, \\ A(t)x = \Big(A_B + h(t)I\Big)x, & x \in D(A(t)),\, h \in P_\omega(\mathbb{R}, \mathbb{R}). \end{cases}$$

Then, $\{A(t)\}_{t\in\mathbb{R}}$ generates an evolution family $\{U(t,s)\}_{t\geq s}$ given by

$$U(t,s)x = T(t-s)e^{\int_s^t h(\sigma)d\sigma}x,$$

that is,

$$U(t,s)x(r) = \sum_{n=1}^{\infty} e^{\int_s^t [h(\sigma) - n^2\pi^2]d\sigma}\langle x, e_n\rangle_{L^2} e_n(r).$$

Note that, for $t > s$, the exponential terms in the sum vanish as $n \to \infty$. Let $N \in \mathbb{N}$ such that $\sum_1^N$ is the unstable part and $\sum_{N+1}^{\infty}$ the stable part.

Let $I - P = diag(1, 1, 1, \ldots, 1, 0, 0, 0, \ldots)$ with N numbers 1 at the diagonal. Then, $rank(I - P) = N$ and $rank(P) = \infty$. Moreover, the authors in ([21], Example 22) proved that

$$|U(t,s)P| \leq Ce^{-\int_s^t \mu_1(r)dr}, \quad t \geq s,$$
$$|U(t,s)(I-P)| \leq Ce^{\int_s^t \mu_2(r)dr}, \quad s > t,$$

where $C > 0$, μ_1 and μ_2 are positive, locally integrable functions with $\int_{-\infty}^{\infty}\mu_1(r)dr = \infty$ and $\int_{-\infty}^{\infty}\mu_2(r)dr = \infty$. Thus, $x'(t) = A(t)x(t)$ has an integrable dichotomy (λ, P), where λ is given by

$$\lambda(t,s) = \begin{cases} Ce^{-\int_s^t \mu_1(r)dr}, & t \geq s, \\ Ce^{\int_s^t \mu_2(r)dr}, & s > t. \end{cases}$$

On the other hand, we claim that the Green function Γ is ω-bi-periodic. Indeed, for $t \geq s$,

$$\Gamma(t+\omega, s+\omega)x = U(t+\omega, s+\omega)Px = T(t-s)Pe^{\int_{s+\omega}^{t+\omega} h(\sigma)d\sigma}x = \Gamma(t,s)x,$$

where we have used that h is ω-periodic. The case $\Gamma(t+\omega, s+\omega) = \Gamma(t,s)$ for $s > t$ is similar.

Let $a \in P_{\omega c}(\mathbb{R}, \mathbb{R})$ and $b \in P_{\omega\frac{1}{c}}(\mathbb{R}, \mathbb{R})$ be given. We consider f, α, φ given by

$$f(t,x,y) = a(t)[\cos(b(t)x) + \cos(b(t)y)],$$

$$\alpha(t,x) = a(t)\sin(b(t)x),$$

and

$$\varphi(t) = t.$$

We claim that f, α and φ satisfy conditions (C1)–(C3) *and* (a)–(c) *with* $L_\varphi = 1$, $C_1 = C_2 = M_1 := \sup_{t\in\mathbb{R}} |a(t)b(t)|$ *(which exists because the function* $L(t) = |a(t)b(t)|$ *is ω-periodic on $\mathbb{R}$ and therefore bounded on $\mathbb{R}$). Indeed,*

$$\begin{aligned}\|f(t,x_1,y_1) - f(t,x_2,y_2)\|_2^2 &\leq \int_0^1 |f(s,x_1(s),y_1(s)) - f(s,x_2(s),y_2(s))|^2 ds \\ &\leq \int_0^1 |a(s)|^2|b(s)|^2(|x_1(s)-y_1(s)| + |x_2(s)-y_2(s)|)^2 ds \\ &\leq M_1^2(\|x_1-y_1\|_2 + \|x_2-y_2\|_2)^2.\end{aligned}$$

Analogously, $\|\alpha(t,x) - \alpha(t,y)\|_2^2 \leq M_1^2\|x-y\|_2^2$.
Therefore, by Theorem 23 we have that the problem

$$u'(t) = A(t)u(t) + f(t,x,\varphi[\alpha(t,u(t))]), \quad t\in\mathbb{R},$$

has a unique (ω,c)-periodic mild solution whenever (10) *and* (11) *are satisfied for this c and* $M_1(M_1+1)I^{\sim} < 1$.

4.3. Semi-Linear Problem with Delay

Finally, in this last subsection, we study (ω,c)-periodic mild solutions of (3), that is,

$$x(t) = A(t)x(t) + f(t,x(t),x(t-h)), \quad t\in\mathbb{R},\, h\geq 0,$$

where $f:\mathbb{R}\times X\times X\to X$ is a continuous function and $A(t)$ for $t\in\mathbb{R}$ is as above.

In the following, we consider the mild solution concept for the previous equation.

Definition 26. *A continuous function $x:\mathbb{R}\to X$ is called a (ω,c)-periodic mild solution to* (3) *on $\mathbb{R}$ if $x\in P_{\omega c}(\mathbb{R},X)$ and it satisfies the integral equation*

$$x(t) = \int_{\mathbb{R}} U(t,\sigma)f(\sigma,x(\sigma),x(\sigma-h))d\sigma$$

for all $t,s\in\mathbb{R}$.

Before stating the main result of this subsection, we show the following composition lemma.

Lemma 27. *Assume that $f\in C(\mathbb{R}\times X\times X)$ satisfies* (C1). *If $x\in P_{\omega c}(\mathbb{R},X)$ then $f(\cdot,x(\cdot),x_h(\cdot))\in P_{\omega c}(\mathbb{R},X)$.*

Proof. Similar to the proof of Theorem 21. □

Theorem 28. *Assume that* (C1), (H1)–(H2) *and* (A3) *hold. Let Γ be a ω-bi-periodic function, where ω is given in* (C1). *Suppose that there exists $C_1 > 0$ such that*

$$\|f(t,u_1,v_1) - f(t,u_2,v_2)\| \leq C_1(\|u_1-v_1\| + \|u_2-v_2\|)$$

for all $t\in\mathbb{R}$ and for all $u_1,u_2,v_1,v_2\in X$. If $C_1(1+|c|^\wedge(-h))I^{\sim} < 1$, then (3) *has a unique (ω,c)-periodic mild solution.*

Proof. Let us define the operator $\mathcal{G}$ on $P_{\omega c}(\mathbb{R},X)$ by

$$\mathcal{G}x(t) := \int_{\mathbb{R}} \Gamma(t,s)f(s,x(s),x(s-h))ds.$$

Remark 4 and Lemma 27 imply that $f(\cdot,x(\cdot),x_h(\cdot))\in P_{\omega c}(\mathbb{R},X)$ whenever $x\in P_{\omega c}(\mathbb{R},X)$. The rest of the proof is similar to the one of Theorem 23. □

The next result is a straightforward consequence of Theorem 28 and Remark 16.

Corollary 29. *Assume that* $(C1)$ *and* $(H1)$*–*$(H2)$ *hold. Let* Γ *be a* ω*-bi-periodic function, where* ω *is given in* $(C1)$*. Furthermore, suppose the evolution family generated by* $A(t)$ *has an exponential dichotomy as in Remark 16, and there exists* $C_1 > 0$ *such that*

$$\|f(t,u_1,v_1) - f(t,u_2,v_2)\| \le C_1(\|u_1 - v_1\| + \|u_2 - v_2\|)$$

for all $t \in \mathbb{R}$ *and for all* $u_1, u_2, v_1, v_2 \in X$*. Then* (3) *has a unique* (ω, c)*-periodic solution whenever*

$$\frac{MC_1(1 + |c|\hat{}(-h))(2\omega\varpi + \ln|c|)}{\varpi(\ln|c| + \omega\varpi)} < 1,$$

with $e^{-\omega\varpi} < |c| \le 1$*. In the particular case that* $P = I$*, the result holds, provided* $\frac{MC_1(1+|c|\hat{}(-h))\omega}{\ln|c|+\omega\varpi} < 1$*, with* $e^{-\omega\varpi} < |c|$*.*

5. Conclusions

In this work, we obtained regularity of (ω, c)-periodic solutions of

$$x'(t) = A(t)x(t) + f(t, x(t), \varphi[\alpha(t, x(t))]), \quad t \in \mathbb{R}$$

where $A(t)$ satisfies the Acquistapace–Terreni conditions, f satisfies suitable conditions, and the associated homogeneous problem has an integrable dichotomy.

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References

1. Treanţă, S. Differential Variational Inequalities in Infinite-Dimensional Spaces. *Mathematics* **2021**, *9*, 266. [CrossRef]
2. Treanţă, S. Gradient Structures Associated with a Polynomial Differential Equation. *Mathematics* **2020**, *8*, 535. [CrossRef]
3. Treanţă, S. On the Kernel of a Polynomial of Scalar Derivations. *Mathematics* **2020**, *8*, 515. [CrossRef]
4. Alvarez, E.; Gómez, A.; Pinto, M. (ω, c)-periodic functions and mild solutions to abstract fractional integro-differential equations. *Electron. J. Qual. Theory Differ. Equ.* **2018**, *16*, 1–8. [CrossRef]
5. Li, M.; Wang, J.R.; Feckan, M. (ω, c)-periodic solutions for impulsive differential systems. *Commun. Math. Anal.* **2018**, *21*, 35–46.
6. Wang, J.R.; Ren, L.; Zhou, Y. (ω, c)-Periodic solutions for time varying impulsive differential equations. *Adv. Differ. Equ.* **2019**, *259*, 1–9. [CrossRef]
7. Agaoglou, M.; Feckan, M.; Panagiotidou, A.P. Existence and uniqueness of (ω, c)-periodic solutions of semi-linear evolution equations. *Int. Dyn. Syst. Differ. Equ.* **2020**, *10*, 149–166.
8. Mophou, G.; N'Guérékata, G.M. An existence result of (ω, c)-periodic mild solutions to some fractional differential equation. *Nonlinear Stud.* **2002**, *27*, 1–8.
9. Chicone, C.; Latushkin, Y. *Evolution Semigroups in Dynamical Systems and Differential Equations*; American Mathematical Society: Providence, RI, USA, 1999.
10. Engel, K.J.; Nagel, R. *One-Parameter Semigroups for Linear Evolution Equations*; Springer Science & Business Media: Berlin, Germany, 1999; Volume 194.
11. Pazy, A. Semigroups of Linear Operators and Applications to Partial Differential Equations. In *Applied Mathematical Sciences*; Springer: New York, NY, USA, 1983; Volume 44.
12. Acquistapace, P. Evolution operators and strong solution for abstract parabolic equations. *Differ. Integral Equ.* **1988**, *1*, 433–457.

13. Acquistapace, P.; Terreni, B. A unified approach to abstract linear parabolic equations. *Rendiconti del Seminario Matematico della Università di Padova* **1987**, *78*, 47–107.
14. Chang, Y.K.; Zhang, R.; N'Guérékata, G.M. Weighted pseudo almost automorphic solutions to non-autonomous semi-linear evolution equations with delay and S^p-weighted pseudo almost automorphic coefficients. *Topol. Methods Nonlinear Anal.* **2014**, *43*, 69–88. [CrossRef]
15. Diagana, T. Existence of almost periodic solutions to some third-order non-autonomous differential equations. *Electron. J. Qual. Theory Differ. Equ.* **2011**, *65*, 1–15.
16. Diagana, T.; Hernandez, E.; Rabello, M. Pseudo almost periodic solutions to some non-autonomous neutral functional differential equations with unbounded delay. *Math. Comput. Model.* **2007**, *45*, 1241–1252. [CrossRef]
17. Xia, Z. Pseudo almost periodic mild solution of non-autonomous impulsive integro-differential equations. *Mediterr. J. Math.* **2016**, *13*, 1065–1086. [CrossRef]
18. Xia, Z. Pseudo asymptotic behavior of mild solution for non-autonomous integrodifferential equations with nondense domain. *J. Appl. Math.* **2014**, *2014*, 1–10.
19. Xiao, T.-J.; Zhu, X.-X.; Liang, J. Pseudo-almost automorphic mild solutions to non-autonomous differential equations and applications. *Nonlinear Anal.* **2009**, *70*, 4079–4085. [CrossRef]
20. Pinto, M. Bounded and periodic solutions of nonlinear integro-differential equations with infinite delay. *Electron. J. Qual. Theory Differ. Equ.* **2009**, *46*, 1–20. [CrossRef]
21. Pinto, M.; Vidal, C. Pseudo-almost-periodic solutions for delayed differential equations with integrable dichotomies and bi-almost-periodic Green functions. *Math. Meth. Appl. Sci.* **2017**, *40*, 6998–7012. [CrossRef]
22. Lunardi, A. Analytic Semigroups and Optimal Regularity in Parabolic Problems. In *PNLDE*; Birkhäuser: Basel, Switzerland, 1995; Volume 16.