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Analysis of a Discrete-Time Queueing Model with Disasters

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Abstract: Queueing models with disasters can be used to evaluate the impact of a breakdown or a system reset in a service facility. In this paper, we consider a discrete-time single-server queueing system with general independent arrivals and general independent service times and we study the effect of the occurrence of disasters on the queueing behavior. Disasters occur independently from time slot to time slot according to a Bernoulli process and result in the simultaneous removal of all customers from the queueing system. General probability distributions are allowed for both the number of customer arrivals during a slot and the length of the service time of a customer (expressed in slots). Using a two-dimensional Markovian state description of the system, we obtain expressions for the probability, generating functions, the mean values, variances and tail probabilities of both the system content and the sojourn time of an arbitrary customer under a first-come-first-served policy. The customer loss probability due to a disaster occurrence is derived as well. Some numerical illustrations are given.

Keywords: queueing theory; discrete-time model; disasters; general service times

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1. Introduction

Queueing models with negative arrivals have been studied extensively over the last decades, owing to their applicability in the performance analysis of a wide range of systems, such as computers, telecommunication systems and manufacturing systems. The basic version of such models, known as the G-queue, is due to Gelenbe [1] and considers the notion of a negative customer that, upon arrival, removes one ordinary (or positive) customer from the queueing system according to some killing strategy, such as the removal of the customer in service or the removal of the customer that arrived most recently, if any. Another type of negative arrival, introduced by Towsley and Tripathi [2], is disasters that upon occurrence result in the simultaneous removal of all customers from the queueing system. As such, queueing models with disasters can be used to evaluate the impact of a machine breakdown in a production system, a system reset in a service facility or a virus infection affecting a computer system. Queues with disasters are also referred to in the literature as queues with mass exodus, catastrophes, queue flushing or stochastic clearing [3].

In this paper, our focus is on queueing models with disasters in the discrete-time domain, which have so far been analyzed to a lesser extent than their continuous-time counterparts. The first study [3] on discrete-time queues with disasters considered the Geo/Geo/1 queueing model with a Bernoulli distribution of the number of customer arrivals per slot and geometric service times under the impact of Bernoulli disasters. A similar disaster model was used in [4] to model the behavior of an email contact center. A transient analysis of the system content in the Geo/Geo/1 disaster model was performed in [5]. The extension to a discrete-time Geo/G/1 disaster model with Bernoulli arrivals and general independent service times was considered in [6] (system content) and [7,8] (sojourn time). The Geo/G/1 disaster model was also analyzed in [9] under an N-policy operation and in [10–12] under repair times after a disaster. The queue length and sojourn time in a G/Geo/1 disaster queue with general independent interarrival times between customers and, at most, 1 customer arrival per slot were investigated in [13], while disaster queues with bursty Bernoulli arrivals and geometric service times were considered in [14,15].

Our current work further extends the existing results in the literature to a discrete-time disaster model, where general probability distributions are allowed for both the number of customer arrivals during a slot and the length of a customer service time. We present a full queueing analysis of this queueing model with disasters and derive expressions for the probability generating functions as well as the mean values, variances and tail probabilities of both the system content and the sojourn time under a first-come-first-served (FCFS) policy.

For related work on discrete-time G-queues with negative customers, the reader is referred to [3,7,14,16–23]. For work on continuous-time queueing models with negative customers and/or disasters, we refer to the bibliography in [24,25] and the more recent papers [26–36]. Additionally, somewhat related to this paper in the sense that customers may leave the system before their service is completed are queueing models with customer impatience or deadlines; we refer to [37] and the references therein for an overview of such models.

The paper is organized as follows. The specific assumptions of the considered queueing model are detailed in Section 2. In Section 3, system equations are established that describe the evolution of the state of the queueing system from slot to slot. Based on these system equations, an expression for the steady-state joint probability generating function (pgf) of the system state variables is obtained in Section 4 in terms of the unknown probability of having an empty system. A technique to calculate this remaining unknown is presented in Section 5, together with the analysis of the main characteristics of the system content. Section 6 then derives the pgf of the unfinished work in the system, as an intermediate step for the analysis of the characteristics of the sojourn time in Section 7. The customer loss probability due to a disaster occurrence is derived in Section 8. Some numerical examples to illustrate the analysis are given in Section 9, before the paper is concluded in Section 10.

2. Queueing Model

We consider a discrete-time queueing system with one server and an infinite waiting room for customers. The time axis is divided into fixed-length slots. New customers arrive at the system in a stochastic way, according to a general independent arrival process, i.e., the numbers of customers arriving during the consecutive slots are assumed to be independent and identically distributed (i.i.d.) discrete random variables. Their common probability mass function (pmf) is indicated as

$$a(n) = \text{Prob}[n \text{ customer arrivals during a slot}]\,, \quad n \geq 0\,,$$

with corresponding pgf

$$A(z) = \sum_{n=0}^{\infty} a(n) z^n\,.$$

The mean arrival rate, i.e., the mean number of customer arrivals during a slot, is given by

$$\lambda = A'(1)\,.$$

The service of a customer is assumed to require a positive integer number of slots and can start or end at slot boundaries only. More specifically, the service times of the customers are assumed to constitute a sequence of i.i.d. positive discrete random variables with common pmf

$$s(n) = \text{Prob}[\text{service of a customer takes } n \text{ slots}]\,, \quad n \geq 1\,,$$

corresponding pgf

$$S(z) = \sum_{n=1}^{\infty} s(n) z^n$$

and mean service time

$$S'(1) = \frac{1}{\mu}\,,$$

where μ is the so-called mean service rate, i.e., the mean number of customers that can be served per slot. The service times are also assumed to be independent of the random variables used in the description of the arrival process.

The queueing system is subject to so-called independent Bernoulli disasters, i.e., during any slot, either a disaster occurs with probability σ ($\sigma > 0$) or no disaster occurs with probability $1 - \sigma$, independently from slot to slot. When such a disaster occurs during a slot, all customers in the system as well as all new arrivals during the slot get lost. In the sequel, we specifically consider that in case of a disaster, all customers are removed from the system at the end of the disaster slot, thus leaving the system empty at the end of that slot.

3. System Equations

Let the random variable u_k denote the system content, i.e., the total number of customers in the system, at the beginning of slot k. Let a_k be the number of customer arrivals during slot k, and let d_k indicate the number of disasters occurring during slot k. Clearly, to describe the evolution of the system content from slot k to slot $k+1$, some information is also needed about the still remaining part of the service time of the customer in service, if any, at the beginning of slot k. We therefore define the random variable h_k as follows: h_k denotes the remaining number of slots needed to complete the service of the customer currently in service at the beginning of slot k, if $u_k \geq 1$, and $h_k = 0$ if $u_k = 0$. Note that this definition implies that $h_k > 0$ if and only if $u_k > 0$. Similarly, $h_k = 0$ if and only if $u_k = 0$. Finally, we let s^* indicate the service time of the next customer to receive service at the beginning of slot k.

With these definitions, the behavior of the queueing system is then characterized by the following system equations:

(a) If $d_k = 1$:

$$h_{k+1} = u_{k+1} = 0\,. \tag{1}$$

(b) If $d_k = 0$ and $h_k = 0$:

$$u_{k+1} = a_k\,, \tag{2}$$

$$h_{k+1} = \begin{cases} 0\,, & \text{if } a_k = 0\,, \\ s^*\,, & \text{if } a_k > 0\,. \end{cases} \tag{3}$$

(c) If $d_k = 0$ and $h_k = 1$:

$$u_{k+1} = u_k - 1 + a_k\,, \tag{4}$$

$$h_{k+1} = \begin{cases} 0\,, & \text{if } u_k = 1 \text{ and } a_k = 0\,, \\ s^*\,, & \text{if } u_k - 1 + a_k > 0\,. \end{cases} \tag{5}$$

(d) If $d_k = 0$ and $h_k > 1$:

$$u_{k+1} = u_k + a_k\,, \tag{6}$$

$$h_{k+1} = h_k - 1\,. \tag{7}$$

Equations (1)–(7) are based on the following observations. If there is a disaster in slot k, all customers (including new arrivals during slot k) are removed from the system, so we have an empty system at the beginning of slot $k+1$. In case no disaster occurs during slot

k and the system is empty at the beginning of slot k, then at the beginning of slot $k+1$, the system only contains the new arrivals during slot k, and, if any, one of these new arrivals are taken into service. If $h_k = 1$ and there is no disaster in slot k, the customer in service leaves the system at the end of slot k; moreover, the service of a new customer starts at the beginning of slot $k+1$ unless the system has become empty. Finally, if $h_k > 1$ and there is no disaster in slot k, no customer leaves the system at the end of slot k, and the remaining service time of the customer in service decreases by one slot.

It is obvious from the system equations that knowledge of the values of h_k and u_k suffices to determine the joint probability distribution of h_{k+1} and u_{k+1}. The sequence of pairs $\{(h_k, u_k)\}$, therefore, forms a two-dimensional first-order Markov chain and the state of the queueing system in slot k is fully characterized by the pair (h_k, u_k).

4. Queueing Analysis

By means of the system Equations (1)–(7), we can now analyze the queueing behavior. To do so, we first define $P_k(x,z)$ as the joint pgf of the state vector (h_k, u_k) at the beginning of slot k:

$$P_k(x,z) \triangleq \mathrm{E}\Big[x^{h_k} z^{u_k}\Big] = \sum_{i=0}^{\infty}\sum_{n=0}^{\infty} \mathrm{Prob}[h_k = i, u_k = n] x^i z^n ,$$

where the operator $\mathrm{E}[\ldots]$ indicates the expected value of the random expression between the square brackets.

The next step in our analysis is then to derive a relationship between the pgfs $P_k(x,z)$ and $P_{k+1}(x,z)$ of the state vectors at the beginning of two consecutive slots. Using Equations (1)–(7), we write the function $P_{k+1}(x,z)$ as

$$\begin{aligned} P_{k+1}(x,z) &\triangleq \mathrm{E}\Big[x^{h_{k+1}} z^{u_{k+1}}\Big] \\ &= \mathrm{Prob}[d_k = 1]\mathrm{E}\Big[x^0 z^0 | d_k = 1\Big] \\ &\quad + \mathrm{Prob}[h_k = 0, a_k = 0, d_k = 0]\mathrm{E}\Big[x^0 z^0 | h_k = 0, a_k = 0, d_k = 0\Big] \\ &\quad + \mathrm{Prob}[h_k = 0, a_k > 0, d_k = 0]\mathrm{E}\Big[x^{s^*} z^{a_k} | h_k = 0, a_k > 0, d_k = 0\Big] \\ &\quad + \mathrm{Prob}[h_k = 1, u_k = 1, a_k = 0, d_k = 0] \\ &\qquad \cdot \mathrm{E}\Big[x^0 z^0 | h_k = 1, u_k = 1, a_k = 0, d_k = 0\Big] \\ &\quad + \mathrm{Prob}[h_k = 1, u_k - 1 + a_k > 0, d_k = 0] \\ &\qquad \cdot \mathrm{E}\Big[x^{s^*} z^{u_k - 1 + a_k} | h_k = 1, u_k - 1 + a_k > 0, d_k = 0\Big] \\ &\quad + \mathrm{Prob}[h_k > 1, d_k = 0]\mathrm{E}\Big[x^{h_k - 1} z^{u_k + a_k} | h_k > 1, d_k = 0\Big] . \end{aligned} \tag{8}$$

Note that the system state variables h_k and u_k are statistically independent of the variables a_k, d_k and s^* due to the uncorrelated nature of both the customer arrival process and the occurrence of disasters from slot to slot and the i.i.d. nature of the service times of the customers. This allows us to further rewrite $P_{k+1}(x,z)$ as follows:

$$\begin{aligned} P_{k+1}(x,z) = \mathrm{Prob}[d_k = 1] + \mathrm{Prob}[d_k = 0]\Big\{&\mathrm{Prob}[h_k = 0]\mathrm{Prob}[a_k = 0] \\ &+ \mathrm{Prob}[h_k = 0]\mathrm{Prob}[a_k > 0]S(x)\mathrm{E}[z^{a_k} | a_k > 0] \\ &+ \mathrm{Prob}[h_k = 1, u_k = 1]\mathrm{Prob}[a_k = 0] \\ &+ \mathrm{Prob}[h_k = 1, u_k - 1 + a_k > 0]S(x)\mathrm{E}\Big[z^{u_k - 1 + a_k} | h_k = 1, u_k - 1 + a_k > 0\Big] \\ &+ \mathrm{Prob}[h_k > 1]\frac{1}{x}A(z)\mathrm{E}\Big[x^{h_k} z^{u_k} | h_k > 1\Big]\Big\} . \end{aligned} \tag{9}$$

Using the property that $\text{Prob}[h_k = 0] = P_k(0,0)$ and the law of total expectation, we then obtain

$$\begin{aligned} P_{k+1}(x,z) = \sigma + (1-\sigma)\Big\{ & P_k(0,0)A(0) + P_k(0,0)S(x)[A(z) - A(0)] \\ & + \text{Prob}[h_k = 1, u_k = 1]A(0) \\ & + S(x)\{\text{Prob}[h_k = 1]\text{E}\Big[z^{u_k - 1 + a_k} | h_k = 1\Big] \\ & - \text{Prob}[h_k = 1, u_k = 1]\text{Prob}[a_k = 0]\text{E}\Big[z^{u_k - 1 + a_k} | h_k = 1, u_k = 1, a_k = 0\Big]\} \\ & + \frac{1}{x}A(z)\{P_k(x,z) - \text{Prob}[h_k = 1]\text{E}\Big[x^{h_k} z^{u_k} | h_k = 1\Big] \\ & - \text{Prob}[h_k = 0]\text{E}\Big[x^{h_k} z^{u_k} | h_k = 0\Big]\}\Big\} . \end{aligned} \tag{10}$$

Let us now define the function $R_k(z)$ as

$$R_k(z) \triangleq \text{Prob}[h_k = 1]\text{E}\Big[z^{u_k - 1} | h_k = 1\Big] = \sum_{n=1}^{\infty} \text{Prob}[h_k = 1, u_k = n] z^{n-1} , \tag{11}$$

such that $R_k(0) = \text{Prob}[h_k = 1, u_k = 1]$. Then we finally find the following relationship between $P_k(x,z)$ and $P_{k+1}(x,z)$:

$$\begin{aligned} P_{k+1}(x,z) = \sigma + (1-\sigma)\Big\{ & P_k(0,0)A(0)[1 - S(x)] + P_k(0,0)S(x)A(z) \\ & + R_k(0)A(0) + S(x)A(z)R_k(z) - S(x)A(0)R_k(0) \\ & + \frac{1}{x}A(z)\{P_k(x,z) - xzR_k(z) - P_k(0,0)\}\Big\} . \end{aligned} \tag{12}$$

Since we are interested in the steady-state behavior of the queueing system, we let the time index k go to ∞. In steady state (for $k \to \infty$), the pgfs $P_k(x,z)$ and $P_{k+1}(x,z)$ both converge to a common limiting function

$$P(x,z) \triangleq \lim_{k \to \infty} P_k(x,z) . \tag{13}$$

Note that due to the possible occurrence of disasters ($\sigma > 0$), such a steady state will always exist. Equation (12) then leads to a linear equation for $P(x,z)$ with the following solution:

$$\begin{aligned} P(x,z) = \frac{1}{x - (1-\sigma)A(z)} \Big\{ & \sigma x + A(0)x(1-\sigma)[1 - S(x)][P(0,0) + R(0)] \\ & + P(0,0)A(z)(1-\sigma)[xS(x) - 1] + A(z)x(1-\sigma)R(z)[S(x) - z]\Big\} , \end{aligned} \tag{14}$$

where

$$R(z) \triangleq \lim_{k \to \infty} R_k(z) .$$

It now remains to determine the unknown function $R(z)$ and the two unknown probabilities $R(0)$ and $P(0,0)$. This can be done as follows. First note that, due to the fact that $h_k = 0$ if and only if $u_k = 0$, the following property holds:

$$P(x,0) = P(0,0) , \quad \text{for all } x .$$

In particular, this means that $P(1,0)$ as obtained from (14) should equal $P(0,0)$, which leads to the following relationship between $P(0,0)$ and $R(0)$:

$$P(0,0) = \sigma + A(0)(1-\sigma)[P(0,0) + R(0)] . \tag{15}$$

Next, we notice that the pgf $P(x,z)$ must be bounded for all values of its arguments x and z such that $|x| \leq 1$ and $|z| \leq 1$. In particular, this should be true for $x = (1-\sigma)A(z)$ and $|z| \leq 1$, since $(1-\sigma)|A(z)| \leq 1$ for all such z, as $A(z)$ is a pgf. If we now choose $x = (1-\sigma)A(z)$ in Equation (14), where $|z| \leq 1$, it is clear that the denominator of $P(x,z)$ vanishes. Of course, the numerator of $P(x,z)$ in (14) must then also be equal to zero for $x = (1-\sigma)A(z)$ with $|z| \leq 1$. This requirement together with the relation (15) then leads to the following equation for $R(z)$:

$$(1-\sigma)A(z)R(z) = S((1-\sigma)A(z))\,\frac{P(0,0)[(1-\sigma)A(z)-1]+\sigma}{z-S((1-\sigma)A(z))}\,. \tag{16}$$

From (14) together with Equations (15) and (16), an expression for $P(x,z)$ can then be derived in terms of the single unknown probability $P(0,0)$:

$$\begin{aligned} P(x,z) = \frac{1}{x-(1-\sigma)A(z)} \Big\{ & \sigma x + x[1-S(x)][P(0,0)-\sigma] \\ & + P(0,0)A(z)(1-\sigma)[xS(x)-1] \\ & + xS((1-\sigma)A(z))\,\frac{P(0,0)[(1-\sigma)A(z)-1]+\sigma}{z-S((1-\sigma)A(z))}\,[S(x)-z] \Big\}\,. \end{aligned} \tag{17}$$

The classical approach to determine the final remaining unknown $P(0,0)$ would now be to express the normalization condition for the joint distribution of the state vector. However, in our case it turns out that $P(1,1) = 1$, irrespective of the value of $P(0,0)$. So, a different approach is needed to obtain $P(0,0)$, which is presented in the next section. Once $P(0,0)$ is determined and, hence, the joint probability generating function $P(x,z)$ is fully known, all main performance measures of the queueing system (namely the moments and tail probabilities of the system content and the sojourn time as well as the customer loss probability) can be derived *directly* from the function $P(x,z)$, i.e., without any need for inversion of this joint pgf or calculation of joint probabilities. The methodology is explained in the next sections.

5. System Content

The pgf $U(z)$ of the system content u observed at the beginning of a random slot in the steady state can be obtained from $P(x,z)$ by simply putting $x = 1$: $U(z) = P(1,z)$. After rearranging some terms, we obtain

$$U(z) = \frac{S((1-\sigma)A(z))[1-(1-\sigma)A(z)](z-1)P(0,0)+\sigma z[1-S((1-\sigma)A(z))]}{[1-(1-\sigma)A(z)][z-S((1-\sigma)A(z))]}\,. \tag{18}$$

We can now find the unknown $P(0,0) = U(0)$ by noting that the function $U(z)$, as a pgf, must be bounded for all z with $|z| \leq 1$. Using Rouché's theorem, it can be shown (see Appendix A, Property A1) that the factor $z - S((1-\sigma)A(z))$ in the denominator of $U(z)$ in (18) has exactly one zero inside the unit circle in the complex z-plane. Let us denote this zero by z^*. It satisfies the following equation:

$$z^* - S((1-\sigma)A(z^*)) = 0\,, \quad \text{with } |z^*| < 1\,. \tag{19}$$

Clearly, the zero z^* of the denominator must then also be a zero of the numerator of $U(z)$. This property then yields the following linear equation for $P(0,0)$:

$$z^*[1-(1-\sigma)A(z^*)](z^*-1)P(0,0) + \sigma z^*(1-z^*) = 0\,, \tag{20}$$

where we also use Equation (19). Since $z^* \neq 1$ and under the assumption that $z^* \neq 0$, the probability $P(0,0)$ then directly follows from (20) as

$$P(0,0) = \frac{\sigma}{1-(1-\sigma)A(z^*)} \,. \tag{21}$$

It is worth noting here that in view of (19), $z^* = 0$ is only possible in the case that $A(0) = 0$. In such a case, new arrivals occur in each slot and the system can only be empty at the beginning of a slot if there is a disaster during the previous slot, so $P(0,0) = \sigma$. The latter is, in fact, in full agreement with the result (21), so the expression (21) for $P(0,0)$ turns out to be generally valid. The value of z^* in this expression for $P(0,0)$ needs to be determined numerically from (19), e.g., by means of Newton–Raphson's method.

Based on the pgf $U(z)$, the moments and tail probabilities of the system content can now be derived, as explained below.

5.1. Mean and Variance of the System Content

In general, any moment of the system-content distribution can be obtained by expressing the desired moment of u as a function of the consecutive derivatives of the pgf $U(z)$ with respect to z for $z = 1$. Here, we give the results obtained from Equation (18) for the mean value $\mathrm{E}[u]$ and the variance $\mathrm{var}[u]$ of the system content:

$$\mathrm{E}[u] = U'(1) = \frac{1-\sigma}{\sigma}\, A'(1) + \frac{S(1-\sigma)}{1-S(1-\sigma)}\,[P(0,0)-1] \tag{22}$$

and

$$\begin{aligned}
\mathrm{var}[u] &= U''(1) + U'(1) - U'(1)^2 \\
&= \frac{1-\sigma}{\sigma}\left\{A'(1) + A''(1) + 2(A'(1))^2\,\frac{1-\sigma}{\sigma}\right\} - \frac{2(1-\sigma)A'(1)S(1-\sigma)}{\sigma(1-S(1-\sigma))} \\
&\quad + \frac{P(0,0)-1}{[1-S(1-\sigma)]^2}\left\{2(1-\sigma)A'(1)S'(1-\sigma) - S(1-\sigma)[1+S(1-\sigma)]\right\} \\
&\quad - \left\{\frac{1-\sigma}{\sigma}\, A'(1) + \frac{S(1-\sigma)}{1-S(1-\sigma)}\,[P(0,0)-1]\right\}^2 .
\end{aligned} \tag{23}$$

5.2. Tail Distribution of the System Content

Another important characteristic is the tail distribution of the system content. We use here an approximation technique as described, for example, in [38]. Specifically, from the inversion formula for z-transforms, it follows that the pmf of the system content can be expressed as a weighted sum of negative nth powers of the poles of the pgf $U(z)$. Since all these poles have a modulus larger than 1, it is clear that for n sufficiently large, $\mathrm{Prob}[u = n]$ is dominated by the contribution of the pole having the smallest modulus. It can be argued (see, for example, [38]) that this dominant pole must be real and positive to ensure that the tail probabilities are non-negative anywhere. Moreover, it can be shown (see Appendix A, Property A2) that the dominant pole of $U(z)$ has multiplicity 1. As such, $\mathrm{Prob}[u = n]$ can be approximated as

$$\mathrm{Prob}[u = n] \approx -\frac{b_u}{z_u}(z_u)^{-n} \,, \tag{24}$$

for large n, where z_u is the dominant pole of $U(z)$ and b_u is the residue of $U(z)$ in the point $z = z_u$. From the expression (18) for $U(z)$, it follows (see Appendix A, Property A2) that z_u is the unique real root larger than 1 of the equation

$$z - S((1-\sigma)A(z)) = 0 \,. \tag{25}$$

Note that z_u and z^* are roots of the same equation; see also (19). The value of z_u can be calculated numerically from (25) via the Newton–Raphson procedure. The residue b_u can be calculated as

$$\begin{aligned} b_u &= \lim_{z \to z_u} (z - z_u) U(z) \\ &= \frac{z_u(z_u-1)\left\{[1-(1-\sigma)A(z_u)]P(0,0)-\sigma\right\}}{[1-(1-\sigma)A(z_u)][1-S'((1-\sigma)A(z_u))(1-\sigma)A'(z_u)]} . \end{aligned} \tag{26}$$

6. Unfinished Work

As an intermediate step in the study of the sojourn time of an arbitrary customer, we determine the pgf $W(z)$ of the unfinished work in the queueing system at the beginning of a slot in the steady state. Let the random variable w_k indicate the unfinished work at the beginning of slot k, i.e., the remaining number of slots required to complete the service of all customers present in the system at the beginning of slot k. The unfinished work w_k can then be expressed in terms of the system state variables h_k and u_k as follows:

$$w_k = \begin{cases} 0 & , \quad \text{if } h_k = 0 , \\ h_k + \sum\limits_{i=1}^{u_k - 1} s_i , & \quad \text{if } h_k \geq 1 , \end{cases} \tag{27}$$

where the variables s_i are the full service times of the $u_k - 1$ customers still awaiting service at the beginning of slot k. Next, from (27), we derive an expression for the steady-state pgf $W(z)$ of the unfinished work in terms of the steady-state joint pgf of the system state variables:

$$\begin{aligned} W(z) &= \lim_{k \to \infty} \mathrm{E}[z^{w_k}] \\ &= \lim_{k \to \infty} \left\{ \mathrm{Prob}[h_k = 0] + \mathrm{Prob}[h_k \geq 1] \mathrm{E}\left[z^{h_k + \sum\limits_{i=1}^{u_k-1} s_i} \mid h_k \geq 1 \right] \right\} \\ &= \lim_{k \to \infty} \left\{ P_k(0,0) + \mathrm{Prob}[h_k \geq 1] \mathrm{E}\left[z^{h_k} S(z)^{u_k - 1} \mid h_k \geq 1 \right] \right\} \\ &= \lim_{k \to \infty} \left\{ P_k(0,0) + \frac{1}{S(z)} \mathrm{E}\left[z^{h_k} S(z)^{u_k} \right] \right. \\ &\qquad \left. - \frac{1}{S(z)} \mathrm{Prob}[h_k = 0] \mathrm{E}\left[z^{h_k} S(z)^{u_k} \mid h_k = 0 \right] \right\} \\ &= P(0,0) + \frac{P(z, S(z)) - P(0,0)}{S(z)} . \end{aligned} \tag{28}$$

Finally, using (17) and after some further mathematical manipulations, we obtain the following result:

$$W(z) = \frac{\sigma z + P(0,0)(1-\sigma)A(S(z))(z-1)}{z - (1-\sigma)A(S(z))} , \tag{29}$$

where $P(0,0)$ is given by (21). This result will prove useful to derive the pgf of the customer sojourn time in the next section.

7. Sojourn Time

We define the sojourn time of a customer as the total (integer) number of slots between the end of the arrival slot of the customer and the departure instant of the customer from the system. In this section, we analyze the sojourn time of an arbitrary customer under the assumption of a FCFS queueing discipline.

Let us consider an arbitrary customer, say customer C, that arrives in the system during some slot in the steady state, referred to as slot I. Let t with pgf $T(z)$ denote the sojourn time of C. To derive $T(z)$, we use a two-step approach. Firstly, we focus on the number of slots $t_{\max}$ that C would spend in the system until its service is completed in case no disasters occur while C is in the system. Note that, due to the occurrence of disasters, C can actually be removed from the system before its service is completed, so $t_{\max}$ is an upper bound on the actual sojourn time t of C. If we define $\tilde{w}$ as the unfinished work observed at the beginning of slot I and f as the number of customers arriving during slot I before customer C, then the maximum sojourn time $t_{\max}$ is expressed as follows:

$$t_{\max} = (\tilde{w} - 1)^+ + \sum_{i=1}^{f+1} \tilde{s}_i \,, \tag{30}$$

where $(\ldots)^+$ denotes $\max(0, \ldots)$ and the variables $\tilde{s}_i$ are the service times of C and the customers arriving during slot I and to be served before C. The pgf $F(z)$ of f is known to be given by (see, for example, [39])

$$F(z) = \frac{A(z) - 1}{A'(1)(z-1)} \,. \tag{31}$$

Moreover, due to the uncorrelated nature of the arrival process from slot to slot, $\tilde{w}$ has the same pgf $W(z)$ as the unfinished work at the beginning of an arbitrary slot and the variables $\tilde{w}$ and f are statistically independent. The variables $\tilde{s}_i$ are i.i.d. with common pgf $S(z)$. Translating (30) to pgfs, we then find

$$T_{\max}(z) = \frac{W(z) + (z-1)W(0)}{z} \, S(z) F(S(z)) \,, \tag{32}$$

or, using Equations (29) and (31),

$$T_{\max}(z) = \frac{[\sigma + (z-1)P(0,0)] S(z) [A(S(z)) - 1]}{[z - (1-\sigma) A(S(z))] A'(1) [S(z) - 1]} \,. \tag{33}$$

Secondly, we note that for a given value of $t_{\max}$, the actual sojourn time t of C cannot be larger than $t_{\max}$, and its specific value depends on the occurrence process of disasters. In particular, in view of the independent Bernoulli nature of the disaster process, it can be shown that the conditional pmf of t, given that $t_{\max} = i$ $(i \geq 1)$, equals

$$\mathrm{Prob}[t = n | t_{\max} = i] = \begin{cases} (1-\sigma)^n \sigma \,, & 0 \leq n < i \,, \\ (1-\sigma)^i \,, & n = i \,, \\ 0 \,, & n > i \,. \end{cases} \tag{34}$$

Indeed, the sojourn time t is zero if a disaster occurs during slot I. For $t_{\max} = i$, the variable t takes a value n $(0 < n < i)$ if there are no disasters in slot I nor in the $n-1$ slots following slot I, and a disaster occurs in the nth slot after slot I. Finally, as long as there are no disasters in slot I or in the $i-1$ slots after slot I, the customer spends the maximum number of i slots in the system. Using the above conditional pmf (34) and the law of total expectation, the pgf $T(z)$ of t is then found as

$$\begin{aligned} T(z) \triangleq \mathrm{E}[z^t] &= \sum_{i=1}^{\infty} \mathrm{Prob}[t_{\max} = i] \mathrm{E}[z^t | t_{\max} = i] \\ &= \sum_{i=1}^{\infty} \mathrm{Prob}[t_{\max} = i] \left\{ \sum_{n=0}^{i-1} (1-\sigma)^n \sigma z^n + (1-\sigma)^i z^i \right\} \\ &= \sum_{i=1}^{\infty} \mathrm{Prob}[t_{\max} = i] \left\{ \sigma \, \frac{1 - [(1-\sigma)z]^i}{1 - (1-\sigma)z} + [(1-\sigma)z]^i \right\} \\ &= \frac{\sigma + (1-\sigma)(1-z) \, T_{\max}((1-\sigma)z)}{1 - (1-\sigma)z} \,. \end{aligned} \tag{35}$$

The combination of Equations (33) and (35) finally leads to the following expression for $T(z)$:

$$\begin{aligned}T(z) = &\frac{1}{[1-(1-\sigma)z]A'(1)[S((1-\sigma)z)-1][z-A(S((1-\sigma)z))]}\\&\cdot\Big\{\sigma A'(1)[S((1-\sigma)z)-1][z-A(S((1-\sigma)z))]\\&+(1-z)[\sigma+((1-\sigma)z-1)P(0,0)]S((1-\sigma)z)[A(S((1-\sigma)z))-1]\Big\}.\end{aligned} \quad (36)$$

7.1. Mean and Variance of the Sojourn Time

Based on the moment-generating property of pgfs, the following expressions for the mean value and the variance of the sojourn time are obtained from the pgf $T(z)$:

$$\mathrm{E}[t] = \frac{1-\sigma}{\sigma} + \frac{S(1-\sigma)}{A'(1)[1-S(1-\sigma)]}\,[P(0,0)-1] \quad (37)$$

and

$$\begin{aligned}\mathrm{var}[t] = &\frac{1-\sigma}{\sigma^2} + \frac{2(1-\sigma)S(1-\sigma)}{\sigma A'(1)[S(1-\sigma)-1]}\\&+\frac{P(0,0)-1}{A'(1)[S(1-\sigma)-1]}\left\{\frac{2(1-\sigma)S(1-\sigma)}{\sigma}+\frac{2(1-\sigma)S'(1-\sigma)}{S(1-\sigma)-1}\right.\\&\left.+\frac{2S(1-\sigma)}{1-A(S(1-\sigma))}-S(1-\sigma)\right\}\\&-\left\{\frac{[P(0,0)-1]S(1-\sigma)}{A'(1)[S(1-\sigma)-1]}\right\}^2.\end{aligned} \quad (38)$$

It was also verified that the above result for $\mathrm{E}[t]$ is in full agreement with Little's law: $\mathrm{E}[u] = A'(1)\mathrm{E}[t]$.

7.2. Tail Distribution of the Sojourn Time

Similar to Section 5.2, we use a dominant-pole approximation for the tail distribution of the sojourn time. The pmf $\mathrm{Prob}[t=n]$ is approximated by the following geometric form:

$$\mathrm{Prob}[t=n] \approx -\frac{b_t}{z_t}(z_t)^{-n}\,, \quad (39)$$

for n that is sufficiently large. It can be proved (see Appendix A, Property A3) that the dominant pole z_t of $T(z)$ is the unique real root larger than $\frac{1}{1-\sigma}$ with multiplicity 1 of the equation

$$z - A(S((1-\sigma)z)) = 0\,. \quad (40)$$

The residue b_t then follows from (36) as

$$\begin{aligned}b_t &= \lim_{z\to z_t}(z-z_t)T(z)\\&=\frac{[\sigma+((1-\sigma)z_t-1)P(0,0)]S((1-\sigma)z_t)(z_t-1)^2}{A'(1)[1-(1-\sigma)z_t][1-S((1-\sigma)z_t)][1-A'(S((1-\sigma)z_t))S'((1-\sigma)z_t)(1-\sigma)]}\,.\end{aligned} \quad (41)$$

8. Loss Probability

Due to the occurrence of disasters, some customers are removed from the system and get lost without receiving complete service. Let the random variable ℓ_k denote the number of customers that get lost due to a disaster occurring during slot k. Then ℓ_k can be expressed as follows:

$$\ell_k = \begin{cases} a_k + u_k\,, & \text{if } d_k = 1\,, \\ 0 \quad\quad\;\;, & \text{if } d_k = 0\,. \end{cases} \tag{42}$$

The steady-state pgf $L(z)$ of the number of lost customers during a slot immediately follows from (42) as

$$L(z) = 1 - \sigma + \sigma A(z)U(z)\,. \tag{43}$$

Finally, the customer loss probability CLP, i.e., the fraction of the arriving customers that get lost due to a disaster, is then obtained from (43) as

$$\mathrm{CLP} = \frac{L'(1)}{A'(1)} = 1 + \frac{\sigma S(1-\sigma)[P(0,0)-1]}{[1-S(1-\sigma)]A'(1)}\,. \tag{44}$$

9. Numerical Examples

In order to illustrate the results obtained above, let us consider a number of numerical examples. In a first set of examples, in Figures 1–4, we consider a Poisson distribution for the number of customer arrivals during a slot, i.e.,

$$A(z) = e^{\lambda\,(z-1)}\,,$$

and a (shifted) geometric distribution for the service times, i.e.,

$$S(z) = \frac{\mu\, z}{1-(1-\mu)\, z}\,,$$

with $\mu = 0.75$ ($S'(1) = \frac{1}{\mu} = 1.33$). In Figure 1, the mean system content $\mathrm{E}[u]$ is plotted versus the arrival rate λ, for different values of the disaster probability σ. Similarly, Figure 2 shows the variance of the system content $\mathrm{var}[u]$, the mean sojourn time $\mathrm{E}[t]$ is plotted in Figure 3 and the customer loss probability is shown in Figure 4, all versus λ for different values of σ. We observe that for an increasing disaster probability, $\mathrm{E}[u]$, $\mathrm{var}[u]$ and $\mathrm{E}[t]$ are decreasing, while CLP is increasing. This is clearly as intuitively expected. Indeed, the more often the system gets emptied due to a disaster occurrence (higher σ), the more customers are expected to get lost and the lower and less variable the system content and sojourn time thus become.

Next, in Figure 5, a number of different distributions are considered for the customer service times, all with the same mean service time $S'(1) = \frac{1}{\mu} = 5$: a (shifted) geometric distribution, a (shifted) Poisson distribution with

$$S(z) = z\, e^{(\frac{1}{\mu}-1)\,(z-1)}$$

and constant service times with

$$S(z) = z^{\frac{1}{\mu}}\,.$$

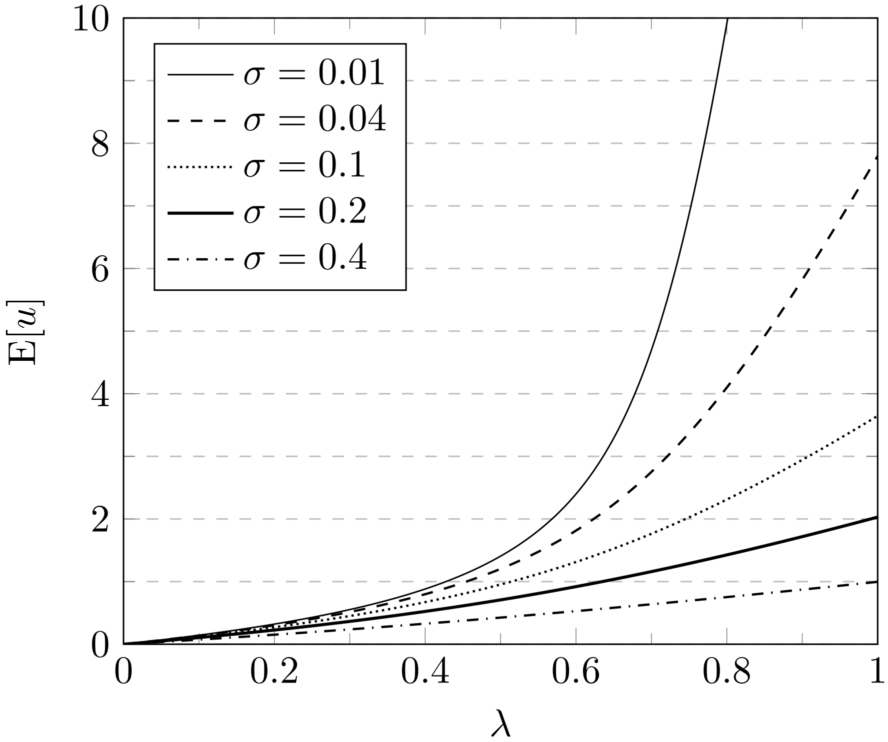

Figure 1. Mean system content $\mathrm{E}[u]$ versus λ, for Poisson arrivals, geometric service times with $\mu = 0.75$ and different values of σ.

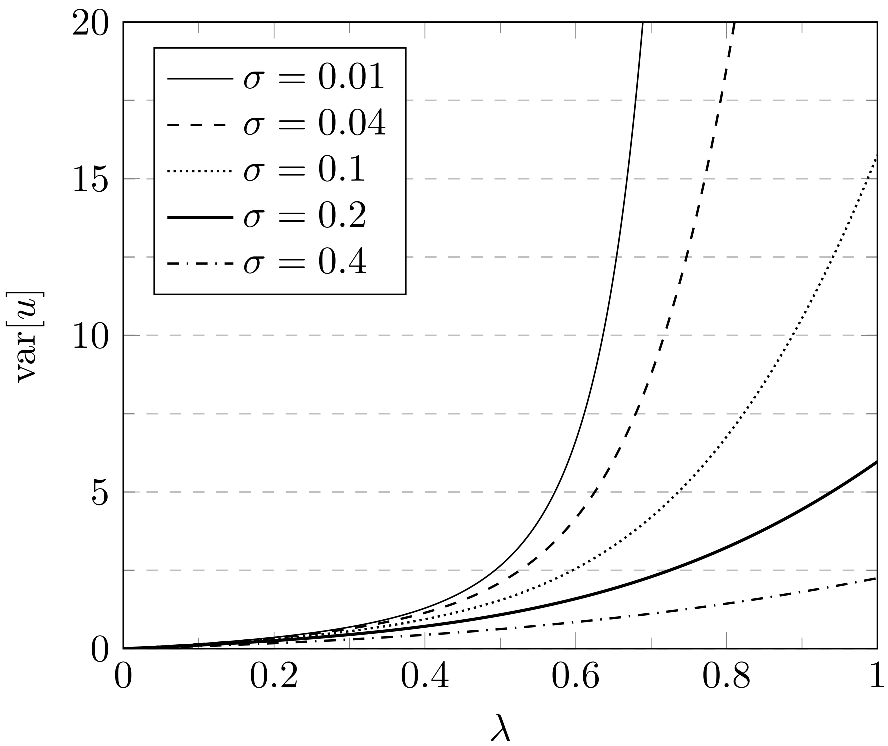

Figure 2. Variance of the system content $\mathrm{var}[u]$ versus λ, for Poisson arrivals, geometric service times with $\mu = 0.75$ and different values of σ.

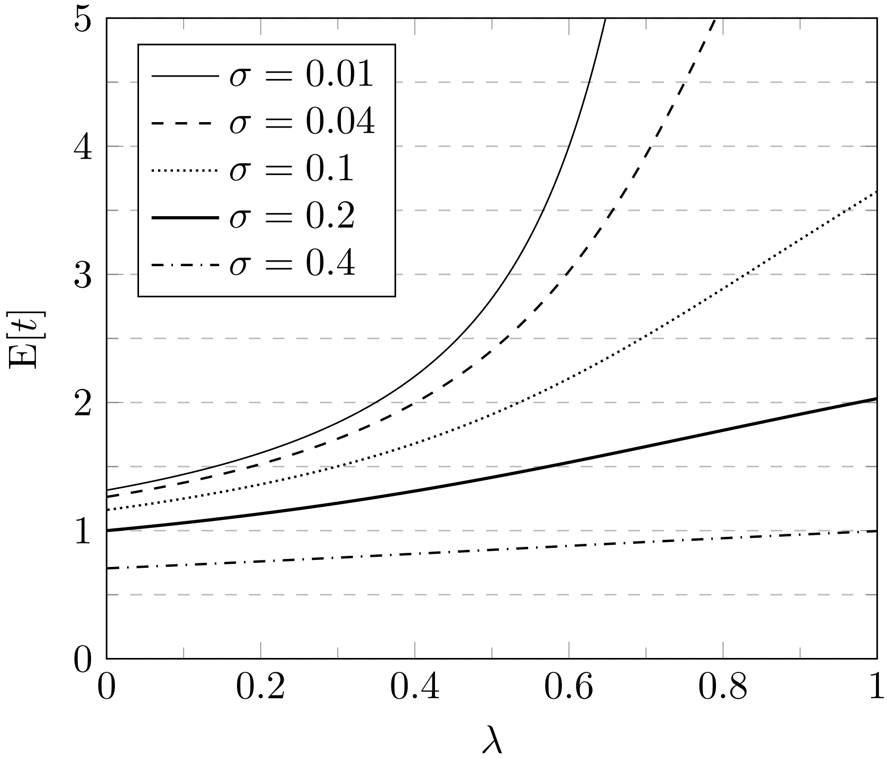

Figure 3. Mean sojourn time E[t] versus λ, for Poisson arrivals, geometric service times with $\mu = 0.75$ and different values of σ.

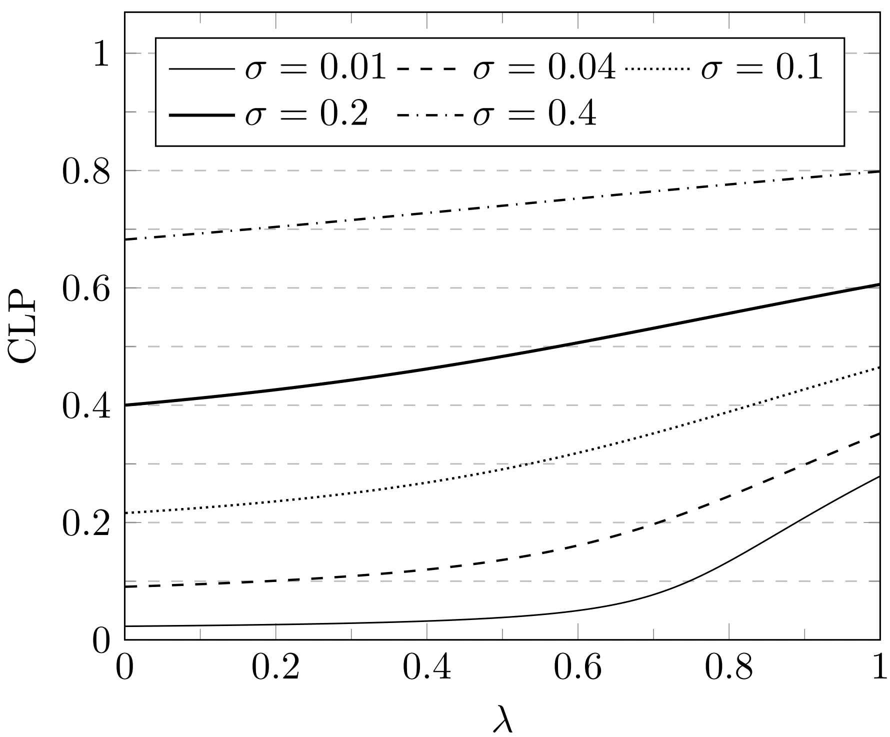

Figure 4. Customer loss probability CLP versus λ for Poisson arrivals, geometric service times with $\mu = 0.75$ and different values of σ.

Figure 5 shows the mean system content versus λ, for Poisson arrivals and a disaster probability $\sigma = 0.4$. For $S'(1) = 5$, the variance of the service times increases from 0 in the case of constant service times, to $\frac{1}{\mu} - 1 = 4$ in the case of Poisson service times, and finally to $\frac{1-\mu}{\mu^2} = 20$ in the case of geometric service times. We observe that $\mathrm{E}[u]$ decreases as the variance of the service times increases. Note that this is different from what is seen in classical queueing systems without disasters, where the mean system content typically increases with higher irregularity in the service times. The behavior of Figure 5 then follows from the fact that in the case of disaster occurrence, typically more customers are removed from the system when the service times are more variable, resulting in a lower system content on average.

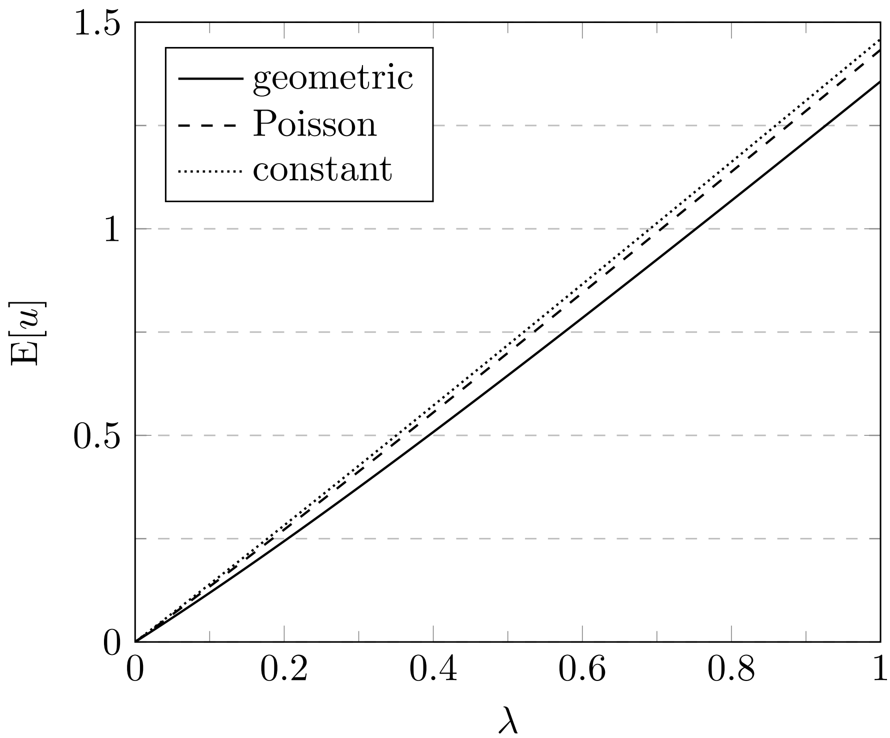

Figure 5. Mean system content $\mathrm{E}[u]$ versus λ, for Poisson arrivals, $\sigma = 0.4$ and various distributions for the service times with $S'(1) = 5$.

In Figures 6–8, we consider Poisson arrivals and a (shifted) Poisson distribution for the service times. In Figure 6, the mean system content is shown versus the mean service time $S'(1)$ for $\lambda = 0.4$ and various values of σ. Similarly, Figure 7 shows the variance $\mathrm{var}[u]$ of the system content, and Figure 8 shows the customer loss probability. From these figures, it can be seen that for a given value of λ, $\mathrm{E}[u]$, $\mathrm{var}[u]$ and CLP all increase with increasing values of $S'(1)$, which could be expected due to the increasing system load $\lambda S'(1)$. For increasing values of σ, we observe again that $\mathrm{E}[u]$ and $\mathrm{var}[u]$ are decreasing, while the loss probability increases, as is intuitively clear.

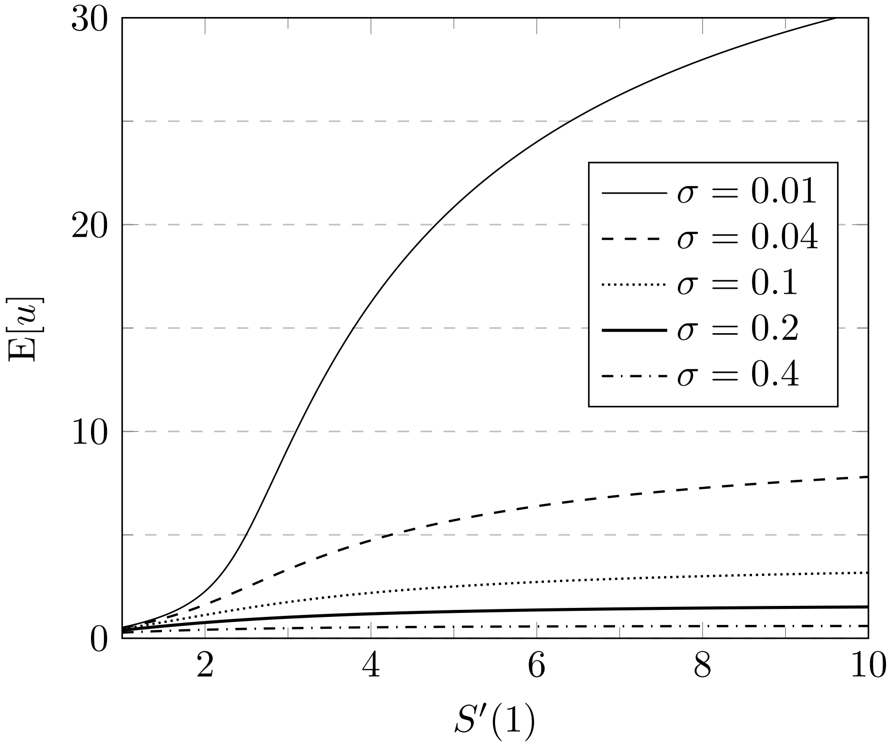

Figure 6. Mean system content E[u] versus $S'(1)$ for Poisson service times, Poisson arrivals with $\lambda = 0.4$ and different values of σ.

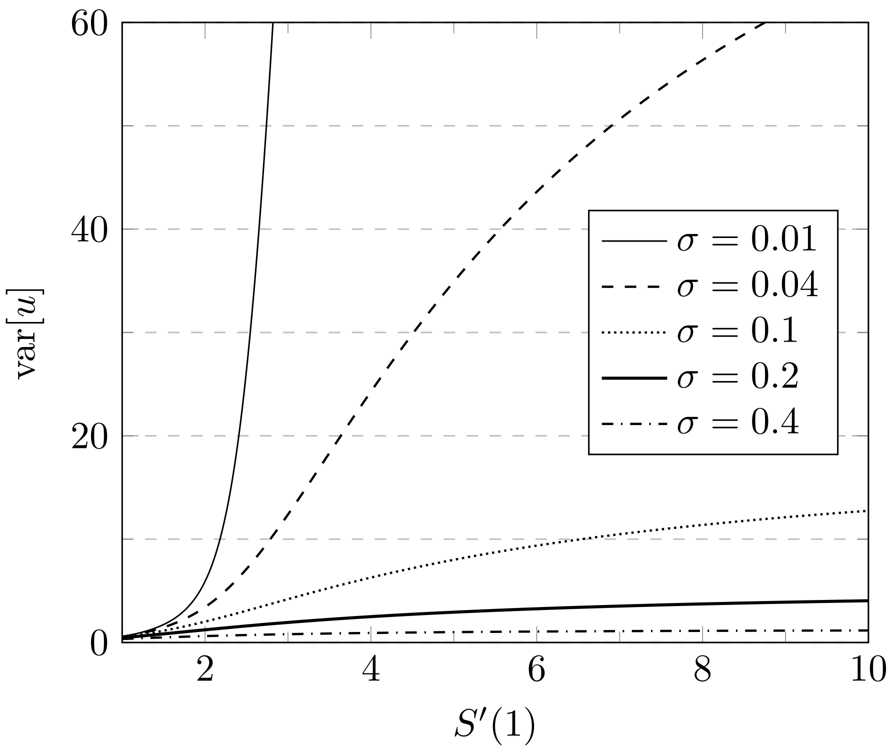

Figure 7. Variance of the system content var[u] versus $S'(1)$ for Poisson service times, Poisson arrivals with $\lambda = 0.4$ and different values of σ.

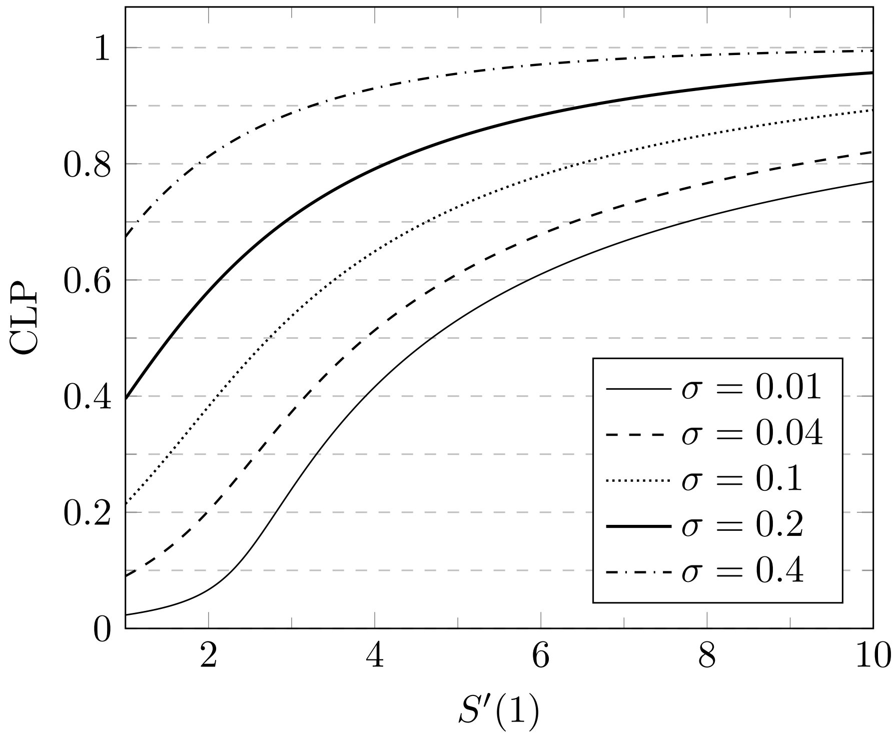

Figure 8. Customer loss probability CLP versus $S'(1)$ for Poisson service times, Poisson arrivals with $\lambda = 0.4$ and different values of σ.

In Figures 9–11, (shifted) geometric service times are considered with $\mu = 0.75$. A number of different arrival distributions are considered, all with the same mean arrival rate λ: Poisson arrivals, Bernoulli arrivals, i.e.,

$$A(z) = 1 - \lambda + \lambda z\,,$$

geometric arrivals, i.e.,

$$A(z) = \frac{1}{1 + \lambda - \lambda z},$$

and binomial arrivals, i.e.,

$$A(z) = \left(1 - \frac{\lambda}{N} + \frac{\lambda}{N} z\right)^N,$$

where $N = 2$. Figures 9 and 10 show the mean system content $\mathrm{E}[u]$ and the variance of the system content $\mathrm{var}[u]$ versus λ, for a fixed disaster probability $\sigma = 0.1$. We observe that both $\mathrm{E}[u]$ and $\mathrm{var}[u]$ decrease in the order of geometric, Poisson, binomial and Bernoulli arrivals. This means that for given values of λ and μ, $\mathrm{E}[u]$ and $\mathrm{var}[u]$ decrease as the variance of the number of arrivals per slot decreases.

In Figure 11, the variance of the system content is plotted versus σ for the same four arrival distributions with a fixed value of $\lambda = 0.9$. For all arrival distributions, the variance $\mathrm{var}[u]$ decreases while the value of σ is increasing, in accordance with the observations of Figure 2.

In Figure 12, the tail distribution of the system content is plotted on a logarithmic scale for (shifted) Poisson service times with $\mu = 0.75$, Poisson arrivals with arrival rate $\lambda = 1$ and different values of the disaster probability σ. For the same setting, Figure 13 shows the tail distribution of the sojourn time. We observe that, similar to the moments of the system content and the sojourn time, also the corresponding tail probabilities are decreasing functions of the disaster probability.

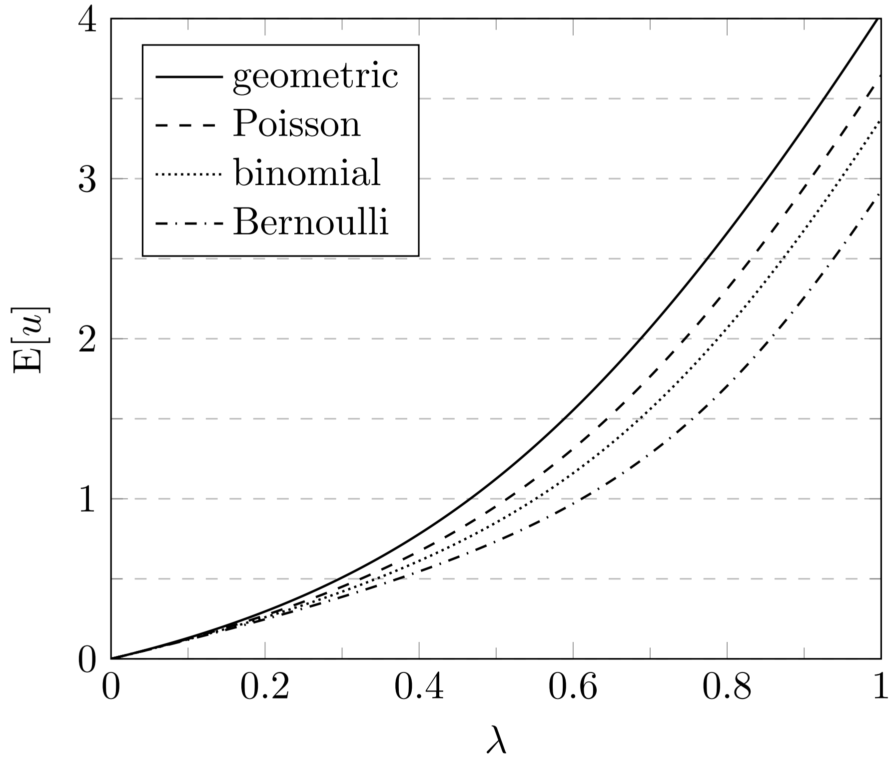

Figure 9. Mean system content E[u] versus λ for geometric service times with $\mu = 0.75$, $\sigma = 0.1$ and various distributions for the number of arrivals per slot.

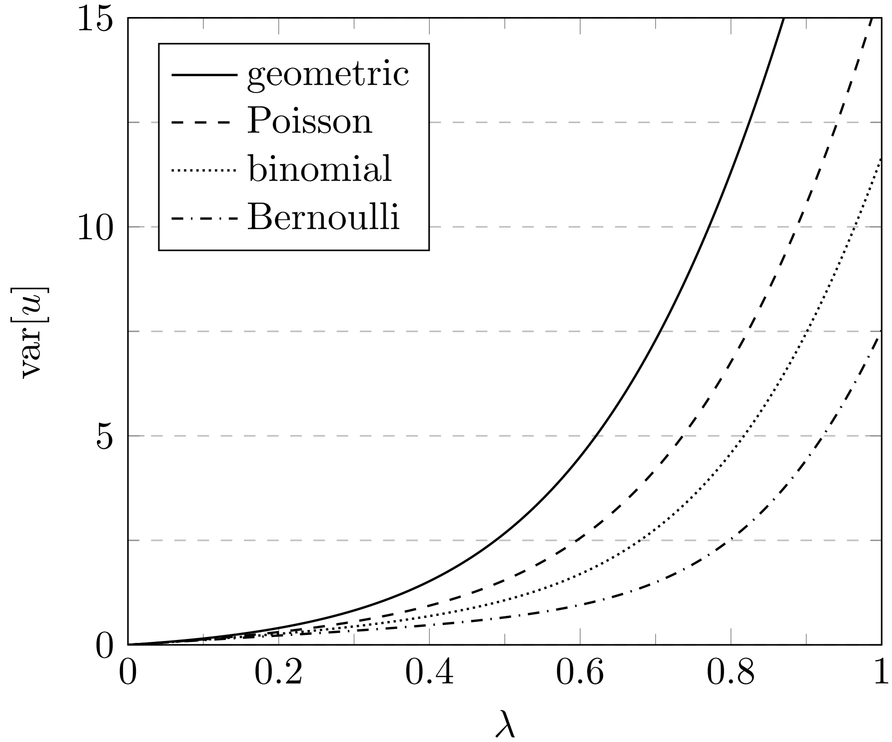

Figure 10. Variance of the system content var[u] versus λ for geometric service times with $\mu = 0.75$, $\sigma = 0.1$ and various distributions for the number of arrivals per slot.

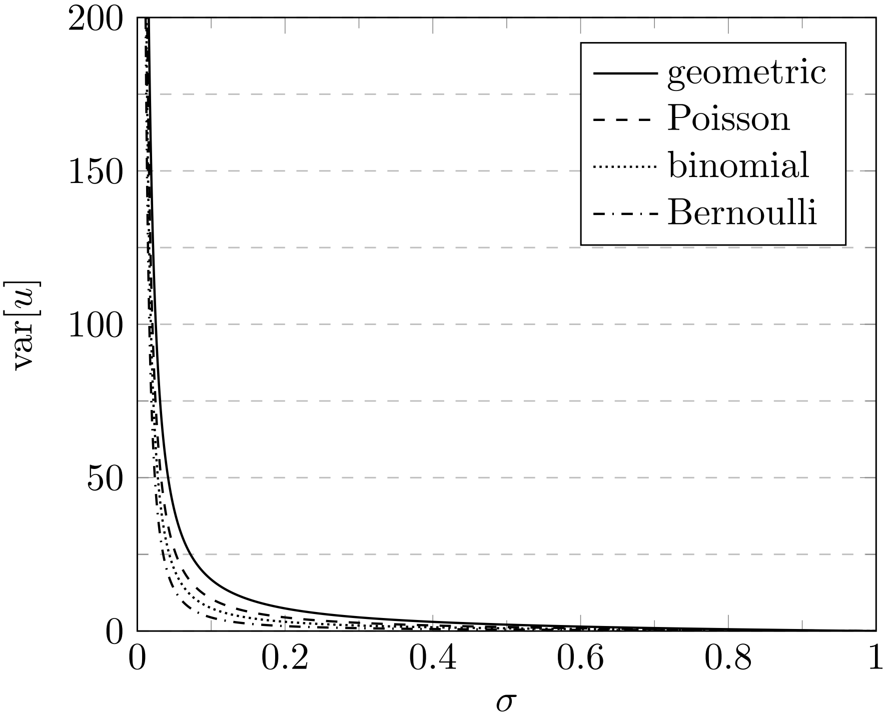

Figure 11. Variance of the system content var$[u]$ versus σ, for geometric service times with $\mu = 0.75$, $\lambda = 0.9$ and various distributions for the number of arrivals per slot.

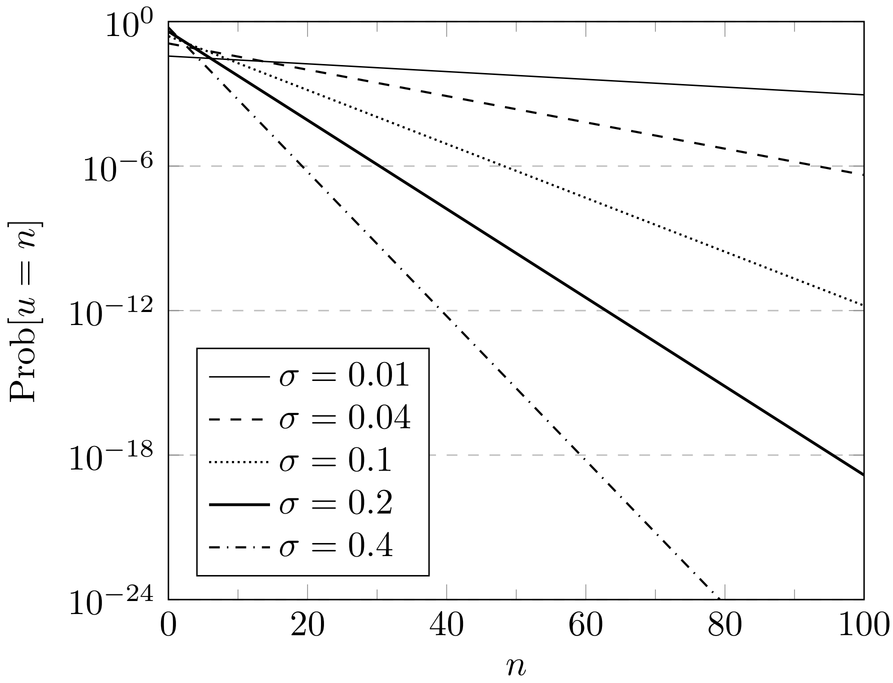

Figure 12. Tail distribution of the system content Prob$[u = n]$ versus n for Poisson service times with $\mu = 0.75$, Poisson arrivals with $\lambda = 1$ and different values of σ.

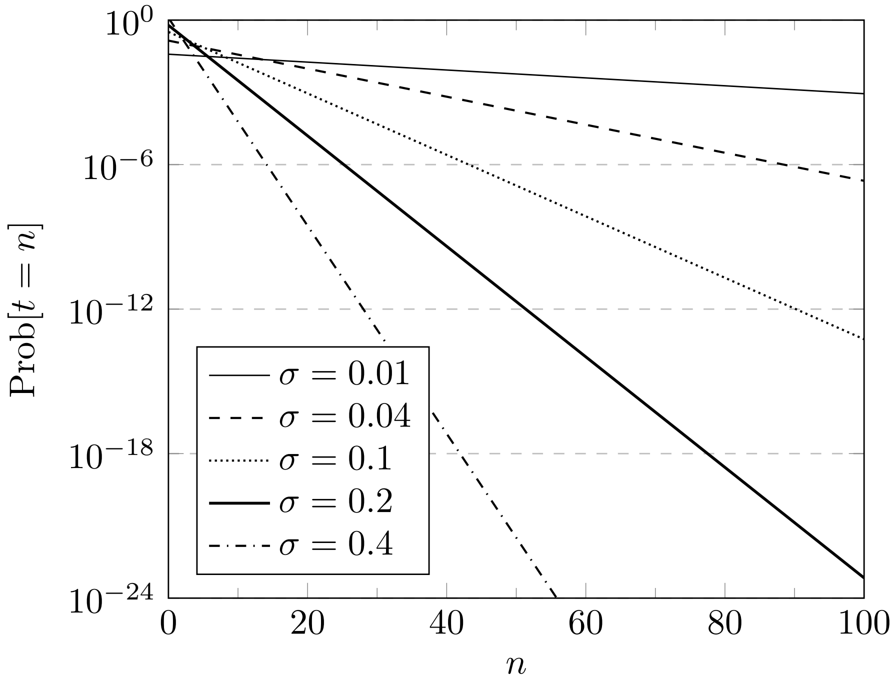

Figure 13. Tail distribution of the sojourn time Prob$[t = n]$ versus n for Poisson service times with $\mu = 0.75$, Poisson arrivals with $\lambda = 1$ and different values of σ.

Finally, Table 1 presents some numerical results on the tail probabilities, the mean value and the variance of both the system content u and the sojourn time t for two different values of the disaster probability σ. We consider Poisson arrivals with $\lambda = 0.4$ and Poisson service times with $\mu = 0.2$ ($S'(1) = \frac{1}{\mu} = 5$).

Table 1. Tail probabilities, mean and variance of the system content u and the sojourn time t for Poisson arrivals with $\lambda = 0.4$, Poisson service times with $\mu = 0.2$ and different values of σ.

	$\sigma = 0.05$		$\sigma = 0.2$	
n	**Prob$[u = n]$**	**Prob$[t = n]$**	**Prob$[u = n]$**	**Prob$[t = n]$**
10	2.5647668×10^{-2}	3.9904988×10^{-2}	1.2037267×10^{-3}	1.4136466×10^{-2}
20	3.2333712×10^{-3}	1.5896751×10^{-2}	2.5425991×10^{-6}	4.6894762×10^{-4}
30	4.0762729×10^{-4}	6.3327096×10^{-3}	5.3706629×10^{-9}	1.5556354×10^{-5}
40	5.1389091×10^{-5}	2.5227300×10^{-3}	$1.1344305 \times 10^{-11}$	5.1604946×10^{-7}
50	6.4785619×10^{-6}	1.0049674×10^{-3}	$2.3962267 \times 10^{-14}$	1.7118859×10^{-8}
60	8.1674464×10^{-7}	4.0034385×10^{-4}	$5.0614846 \times 10^{-17}$	$5.6788228 \times 10^{-10}$
70	1.0296603×10^{-7}	1.5948299×10^{-4}	$1.0691236 \times 10^{-19}$	$1.8838304 \times 10^{-11}$
80	1.2980806×10^{-8}	6.3532442×10^{-5}	$2.2582808 \times 10^{-22}$	$6.2492127 \times 10^{-13}$
90	1.6364749×10^{-9}	2.5309103×10^{-5}	$4.7701050 \times 10^{-25}$	$2.0730453 \times 10^{-14}$
100	$2.0630845 \times 10^{-10}$	1.0082261×10^{-5}	$1.0075763 \times 10^{-27}$	$6.8768935 \times 10^{-16}$
	E$[u] = 4.663592380$ var$[u] = 23.95418492$ E$[t] = 11.65898095$ var$[t] = 120.6749369$		E$[u] = 1.292139933$ var$[u] = 2.933050427$ E$[t] = 3.230349832$ var$[t] = 10.28852063$	

10. Conclusions

In this paper, we studied the impact of disasters on the behavior of a discrete-time single-server queueing system under general probability distributions for both the number of customer arrivals during a slot and the length of the service time of a customer. Using a supplementary variable technique, we derived expressions for the pgfs, moments and tail probabilities of the system content and the sojourn time, and an expression for the customer loss probability due to disasters. Through numerical examples, the impact of disasters on the system characteristics was assessed. In contrast to classical queueing systems without disasters, we observed that for systems with disasters under a given system load, the mean system content decreases with increasing irregularity in the service times.

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Appendix A

In this appendix, we show some properties with respect to the zeros of the denominators of the pgfs $U(z)$ and $T(z)$.

Property A1. *The factor $z - S((1-\sigma)A(z))$ in the denominator of $U(z)$ has exactly one zero inside the unit circle in the complex z-plane.*

Proof of Property A1. We define

$$
\begin{aligned}
f(z) &\triangleq z\,,\\
g(z) &\triangleq -S((1-\sigma)A(z))\,.
\end{aligned}
$$

It is clear that the functions $f(z)$ and $g(z)$ are analytic functions of z at least inside and on the unit circle in the complex z-plane. For $|z| = 1$, we also have that $|f(z)| = |z| = 1$ and $|g(z)| = |S((1-\sigma)A(z))| < 1$. The latter follows from $|S(z)| < 1$ for $|z| < 1$, and $|(1-\sigma)A(z)| < 1$ for $|z| = 1$ and $\sigma > 0$. Thus $|g(z)| < |f(z)|$ for all z with $|z| = 1$. We can then apply Rouché's theorem from complex analysis (see, for example, [39,40]) to conclude that $f(z)$ and $f(z) + g(z) = z - S((1-\sigma)A(z))$ have the same number of zeros inside the unit circle, i.e., the denominator factor $z - S((1-\sigma)A(z))$ has exactly one zero within $\{z : |z| < 1\}$. □

Property A2. *The dominant pole z_u of $U(z)$ has multiplicity 1 and is the unique real positive root with modulus larger than 1 of the equation $z - S((1-\sigma)A(z)) = 0$.*

Proof of Property A2. First, we note that any zero of the factor $1 - (1-\sigma)A(z)$ in the denominator of $U(z)$ is also a zero of the numerator of $U(z)$ and therefore cannot be a pole of the pgf $U(z)$. Since the dominant pole z_u must be real and positive, we now look at the factor $m(z) \triangleq z - S((1-\sigma)A(z))$ in the denominator for real values of z. Clearly, $m(1) = 1 - S(1-\sigma) > 0$ for a disaster system ($\sigma > 0$). In addition, since $A(z)$ and $S(z)$ are both pgfs of non-negative random variables, it is easily seen that $m'(z) = 1 - S'((1-\sigma)A(z))(1-\sigma)A'(z) < 0$ for sufficiently large real values of z. Moreover, $m''(z) = -S''((1-\sigma)A(z))(1-\sigma)^2A'(z)^2 - S'((1-\sigma)A(z))(1-\sigma)A''(z) \geq 0$ for all real $z > 0$. These properties imply that $m(z)$ has exactly one positive real zero outside the unit circle, which has multiplicity 1 and is the dominant pole z_u we are looking for. □

Property A3. *The dominant pole z_t of $T(z)$ has multiplicity 1 and is the unique real positive root with modulus larger than $\frac{1}{1-\sigma}$ of the equation $z - A(S((1-\sigma)z)) = 0$.*

Proof of Property A3. In this proof, we only consider real values of z since the dominant pole z_t must be real and positive. First, we observe that $z = \frac{1}{1-\sigma}$ is a zero of both the factors $1-(1-\sigma)z$ and $S((1-\sigma)z)-1$ in the denominator of $T(z)$. However, it is easily verified that both the numerator of $T(z)$ and the first derivative of this numerator vanish for $z = \frac{1}{1-\sigma}$ as well. We conclude that $z = \frac{1}{1-\sigma}$ is a zero of both the denominator and the numerator of $T(z)$ with the same multiplicity 2 and therefore cannot be a pole of $T(z)$. We then look at the factor $r(z) \triangleq z - A(S((1-\sigma)z))$ in the denominator of $T(z)$. For $1 < z \leq \frac{1}{1-\sigma}$, we have that $(1-\sigma)z \leq 1$, hence also $S((1-\sigma)z) \leq 1$, and therefore $A(S((1-\sigma)z)) \leq 1 < z$. This means that $r(z) > 0$ for all real values of z with $1 < z \leq \frac{1}{1-\sigma}$. We also have that $r'(z) < 0$ for sufficiently large real values of z and $r''(z) \geq 0$ for all $z > 0$. We may conclude that $r(z)$ has exactly one positive real zero with modulus larger than $\frac{1}{1-\sigma}$ and multiplicity 1; this zero of $r(z)$ is the dominant pole z_t of $T(z)$. □

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