mathematics

MDPI

Article

Solving an Integral Equation via Fuzzy Triple Controlled Bipolar Metric Spaces

Gunaseelan Mani [1,†], Arul Joseph Gnanaprakasam [2,†], Zoran D. Mitrović [3,†] and Monica-Felicia Bota [4,*,†]

[1] Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Saveetha University, Chennai 602105, Tamil Nadu, India; mathsguna@yahoo.com
[2] Department of Mathematics, College of Engineering and Technology, Faculty of Engineering and Technology, SRM Institute of Science and Technology, SRM Nagar, Kattankulathur, Kanchipuram, Chennai 603203, Tamil Nadu, India; aruljoseph.alex@gmail.com
[3] Faculty of Electrical Engineering, University of Banja Luka, Patre 5, 78000 Banja Luka, Bosnia and Herzegovina; zoran.mitrovic@etf.unibl.org
[4] Department of Mathematics, Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania
* Correspondence:bmonica@math.ubbcluj.ro or monica.bota@ubbcluj.ro
† All authors contributed equally to this work.

Abstract: In this paper, motivated by the recent result of Sezen, we introduce the notion of fuzzy triple controlled bipolar metric space and prove some fixed point results in this framework. Our results generalize and extend some of the well-known results from the literature. We also explore some of the applications of our key results to integral equations.

Keywords: fuzzy metric space; fuzzy bipolar metric space; fuzzy triple controlled bipolar metric space; fixed points

Citation: Mani, G.; Gnanaprakasam, A.J.; Mitrović, Z.D.; Bota, M.-F. Solving an Integral Equation via Fuzzy Triple Controlled Bipolar Metric Spaces. *Mathematics* **2021**, *9*, 3181. https://doi.org/10.3390/math9243181

Academic Editors: Zhen-Song Chen, Witold Pedrycz, Lesheng Jin, Rosa M. Rodriguez and Luis Martínez López

Received: 16 November 2021
Accepted: 7 December 2021
Published: 9 December 2021

1. Introduction

In their pioneering works, Schweizer and Sklar [1] introduced the notion of continuous triangular norm, and Zadeh [2] provided the theory of fuzzy sets. Using the ideas of Zadeh, in the paper [3], Kramosil and Michalek defined the fuzzy metric space. A modified definition of fuzzy metric spaces is given in the paper of George and Veeramani [4]. After that, Grabiec [5] obtained the well-known fixed-point theorem of Banach to fuzzy metric spaces in the sense of Karamosil and Michalek. Similarly, an extension of the fuzzy Banach contraction theorem to fuzzy metric space in the sense of George and Veeramani was obtained by Gregori and Sapena [6]. Recently Mutlu and Gürdal [7] introduced bipolar metric spaces. Bartwal, Dimri and Prasad [8] introduced fuzzy bipolar metric space and proved some fixed-point theorems in this context. On bipolar metric spaces and fuzzy metric spaces, see [2,9–16].

Recently, Sezen [17] considered some fixed-point results in controlled fuzzy metric spaces. Motivated by Sezen [17], in this paper, we prove fixed-point theorems on fuzzy triple controlled bipolar metric spaces. Using the obtained results, we give an application to the existence and uniqueness of the solution of some classes of integral equations.

2. Preliminaries

In this section, we introduce some basic definitions and auxiliary results.

Definition 1 ([4]). *Let Θ be a non-empty set. An ordered triple $(\Theta, \Pi, *)$ is called a fuzzy metric space if Π is a fuzzy set on $\Theta^2 \times (0, \infty)$ and $*$ is a continuous τ-norm satisfying the following conditions for all $\vartheta, \varphi, \zeta \in \Theta$ and $\tau, \varsigma > 0$:*

(i) $\Pi(\vartheta, \varphi, \tau) > 0$;

(ii) $\Pi(\vartheta, \varphi, \tau) = 1$ *iff* $\vartheta = \varphi$;
(iii) $\Pi(\vartheta, \varphi, \tau) = \Pi(\varphi, \vartheta, \tau)$;
(iv) $\Pi(\vartheta, \zeta, \tau + \varsigma) \geq \Pi(\vartheta, \varphi, \tau) * \Pi(\varphi, \zeta, \varsigma)$;
(v) $\Pi(\vartheta, \varphi, .) : (0, \infty) \longrightarrow (0, 1]$ *is continuous.*

Definition 2 ([8]). *Let Λ and Θ be two non-empty sets. A quadruple $(\Lambda, \Theta, \Pi_\varrho, *)$ is said to be fuzzy bipolar metric space, where $*$ is a continuous τ-norm and Π_ϱ is a fuzzy set on $\Lambda \times \Theta \times (0, \infty)$, satisfying the following conditions for all $\tau, \varsigma, \gamma > 0$:*

(FB1) $\Pi_\varrho(\vartheta, \varphi, \tau) > 0$ *for all* $(\vartheta, \varphi) \in \Lambda \times \Theta$;
(FB2) $\Pi_\varrho(\vartheta, \varphi, \tau) = 1$ *iff* $\vartheta = \varphi$ *for* $\vartheta \in \Lambda$ *and* $\varphi \in \Theta$;
(FB3) $\Pi_\varrho(\vartheta, \varphi, \tau) = \Pi_\varrho(\varphi, \vartheta, \tau)$ *for all* $\vartheta, \varphi \in \Lambda \cap \Theta$;
(FB4) $\Pi_\varrho(\vartheta_1, \varphi_2, \tau + \varsigma + \gamma) \geq \Pi_\varrho(\vartheta_1, \varphi_1, \tau) * \Pi_\varrho(\vartheta_2, \varphi_1, \varsigma) * \Pi_\varrho(\vartheta_2, \varphi_2, \gamma)$ *for all* $\vartheta_1, \vartheta_2 \in \Lambda$ *and* $\varphi_1, \varphi_2 \in \Theta$;
(FB5) $\Pi_\varrho(\vartheta, \varphi, .) : [0, \infty) \longrightarrow [0, 1]$ *is left continuous;*
(FB6) $\Pi_\varrho(\vartheta, \varphi, .)$ *is non-decreasing for all* $\vartheta \in \Lambda$ *and* $\varphi \in \Theta$.

Definition 3. *Let Λ and Θ be two non-empty sets. Let $\lambda, \alpha, \beta : \Lambda \times \Theta \to [1, \infty)$ be three non-comparable functions. A quadruple $(\Lambda, \Theta, \Pi_\varrho, *)$ is said to be fuzzy triple controlled bipolar metric space, where $*$ is a continuous τ-norm and Π_ϱ is a fuzzy set on $\Lambda \times \Theta \times (0, \infty)$, satisfying the following conditions for all $\tau, \varsigma, \gamma > 0$:*

(FCB1) $\Pi_\varrho(\vartheta, \varphi, \tau) > 0$ *for all* $(\vartheta, \varphi) \in \Lambda \times \Theta$;
(FCB2) $\Pi_\varrho(\vartheta, \varphi, \tau) = 1$ *iff* $\vartheta = \varphi$ *for* $\vartheta \in \Lambda$ *and* $\varphi \in \Theta$;
(FCB3) $\Pi_\varrho(\vartheta, \varphi, \tau) = \Pi_\varrho(\varphi, \vartheta, \tau)$ *for all* $\vartheta, \varphi \in \Lambda \cap \Theta$;
(FCB4) $\Pi_\varrho(\vartheta_1, \varphi_2, \tau + \varsigma + \gamma) \geq \Pi_\varrho(\vartheta_1, \varphi_1, \frac{\tau}{\lambda(\vartheta_1,\varphi_1)}) * \Pi_\varrho(\vartheta_2, \varphi_1, \frac{\varsigma}{\alpha(\vartheta_2,\varphi_1)})$ $* \Pi_\varrho(\vartheta_2, \varphi_2, \frac{\gamma}{\beta(\vartheta_2,\varphi_2)})$ *for all* $\vartheta_1, \vartheta_2 \in \Lambda$ *and* $\varphi_1, \varphi_2 \in \Theta$;
(FCB5) $\Pi_\varrho(\vartheta, \varphi, .) : [0, \infty) \longrightarrow [0, 1]$ *is left continuous;*
(FCB6) $\Pi_\varrho(\vartheta, \varphi, .)$ *is non-decreasing for all* $\vartheta \in \Lambda$ *and* $\varphi \in \Theta$.

Remark 1. *Taking $\lambda(\vartheta_1, \varphi_1) = \alpha(\vartheta_2, \varphi_1) = \beta(\vartheta_2, \varphi_2) = 1$, we obtain the definition of fuzzy bipolar metric space [8].*

Example 1. *Let $\Lambda = \{1, 2, 3, 4\}$, $\Theta = \{2, 4, 5, 6\}$ and $\lambda, \alpha, \beta : \Lambda \times \Theta \to [1, \infty)$ be three non-comparable mappings defined as $\lambda(\vartheta, \varphi) = \vartheta + \varphi + 1$, $\alpha(\vartheta, \varphi) = \vartheta^2 + \varphi + 1$ and $\beta(\vartheta, \varphi) = \vartheta^2 + \varphi - 1$. Let $\Pi_\varrho : \Lambda \times \Theta \times (0, +\infty) \to [0, 1]$ be defined by*

$$\Pi_\varrho(\vartheta, \varphi, v) = \frac{\min\{\vartheta, \varphi\} + v}{\max\{\vartheta, \varphi\} + v},$$

for all $\vartheta \in \Lambda$, $\varphi \in \Theta$ and $v \in (0, +\infty)$. Then $(\Lambda, \Theta, \Pi_\varrho, \star)$ is a fuzzy triple controlled bipolar metric space with the continuous τ-norm $\star$ such that $\sigma * \varrho = \sigma\varrho$. Now,

$$\begin{aligned}\lambda(1,2) = 4, \lambda(1,4) = 6, \lambda(1,5) = 7, \lambda(1,6) = 8, \lambda(2,2) = 5, \lambda(2,4) = 7,\\ \lambda(2,5) = 8, \lambda(2,6) = 9, \lambda(3,2) = 6, \lambda(3,4) = 8, \lambda(3,5) = 9, \lambda(3,6) = 10,\\ \lambda(4,2) = 7, \lambda(4,4) = 9, \lambda(4,5) = 10, \lambda(4,6) = 11,\end{aligned}$$

$$\begin{aligned}\alpha(1,2) = 4, \alpha(1,4) = 6, \alpha(1,5) = 7, \alpha(1,6) = 8, \alpha(2,2) = 7, \alpha(2,4) = 9,\\ \alpha(2,5) = 10, \alpha(2,6) = 11, \alpha(3,2) = 12, \alpha(3,4) = 14, \alpha(3,5) = 15,\\ \alpha(3,6) = 16, \alpha(4,2) = 19, \alpha(4,4) = 21, \alpha(4,5) = 22, \alpha(4,6) = 23,\end{aligned}$$

$$\beta(1,2)=2, \beta(1,4)=4, \beta(1,5)=5, \beta(1,6)=6, \beta(2,2)=5, \beta(2,4)=7,$$
$$\beta(2,5)=8, \beta(2,6)=9, \beta(3,2)=10, \beta(3,4)=12, \beta(3,5)=13, \beta(3,6)=14,$$
$$\beta(4,2)=17, \beta(4,4)=19, \beta(4,5)=20, \beta(4,6)=21,$$

Axioms (FCB1)–(FCB3), (FCB5) and (FCB6) are easy to verify; we only prove (FCB4). Let $\vartheta_1=1, \varphi_2=4, \varphi_1=2$ and $\vartheta_2=3$. Then

$$\begin{aligned}\Pi_\varrho(1,4,v+\varsigma+\gamma) &= \frac{\min\{1,4\}+v+\varsigma+\gamma}{\max\{1,4\}+v+\varsigma+\gamma} \\ &= \frac{1+v+\varsigma+\gamma}{4+v+\varsigma+\gamma}.\end{aligned}$$

Now,

$$\begin{aligned}\Pi_\varrho(1,2,\frac{v}{\lambda(1,2)}) &= \frac{\min\{1,2\}+\frac{v}{\lambda(1,2)}}{\max\{1,2\}+\frac{v}{\lambda(1,2)}} \\ &= \frac{1+\frac{v}{4}}{2+\frac{v}{4}} \\ &= \frac{4+v}{8+v},\end{aligned}$$

$$\begin{aligned}\Pi_\varrho(3,2,\frac{\varsigma}{\alpha(3,2)}) &= \frac{\min\{3,2\}+\frac{\varsigma}{\alpha(3,2)}}{\max\{3,2\}+\frac{\varsigma}{\alpha(3,2)}} \\ &= \frac{2+\frac{\varsigma}{12}}{3+\frac{\varsigma}{12}} \\ &= \frac{24+\varsigma}{36+\varsigma}\end{aligned}$$

and

$$\begin{aligned}\Pi_\varrho(3,4,\frac{\gamma}{\beta(3,4)}) &= \frac{\min\{3,4\}+\frac{\gamma}{\beta(3,4)}}{\max\{3,4\}+\frac{\gamma}{\beta(3,4)}} \\ &= \frac{3+\frac{\gamma}{12}}{4+\frac{\gamma}{12}} \\ &= \frac{36+\gamma}{48+\gamma}.\end{aligned}$$

Clearly,

$$\frac{1+v+\varsigma+\gamma}{4+v+\varsigma+\gamma} \geq \left(\frac{4+v}{8+v}\right)\left(\frac{24+\varsigma}{36+\varsigma}\right)\left(\frac{36+\gamma}{48+\gamma}\right),$$

for all $v, \varsigma, \gamma > 0$. So,

$$\Pi_\varrho(1,4,v+\varsigma+\gamma) \geq \Pi_\varrho(1,2,\frac{v}{\lambda(1,2)}) \star \Pi_\varrho(3,2,\frac{\varsigma}{\alpha(3,2)}) \star \Pi_\varrho(3,4,\frac{\gamma}{\beta(3,4)}).$$

On the same steps, one can prove the remaining cases. Hence, $(\Lambda, \Theta, \Pi_\varrho, \star)$ is a fuzzy triple controlled bipolar metric space.

Example 2. *If we take minimum τ-norm instead of product τ-norm in Example 1, then $(\Lambda, \Theta, \Pi_\varrho, \star)$ is not a fuzzy triple controlled bipolar metric space. For instance, let $\vartheta_1=1, \varphi_2=4, \varphi_1=2$,*

$\vartheta_2 = 3$ and $\tau = 0.02, \varsigma = 0.03, \gamma = 0.04$ with $\lambda, \alpha, \beta : \Lambda \times \Theta \to [1, \infty)$ be three non-comparable mappings defined as $\lambda(\vartheta, \varphi) = \vartheta + \varphi + 1$,$\alpha(\vartheta, \varphi) = \vartheta^2 + \varphi + 1$ and $\beta(\vartheta, \varphi) = \vartheta^2 + \varphi - 1$, then

$$\Pi_\varrho(1, 4, 0.02 + 0.03 + 0.04) = \frac{1 + 0.09}{4 + 0.09} = 0.2665,$$

and

$$\Pi_\varrho(1, 2, \frac{0.02}{\lambda(1,2)}) = \frac{1 + \frac{0.02}{4}}{2 + \frac{0.02}{4}} = 0.50124,$$

$$\Pi_\varrho(3, 2, \frac{0.03}{\alpha(3,2)}) = \frac{2 + \frac{0.03}{12}}{3 + \frac{0.03}{12}} = 0.6669,$$

$$\Pi_\varrho(3, 4, \frac{0.04}{\beta(3,4)}) = \frac{3 + \frac{0.04}{12}}{4 + \frac{0.04}{12}} = 0.7502.$$

Clearly,

$$\Pi_\varrho(1, 4, 0.02 + 0.03 + 0.04) \ngeq \Pi_\varrho(1, 2, \frac{0.02}{\lambda(1,2)}) \star \Pi_\varrho(3, 2, \frac{0.03}{\alpha(3,2)}) \star \Pi_\varrho(3, 4, \frac{0.04}{\beta(3,4)}).$$

Hence, $(\Lambda, \Theta, \Pi_\varrho, \star)$ is not a fuzzy triple controlled bipolar metric space with minimum τ-norm.

Definition 4. *Let $(\Lambda, \Theta, \Pi_\varrho, *)$ be a fuzzy triple controlled bipolar metric space. The points belong to Λ, Θ and $\Lambda \cap \Theta$ are called left, right and central points, respectively, and sequences that belong to Λ, Θ and $\Lambda \cap \Theta$ are named left, right and central sequences, respectively.*

Lemma 1. *Let $(\Lambda, \Theta, \Pi_\varrho, *)$ be a fuzzy triple controlled bipolar metric space such that*

$$\Pi_\varrho(\vartheta, \varphi, \kappa\tau) \geq \Pi_\varrho(\vartheta, \varphi, \tau)$$

for $\vartheta \in \Lambda, \varphi \in \Theta$, $\tau \in (0, +\infty)$ and $\kappa \in (0, 1)$. Then $\vartheta = \varphi$.

Proof. We have

$$\Pi_\varrho(\vartheta, \varphi, \kappa\tau) \geq \Pi_\varrho(\vartheta, \varphi, \tau) \text{ for } \tau > 0. \tag{1}$$

Since $\kappa\tau < \tau$ for all $\tau > 0$ and $\kappa \in (0, 1)$, by (FCB6), we have

$$\Pi_\varrho(\vartheta, \varphi, \kappa\tau) \leq \Pi_\varrho(\vartheta, \varphi, \tau). \tag{2}$$

From (2), (3) and the definition of fuzzy triple controlled bipolar metric space (see (FCB2)), we obtain $\vartheta = \varphi$. □

Definition 5. *Let $(\Lambda, \Theta, \Pi_\varrho, *)$ be a fuzzy triple controlled bipolar metric space. A sequence $\{\vartheta_\eta\} \in \Lambda$ converges to a right point φ if $\Pi_\varrho(\vartheta_\eta, \varphi, \tau) \to 1$ as $\eta \to +\infty$, for all $\tau > 0$. Similarly, a right sequence $\{\varphi_\eta\}$ converges to a left point ϑ if $\Pi_\varrho(\vartheta, \varphi_\eta, \tau) \to 1$ as $\eta \to +\infty$, for all $\tau > 0$.*

Definition 6. *Let $(\Lambda, \Theta, \Pi_\varrho, *)$ be a fuzzy triple controlled bipolar metric space. Then, we have the following:*

(i) *Sequence $(\vartheta_\eta, \varphi_\eta) \in \Lambda \times \Theta$ is named as a bisequence on $(\Lambda, \Theta, \Pi_\varrho, *)$.*

(ii) *If both ϑ_η and φ_η converge, the sequence $(\vartheta_\eta, \varphi_\eta) \in \Lambda \times \Theta$ is said to be a convergent sequence. If both ϑ_η and φ_η converge to some center point, bisequence $(\vartheta_\eta, \varphi_\eta)$ is said to be a biconvergent sequence.*

(iii) *A bisequence $(\vartheta_\eta, \varphi_\eta)$ on fuzzy triple controlled bipolar metric space $(\Lambda, \Theta, \Pi_\varrho, *)$ is said to be a Cauchy bisequence $\Pi_\varrho(\vartheta_\eta, \varphi_\omega, \tau) \to 1$ as $\eta, \omega \to +\infty$, for all $\tau > 0$.*

Definition 7. *The fuzzy triple controlled bipolar metric space $(\Lambda, \Theta, \Pi_\varrho, *)$ is said to be complete if every Cauchy bisequence in $\Lambda \times \Theta$ is convergent in it.*

Proposition 1. *In a fuzzy triple controlled bipolar metric space, every convergent Cauchy bisequence is biconvergent.*

Proof. Let $(\Lambda, \Theta, \Pi_\varrho, *)$ be a fuzzy triple controlled bipolar metric space and a bisequence $(\vartheta_\eta, \varphi_\eta) \in \Lambda \times \Theta$ such that $\vartheta_\eta \to \varphi$ as $\eta \to +\infty$ and $\varphi_\eta \to \vartheta$ as $\eta \to +\infty$, where $\eta \in \Theta$ and $\vartheta \in \Lambda$. Since $(\vartheta_\eta, \varphi_\eta)$ is a convergent Cauchy bisequence, we obtain

$$\Pi_\varrho(\vartheta_\eta, \varphi_\omega, \tau) \to 1 \text{ as } \eta \to \infty, \tag{3}$$

for all $\tau > 0$. Now, from (3), we conclude that

$$\Pi_\varrho(\vartheta, \varphi, \tau) = 1,$$

for all $\tau > 0$. Therefore, by (FCB2), we obtain that the bisequence $(\vartheta_\eta, \varphi_\eta)$ is biconvergent. □

Proposition 2. *In a fuzzy triple controlled bipolar metric space, every biconvergent bisequence is a Cauchy bisequence.*

Proof. Let $(\Lambda, \Theta, \Pi_\varrho, *)$ be a fuzzy triple controlled bipolar metric space, and bisequence $(\vartheta_\eta, \varphi_\omega) \in \Lambda \times \Theta$ converges to a point $\vartheta_0 \in \Lambda \cap \Theta$ for all $\eta, \omega \in \mathbb{N}$ and $\tau > 0$. By (FCB4), we have

$$\begin{aligned}\Pi_\varrho(\vartheta_\eta, \varphi_\omega, \tau) \geq \Pi_\varrho(\vartheta_\eta, \vartheta_0, \frac{\tau}{3\lambda(\vartheta_\eta, \vartheta_0)}) &* \Pi_\varrho(\vartheta_0, \vartheta_0, \frac{\tau}{3\lambda(\vartheta_0, \vartheta_0)}) \\ &* \Pi_\varrho(\vartheta_0, \varphi_\omega, \frac{\tau}{3\lambda(\vartheta_0, \varphi_\omega)})\end{aligned}$$

As $\eta, \omega \to \infty$, we obtain

$$\Pi_\varrho(\vartheta_\eta, \varphi_\omega, \tau) \geq 1 \text{ for all } \tau > 0.$$

This implies that $\Pi_\varrho(\vartheta_\eta, \varphi_\omega, \tau) \to 1$ for all $\tau > 0$. Hence, $(\vartheta_\eta, \varphi_\eta)$ is a Cauchy bisequence. □

Lemma 2. *Let $(\Lambda, \Theta, \Pi_\varrho, *)$ be a fuzzy triple controlled bipolar metric space and $\mu \in \Lambda \cap \Theta$ is a limit of a sequence then it is a unique limit of the sequence.*

Proof. Let $\{\vartheta_\eta\} \in \Lambda$ be a sequence. Suppose that $\{\vartheta_\eta\} \to \varphi \in \Theta$ and also $\{\vartheta_\eta\} \to \mu \in \Lambda \cap \Theta$; then, for $\tau, \varsigma, \gamma > 0$, we have

$$\Pi_\varrho(\mu, \varphi, \tau + \varsigma + \gamma) \geq \Pi_\varrho(\mu, \mu, \tau) * \Pi_\varrho(\vartheta_\eta, \mu, \varsigma) * \Pi_\varrho(\vartheta_\eta, \varphi, \gamma)$$

As $\eta \to \infty$, we obtain

$$\Pi_\varrho(\mu, \varphi, \tau + \varsigma + \gamma) \geq 1,$$

which implies that $\mu = \varphi$, i.e., sequence $\{\vartheta_\eta\}$, has a unique limit. □

Definition 8. *A point $\vartheta \in \Lambda \cap \Theta$ is said to be a fixed-point of the mapping Γ if $\vartheta = \Gamma\vartheta$.*

3. Main Results

In this section, we prove the extension of some well-known fixed-point theorems to fuzzy triple controlled bipolar metric spaces.

Theorem 1. *Let $(\Lambda, \Theta, \Pi_\varrho, *)$ be a complete fuzzy triple controlled bipolar metric space with three non-comparable functions $\lambda, \alpha, \beta : \Lambda \times \Theta \to [1, \infty)$ such that*

$$\lim_{\tau \to \infty} \Pi_\varrho(\vartheta, \varphi, \tau) = 1 \text{ for all } \vartheta \in \Lambda, \varphi \in \Theta. \tag{4}$$

Let $\Gamma : \Lambda \cup \Theta \to \Lambda \cup \Theta$ be a mapping satisfying

(i) $\Gamma(\Lambda) \subseteq \Lambda$ *and* $\Gamma(\Theta) \subseteq \Theta$*;*
(ii) $\Pi_\varrho(\Gamma(\vartheta), \Gamma(\varphi), \kappa\tau) \geq \Pi_\varrho(\vartheta, \varphi, \tau)$ *for all $\vartheta \in \Lambda, \varphi \in \Theta$ and $\tau > 0$, where $\kappa \in (0, 1)$.*
Additionally, assume that for every $\vartheta \in \Lambda$, we obtain that

$$\lim_{\eta \to \infty} \lambda(\vartheta_\eta, \varphi) \text{ and } \lim_{\eta \to \infty} \lambda(\varphi, \vartheta_\eta),$$

exist and are finite. Then Γ has a unique fixed point.

Proof. Fix $\vartheta_0 \in \Lambda$ and $\varphi_0 \in \Theta$ and assume that $\Gamma(\vartheta_\eta) = \vartheta_{\mathtt{n}+1}$ and $\Gamma(\varphi_\eta) = \varphi_{\mathtt{n}+1}$ for all $\eta \in \mathbb{N} \cup \{0\}$. Then we obtain $(\vartheta_\eta, \varphi_\eta)$ as a bisequence on fuzzy triple controlled bipolar metric space $(\Lambda, \Theta, \Pi_\varrho, *)$. Now, we have

$$\Pi_\varrho(\vartheta_1, \varphi_1, \tau) = \Pi_\varrho(\Gamma(\vartheta_0), \Gamma(\varphi_0), \tau) \geq \Pi_\varrho(\vartheta_0, \varphi_0, \frac{\tau}{\kappa})$$

for all $\tau > 0$ and $\eta \in \mathbb{N}$. By simple induction, we obtain

$$\Pi_\varrho(\vartheta_\eta, \varphi_\eta, \tau) = \Pi_\varrho(\Gamma(\vartheta_{\mathtt{n}-1}), \Gamma(\varphi_{\mathtt{n}-1}), \tau) \geq \Pi_\varrho(\vartheta_0, \varphi_0, \frac{\tau}{\kappa^\eta}) \tag{5}$$

and

$$\Pi_\varrho(\vartheta_{\mathtt{n}+1}, \varphi_\eta, \tau) = \Pi_\varrho(\Gamma(\vartheta_\eta), \Gamma(\varphi_{\mathtt{n}-1}), \tau) \geq \Pi_\varrho(\vartheta_1, \varphi_0, \frac{\tau}{\kappa^\eta}) \tag{6}$$

for all $\tau > 0$ and $\eta \in \mathbb{N}$. Let $\eta < \omega$ for $\eta, \omega \in \mathbb{N}$. Then,

$$\begin{aligned}
\Pi_\varrho(\vartheta_\eta, \varphi_\omega, \tau) \geq &\Pi_\varrho(\vartheta_\eta, \varphi_\eta, \frac{\tau}{3\lambda(\vartheta_\eta, \varphi_\eta)}) \star \Pi_\varrho(\vartheta_{\eta+1}, \varphi_\eta, \frac{\tau}{3\alpha(\vartheta_{\eta+1}, \varphi_\eta)}) \\
&\star \Pi_\varrho(\vartheta_{\eta+1}, \varphi_\omega, \frac{\tau}{3\beta(\vartheta_{\eta+1}, \varphi_\omega)}) \\
&\geq \Pi_\varrho(\vartheta_\eta, \varphi_\eta, \frac{\tau}{3\lambda(\vartheta_\eta, \varphi_\eta)}) \star \Pi_\varrho(\vartheta_{\eta+1}, \varphi_\eta, \frac{\tau}{3\alpha(\vartheta_{\eta+1}, \varphi_\eta)}) \\
&\star \Pi_\varrho(\vartheta_{\eta+1}, \varphi_{\eta+1}, \frac{\tau}{3^2\beta(\vartheta_{\eta+1}, \varphi_\omega)\lambda(\vartheta_{\eta+1}, \varphi_{\eta+1})}) \\
&\star \Pi_\varrho(\vartheta_{\eta+2}, \varphi_{\eta+1}, \frac{\tau}{3^2\beta(\vartheta_{\eta+1}, \varphi_\omega)\alpha(\vartheta_{\eta+2}, \varphi_{\eta+1})}) \\
&\star \Pi_\varrho(\vartheta_{\eta+2}, \varphi_\omega, \frac{\tau}{3^2\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)}) \\
&\geq \Pi_\varrho(\vartheta_\eta, \varphi_\eta, \frac{\tau}{3\lambda(\vartheta_\eta, \varphi_\eta)}) \star \Pi_\varrho(\vartheta_{\eta+1}, \varphi_\eta, \frac{\tau}{3\alpha(\vartheta_{\eta+1}, \varphi_\eta)}) \\
&\star \Pi_\varrho(\vartheta_{\eta+1}, \varphi_{\eta+1}, \frac{\tau}{3^2\beta(\vartheta_{\eta+1}, \varphi_\omega)\lambda(\vartheta_{\eta+1}, \varphi_{\eta+1})}) \\
&\star \Pi_\varrho(\vartheta_{\eta+2}, \varphi_{\eta+1}, \frac{\tau}{3^2\beta(\vartheta_{\eta+1}, \varphi_\omega)\alpha(\vartheta_{\eta+2}, \varphi_{\eta+1})}) \\
&\star \Pi_\varrho(\vartheta_{\eta+2}, \varphi_{\eta+2}, \frac{\tau}{3^3\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)\lambda(\vartheta_{\eta+2}, \varphi_{\eta+2})}) \\
&\star \Pi_\varrho(\vartheta_{\eta+3}, \varphi_{\eta+2}, \frac{\tau}{3^3\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)\alpha(\vartheta_{\eta+3}, \varphi_{\eta+2})}) \\
&\star \Pi_\varrho(\vartheta_{\eta+3}, \varphi_\omega, \frac{\tau}{3^3\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)\beta(\vartheta_{\eta+3}, \varphi_\omega)}) \\
&\geq \Pi_\varrho(\vartheta_\eta, \varphi_\eta, \frac{\tau}{3\lambda(\vartheta_\eta, \varphi_\eta)}) \star \Pi_\varrho(\vartheta_{\eta+1}, \varphi_\eta, \frac{\tau}{3\lambda(\vartheta_{\eta+1}, \varphi_\eta)}) \star \cdots \\
&\star \Pi_\varrho(\vartheta_{\omega-1}, \varphi_{\omega-1}, \frac{\tau}{3^{\omega-1}\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)\cdots\lambda(\vartheta_{\omega-1}, \varphi_{\omega-1})}) \\
&\star \Pi_\varrho(\vartheta_\omega, \varphi_{\omega-1}, \frac{\tau}{3^{\omega-1}\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)\cdots\alpha(\vartheta_\omega, \varphi_{\omega-1})}) \\
&\star \Pi_\varrho(\vartheta_\omega, \varphi_\omega, \frac{\tau}{3^{\omega-1}\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)\cdots\beta(\vartheta_\omega, \varphi_\omega)}).
\end{aligned}$$

Now, applying (5) and (6) on each term of the right-hand side of the above inequality, we obtain

$$\begin{aligned}
\Pi_\varrho(\vartheta_\eta, \varphi_\omega, \tau) \geq \Pi_\varrho(\vartheta_0, \varphi_0, \frac{\tau}{3\kappa^\eta\lambda(\vartheta_\eta, \varphi_\eta)}) \star \Pi_\varrho(\vartheta_1, \varphi_0, \frac{\tau}{3\kappa^{\eta+1}\lambda(\vartheta_{\eta+1}, \varphi_\eta)}) \\
\star \cdots\cdots \star \Pi_\varrho(\vartheta_0, \varphi_0, \frac{\tau}{3^{\omega-1}\kappa^\omega\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)\cdots\beta(\vartheta_\omega, \varphi_\omega)}).
\end{aligned}$$

From (4), as $\eta, \omega \to \infty$, we obtain

$$\Pi_\varrho(\vartheta_\eta, \varphi_\omega, \tau) \geq 1 \text{ for all } \tau > 0.$$

Therefore, bisequence $(\vartheta_\eta, \varphi_\eta)$ is a Cauchy bisequence. Since $(\Lambda, \Theta, \Pi_\varrho, *)$ is a complete space, bisequence $(\vartheta_\eta, \varphi_\eta)$ is a convergent Cauchy bisequence. According to Proposition 1, the bisequence $(\vartheta_\eta, \varphi_\eta)$ is a biconvergent sequence.

As bisequence $(\vartheta_\eta, \varphi_\eta)$ is biconvergent, there exists a point $\mu \in \Lambda \cap \Theta$ which is a limit of both sequences $\{\vartheta_\eta\}$ and $\{\varphi_\eta\}$. By Lemma 2, both sequences $\{\vartheta_\eta\}$ and $\{\varphi_\eta\}$ have a unique limit. From (FCB4), consider

$$\Pi_\varrho(\Gamma(\mu), \mu, \tau) \geq \Pi_\varrho(\Gamma(\mu), \Gamma(\varphi_\eta), \frac{\tau}{3\lambda(\mu, \varphi_\eta)}) * \Pi_\varrho(\Gamma(\vartheta_\eta), \Gamma(\varphi_\eta), \frac{\tau}{3\alpha(\vartheta_\eta, \varphi_\eta)}) * \Pi_\varrho(\Gamma(\vartheta_\eta), \mu, \frac{\tau}{3\beta(\vartheta_\eta, \mu)})$$

for all $\eta \in \mathbb{N}$ and $\tau > 0$. As $\eta \to \infty$, we have

$$\Pi_\varrho(\Gamma(\mu), \mu, \tau) \to 1*1*1 = 1.$$

From (FCB2), we obtain $\Gamma(\mu) = \mu$. Let $\nu \in \Lambda \cap \Theta$ be another fixed point of Γ. Then

$$\Pi_\varrho(\mu, \nu, \tau) = \Pi_\varrho(\Gamma(\mu), \Gamma(\nu), \tau) \geq \Pi_\varrho(\mu, \nu, \frac{\tau}{\kappa})$$

for $\kappa \in (0,1)$ and for all $\tau > 0$. By Lemma 1, we obtain $\mu = \nu$. □

Example 3. *Let $\Lambda = [0,1]$, $\Theta = \{0\} \cup \mathbb{N} - \{1\}$ and $\lambda, \alpha, \beta : \Lambda \times \Theta \to [1, \infty)$ be three non-comparable mappings defined as $\lambda(\vartheta, \varphi) = \vartheta + \varphi + 1, \alpha(\vartheta, \varphi) = \vartheta^2 + \varphi + 1$ and $\beta(\vartheta, \varphi) = \vartheta^2 + \varphi - 1$. Define $\Pi_\varrho(\vartheta, \varphi, \tau) = \frac{\tau}{\tau + |\vartheta - \varphi|}$ for all $\tau > 0$ and $\vartheta \in \Lambda$ and $\varphi \in \Theta$. Clearly, $(\Lambda, \Theta, \Pi_\varrho, *)$ is a complete fuzzy triple controlled bipolar metric space, where $*$ is a continuous τ-norm defined as $\sigma * \varrho = \sigma\varrho$.*

Define $\Gamma : \Lambda \cup \Theta \to \Lambda \cup \Theta$ by

$$\Gamma(\mu) = \begin{cases} \frac{\mu}{2}, & \text{if } \mu \in [0,1], \\ 0, & \text{if } \mu \in \mathbb{N} - \{1\}, \end{cases}$$

for all $\mu \in \Lambda \cup \Theta$. Clearly, all the hypotheses of Theorem 1 are satisfied. Hence Γ has a unique fixed point, i.e., $\mu = 0$.

Theorem 2. *Let $(\Lambda, \Theta, \Pi_\varrho, *)$ be a complete fuzzy triple controlled bipolar metric space with three non-comparable functions $\lambda, \alpha, \beta : \Lambda \times \Theta \to [1, \infty)$ such that*

$$\lim_{\tau \to \infty} \Pi_\varrho(\vartheta, \varphi, \tau) = 1 \text{ for all } \vartheta \in \Lambda, \varphi \in \Theta. \tag{7}$$

Let $\Gamma : \Lambda \cup \Theta \to \Lambda \cup \Theta$ be a mapping satisfying

(i) $\Gamma(\Lambda) \subseteq \Theta$ *and* $\Gamma(\Theta) \subseteq \Lambda$;

(ii) $\Pi_\varrho(\Gamma(\varphi), \Gamma(\vartheta), \kappa\tau) \geq \Pi_\varrho(\vartheta, \varphi, \tau)$, *for all $\vartheta \in \Lambda, \varphi \in \Theta$ and $\tau > 0$, where $\kappa \in (0,1)$.*

Then Γ has a unique fixed point.

Proof. Fix $\vartheta_0 \in \Lambda$ and assume that $\Gamma(\vartheta_\eta) = \varphi_\eta$ and $\Gamma(\varphi_\eta) = \vartheta_{\eta+1}$ for all $\eta \in \mathbb{N} \cup \{0\}$. Then, we obtain $(\vartheta_\eta, \varphi_\eta)$ as a bisequence on fuzzy triple controlled bipolar metric space $(\Lambda, \Theta, \Pi_\varrho, *)$. Now, we have

$$\Pi_\varrho(\vartheta_1, \varphi_0, \tau) = \Pi_\varrho(\Gamma(\varphi_0), \Gamma(\vartheta_0), \tau) \geq \Pi_\varrho(\vartheta_0, \varphi_0, \frac{\tau}{\kappa})$$

for all $\tau > 0$ and $\eta \in \mathbb{N}$. By simple induction, we obtain

$$\Pi_\varrho(\vartheta_\eta, \varphi_\eta, \tau) = \Pi_\varrho(\Gamma(\varphi_{\eta-1}), \Gamma(\vartheta_\eta), \tau) \geq \Pi_\varrho(\vartheta_0, \varphi_0, \frac{\tau}{\kappa^{2\eta}}) \tag{8}$$

and

$$\Pi_\varrho(\vartheta_{\eta+1}, \varphi_\eta, \tau) = \Pi_\varrho(\Gamma(\varphi_\eta), \Gamma(\vartheta_\eta), \tau) \geq \Pi_\varrho(\vartheta_0, \varphi_0, \frac{\tau}{\kappa^{2\eta+1}}) \tag{9}$$

for all $\tau > 0$ and $\eta \in \mathbb{N}$. Let $\eta < \omega$, for $\eta, \omega \in \mathbb{N}$. Then,

$$\begin{aligned}
\Pi_\varrho(\vartheta_\eta, \varphi_\omega, \tau) \geq & \Pi_\varrho(\vartheta_\eta, \varphi_\eta, \frac{\tau}{3\lambda(\vartheta_\eta, \varphi_\eta)}) \star \Pi_\varrho(\vartheta_{\eta+1}, \varphi_\eta, \frac{\tau}{3\alpha(\vartheta_{\eta+1}, \varphi_\eta)}) \\
& \star \Pi_\varrho(\vartheta_{\eta+1}, \varphi_\omega, \frac{\tau}{3\beta(\vartheta_{\eta+1}, \varphi_\omega)}) \\
& \geq \Pi_\varrho(\vartheta_\eta, \varphi_\eta, \frac{\tau}{3\lambda(\vartheta_\eta, \varphi_\eta)}) \star \Pi_\varrho(\vartheta_{\eta+1}, \varphi_\eta, \frac{\tau}{3\alpha(\vartheta_{\eta+1}, \varphi_\eta)}) \\
& \star \Pi_\varrho(\vartheta_{\eta+1}, \varphi_{\eta+1}, \frac{\tau}{3^2\beta(\vartheta_{\eta+1}, \varphi_\omega)\lambda(\vartheta_{\eta+1}, \varphi_{\eta+1})}) \\
& \star \Pi_\varrho(\vartheta_{\eta+2}, \varphi_{\eta+1}, \frac{\tau}{3^2\beta(\vartheta_{\eta+1}, \varphi_\omega)\alpha(\vartheta_{\eta+2}, \varphi_{\eta+1})}) \\
& \star \Pi_\varrho(\vartheta_{\eta+2}, \varphi_\omega, \frac{\tau}{3^2\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)}) \\
& \geq \Pi_\varrho(\vartheta_\eta, \varphi_\eta, \frac{\tau}{3\lambda(\vartheta_\eta, \varphi_\eta)}) \star \Pi_\varrho(\vartheta_{\eta+1}, \varphi_\eta, \frac{\tau}{3\alpha(\vartheta_{\eta+1}, \varphi_\eta)}) \\
& \star \Pi_\varrho(\vartheta_{\eta+1}, \varphi_{\eta+1}, \frac{\tau}{3^2\beta(\vartheta_{\eta+1}, \varphi_\omega)\lambda(\vartheta_{\eta+1}, \varphi_{\eta+1})}) \\
& \star \Pi_\varrho(\vartheta_{\eta+2}, \varphi_{\eta+1}, \frac{\tau}{3^2\beta(\vartheta_{\eta+1}, \varphi_\omega)\alpha(\vartheta_{\eta+2}, \varphi_{\eta+1})}) \\
& \star \Pi_\varrho(\vartheta_{\eta+2}, \varphi_{\eta+2}, \frac{\tau}{3^3\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)\lambda(\vartheta_{\eta+2}, \varphi_{\eta+2})}) \\
& \star \Pi_\varrho(\vartheta_{\eta+3}, \varphi_{\eta+2}, \frac{\tau}{3^3\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)\alpha(\vartheta_{\eta+3}, \varphi_{\eta+2})}) \\
& \star \Pi_\varrho(\vartheta_{\eta+3}, \varphi_\omega, \frac{\tau}{3^3\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)\beta(\vartheta_{\eta+3}, \varphi_\omega)}) \\
& \geq \Pi_\varrho(\vartheta_\eta, \varphi_\eta, \frac{\tau}{3\lambda(\vartheta_\eta, \varphi_\eta)}) \star \Pi_\varrho(\vartheta_{\eta+1}, \varphi_\eta, \frac{\tau}{3\lambda(\vartheta_{\eta+1}, \varphi_\eta)}) \star \cdots \\
& \star \Pi_\varrho(\vartheta_{\omega-1}, \varphi_{\omega-1}, \frac{\tau}{3^{\omega-1}\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)\cdots\lambda(\vartheta_{\omega-1}, \varphi_{\omega-1})}) \\
& \star \Pi_\varrho(\vartheta_\omega, \varphi_{\omega-1}, \frac{\tau}{3^{\omega-1}\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)\cdots\alpha(\vartheta_\omega, \varphi_{\omega-1})}) \\
& \star \Pi_\varrho(\vartheta_\omega, \varphi_\omega, \frac{\tau}{3^{\omega-1}\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)\cdots\beta(\vartheta_\omega, \varphi_\omega)}).
\end{aligned}$$

Now applying (5) and (6) on each term of the right-hand side of the above inequality, we obtain

$$\begin{aligned}
\Pi_\varrho(\vartheta_\eta, \varphi_\omega, \tau) \geq & \Pi_\varrho(\vartheta_0, \varphi_0, \frac{\tau}{3\kappa^{2\eta}\lambda(\vartheta_\eta, \varphi_\eta)}) \star \Pi_\varrho(\vartheta_0, \varphi_0, \frac{\tau}{3\kappa^{2\eta+1}\lambda(\vartheta_{\eta+1}, \varphi_\eta)}) \\
& \star \cdots\cdots \star \Pi_\varrho(\vartheta_0, \varphi_0, \frac{\tau}{3^{\omega-1}\kappa^{2\omega}\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)\cdots\beta(\vartheta_\omega, \varphi_\omega)}).
\end{aligned}$$

From (7), as $\eta, \omega \to \infty$, we obtain

$$\Pi_\varrho(\vartheta_\eta, \varphi_\omega, \tau) \geq 1 \text{ for all } \tau > 0.$$

Therefore, bisequence $(\vartheta_\eta, \varphi_\eta)$ is a Cauchy bisequence. Since $(\Lambda, \Theta, \Pi_\varrho, *)$ is a complete space, bisequence $(\vartheta_\eta, \varphi_\eta)$ is a convergent Cauchy bisequence. According to Proposition 1, the bisequence $(\vartheta_\eta, \varphi_\eta)$ is a biconvergent sequence. As bisequence $(\vartheta_\eta, \varphi_\eta)$ is biconvergent, there exists a point $\mu \in \Lambda \cap \Theta$ which is a limit of both sequences $\{\vartheta_\eta\}$ and $\{\varphi_\eta\}$. By Lemma 2, both sequences $\{\vartheta_\eta\}$ and $\{\varphi_\eta\}$ have a unique limit. From (FCB4), consider

$$\begin{aligned}\Pi_\varrho(\Gamma(\mu), \mu, \tau) \geq \Pi_\varrho(\Gamma(\mu), \Gamma(\vartheta_\eta), &\frac{\tau}{3\lambda(\Gamma(\mu), \Gamma(\vartheta_\eta))}) \\ *\Pi_\varrho(\Gamma(\varphi_\eta), \Gamma(\vartheta_\eta), &\frac{\tau}{3\alpha(\Gamma(\varphi_\eta), \Gamma(\vartheta_\eta))}) \\ *\Pi_\varrho(\mu, \Gamma(\vartheta_\eta), &\frac{\tau}{3\beta(\mu, \Gamma(\vartheta_\eta))}),\end{aligned}$$

for all $\eta \in \mathbb{N}$ and $\tau > 0$. As $\eta \to \infty$, we have

$$\Pi_\varrho(\Gamma(\mu), \mu, \tau) \to 1*1*1 = 1.$$

From (FCB2), we get $\Gamma(\mu) = \mu$. Let $\nu \in \Lambda \cap \Theta$ be another fixed point of Γ. Then

$$\Pi_\varrho(\mu, \nu, \tau) = \Pi_\varrho(\Gamma(\nu), \Gamma(\mu), \tau) \geq \Pi_\varrho(\mu, \nu, \frac{\tau}{\kappa})$$

for $\kappa \in (0, 1)$ and for all $\tau > 0$. By Lemma 1, we obtain $\mu = \nu$. □

Example 4. *Let $\Lambda = [0, 1]$, $\Theta = \{0\} \cup \mathbb{N} - \{1\}$ and $\lambda, \alpha, \beta : \Lambda \times \Theta \to [1, \infty)$ be three non-comparable mappings defined as $\lambda(\vartheta, \varphi) = \vartheta + \varphi + 1$,$\alpha(\vartheta, \varphi) = \vartheta^2 + \varphi + 1$ and $\beta(\vartheta, \varphi) = \vartheta^2 + \varphi - 1$. Define*

$$\Pi_\varrho(\vartheta, \varphi, \tau) = e^{-\frac{(\vartheta-\varphi)^2}{\tau}}, \ \forall\ \vartheta \in \Lambda, \varphi \in \Theta, \tau > 0.$$

Then $(\Lambda, \Theta, \Pi_\varrho, *)$ is a complete fuzzy triple controlled bipolar metric space with product τ-norm. Define $\Gamma : \Lambda \cup \Theta \to \Lambda \cup \Theta$ by

$$\Gamma(\mu) = \begin{cases} \frac{1-2^{-\mu}}{3}, & \text{if } \mu \in [0, 1], \\ 0, & \text{if } \mu \in \mathbb{N} - \{1\}, \end{cases}$$

for all $\mu \in \Lambda \cup \Theta$. Let $\vartheta \in [0, 1]$ and $\varphi \in \mathbb{N} - \{1\}$, then

$$\begin{aligned}\Pi_\varrho(\Gamma(\vartheta), \Gamma(\varphi), \kappa\tau) &= \Pi_\varrho\left(\frac{1-2^{-\vartheta}}{3}, 0, \kappa\tau\right) \\ &= e^{-\frac{\left(\frac{1-2^{-\vartheta}}{3}\right)^2}{\kappa\tau}} \\ &\geq e^{-\frac{(\vartheta-\varphi)^2}{\tau}} \\ &= \Pi_\varrho(\vartheta, \varphi, \tau).\end{aligned}$$

Therefore, all the hypotheses of Theorem 2 are satisfied. Hence Γ has a unique fixed point, i.e., $\mu = 0$.

Theorem 3. *Let $(\Lambda, \Theta, \Pi_\varrho, *)$ be a complete fuzzy triple controlled bipolar metric space with three non-comparable functions $\lambda, \alpha, \beta : \Lambda \times \Theta \to [1, \infty)$ and $\Gamma : \Lambda \cup \Theta \to \Lambda \cup \Theta$, a mapping satisfying*

(i) $\Gamma(\Lambda) \subseteq \Lambda$ *and* $\Gamma(\Theta) \subseteq \Theta$;
(ii) *For* $\vartheta \in \Lambda, \varphi \in \Theta$ *and* $\tau > 0, \Pi_\varrho(\vartheta, \varphi, \tau) > 0 \Rightarrow \Pi_\varrho(\Gamma(\vartheta), \Gamma(\varphi), \tau) \geq \Pi(\Pi_\varrho(\vartheta, \varphi, \tau))$, *where* $\Pi : (0, 1] \to (0, 1]$ *is an increasing function such that* $\lim_{\eta\to\infty} \Pi^\eta(\kappa) = 1$ *and* $\Pi(\kappa) \geq \kappa$ *for all* $\kappa \in (0, 1]$.

Then Γ has a fixed point.

Proof. Let $\vartheta_0 \in \Lambda$ and $\varphi_0 \in \Theta$ such that $\Gamma(\vartheta_\eta) = \vartheta_{\eta+1}$ and $\Gamma(\varphi_\eta) = \varphi_{\eta+1}$ for all $\eta \in \mathbb{N} \cup \{0\}$, then $(\vartheta_\eta, \varphi_\eta)$ be a bisequence on fuzzy triple controlled bipolar metric space $(\Lambda, \Theta, \Pi_\varrho, *)$. By the definition of (FCB2) for all $\tau > 0$ and condition (ii), we have

$$\Pi_\varrho(\vartheta_\eta, \varphi_\eta, \tau) \geq \Pi^\eta(\Pi_\varrho(\vartheta_0, \varphi_0, \tau)) \tag{10}$$

and

$$\Pi_\varrho(\vartheta_{\eta+1}, \varphi_\eta, \tau) \geq \Pi^\eta(\Pi_\varrho(\vartheta_1, \varphi_0, \tau)). \tag{11}$$

Let $\eta < \omega$, for $\eta, \omega \in \mathbb{N}$. Then

$$\begin{aligned}
\Pi_\varrho(\vartheta_\eta, \varphi_\omega, \tau) \geq & \Pi_\varrho(\vartheta_\eta, \varphi_\eta, \frac{\tau}{3\lambda(\vartheta_\eta, \varphi_\eta)}) \star \Pi_\varrho(\vartheta_{\eta+1}, \varphi_\eta, \frac{\tau}{3\alpha(\vartheta_{\eta+1}, \varphi_\eta)}) \\
& \star \Pi_\varrho(\vartheta_{\eta+1}, \varphi_\omega, \frac{\tau}{3\beta(\vartheta_{\eta+1}, \varphi_\omega)}) \\
& \geq \Pi_\varrho(\vartheta_\eta, \varphi_\eta, \frac{\tau}{3\lambda(\vartheta_\eta, \varphi_\eta)}) \star \Pi_\varrho(\vartheta_{\eta+1}, \varphi_\eta, \frac{\tau}{3\alpha(\vartheta_{\eta+1}, \varphi_\eta)}) \\
& \star \Pi_\varrho(\vartheta_{\eta+1}, \varphi_{\eta+1}, \frac{\tau}{3^2\beta(\vartheta_{\eta+1}, \varphi_\omega)\lambda(\vartheta_{\eta+1}, \varphi_{\eta+1})}) \\
& \star \Pi_\varrho(\vartheta_{\eta+2}, \varphi_{\eta+1}, \frac{\tau}{3^2\beta(\vartheta_{\eta+1}, \varphi_\omega)\alpha(\vartheta_{\eta+2}, \varphi_{\eta+1})}) \\
& \star \Pi_\varrho(\vartheta_{\eta+2}, \varphi_\omega, \frac{\tau}{3^2\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)}) \\
& \geq \Pi_\varrho(\vartheta_\eta, \varphi_\eta, \frac{\tau}{3\lambda(\vartheta_\eta, \varphi_\eta)}) \star \Pi_\varrho(\vartheta_{\eta+1}, \varphi_\eta, \frac{\tau}{3\alpha(\vartheta_{\eta+1}, \varphi_\eta)}) \\
& \star \Pi_\varrho(\vartheta_{\eta+1}, \varphi_{\eta+1}, \frac{\tau}{3^2\beta(\vartheta_{\eta+1}, \varphi_\omega)\lambda(\vartheta_{\eta+1}, \varphi_{\eta+1})}) \\
& \star \Pi_\varrho(\vartheta_{\eta+2}, \varphi_{\eta+1}, \frac{\tau}{3^2\beta(\vartheta_{\eta+1}, \varphi_\omega)\alpha(\vartheta_{\eta+2}, \varphi_{\eta+1})}) \\
& \star \Pi_\varrho(\vartheta_{\eta+2}, \varphi_{\eta+2}, \frac{\tau}{3^3\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)\lambda(\vartheta_{\eta+2}, \varphi_{\eta+2})}) \\
& \star \Pi_\varrho(\vartheta_{\eta+3}, \varphi_{\eta+2}, \frac{\tau}{3^3\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)\alpha(\vartheta_{\eta+3}, \varphi_{\eta+2})}) \\
& \star \Pi_\varrho(\vartheta_{\eta+3}, \varphi_\omega, \frac{\tau}{3^3\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)\beta(\vartheta_{\eta+3}, \varphi_\omega)}) \\
& \geq \Pi_\varrho(\vartheta_\eta, \varphi_\eta, \frac{\tau}{3\lambda(\vartheta_\eta, \varphi_\eta)}) \star \Pi_\varrho(\vartheta_{\eta+1}, \varphi_\eta, \frac{\tau}{3\lambda(\vartheta_{\eta+1}, \varphi_\eta)}) \star \cdots \\
& \star \Pi_\varrho(\vartheta_{\omega-1}, \varphi_{\omega-1}, \frac{\tau}{3^{\omega-1}\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)\cdots\lambda(\vartheta_{\omega-1}, \varphi_{\omega-1})}) \\
& \star \Pi_\varrho(\vartheta_\omega, \varphi_{\omega-1}, \frac{\tau}{3^{\omega-1}\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)\cdots\alpha(\vartheta_\omega, \varphi_{\omega-1})}) \\
& \star \Pi_\varrho(\vartheta_\omega, \varphi_\omega, \frac{\tau}{3^{\omega-1}\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)\cdots\beta(\vartheta_\omega, \varphi_\omega)}).
\end{aligned}$$

Now applying (10) and (11) on each term of the right-hand side of the above inequality, we obtain

$$\begin{aligned}
\Pi_\varrho(\vartheta_\eta, \varphi_\omega, \tau) \geq & \Pi^\eta(\Pi_\varrho(\vartheta_0, \varphi_0, \frac{\tau}{3\lambda(\vartheta_\eta, \varphi_\eta)})) \star \Pi^\eta(\Pi_\varrho(\vartheta_1, \varphi_0, \frac{\tau}{3\lambda(\vartheta_{\eta+1}, \varphi_\eta)})) \\
& \star \cdots\cdots \star \Pi^\eta(\Pi_\varrho(\vartheta_0, \varphi_0, \frac{\tau}{3^{\omega-1}\beta(\vartheta_{\eta+1}, \varphi_\omega)\beta(\vartheta_{\eta+2}, \varphi_\omega)\cdots\beta(\vartheta_\omega, \varphi_\omega)})).
\end{aligned}$$

As $\eta, \omega \to \infty$, we have $\Pi_\varrho(\vartheta_\eta, \varphi_\omega, \tau) \to 1$ for all $\tau > 0$. Apply the same lines of the proof of Theorem 1 here. We have, if $\mu \in \Lambda \cap \Theta$ is a unique limit of sequences, $\{\vartheta_\eta\}$ and

$\{\varphi_\eta\}$, then we have to show μ is a fixed point of Γ. Since we have $\Pi_\varrho(\vartheta_\eta, \mu, \tau) \to \tau$ for all $\tau > 0$ and $\Pi_\varrho(\vartheta_{\eta+1}, \Gamma(\mu), \tau) = \Pi_\varrho(\Gamma(\vartheta_\eta), \Gamma(\mu), \tau) \geq \Pi(\Pi_\varrho(\vartheta_\eta, \mu, \tau)) \geq \Pi_\varrho(\vartheta_\eta, \mu, \tau)$, it follows that $\vartheta_{\eta+1} \to \Gamma(\mu)$, which implies that $\Gamma(\mu) = \mu$. □

Example 5. *Let $\Lambda = \{2,4,5,6\}, \Theta = \{1,2\}, \sigma * \varrho = \sigma\varrho$ for all $\sigma, \varrho \in [0,1]$ and $\lambda, \alpha, \beta : \Lambda \times \Theta \to [1, \infty)$ be three non-comparable mappings defined as $\lambda(\vartheta, \varphi) = \vartheta + \varphi + 1, \alpha(\vartheta, \varphi) = \vartheta^2 + \varphi + 1$and $\beta(\vartheta, \varphi) = \vartheta^2 + \varphi - 1$. Define*

$$\Pi_\varrho(\vartheta, \varphi, \tau) = \frac{\min\{\vartheta, \varphi\} + \tau}{\max\{\vartheta, \varphi\} + \tau} \text{ for all } \vartheta \in \Lambda, \varphi \in \Theta \text{ and for all } \tau > 0.$$

Then $(\Lambda, \Theta, \Pi_\varrho, *)$ is a complete fuzzy triple controlled bipolar metric space. Now, define $\Pi : (0,1] \to (0,1]$ such that $\Pi(\kappa) = \sqrt{\kappa}$. Clearly, $\Pi(\kappa) = \sqrt{\kappa}$ satisfies the conditions of the Π function.

Let $\Gamma : \Lambda \cup \Theta \to \Lambda \cup \Theta$ be a mapping such that $\Gamma(2) = \Gamma(4) = \Gamma(1) = 2, \Gamma(5) = \Gamma(6) = 4$. Then all the conditions of Theorem 3 are satisfied. The fixed point of Γ is $\vartheta = 2$.

Theorem 4. *Let $(\Lambda, \Theta, \Pi_\varrho, *)$ be a complete fuzzy triple controlled bipolar metric space with three noncomparable functions $\lambda, \alpha, \beta : \Lambda \times \Theta \to [1, \infty)$ and $\Gamma : \Lambda \cup \Theta \to \Lambda \cup \Theta$, a mapping satisfying the following:*

(i) $\Gamma(\Lambda) \subseteq \Theta$ *and* $\Gamma(\Theta) \subseteq \Lambda$;

(ii) *For $\vartheta \in \Lambda, \varphi \in \Theta$ and $\tau > 0, \Pi_\varrho(\vartheta, \varphi, \tau) > 0 \Longrightarrow \Pi_\varrho(\Gamma(\varphi), \Gamma(\vartheta), \tau) \geq \Pi(\Pi_\varrho(\vartheta, \varphi, \tau))$.*

Then Γ has a fixed point.

Proof. The proof of the theorem follows along the lines of the proof of Theorem 3 and Theorem 2. □

4. Application

In this section, we study the existence and uniqueness of the solution of an integral equation as an application of Theorem 1.

Theorem 5. *Let us consider the integral equation*

$$\vartheta(\rho) = \varrho(\rho) + \int_{\Xi_1 \cup \Xi_2} \Omega(\rho, \varsigma, \vartheta(\varsigma)) d\varsigma, \ \rho \in \Xi_1 \cup \Xi_2,$$

where $\Xi_1 \cup \Xi_2$ is a Lebesgue measurable set. Suppose

(T1) $\Omega : (\Xi_1^2 \cup \Xi_2^2) \times [0, \infty) \to [0, \infty)$ *and* $b \in L^\infty(\Xi_1) \cup L^\infty(\Xi_2)$;

(T2) *There is a continuous function $\theta : \Xi_1^2 \cup \Xi_2^2 \to [0, \infty)$ and $\kappa \in (0,1)$ such that*

$$|\Omega(\rho, \varsigma, \vartheta(\varsigma)) - \Omega(\rho, \varsigma, \varphi(\varsigma))| \leq \kappa\theta(\rho, \varsigma)(|\vartheta(\rho) - \varphi(\rho)|),$$

for $\rho, \varsigma \in \Xi_1^2 \cup \Xi_2^2$,

(T3) $\sup_{\rho \in \Xi_1 \cup \Xi_2} \int_{\Xi_1 \cup \Xi_2} \theta(\rho, \varsigma) d\varsigma \leq 1$.

Then the integral equation has a unique solution in $L^\infty(\Xi_1) \cup L^\infty(\Xi_2)$.

Proof. Let $\Lambda = L^\infty(\Xi_1)$ and $\Theta = L^\infty(\Xi_2)$ be two normed linear spaces, where Ξ_1, Ξ_2 are Lebesgue measurable sets and $m(\Xi_1 \cup \Xi_2) < \infty$.
Consider $\Pi_\varrho : \Lambda \times \Theta \times (0, \infty) \to [0,1]$ by

$$\Pi_\varrho(\vartheta, \varphi, \tau) = e^{-\frac{\sup_{\rho \in \Xi_1 \cup \Xi_2} |\vartheta(\rho) - \varphi(\rho)|}{\tau}}.$$

for all $\vartheta \in \Lambda, \varphi \in \Theta$. Define $\lambda, \alpha, \beta : \Lambda \times \Theta \to [1, \infty)$ as three non-comparable mappings defined as $\lambda(\vartheta, \varphi) = \vartheta + \varphi + 1, \alpha(\vartheta, \varphi) = \vartheta^2 + \varphi + 1$ and $\beta(\vartheta, \varphi) = \vartheta^2 + \varphi - 1$. Then

$(\mathit{\Lambda}, \mathit{\Theta}, \Pi_\varrho, \star)$ is a complete fuzzy triple controlled bipolar metric space. Define a mapping $\Gamma : L^\infty(\mathit{\Xi}_1) \cup L^\infty(\mathit{\Xi}_2) \to L^\infty(\mathit{\Xi}_1) \cup L^\infty(\mathit{\Xi}_2)$ by

$$\Gamma(\vartheta(\rho)) = \varrho(\rho) + \int_{\mathit{\Xi}_1 \cup \mathit{\Xi}_2} \mathit{\Omega}(\rho, \varsigma, \vartheta(\varsigma)) d\varsigma, \ \rho \in \mathit{\Xi}_1 \cup \mathit{\Xi}_2.$$

Now, we have

$$\begin{aligned}
\Pi_\varrho(\Gamma\vartheta(\rho), \Gamma\varphi(\rho), \kappa\tau) &= e^{-\sup_{\rho\in\mathit{\Xi}_1\cup\mathit{\Xi}_2} \frac{|\Gamma\vartheta(\rho) - \Gamma\varphi(\rho)|}{\kappa\tau}} \\
&= e^{-\sup_{\rho\in\mathit{\Xi}_1\cup\mathit{\Xi}_2} \frac{|\varrho(\rho) + \int_{\mathit{\Xi}_1\cup\mathit{\Xi}_2} \mathit{\Omega}(\rho,\varsigma,\vartheta(\varsigma))d\varsigma - \varrho(\rho) - \int_{\mathit{\Xi}_1\cup\mathit{\Xi}_2} \mathit{\Omega}(\rho,\varsigma,\varphi(\varsigma))d\varsigma)|}{\kappa\tau}} \\
&= e^{-\sup_{\rho\in\mathit{\Xi}_1\cup\mathit{\Xi}_2} \frac{|\varrho(\int_{\mathit{\Xi}_1\cup\mathit{\Xi}_2} \mathit{\Omega}(\tau,\varsigma,\vartheta(\varsigma))d\varsigma - \int_{\mathit{\Xi}_1\cup\mathit{\Xi}_2} \mathit{\Omega}(\tau,\varsigma,\varphi(\varsigma))d\varsigma|}{\kappa\tau}} \\
&\geq e^{-\sup_{\rho\in\mathit{\Xi}_1\cup\mathit{\Xi}_2} \frac{\int_{\mathit{\Xi}_1\cup\mathit{\Xi}_2} |\mathit{\Omega}(\rho,\varsigma,\vartheta(\varsigma)) - \mathit{\Omega}(\rho,\varsigma,\varphi(\varsigma))| d\varsigma}{\kappa\tau}} \\
&\geq e^{-\sup_{\rho\in\mathit{\Xi}_1\cup\mathit{\Xi}_2} \frac{\int_{\mathit{\Xi}_1\cup\mathit{\Xi}_2} \kappa\theta(\rho,\varsigma)(|\vartheta(\rho) - \varphi(\rho)|) d\varsigma}{\kappa\tau}} \\
&\geq e^{-\sup_{\rho\in\mathit{\Xi}_1\cup\mathit{\Xi}_2} \frac{\int_{\mathit{\Xi}_1\cup\mathit{\Xi}_2} \kappa\theta(\rho,\varsigma)(|\vartheta(\rho) - \varphi(\rho)|) d\varsigma}{\kappa\tau}} \\
&\geq e^{-\sup_{\rho\in\mathit{\Xi}_1\cup\mathit{\Xi}_2} \frac{|\vartheta(\rho) - \varphi(\rho)|}{\tau}} \\
&= \Pi_\varrho(\vartheta, \varphi, \tau).
\end{aligned}$$

Hence, all the hypotheses of Theorem 1 are verified, and consequently, the integral equation has a unique solution. □

5. Conclusions

Motivated by the recent result of Sezen [17], we introduce the notion of fuzzy triple controlled bipolar metric space. In these spaces, we proved some fixed-point results in this framework. Our results generalize and extend some of the well-known results from the literature. In addition, as an application of our results, we show that some classes of integral equations have a unique solution. The examples that are given have the role of strengthening the obtained results.

We find it interesting to research, in future works, other conditions that would guarantee the existence of fixed points in fuzzy triple controlled bipolar metric spaces.

Author Contributions: All authors contributed equally. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The last author would like to thank Babeş-Bolyai University for the support.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Schweizer, B.; Sklar, A. Statistical metric spaces. *Pac. J. Math.* **1960**, *10*, 313–334. [CrossRef]
2. Zadeh, L. A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
3. Kramosil, I.; Michalek, J. Fuzzy metric and statistical metric spaces. *Kybernetica* **1975**, *11*, 326–334.
4. George, A.; Veeramani, P. On some results in fuzzy metric spaces. *Fuzzy Sets Syst.* **1994**, *64*, 395–399.
5. Grabiec, M. Fixed points in fuzzy metric spaces. *Fuzzy Sets Syst.* **1988**, *27*, 385–389. [CrossRef]
6. Gregori, V.; Sapena, A. On fixed point theorems in fuzzy metric spaces. *Fuzzy Sets Syst.* **2002**, *125*, 245–252. [CrossRef]
7. Mutlu, A.; Gürdal, U. Bipolar metric spaces and some fixed point theorems. *J. Nonlinear Sci. Appl.* **2016**, *9*, 5362–5373. [CrossRef]

8. Bartwal, A.; Dimri, R.C.; Prasad, G. Some fixed point theorems in fuzzy bipolar metric spaces. *J. Nonlinear Sci. Appl.* **2020**, *13*, 196–204. [CrossRef]
9. Gupta, V.; Mani, N.; Saini, A. Fixed point theorems and its applications in fuzzy metric spaces. *Conf. Pap.* **2013**, *AEMDS-2013*, 961–964.
10. Mehmood, F.; Ali, E.; Ionescu, C.; Kamran, T. Extended fuzzy b-metric spaces. *J. Math. Anal.* **2017**, *8*, 124–131.
11. Rakić, D.; Došenović, T.; Mitrović, Z. D.; de la Sen, M.; Radenović, S. Some fixed point theorems of Ćirić type in fuzzy metric spaces. *Mathematics* **2020**, *8*, 297. [CrossRef]
12. Rakić, D.; Mukheimer, A.; Došenović, T.; Mitrović, Z.D.; Radenović, S. On some new fixed point results in fuzzy b-metric spaces. *J. Inequal. Appl.* **2020**, *99*. [CrossRef]
13. Bajović, D.; Mitrović, Z. D.; Saha, M. Remark on contraction principle in cone$_{tvs}$ b-metric spaces. *J. Anal.* **2021**, *29*, 273–280. [CrossRef]
14. Kishore, G.N.V.; Agarwal, Ravi P.; Rao, B.; Srinuvasa, R.V.N.; Srinivasa, R. Caristi type contraction and common fixed point theorems in bipolar metric spaces with applications. *Fixed Point Theory Appl.* **2018**, 21. [CrossRef]
15. Mutlu, A.; Özkan, K.; Gürdal, U. Coupled fixed point theorems on bipolar metric spaces. *Eur. J. Pure Appl. Math.* **2017**, *10*, 655–667.
16. Mutlu, A.; Özkan, K.; Gürdal, U. Locally and weakly contractive principle in bipolar metric spaces. *TWMS J. App. Eng. Math.* **2020**, *10*, 379–388.
17. Sezen, M.S. Controlled fuzzy metric spaces and some related fixed point results. *Numer. Methods Partial Differ. Equ.* **2021**, *37*, 583–593. [CrossRef]