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Semiring-Valued Fuzzy Sets and F-Transform

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Abstract: The notion of a semiring-valued fuzzy set is introduced for special commutative partially pre-ordered semirings, including basic operations with these fuzzy structures. It is showed that many standard MV-algebra-valued fuzzy type structures with standard operations, such as hesitant, intuitionistic, neutrosophic or fuzzy soft sets are, for appropriate semirings, isomorphic to semiring-valued fuzzy sets with operations defined. F-transform and inverse F-transform are introduced for semiring-valued fuzzy sets and properties of these transformations are investigated. Using the transformation of MV-algebra-valued fuzzy type structures to semiring-valued fuzzy sets, the F-transforms for these fuzzy type structures is introduced. The advantage of this procedure is, among other things, that the properties of this F-transform are analogous to the properties of the classical F-transform and because these properties are proven for any semiring-valued fuzzy sets, it is not necessary to prove them for individual fuzzy type structures.

Keywords: partially pre-ordered semiring; semiring-valued fuzzy set; adjoint pair of semirings; F-transform

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1. Introduction

Shortly after the introduction of fuzzy sets and their lattice-valued variants by L. A. Zadeh, two new tendencies appear, the development of which continues to this day. The first of these tendencies consists in creating various generalizations of fuzzy sets and their lattice-valued variants, motivated mainly by the application possibilities of these new structures. In that way, new fuzzy type structures are created, which include, for example, intuitionistic fuzzy sets, neutrosophic fuzzy sets, hesitant fuzzy sets or fuzzy soft sets. For basic information about these structures and their possible applications see, e.g., [1–6] for intuitionistic fuzzy sets, refs. [7–14] for fuzzy soft sets, refs. [15–18] for hesitant fuzzy sets and [19–21] for neutrosophic sets. Some of these structures can be relatively easily approximated using classical fuzzy sets, others create completely new structures with their own theory. In addition to these "primary" new structures, their various clones, formed by mutual combinations of different structures, very often begin to appear. These new structures include, for example, intuitionistic hesitant fuzzy sets [22–24], intuitionistic fuzzy soft sets [25,26], hesitant fuzzy soft sets [27–30] and many others. In most cases, only a very minimal theory is developed for these new fuzzy type structures, but their application possibilities are very extensive, as evidenced by, among other things, the extensive citation of these structures on Google Scholar.

Simultaneously with the development of new fuzzy type structures and their applications, for classical fuzzy sets and lattice-valued fuzzy sets new methods simplifying calculations with these structures are also being developed. This trend is related, among other things, to the expansion of the use of fuzzy sets both for applications using large data sets, such as image processing, and for applications working in real time. In order to use these applications effectively, either powerful computing technology or simplification of the task is needed.

A natural response to these requirements was the introduction of a number of theoretical tools in fuzzy set theory, which deal with the transformation of a given fuzzy sets space. If we consider, for example, fuzzy sets with a lattice L as the value-set structure, most of these transformations can be characterized as a special mapping $T : L^X \to L^Y$ of fuzzy sets spaces, where X and Y are basic sets and $Y \subseteq X$. An important feature of these transformations is, in particular, that for each of these transformations T there is a so-called inverse transformation $T^* : L^Y \to L^X$, such that the composition $T^*T(f)$ is an approximation of $f \in L^X$.

One of the important transformation methods for both classical $[0,1]$-valued fuzzy sets and for fuzzy sets with complete residuated lattices as value set is the so-called F-transform method, which was defined by I. Perfilieva [31]. Fuzzy transform (F-transform, shortly) represents a method in fuzzy set theory, which is used in many applications in signal and image processing [32–34], signal compressions [35,36], numerical solutions of ordinary and partial differential equations [37–39], data analysis [40–42] and many other applications. The F-transform method represents a special transformation map based on a system of fuzzy sets defined on a given universe, which is called a *fuzzy partition*. F-transform defined by a fuzzy partition $\mathcal{A}$ significantly reduces the computational complexity of operations with fuzzy sets, because instead of fuzzy sets defined on the original set X, it allows to work with fuzzy sets on the index set of the fuzzy partition $\mathcal{A}$ and then transform the result using the inverse F-transform to the original fuzzy sets space L^X.

Typical tasks of this type are algorithms used in image processing, time series analysis, or even the solutions of differential equations with uncertainties. For many of these tasks, methods that use other fuzzy type structures, including hesitant, intuitionistic, or fuzzy soft sets, have also been used successfully. For illustration of these methods in fuzzy type structures see [43–47]. To extend the application potential of these new fuzzy type methods, it is therefore natural to address the issue of reducing the computational complexity of operations with these new fuzzy type structures. One of the possible approaches to reduce this complexity seems to be the use of the F-transform analogy for these new fuzzy type structures.

The lattice-based F-transform is defined by default for fuzzy sets with a complete residuated lattices L as value-sets. This lattice allows both the construction of F-transform and the inverse F-transform. The key assumption of these constructions is the fact that the F-transform concerns standard fuzzy sets, i.e., mappings $X \to L$. Unfortunately, many of the new fuzzy type structures cannot be easily transformed into mappings of this type. For example, if we consider fuzzy soft sets in the space (X, K), where X is the basic set and K is the set of criteria, a fuzzy soft set is a pair (E, s), where $E \subseteq K$ and $s : E \to L^X$. In that case (E, s) is not expressed as the mapping $X \to L'$, where L' is a complete residuated lattice.

This problem led, among other things, to the gradual use of values structures other than complete residuated lattices and, subsequently, the definition of transformation maps associated with these new structures. An example of these modified transformation maps are the so called Q-module transforms, where Q stands for unitale quantale [48,49]. From the algebraic point of view, this structure is a bit more general than that of a residuated lattice and it allows to express the lattice-based F-transforms with the help of two residuated homomorphisms between Q-modules. Another approach we use in our previous paper [50], where the F-transform is defined as a semimodule homomorphism of free semimodules defined over special semirings, based on the residuated lattice L.

Although the F-transform method is very successful in applications, it is surprising that no full analogy of the F-transform has yet been defined for other types of fuzzy structures, which were mentioned in the introduction. In our previous paper [51] we tried to fill this gap by introducing the concept of the F-transform for hesitant, intuitionistic and fuzzy soft sets with special lattice-valued structures. However, for the possibility of the use of this theory in applications, the theory of inverse F-transform was still missing in these structures.

Our objective is to show how some of the methods successfully used for classical $\mathcal{L}$-fuzzy sets can be universally transformed to analogical methods used in new fuzzy type structures, such as intutionistic, hesitant, neutrosophic of $\mathcal{L}$-fuzzy soft sets and their mutual combinatons. Even if these fuzzy type structures have a completely different forms than the standard mappings $X \to \mathcal{L}$ typical for $\mathcal{L}$-fuzzy sets and, therefore, the methods of classical $\mathcal{L}$-fuzzy sets cannot be applied directly to them. An integral part of this transformation must be the fact that these transformed methods applied in fuzzy type structures have, as far as possible, properties analogous to the original methods.

In this paper, we try to meet this goal for the direct and inverse F-transform methods, which are one of the most commonly used methods for $\mathcal{L}$-valued fuzzy sets, both in theory and in applications. We adapted this method to use for a large part of the new $\mathcal{L}$-fuzzy type structures, including intuitionistic, hesitant, neutrosophic or $\mathcal{L}$-fuzzy soft sets. Although a large part of the new fuzzy type structures deal with, among other things, the issue of image and signal processing and data analysis, the F-transform methods have not yet been used for these $\mathcal{L}$-fuzzy type structures, in contrast to classical fuzzy sets, where for these tasks the F-transform methods are often used. The advantage of such adapted method is its universal use in various fuzzy type structures, including the above mentioned $\mathcal{L}$-fuzzy type structures. Moreover, the properties of this adapted F-transform method in these new $\mathcal{L}$-fuzzy type structures will be analogous to the properties of the F-transform for classical $\mathcal{L}$-fuzzy sets and it will not be necessary to prove them separately for individual fuzzy type structures. Hence, it seems natural that this adapted F-transform method can be used for improving the present applications of new $\mathcal{L}$-fuzzy type structures in image and signal processing and data analysis.

This method is based on a simple principle: we transform each of the mentioned $\mathcal{L}$-valued fuzzy type structures in a set X into mappings $X \to \mathcal{R}$, where $\mathcal{R}$ is a suitable partially ordered semiring. Moreover, we suppose that for the ring $\mathcal{R}$ there exists another ring $\mathcal{R}^*$ with the same underlying set R and with non-trivial involutorial isomorphism $\Phi : \mathcal{R} \to \mathcal{R}^*$. In that case we say that a new $\mathcal{L}$-fuzzy type structure is transformable to $\mathcal{R}$-fuzzy set

The F-transform for these $\mathcal{R}$-fuzzy sets is then the special mapping $\mathcal{R}^X \to \mathcal{R}^Y$ from the set of all $\mathcal{R}$-fuzzy sets in X to the set of all $\mathcal{R}$-fuzzy sets in Y and it can be defined as a formal transcription of the classical $\mathcal{L}$-valued F-transform formulas using operations of semirings $\mathcal{R}$ and $\mathcal{R}^*$ instead of lattice operations. The advantage of this procedure is also that it allows to introduce the concept of the inverse F-transform for fuzzy type structures and thus expand the application possibilities of these fuzzy type structures.

In summary, the aim of our paper is as follows:

- To introduce the notion of $\mathcal{R}$-fuzzy sets and to show that a significant part of $\mathcal{L}$-fuzzy type structures, where $\mathcal{L}$ is the complete MV-algebra, can be transformed into $\mathcal{R}$-fuzzy sets,
- To show that for $\mathcal{R}$-fuzzy sets it is possible to define analogies of concepts and transformations with analogous properties known from the classical $\mathcal{L}$-fuzzy sets,
- To show that these new concepts and transformations for $\mathcal{R}$-fuzzy sets can be transformed back into concepts and transformations of the original $\mathcal{L}$-fuzzy type structures.

The content of the paper is as follows. After the introductory section, where we repeat some basic definitions from the theory of MV-algebras and pre-ordered semirings, in Section 3.1 we introduce the notion of the $\mathcal{R}$-fuzzy set, where $\mathcal{R}$ is a commutative *po*-semiring. This notion will be based on the notion of the adjoint pair $(\mathcal{R}, \mathcal{R}^*)$ of *po*-semirings with the same underlying set R and such that there exists an involutorial isomorphism $\Phi : \mathcal{R} \to \mathcal{R}^*$ of *po*-semirings. We also show that for any MV-algebra $\mathcal{L}$, the above $\mathcal{L}$-valued fuzzy type structures such as hesitant fuzzy sets, intuitionistic fuzzy sets, neutrosophic fuzzy sets and soft fuzzy sets and their mutual combinations can be transformed to $\mathcal{R}$-fuzzy sets, where $\mathcal{R}$ are suitable *po*-semirings. On the set $\mathcal{R}^X$ of $\mathcal{R}$-fuzzy sets we defined basic operations with $\mathcal{R}$-fuzzy sets and we prove some basic properties of these operations. We also prove that the above mentioned $\mathcal{L}$-fuzzy type structures in a set X with special operations defined for these structures are isomorphic to $\mathcal{R}^X$ with defined operations.

In Section 3.2 we deal with the F-transform theory for $\mathcal{R}$-fuzzy sets. It should be emphasised that the investigation of applications of this theory is not the primary goal of the paper. Our primary goal is to show how the F-transform methods can be extend to $\mathcal{R}$-fuzzy sets and translated into the language of the respective $\mathcal{L}$-fuzzy type structures. We introduce the notions of upper and lower F-transform and upper and lower inverse F-transform for $\mathcal{R}$-fuzzy sets and using the transformation of $\mathcal{L}$-fuzzy type structures to $\mathcal{R}$-fuzzy sets, we show how the F-transform can be defined in these fuzzy type structures. We also investigate some properties of F-transforms for $\mathcal{R}$-fuzzy sets and relationships between direct an inverse F-transforms.

2. Methods and Basic Structures

A basic membership structure of fuzzy sets for lattice-valued F-transform is a *complete residuated lattice* (see e.g., [52]), i.e., a structure $\mathcal{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0_L, 1_L)$ such that $(L, \wedge, \vee)$ is a complete lattice, $(L, \otimes, 1_L)$ is a commutative monoid with operation $\otimes$ isotone in both arguments and $\rightarrow$ is a binary operation which is residuated with respect to $\otimes$. Recall that a negation of an element a in $\mathcal{L}$ is defined by $\neg a = a \rightarrow 0_L$. By the order relation $\leq$ on $\mathcal{L}$ we understand the order relation of the lattice $(L, \wedge, \vee)$.

For our purposes in the paper we use a special variant of residuated lattice, namely, the MV-algebra [53], i.e., the structure $\mathcal{L} = (L, \oplus, \otimes, \neg, 0_L, 1_L)$ satisfying the following axioms for elements of L:

(i) $(L, \otimes, 1_L)$ is a commutative monoid,
(ii) $(L, \oplus, 0_L)$ is a commutative monoid,
(iii) $\neg\neg x = x$, $\neg 0_L = 1_L$,
(iv) $x \oplus 1_L = 1_L$, $x \oplus 0_L = x$, $x \otimes 0_L = 0_L$,
(v) $x \oplus \neg x = 1_L, x \otimes \neg x = 0_L$,
(vi) $\neg(x \oplus y) = \neg x \otimes \neg y$, $\neg(x \otimes y) = \neg x \oplus \neg y$,
(vii) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

If we put

$$x \vee y = (x \oplus \neg y) \otimes y, \quad x \wedge y = (x \otimes \neg y) \oplus y,$$
$$x \rightarrow y = \neg x \oplus y,$$

then $(L, \wedge, \vee, \otimes, \rightarrow, 0_L, 1_L)$ is a residuated lattice. MV-algebra is called *complete*, if that lattice is a complete lattice. The standard example of the MV-algebra is *Łukasiewicz algebra* $\mathcal{L}_L = ([0,1], \oplus, \otimes, \neg, 0, 1)$, where

$$x \otimes y = 0 \vee (x + y - 1), \quad \neg x = 1 - x,$$
$$x \oplus y = 1 \wedge (x + y).$$

In the rest of the paper, $\mathcal{L}$ is the complete MV-algebra. The $\mathcal{L}$-fuzzy set in a set X is a map $f : X \rightarrow L$. The set of all $\mathcal{L}$-fuzzy sets in X is denoted by $\mathcal{L}^X$.

We recall a basic definition of the F-transform and inverse F-transform for $\mathcal{L}$-fuzzy sets.

Definition 1 ([31]). *Let X be a set and let $\mathcal{A} = \{A_y : y \in Y\} \subseteq \mathcal{L}^X$. Then*

1. *$\mathcal{A}$ is called a fuzzy partition, if $\{core(A_y) : y \in Y\}$ is a partition of X, where $core(A) = \{x \in X : A(x) = 1_L\}$, i.e., $\bigcup_{y \in Y} core(A_y) = X$, $core(A_y) \cap core(A_z) = \emptyset$, if $y \neq z$.*
2. *A mapping $F_{X,\mathcal{A}} : \mathcal{L}^X \rightarrow \mathcal{L}^Y$ is called the upper F-transform based on $\mathcal{A}$, if for $s \in \mathcal{L}^X, y \in Y$, $F_{X,\mathcal{A}}(s)(y) = \bigvee_{x \in X} s(x) \otimes A_y(x)$.*
3. *A mapping $G_{X,\mathcal{A}} : \mathcal{L}^Y \rightarrow \mathcal{L}^X$ is called the inverse upper F-transform based on $\mathcal{A}$, if for $g \in \mathcal{L}^Y, x \in X$, $G_{X,\mathcal{A}}(g)(x) = \bigwedge_{y \in Y} \neg A_y(x) \oplus g(y)$.*

In our previous paper [51] we have shown that fuzzy type transformations for various fuzzy type structures can be equivalently defined using two different tools, based on the theory of monads and monadic relations and on the theory of semirings and semimodules. In this paper, we extend this method based on the theory of semirings so that it can be used to define the inverse F-transform, and special examples of this extended structure should be the fuzzy type of structure mentioned in the introduction, i.e., hesitant, intuitionistic or soft fuzzy sets.

The semiring appears for the first time in [54] and this notion was elaborated in [55]. For our purposes, we need to use semirings in which a partially order or pre-order relation is defined. This notion of a partially ordered semiring was first introduced in the [56]. For more information about semimodules and their applications see, e.g., [57,58].

Definition 2 ([54,56])**.** *A partially pre-ordered (or ordered) idempotent commutative semiring* $\mathcal{R} = (R, \leq, +, \times, 0_R, 1_R)$ *(or, shortly, po-semiring) is an algebraic structure with the following properties:*

(i) $(R, +, 0_R)$ *is an idempotent commutative monoid,*
(ii) $(R, \times, 1_R)$ *is a commutative monoid,*
(iii) $x \times (y + z) = x \times y + x \times z$ *holds for all* $x, y, z \in R$*,*
(iv) $0_R \times x = 0_R$ *holds for all* $x \in R$*.*
(v) $(R, \leq)$ *is a partially pre-ordered (or ordered) set such that for all* $a, b, c \in R$ *the following hold*

$$a \leq b \Rightarrow a +_R c \leq b +_R c, \quad a \times_R c \leq b \times_R c,$$
$$a \geq 0_R.$$

If a structure $\mathcal{R}$ satisfies only axioms (i)–(iv), then $\mathcal{R}$ is called only the *semiring*. An important example of a *po*-semiring which seems to be very useful for the F-transform theory was published in the paper of Di Nola and Gerla [59].

Example 1 ([59])**.**

(1) *Let* $\mathcal{L}$ *be a residuates lattice. Then the reduct* $\mathcal{L}^{\vee} = (L, \leq, \vee, \otimes, 0_L, 1_L)$ *is the po-semiring.*
(2) *Let* $\mathcal{L}$ *be a MV-algebra. Then the reduct* $\mathcal{L}^{\wedge} = (L, \leq, \wedge, \oplus, 1_L, 0_L)$ *is the po-semiring.*

The notion of a semimodule over a semiring is taken from [55]. We use the commutative version of this notion only. Moreover, analogously as for semirings, we need to use the notion of a partially pre-ordered (or ordered) semimodule which is introduced in [60].

Definition 3 ([55])**.** *Let* $\mathcal{R} = (R, \leq_R, +_R, \times_R, 0_R, 1_R)$ *be a po-semiring. A partially pre-ordered* $\mathcal{R}$*-semimodule (or, shortly, po-*$\mathcal{R}$*-semimodule) is a structure* $\mathcal{M} = (M, \leq, \boxplus, \star, 0)$ *defined by the following axioms:*

1. $(M, \boxplus, 0)$ *is a commutative monoid,*
2. $\star : M \times R \to M$ *is a mapping (called an external multiplication),*
3. $r, r' \in R, m \in M,\ (r \times_R r') \star m = r \star (r' \star m),$
4. $r \in R, m, m' \in M,\ r \star (m \boxplus m') = r \star m \boxplus r \star m',$
5. $r, r' \in R, m \in M,\ (r +_R r') \star m = r \star m \boxplus r' \star m,$
6. $1_R \star m = m, 0_R \star m = r \star 0 = 0,$
7. $m, n, p \in M, m \leq n \Rightarrow m \boxplus p \leq n \boxplus p,$
8. $m, n \in M, r \in R, r \geq_R 0_R, m \leq n \Rightarrow r \star m \leq r \star n,$
9. $r, s \in R, m \in M, r \leq_R s \Rightarrow r \star m \leq s \star m,$

If the structure $\mathcal{M}$ satisfies only axioms *1.–6.*, it is called a $\mathcal{R}$*-semimodule*. If there can be no misunderstanding, for simplicity, we will sometimes use only the term semiring and semimodule instead of a *po*-semiring and a *po*-semimodule, respectively.

If a semiring $\mathcal{R}$ and $\mathcal{R}$-semimodule $\mathcal{M} = (M, \boxplus_M, 0_M)$ are such that for any subsets $S \subseteq R$ and $N \subseteq M$, there exist sums of elements $r \in S$ and $x \in N$, then $\mathcal{M}$ is called a *complete* $\mathcal{R}$-semimodule. The sum of elements $x \in N$ is denoted by $\boxplus^{\mathcal{M}}_{x \in N} x$ and the sum of elements $r \in S$ is denoted by $\sum^{\mathcal{R}}_{r \in S} r$.

In the paper [61] the following examples of *po*-semimodules were presented.

Example 2 ([61]).

(1) *Let $X \neq \emptyset$, $\mathcal{L}$ be a complete residuated lattice and let $\mathcal{L}^{\vee} = (L, \leq, \vee, \otimes, 0_L, 1_L)$ be the po-semiring from Example 1. For all $f, g \in M = L^X$ define*

$$f \leq g \Leftrightarrow \forall x \in X, f(x) \leq g(x) \text{ in } \mathcal{L}^{\vee},$$
$$(f \oplus_M g)(x) = f(x) \vee g(x),$$
$$p \star_1 f(x) = p \otimes f(x),$$
$$0_M \in M, \quad 0_M(x) = 0_L, \quad x \in X, p \in L.$$

Then $\mathcal{L}^X = (M, \leq, \oplus_M, \star_1, 0_M)$ is the complete po-$\mathcal{L}^{\vee}$-semimodule.

(2) *Let $X \neq \emptyset$, $\mathcal{L}$ be a complete MV-algebra and let $\mathcal{L}^{\wedge} = (L, \leq, \wedge, \oplus, 1_L, 0_L)$ be the po-semiring from Example 1. For all $f, g \in M = L^X$ define*

$$f \leq g \Leftrightarrow \forall x \in X, f(x) \geq g(x) \text{ in } \mathcal{L}^{\wedge},$$
$$(f \oplus_M g)(x) = f(x) \wedge g(x),$$
$$p \star_2 f(x) = p \oplus f(x),$$
$$0_M \in M, \quad 0_M(x) = 1_L, \quad x \in X, p \in L.$$

Then $\mathcal{L}_X = (M, \leq, \oplus_M, \star_2, 0_M)$ is the complete po-$\mathcal{L}^{\wedge}$-semimodule.

The notion of a semiring homomorphism and $\mathcal{R}$-semimodule homomorphism is defined standardly as follows from the following definition.

Definition 4. *Let $\mathcal{R}$ and $\mathcal{S}$ be po-semirings, $\mathcal{M}$ and $\mathcal{N}$ be po-$\mathcal{R}$-semimodules.*

1. *A po-semiring homomorphism $\Phi : \mathcal{R} \to \mathcal{S}$ is a mapping $\Phi : R \to S$ such that*
   (a) *Φ is a homomorphism of semirings,*
   (b) *Φ is order-preserving.*
2. *A $\mathcal{R}$-semimodule homomorphism $\Psi : \mathcal{M} \to \mathcal{N}$ is a mapping $\Psi : M \to N$ such that*
   (a) *$\Psi : \mathcal{M} \to \mathcal{N}$ is an order preserving homomorphism of monoids,*
   (b) *$\Psi(r \star_M m) = r \star_N \Psi(m)$, for all $m \in M, r \in R$,*

Let us consider the following example, which is very important for our purposes.

Example 3. *Let $\mathcal{L}^X = (M, \leq, \otimes_M, 0_M)$ be the po-$\mathcal{L}^{\vee}$-semimodule from Example 2 (1), and let $\mathcal{L}_X = (M, \leq, \oplus_M, 0_M)$ be the po-$\mathcal{L}^{\wedge}$-semimodule from Example 2 (2). Let $(G, \Phi) : \mathcal{L}^X \to \mathcal{L}_X$ be defined by*

$$\Phi : \mathcal{L}^{\vee} \to \mathcal{L}^{\wedge}, \quad \forall \alpha \in L, \Phi(\alpha) = \neg\alpha,$$
$$G : L^X \to L^X, \quad \forall f \in L^X, G(f) = \neg f, \quad (\neg f)(x) = \Phi(f(x)).$$

Then, (G, Φ) is the $(\mathcal{L}^{\vee}, \mathcal{L}^{\wedge})$-semimodule homomorphism.

3. Results

3.1. $\mathcal{R}$-Valued Fuzzy Sets

As we mentioned in the introduction, in order to be able to use analogies of constructions and methods that are standardly used in classical $\mathcal{L}$-fuzzy sets for new $\mathcal{L}$-valued fuzzy type structures, we will transform these $\mathcal{L}$-valued fuzzy type structures in a set X into mappings $X \to \mathcal{R}$, where $\mathcal{R}$ is a suitable *po*-semiring. The specificity of these *po*-semirings, which will be used as value sets of these new $\mathcal{L}$-fuzzy type structures, lies in the fact that instead of one *po*-semiring $\mathcal{R}$ we will use a pair of *po*-semirings $(\mathcal{R}, \mathcal{R}^*)$ with the same underlying sets, which are adjoint in a specific way. The value sets defined in that way for new fuzzy type structures will allow us not only to transform them into $X \to \mathcal{R}$ mappings, but also to introduce operations on a set of $\mathcal{R}$-fuzzy sets, analogous to existing operations on $\mathcal{L}$-valued fuzzy type structures.

We begin this section with the definition of the pair of adjoint *po*-semirings.

Definition 5. *Let $\mathcal{R} = (R, \leq, +, \times, 0, 1)$ and $\mathcal{R}^* = (R, \leq^*, +^*, \times^*, 0^*, 1^*)$ be complete po-semirings with the same underlying set R. The po-semiring isomorphism $\Phi : \mathcal{R} \to \mathcal{R}^*$ is called adjoint and the pair $(\mathcal{R}, \mathcal{R}^*)$ is called the adjoint pair of semirings, if*

1. *Φ is an order-preserving isomorphism of po-semirings,*
2. *Φ is self-inverse, i.e., $\Phi.\Phi = id_R$,*
3. *$\forall a, b \in \mathcal{R}$, $a \leq b \Leftrightarrow a \geq^* b$,*
4. *$\forall a, b, c \in \mathcal{R},\ a \times^* (b + c) = (a \times^* b) + (a \times^* c)$,*
5. *$\forall a, b, c \in \mathcal{R},\ a + (b +^* c) = (a + b) +^* (a + c)$,*

Remark 1.

1. *In the rest of the paper, if $(\mathcal{R}, \mathcal{R}^*)$ will be the adjoint pair of semirings with the adjoint isomorphism Φ, then $\mathcal{R}$ and $\mathcal{R}^*$ are supposed to be complete po-semirings with the same operations as in Definition 5.*
2. *It should be observed that the following statements dual to statements from Definition 5 also holds:*
   - *4′.* $\forall a, b, c \in \mathcal{R},\ a \times (b +^* c) = (a \times b) +^* (a \times c)$,
   - *5′.* $\forall a, b, c \in \mathcal{R},\ a +^* (b + c) = (a +^* b) + (a +^* c)$.

In the next examples we show some non-trivial examples of adjoint pairs of *po*-semirings which, as we will see later, are closely related to mentioned $\mathcal{L}$-fuzzy type structures. All these examples are based on the complete *MV*-algebra $\mathcal{L} = (L, \otimes, \oplus, \neg, 0, 1)$ with sup $\vee$ and inf $\wedge$ defined by these operations..

Example 4. *Let $\mathcal{L}$ be the MV-algebra and let us consider the semirings $\mathcal{L}^\vee$ and $\mathcal{L}^\wedge$ from Example 1. Then $(\mathcal{L}^\vee, \mathcal{L}^\wedge)$ is the adjoint pair of po-semirings and $\Phi : \mathcal{L}^\vee \to \mathcal{L}^\wedge$ is the adjoint po-semiring isomorphism, where*

$$\Phi : \mathcal{L}^\vee \to \mathcal{L}^\wedge, \quad \Phi(\alpha) = \neg\alpha,$$

Example 5.

1. *The partially pre-ordered semiring $\mathcal{R}_1 = (R_1, \leq_1, +_1, \times_1, 0_1, 1_1)$ is defined by*
   - *(a)* $R_1 = \{A : A \subseteq L\} = 2^L$,
   - *(b)* $A, B \in R_1,\ A +_1 B = A \cup B$,
   - *(c)* $A, B \in R_1,\ A \times_1 B := A \otimes B = \{a \otimes b : a \in A, b \in B\},\ A \times_1 \emptyset = \emptyset$,
   - *(d)* $0_1 = \emptyset, 1_1 = \{1_L\}$,
   - *(e)* $A, B \in R_1$, *we set*

   $$A \leq_1 B \Leftrightarrow (\forall\alpha \in A)(\exists\beta \in B)\alpha \leq \beta.$$

2. *The partially pre-ordered semiring $\mathcal{R}_1^* = (R_1, \leq_1^*, +_1^*, \times_1^*, 0_1^*, 1_1^*)$ is defined by*
   - *(a)* $A, B \in R_1$, $A +_1^* B = A \cap B$,
   - *(b)* *For* $A, B \in R_1$, $A, B \neq L$, $A \times_1^* B := A \oplus B = \{a \oplus b : a \in A, b \in B\}$, $A \times_1^* L = L \times_1^* L = L$,
   - *(c)* $0_1^* = L,\ 1_1^* = \{0_L\}$.
   - *(d)* $A \leq_1^* B \Leftrightarrow B \leq_1 A$.

   Let $\Phi : \mathcal{R}_1 \to \mathcal{R}_1^*$ be defined by

$$A \in R_1, A \neq \emptyset, L, \quad \Phi(A) = \{\neg\alpha : \alpha \in A\},$$
$$\Phi(\emptyset) = L, \quad \Phi(L) = \emptyset.$$

Then $(\mathcal{R}_1, \mathcal{R}_1^*)$ is the adjoint pair of po-semirings and Φ is the adjoint po-semiring isomorphism.

Example 6.

1. *The po-semiring $\mathcal{R}_2 = (R_2, \leq_2, +_2, \times_2, 0_2, 1_2)$ is defined by*
   (a) $R_2 = \{(\alpha, \beta) \in L^2 : \neg\alpha \geq \beta\} \subseteq L^2$,
   (b) $(\alpha, \beta) +_2 (\alpha_1, \beta_1) := (\alpha \vee \alpha_1, \beta \wedge \beta_1)$,
   (c) $(\alpha, \beta) \times_2 (\alpha_1, \beta_1) := (\alpha \otimes \alpha_1, \beta \oplus \beta_1)$,
   (d) $0_2 = (0_L, 1_L)$, $1_2 = (1_L, 0_L)$,
   (e) $(\alpha, \beta) \leq_2 (\alpha', \beta') \Leftrightarrow \alpha \leq \alpha', \beta \geq \beta'$.
2. *The po-semiring $\mathcal{R}_2^* = (R_2, \leq_2^*, +_2^*, \times_2^*, 0_2^*, 1_2^*)$ is defined by*
   (a) $(\alpha, \beta) +_2^* (\alpha_1, \beta_1) := (\alpha \wedge \alpha_1, \beta \vee \beta_1)$,
   (b) $(\alpha, \beta) \times_2^* (\alpha_1, \beta_1) := (\alpha \oplus \alpha_1, \beta \otimes \beta_1)$,
   (c) $0_2^* = (1_L, 0_L)$, $1_2^* = (0_L, 1_L)$,
   (d) $(\alpha, \beta) \leq_2^* (\alpha', \beta') \Leftrightarrow (\alpha, \beta) \geq_2 (\alpha', \beta')$.

   Let $\Phi : \mathcal{R}_2 \to \mathcal{R}_2^*$ be defined by

$$(\alpha, \beta) \in R_2, \quad \Phi(\alpha, \beta) = (\beta, \alpha).$$

Then $(\mathcal{R}_2, \mathcal{R}_2^*)$ is the adjoint pair of po-semirings and Φ is the adjoint po-semiring isomorphism.

Example 7.

1. *Let K be the fixed set of criteria. The po-semiring $\mathcal{R}_3 = (R_3, \leq_3, +_3, \times_3, 0_3, 1_3)$ is defined by*
   (a) $R_3 = \{(E, \psi) : E \subseteq K, \psi \in L^K\} \subseteq L^K$, *where $(E, \psi) \in L^K$ is defined by*

$$k \in K, \quad (E, \psi)(k) = \begin{cases} \psi(k), & k \in E, \\ 0_L, & k \notin E \end{cases}.$$

   (b) $(E, \varphi), (F, \psi) \in R_3$, $(E, \varphi) +_3 (F, \psi) := (E \cap F, \varphi \vee \psi)$, *where $\varphi \vee \psi$ is the supremum in L^K,*
   (c) $(E, \varphi), (F, \psi) \in R_3$, $(E, \varphi) \times_3 (F, \psi) = (E \cap F, \varphi \times \psi)$, *where $\varphi \times \psi \in L^K$ is defined by $\varphi \times \psi(k) = \varphi(k) \otimes \psi(k)$,*
   (d) $0_3 = (K, \underline{0}_L)$, $1_3 = (K, \underline{1}_L)$, *where $\underline{\alpha}(k) = \alpha$ for arbitrary $k \in K$, $\alpha \in L$,*
   (e) $(E, \varphi) \leq_3 (F, \psi) \Leftrightarrow (E, \varphi)(k) \leq (F, \psi)(k), \forall k \in E \cap F$.
2. *The po-semiring $\mathcal{R}_3^* = (R_3, \leq_3^*, +_3^*, \times_3^*, 0_3^*, 1_3^*)$ is defined by*
   (a) $(E, \varphi), (F, \psi) \in R_3$, $(E, \varphi) +_3^* (F, \psi) := (E \cap F, \varphi \wedge \psi)$, *where $\varphi \wedge \psi$ is the infimum in L^K,*
   (b) $(E, \varphi), (F, \psi) \in R_3$, $(E, \varphi) \times_3^* (F, \psi) = (E \cap F, \varphi \oplus \psi)$, *where $\oplus$ in L^K is defined component-wise.*
   (c) $0_3^* = (K, \underline{1}_L)$, $1_3^* = (K, \underline{0}_L)$, *where $\underline{\alpha}(k) = \alpha$ for arbitrary $k \in K$, $\alpha \in L$,*
   (d) $(E, \varphi) \leq_3^* (F, \psi) \Leftrightarrow (E, \varphi) \geq_3 (F, \psi)$.

   Let $\Phi : \mathcal{R}_3 \to \mathcal{R}_3^*$ be defined by

$$(E, \psi) \in R_3, \quad \Phi(E, \psi) = (E, \neg\psi),$$

where $\neg\psi$ is defined component-wise. Then $(\mathcal{R}_3, \mathcal{R}_3^*)$ is the adjoint pair of po-semirings and Φ is the adjoint po-semiring isomorphism.

Example 8.

1. *The po-semiring $\mathcal{R}_4 = (R_4, \leq_4, +_4, \times_4, 0_4, 1_0)$ is defined by*
   (a) $R_4 = L^3$,
   (b) $(\alpha, \beta, \gamma) +_4 (\alpha_1, \beta_1, \gamma_1) := (\alpha \vee \alpha_1, \beta \wedge \beta_1, \gamma \wedge \gamma_1)$,

   (c) $(\alpha,\beta,\gamma)\times_2(\alpha_1,\beta_1,\gamma_1) := (\alpha\otimes\alpha_1,\beta\wedge\beta_1,\gamma\oplus\gamma_1)$,
   (d) $0_4 = (0_L,1_L,1_L),\quad 1_4 = (1_L,1_L,0_L)$,
   (e) $(\alpha,\beta,\gamma)\leq_4(\alpha',\beta',\gamma') \Leftrightarrow \alpha\leq\alpha',\beta\geq\beta',\gamma\geq\gamma'$.
2. *The po-semiring $\mathcal{R}_4^* = (R_4,\leq_4^*,+_4^*,\times_4^*,0_4^*,1_4^*)$ is defined by*
   (a) $(\alpha,\beta,\gamma)+_4^*(\alpha_1,\beta_1,\gamma_1) := (\alpha\wedge\alpha_1,\beta\vee\beta_1,\gamma\vee\gamma_1)$,
   (b) $(\alpha,\beta,\gamma)\times_4^*(\alpha_1,\beta_1,\gamma_1) := (\alpha\oplus\alpha_1,\beta\vee\beta_1,\gamma\otimes\gamma_1)$,
   (c) $0_4^* = (1_L,0_L,0_L),\quad 1_4^* = (0_L,0_L,1_L)$,
   (d) $(\alpha,\beta,\gamma)\leq_4^*(\alpha',\beta',\gamma') \Leftrightarrow (\alpha,\beta,\gamma)\geq_4(\alpha',\beta',\gamma')$.

   Let $\Phi:\mathcal{R}_4\to\mathcal{R}_4^*$ be defined by

$$(\alpha,\beta,\gamma)\in R_4,\quad \Phi(\alpha,\beta,\gamma) = (\gamma,\neg\beta,\alpha).$$

   Then $(\mathcal{R}_4,\mathcal{R}_4^*)$ is the adjoint pair of po-semirings and Φ is the adjoint po-semiring isomorphism.

It is clear from the above examples that the pair $(\mathcal{R},\mathcal{R}^*)$ represents a certain generalization of the pair $(\mathcal{L}^\vee,\mathcal{L}^\wedge)$, where $(L,\oplus,\otimes,\neg,0,1)$ is the MV-algebra. It would therefore be possible to expect that just as the original MV-algebra $\mathcal{L}$ can be derived from the pair $(\mathcal{L}^\vee,\mathcal{L}^\wedge)$ by operations $\oplus$, $\otimes$, $\neg$, 0 and 1, another MV-algebra $\mathcal{L}'$ can be analogously derived from the pair $(\mathcal{R},\mathcal{R}^*)$, i.e., $\mathcal{L}' = (R,\times,\times^*,\Phi,0_R,1_R)$. Unfortunately, it is not true, as the following example shows.

Example 9. *Let $(\mathcal{R}_1,\mathcal{R}_1^*)$ be adjoint pair of po-semirings with adjoint isomorphism Φ from Example 5 and let $\mathcal{L}$ be the Łukasiewicz algera. Then $(R_1,\times_1,\times_1^*,\Phi,0_1,1_1)$ is not an MV-algebra. In fact, it is easy to see that, in general, we have*

$$A\times_1\Phi(A)\neq\{1_L\}=1_1,\quad A\times_1^*\Phi(A)\neq 0_1,\quad \Phi(0_1)\neq 1_1,$$
$$A\times_1^*(\Phi(A)\times_1 B)\not\geq_1 B,$$

and it follows that R_1 is not a MV-algebra. In fact, to prove the first row is trivial. Let $A=\{0.5,0.4\}, B=\{0.6\}$. Then

$$A\times_1^*(\Phi(A)\times_1 B) = \{0.5;0.6,0.7\}\not\geq_1\{0.6\} = B,$$

as follows from the definition of $\leq_1$, because $0.5\not\geq 0.6$.

Using the adjoint pair $(\mathcal{R},\mathcal{R}^*)$ of *po*-semirings we can now define the notion of $\mathcal{R}$-fuzzy sets and we can also introduce basic operations with $\mathcal{R}$-fuzzy sets. Finally, using this definition, we show that mentioned $\mathcal{L}$-fuzzy type structures, such as hesitant, intuitionists, neutrosophic or soft $\mathcal{L}$-fuzzy sets and also their mutual combinations are $\mathcal{R}$-fuzzy sets where operations with these $\mathcal{R}$-fuzzy sets are identical to operations defined on these fuzzy type structures.

Definition 6. *Let $(\mathcal{R},\mathcal{R}^*)$ be adjoint pair of po-semirings with adjoint isomorphism Φ. Let X be a set.*

1. *A mapping $s:X\to\mathcal{R}$ is called the $\mathcal{R}$-fuzzy set in X. For $x\in X$, $s(x)\in\mathcal{R}$ is called the $\mathcal{R}$-membership value of s in x.*
2. *The operations with $\mathcal{R}$-fuzzy sets s,t in X and elements $a\in\mathcal{R}$ are defined by*
   (a) *The intersection $s\sqcap t$ is defined by $(s\sqcap t)(x) = s(x)+^*t(x)$, $x\in X$,*
   (b) *The union $s\sqcup t$ is defined by $(s\sqcup t)(x) = s(x)+t(x)$, $x\in X$,*
   (c) *The complement $\neg s$ is defined by $\neg s(x) = \Phi(s(x))$,*
   (d) *The external multiplication $\star$ by elements from $\mathcal{R}$ is defined by* $(a\star s)(x) = a\times s(x)$,

(e) *The pre-order relation $\subseteq$ between $\mathcal{R}$-fuzzy sets is defined by*

$$s \subseteq t \Leftrightarrow (\forall x \in X) s(x) \leq t(x).$$

3. *For arbitrary $x \in X$, by $\eta_X(x)$ we denote the $\mathcal{R}$-fuzzy set in X such that*

$$y \in X, \quad \eta_X(x)(y) = \begin{cases} 1_{\mathcal{R}}, & x = y, \\ 0_{\mathcal{R}}, & otherwise. \end{cases}$$

In the following simple lemma we show some basic properties of operations with $\mathcal{R}$-fuzzy sets. On the other hand, because $\mathcal{R}$-fuzzy sets represent a relatively strong generalization of standard $\mathcal{L}$-fuzzy sets, it cannot be expected that the same properties of operations will apply to them as for classical $\mathcal{L}$-fuzzy sets. We will show these differences in the following examples.

Lemma 1. *Let $(\mathcal{R}, \mathcal{R}^*)$ be adjoint pair of po-semirings with adjoint isomorphism Φ. Let X be a set and s, t, w be $\mathcal{R}$-fuzzy sets in X. Then the following statements hold.*

1. $s \sqcap s = s, s \sqcup s = s$,
2. $s \sqcap t \subseteq s$,
3. $s \sqcap (t \sqcup w) = (s \sqcap t) \sqcup (s \sqcap w)$,
4. $s \sqcup (t \sqcap w) = (s \sqcup t) \sqcap (s \sqcup w)$,
5. $a \star (t \sqcup w) = (a \star t) \sqcup (a \star w)$,
6. $a \star (t \sqcap w) = (a \star t) \sqcap (a \star w)$,
7. $\neg(s \sqcup t) = \neg s \sqcap \neg t, \neg(s \sqcap t) = \neg s \sqcup \neg t$,
8. $s \subseteq t \Rightarrow s \sqcup w \subseteq t \sqcup w, \quad s \sqcap w \subseteq t \sqcap w$,

Proof. For illustration we show only the proof of 2, 4 and 7. The rest of the proof can be done analogously. According to Definition 5, for $x \in X$ we have $(s \sqcap t)(x) = s(x) +^* t(x) \geq^* s(x) +^* 0^* = s(x)$. Therefore, $s(x) +^* t(x) \leq s(x)$ and $s \sqcap t \subseteq s$. Further,

$$\neg(s \sqcup t)(x) = \Phi((s \sqcup t)(x)) = \Phi(s(x) + t(x)) = \\ \Phi(s(x)) +^* \Phi(t(x)) = (\neg s \sqcap \neg t)(x).$$

Finally, we have

$$(s \sqcup (t \sqcap w))(x) = s(x) + (t(x) +^* w(x)) = (s(x) + t(x)) +^* (s(x) + w(x)) = \\ ((s \sqcup t) \sqcap (s \sqcup w))(x).$$

□

From the definition of $\mathcal{R}$-fuzzy sets in a set X it follows that the set of all $\mathcal{R}$-fuzzy sets in a set X is the free $\mathcal{R}$-semimodule $\mathcal{R}^X$. For convenience of readers we recall the definition of this structure.

Definition 7. *Let $\mathcal{R} = (R, \leq_R, +_R, \times_R, 0_R, 1_R)$ be a po-semiring. By the free $\mathcal{R}$-semimodule over a set X we understand the po-$\mathcal{R}$-semimodule $\mathcal{R}^X = (R^X, \leq, \boxplus, 0, \star)$ defined by*

1. *For arbitrary $f, g \in R^X, x \in X$, $(f \boxplus g)(x) = f(x) +_R g(x)$,*
2. *For arbitrary $f \in R^X, r \in R, x \in X$, $(r \star f)(x) = r \times_R f(x)$,*
3. $0_X(x) = 0_R$,
4. $f \leq g \Leftrightarrow \forall x \in X, f(x) \leq_R g(x)$.

Remark 2. *If $(\mathcal{R}, \mathcal{R}^*)$ is an adjoint pair, the operations $\star, \boxplus, \leq$ for po-semiring $\mathcal{R}^*$ are denoted by $\star^*, \boxplus^*, \leq^*$.*

It is clear that any *po*-semiring homomorphism $\Phi : \mathcal{R}_1 \to \mathcal{R}_2$ can be extended to the semimodule homomorphism $\Phi^e : \mathcal{R}_1^X \to \mathcal{R}_2^X$ of corresponding free semimodules, where

$$f \in \mathcal{R}_1^X, x \in X, \quad \Phi^e(f)(x) = \Phi(f(x)).$$

In the next part we show that the mentioned hesitant, intuitionistic, neutrosophic or soft $\mathcal{L}$-fuzzy sets are, in fact, $\mathcal{R}$-fuzzy sets for appropriate *po*-semiring $\mathcal{R}$. For convenience of the readers, we repeat firstly the definitions of these structures. As we mentioned in the Introduction, as the value-lattice in the paper we use the MV-algebra $\mathcal{L}$.

Definition 8 ([2,9,17,20])**.**

1. *A hesitant $\mathcal{L}$-fuzzy set in a set X is a mapping $h : X \to 2^L$, i.e., for $x \in X$, $h(x) \subseteq L$. By $H(X)$ we denote the set of all hesitant fuzzy sets in X.*
2. *An intuitionistic $\mathcal{L}$-fuzzy set in a set X is a pair (u, v) of $\mathcal{L}$-fuzzy sets on X, such that $\neg u \geq v$. By $J(X)$ we denote the set of all intuitionistic fuzzy sets in X.*
3. *A neutrosophic $\mathcal{L}$-fuzzy set in a set X is a triple (u, v, w) of $\mathcal{L}$-fuzzy sets on X, called a truth membership function u, an indeterminancy membership function v and a falsity membership function w. By $N(X)$ we denote the set of all neutrosophis fuzzy sets in X.*
4. *Let K be the fixed set of criteria. A pair (E, s) is called an $\mathcal{L}$-fuzzy soft set in the set X, if $\emptyset \neq E \subseteq K$ and $s : E \to \mathcal{L}^X$. By $S(X)$ we denote the set of all fuzzy soft sets in X.*

Remark 3. *For a fuzzy soft set (E, s), a mapping s can be extended to the mapping $s : K \to \mathcal{L}^X$ such that $s(e)(x) = 0_L$ for $e \in K \setminus E$. In that case (E, s) can be identified with the mapping $(E, s) : X \to L^K$, such that $(E, s)(x) \in L^K$ is defined by*

$$(E, s)(x)(e) := \begin{cases} s(e)(x), & e \in E \\ 0_L, & e \in K \setminus E \end{cases}.$$

In what follows we use this interpretation of $\mathcal{L}$-fuzzy soft sets.

In the following proposition we show that the elements of sets $H(X), J(X), N(X)$ and $S(X)$ can be represented as $\mathcal{R}$-fuzzy sets in a set X for appropriate *po*-semirings $\mathcal{R}$ and sets $H(X), J(X), N(X)$ and $S(X)$ are isomorphic to the free $\mathcal{R}$-semimodules $\mathcal{R}^X$ of all $\mathcal{R}$-fuzzy sets in a set X. This result allows us to interpret the above mentioned $\mathcal{L}$-fuzzy type structures in the universal way as the $\mathcal{R}$-fuzzy sets and to use common tools and methods from the theory of $\mathcal{R}$-fuzzy sets for these structures. In the next Section we will illustrate this procedure on the issue of the F-transform for these fuzzy type structures.

For arbitrary *po*-semiring $\mathcal{R}$, by $(\mathcal{R}^X, \sqcup, \sqcap, \neg, \star)$ we denote the algebraic structure of all $\mathcal{R}$-fuzzy sets with operations from Definition 6. If $T(X)$ is an arbitrary from the sets $H(X), J(X), N(X)$ and $\mathcal{S}(X)$, by $(T(X), \cup, \cap, \neg, \circ)$ we denote this set with operations of union, intersection, negation and external multiplication by element from $\mathcal{L}$ defined on these fuzzy type structure. These operations for intuitionistic, neutrosophis and soft fuzzy sets are defined in [2,9,20]. The original definition [16] of these operations for hesitant $\mathcal{L}$-fuzzy sets is very atypical in comparison with other fuzzy type structures, because, for example, it does not guarantee distributivity between $\cup$ and $\cap$ operations, as is the case of other fuzzy type structures. For our purposes we use a modified definition, where for hesitant $\mathcal{L}$-fuzzy set s, t we define $(s \cup t)(x) = s(x) \cup t(x)$, $(s \cup t)(x) = s(x) \cup t(x)$.

We use the following notation.

Notation 1. *Let $\mathcal{F}(X) = (F(X), \cup, \cap, \neg, \circ)$ be a fuzzy type structure in a set X with the basic operations union, intersection, negation and external product. $\mathcal{F}(X)$ is called to be transformable to $\mathcal{R}$-fuzzy sets, if $\mathcal{F}(X)$ is isomorphic to the structure $(\mathcal{R}^X, \sqcup, \sqcap, \neg, \star)$, where $(\mathcal{R}, \mathcal{R}^*)$ is the adjoint pair of po-semirings with adjoint isomorphism Φ.*

Proposition 1. *Let X be a set.*

1. *The algebraic structure $(H(X), \cap, \cup, \neg, \circ)$ of all hesitant $\mathcal{L}$-fuzzy sets in X is transformable to $\mathcal{R}_1$-fuzzy sets.*
2. *The algebraic structure $(J(X), \cap, \cup, \neg, \circ)$ of all intuitionistic $\mathcal{L}$-fuzzy sets is transformable to $\mathcal{R}_2$-fuzzy sets.*
3. *The algebraic structure $(S(X), \cap, \cup, \neg, \circ)$ of all $\mathcal{L}$-fuzzy soft sets in X is transformable $\mathcal{R}_3$-fuzzy sets.*
4. *The algebraic structure $(N(X), \cap, \cup, \neg, \circ)$ of all neutrosophic $\mathcal{L}$-fuzzy sets in X is transformable to $\mathcal{R}_4$-fuzzy sets.*

Proof.

(1) Any hesitant $\mathcal{L}$-fuzzy set is a mapping $X \to 2^L$ and it follows that $H(X) = \mathcal{R}_1^X$. The isomorphism of operations follows directly from the definitions of operations in $\mathcal{R}_1^X$ and in $H(X)$.

(2) Any intuitionistic $\mathcal{L}$-fuzzy set is a mapping $X \to \{(\alpha, \beta) \in L^2 : \neg\alpha \geq \beta\}$ and it follows that $J(X) = \mathcal{R}_2^X$. The isomorphism of operations follows directly from the definitions of operations in $\mathcal{R}_2^X$ and in $J(X)$.

(3) According to Remark 3, any $\mathcal{L}$-fuzzy soft set $(E, s) \in S(X)$ is a mapping $(E, s) : X \to L^K$, where $E \subseteq K$, $s : K \to L^X$ and

$$k \in K, x \in X, \quad (E, s)(x)(k) = \begin{cases} s(k)(x), & k \in E, \\ 0_L, & k \notin E \end{cases}.$$

We define the mapping $\Lambda : S(X) \to \mathcal{R}_3^X$ such that

$$\begin{gathered} (E, s) \in S(X), \quad \Lambda(E, s) : X \to \mathcal{R}_3, \\ x \in X, \quad \Lambda(E, s)(x) := (E, s_x) \in \mathcal{R}_3, \\ s_x : K \to L, \quad s_x(k) := s(k)(x) \text{ for } k \in K, \\ (E, s_x) : K \to L, \quad (E, s_x)(k) := \begin{cases} s(k)(x), & k \in E, \\ 0_L, & k \notin E. \end{cases} \end{gathered}$$

We prove that $\Lambda(S(X)) = \mathcal{R}_3^X$. Because $\Lambda(E, s) \in \mathcal{R}_3^X$, we have $\Lambda(S(X)) \subseteq \mathcal{R}_3^X$ and we need to prove only the inverse inclusion. Let $g \in \mathcal{R}_3^X$, $g(x) = (E_x, \psi_x) \in \mathcal{R}_3$ for $x \in X$. Let the element $(E, s) \in S(X)$ be defined by

$$\begin{gathered} E = \bigcup_{x \in X} E_x, \\ s : K \to L^X, \quad s(k)(x) = \begin{cases} \psi_x(k), & k \in E_x, \\ 0_L, & k \notin E_x. \end{cases} \end{gathered}$$

For $x \in X$ we have $\Lambda(E, s)(x) = (E, s_x)$ and we show that $\Lambda(E, s)(x)$ and $g(x)$ are equal as mappings $K \to L$. For $k \in K$ we have

$$(E, s_x)(k) = \begin{cases} s(k)(x), & k \in E, \\ 0_L, & k \notin E \end{cases} =$$

$$\begin{cases} \psi_x(k), & k \in E_x, \\ 0_L, & k \in E \setminus E_x \\ 0_L, & k \notin E \end{cases} = \begin{cases} \psi_x(k), & k \in E_x, \\ 0_L, & k \notin E_x \end{cases} = g(x)(k).$$

Therefore, $\Lambda(E, s) = g$ and $\Lambda(S(X)) = \mathcal{R}_3^X$. It is easy to see that Λ is the injective map. The isomorphism of operations in $\mathcal{R}_3^X$ and in $S(X)$ follows directly from definitions of these operations in $\mathcal{R}_3^X$ and $S(X)$. Therefore, the algebraic structure $S(X)$ is isomorphic to the algebraic structure $\mathcal{R}_3^X$.

(4) Any neutrosphic $\mathcal{L}$-fuzzy set is a mapping $X \to L^3$ and it follows that $N(X) = \mathcal{R}_4^X$. The isomorphism of operations follows directly from the definitions of operations in $\mathcal{R}_4^X$ and in $N(X)$.

□

Free $\mathcal{R}$-semimodules $\mathcal{R}^X$ have specific $\mathcal{R}$-base, which will be essential for F-transform constructions.

Lemma 2. *Let $(\mathcal{R}, \mathcal{R}^*)$ be the adjoint pair of complete semirings with adjoin isomorphism Φ and let X be a set.*

1. *The set $B = \{\eta_X(x) : x \in X\}$ is the $\mathcal{R}$-base of the free $\mathcal{R}$-semimodule $\mathcal{R}^X$.*
2. *The set $\neg B = \{\neg\eta_X(x) : x \in X\}$ is the $\mathcal{R}^*$-base of the free $\mathcal{R}^*$-semimodule $(\mathcal{R}^*)^X$, where $\neg(\eta_X(x))(z) = \Phi(\eta_X(x)(z))$, for $z \in X$.*

Proof.

(1) We show firstly that the following identity holds for arbitrary $f \in \mathcal{R}^X$:

$$f = \mathop{\boxplus}_{x \in X}^{\mathcal{R}^X} f(x) \star \eta_X(x). \tag{1}$$

In fact, according to Definition 7, for $a \in X$ we obtain

$$(\mathop{\boxplus}_{x \in X}^{\mathcal{R}^X} f(x) \star \eta_X(x))(a) = \sum_{x \in X}^{\mathcal{R}} f(x) \times_R \eta_X(x)(a) = f(a) \times_R 1_R = f(a)$$

and the identity (1) holds. If for some elements $\{r_x \in R : x \in X\}$ the identity

$$\mathop{\boxplus}_{x \in X}^{\mathcal{R}^X} r_x \star \eta_X(x) = 0$$

holds, according to Definition 7, for arbitrary $a \in X$ we obtain

$$0_R = (\mathop{\boxplus}_{x \in X}^{\mathcal{R}^X} r_x \star \eta_X(x))(a) = \sum_{x \in X}^{\mathcal{R}} r_x \times_R \eta_X(x)(a) = r_a.$$

Therefore, B is the $\mathcal{R}$-base of $\mathcal{R}^X$.

(2) We show that for arbitrary $f \in (\mathcal{R}^*)^X$ we have

$$f = \mathop{\boxplus}_{x \in X}^{(\mathcal{R}^*)^X} f(x) \star^* (\neg\eta_X(x)). \tag{2}$$

In fact, according to Definition 5 and Definition 7, for $a \in X$ we have

$$(\mathop{\boxplus}_{x \in X}^{(\mathcal{R}^*)^X} f(x) \star^* (\neg\eta_X(x)))(a) = \sum_{x \in X}^{\mathcal{R}^*} f(x) \times^* \Phi(\eta_X(x)(a)) =$$
$$f(a) \times^* \Phi(1_{\mathcal{R}}) = f(a) \times^* 1^*_{\mathcal{R}} = f(a)$$

and the identity (2) holds. The rest is similar to the previous case.

□

3.2. F-Transform for $\mathcal{R}$-Fuzzy Sets

The F-transform method for $\mathcal{L}$-fuzzy sets is one of the very effective tools for reducing the difficulty of uncertainties tasks requiring operations with large sets of this uncertain data. Typical tasks of this type are algorithms used in image processing, time series analysis, or even the solutions of differential equations with uncertainties. For many of these tasks, methods that use other fuzzy type structures, including hesitant, intuitionistic, or fuzzy soft sets, have also been used successfully. For illustration of these methods in these fuzzy type structures see [43–47].

A large part of new fuzzy type methods are used, among others, in applications related to image processing. In these applications, individual images are transformed into objects of these fuzzy type structures, for example, intuitionistic fuzzy sets or hesitant fuzzy sets, which are defined on pixels of individual images. Due to the value-sets of these new fuzzy type structures, this in turn leads to the fact that the computational complexity of operations with these objects is higher than in the case of classical fuzzy sets.

It is therefore natural to create a certain F-transform analogy for these fuzzy type structures, which would subsequently make it possible to reduce the computational complexity of these tasks and thus expand the application possibilities of these methods.

The classical theory of F-transform for $\mathcal{L}$-fuzzy sets deals with two types of this transformation, namely with *upper and lower F-transforms* [31], which represent the upper and lower variant (in terms of ordering) of the transformed function. Since both of these transformations preserve the order relation defined on fuzzy sets, the possibility of choosing from two variants of the transformed function expands, among other things, the application possibilities of this theory. The resulting upper or lower transformations of the original function thus represent a significant simplification of the original $\mathcal{L}$-fuzzy set f and thus simplify the overall processing and operations with the original function. However, in order to take full advantage of these transformations of the original function, there must be a possibility to reproduce the original function from these simplified functions, i.e., there must be so-called *inverse F-transform*.

In this section we introduce both these types of the transforms and their inverse variants for $\mathcal{R}$-fuzzy sets, which allows the use of these terms for general fuzzy type structures, including hesitant, intuitionistic, neutrosophic or fuzzy soft sets and their possible combinations and we show some relationships among these structures.

Analogously as for classical $\mathcal{L}$-fuzzy sets, the F-transform for $\mathcal{R}$-fuzzy sets is also based on analogies of fuzzy partitions. Unlike the classic concept of $\mathcal{L}$-fuzzy partition originally defined in [31], fuzzy type partitions for $\mathcal{R}$-fuzzy sets will be defined more generally. The definition chosen in this way then allows to choose different types of fuzzy partitions according to the conditions of a particular task.

Definition 9. *Let $\mathcal{R}$ be a complete po-semiring. A subset $\mathcal{A} = \{p_y : y \in Y\} \subseteq \mathcal{R}^X$ is called the $\mathcal{R}$-partition of X if there exists a binary relation $u \subseteq X \times Y$ such that the following conditions are satisfies:*

$$dom(u) = X, codom(u) = Y,$$
$$(x, y) \in u \Rightarrow p_y(x) = 1_R.$$

In the following definition we introduce both basic types of the F-transform for $\mathcal{R}$-fuzzy sets, i.e., upper and lower F-transform. This notion can be introduced for arbitrary adjoint pair of *po*-semirings with adjoint isomorphism.

Definition 10. *Let $(\mathcal{R}, \mathcal{R}^*)$ be adjoint pair of po-semirings with adjoint isomorphism Φ, $\mathcal{R} = (R, \leq, +, \times, 0, 1)$, $\mathcal{R}^* = (R, \leq^*, +^*, \times^*, 0^*, 1^*)$ and let $\mathcal{A} = \{p_y : y \in Y\}$ be a $\mathcal{R}$-partition of a set X.*

1. *The upper F-transform of $\mathcal{R}$-fuzzy sets in X is a mapping $F^{\uparrow}_{\mathcal{A},X} : \mathcal{R}^X \to \mathcal{R}^Y$ defined by*

$$f \in \mathcal{R}^X, y \in Y, \quad F^{\uparrow}_{\mathcal{A},X}(f)(y) = \sum_{x \in X}^{\mathcal{R}} p_y(x) \times f(x).$$

2. *The lower F-transform of $\mathcal{R}$-fuzzy sets in X is a mapping $F^{\downarrow}_{\mathcal{A},X} : \mathcal{R}^X \to \mathcal{R}^Y$ defined by*

$$f \in \mathcal{R}^X, y \in Y, \quad F^{\downarrow}_{\mathcal{A},X}(f)(y) = \sum_{x \in X}^{\mathcal{R}^*} \Phi(p_y(x)) \times^* f(x).$$

In the following example we illustrate how the F-transform can be defined also for mutual combinations of various fuzzy type structures. As an example we consider *intuitionistic fuzzy soft sets* which were introduced in [62]. We use an extended variant of intuitionistic fuzzy soft sets, where membership values are from the complete MV-algebra $\mathcal{L}$.

Example 10. *Recall the definition of the intuitionistic $\mathcal{L}$-fuzzy soft set. Let K be the fixed set of criteria and let X be a set. An intuitionistic $\mathcal{L}$-fuzzy soft set in a set X is a pair (E, s), where $E \subseteq K$ and $s : K \to \mathcal{J}(X) = \mathcal{R}_2^X$ is such that $s(k) = \underline{0}_{\mathcal{R}_2}$, if $k \in K \setminus E$, where $\underline{0}_{\mathcal{R}_2} : X \to \mathcal{R}_2$ is the constant function with the valued $0_{\mathcal{R}_2}$. We show that any intuitionistic $\mathcal{L}$-fuzzy soft set is a $\mathcal{R}_5$-fuzzy set for appropriate po-semiring $\mathcal{R}_5$. To define the F-transform for intuitionistic $\mathcal{L}$-fuzzy soft sets we need the adjoint pair $(\mathcal{R}_5, \mathcal{R}_5^*)$ of po-semirings and adjoint isomorphism $\Phi_5 : \mathcal{R}_5 \to \mathcal{R}_5^*$. We set*

$$\mathcal{R}_5 = \mathcal{R}_2^K, \quad \mathcal{R}_5^* = (\mathcal{R}_2^*)^K,$$

where $\mathcal{R}_2$ and $\mathcal{R}_2^*$ are from Example 6. Let the operations in $\mathcal{R}_5$ and $\mathcal{R}_5^*$ be defined point-wise from operations in $\mathcal{R}_2$ and $\mathcal{R}_2^*$, respectively.

Let the mapping $\Phi_5 : \mathcal{R}_5 \to \mathcal{R}_5^*$ be defined by

$$f \in \mathcal{R}_5, k \in K, \quad \Phi_5(f)(k) = \Phi_2(f(k)).$$

It is easy to see that $(\mathcal{R}_5, \mathcal{R}_5^*)$ is the adjoint pair of po-semirings and Φ_5 is the adjoint isomorphism. We show the there exists the isomorphism between the structure $\mathcal{IFS}(X) = (IFS(X), \cup, \cap, \neg)$ of all intuitionistic $\mathcal{L}$-fuzzy soft sets in a set X with standard operations $\cup, \cap, \neg$ defined in [62] and the free $\mathcal{R}_5$-po-semimodule $\mathcal{R}_5^X$ with the point-wise operations defined from $+_2, \times_2$ and Φ_2.

In fact, let us defined the map $\Gamma : IFS(X) \to \mathcal{R}_5^X = (\mathcal{R}_2^K)^X$ by

$$(E, s) \in IFS(X), x \in X, k \in K, \quad \Gamma(E, s)(x)(k) = \begin{cases} s(k)(x) \in \mathcal{R}_2, & k \in E, \\ 0_{\mathcal{R}_2}, & k \in K \setminus E. \end{cases}$$

Γ is the surjective map. In fact, for $f \in \mathcal{R}_5^X$, we set

$$x \in X, \quad E_x = \{k \in K : f(x)(k) \neq 0_{\mathcal{R}_2}\},$$
$$E = \bigcup_{x \in X} E_x, \quad s : K \to \mathcal{R}_2^X,$$
$$k \in K, x \in X, \quad s(k)(x) = \begin{cases} f(x)(k), & k \in E_x, \\ 0_{\mathcal{R}_2}, & k \in K \setminus E_x. \end{cases}$$

It follows that $\Gamma(E, s) = f$ and it is easy to see that Γ is the isomorphism.

Finally, we show how the formula for lower F-transform looks for intuitionistic $\mathcal{L}$-fuzzy soft sets. Using the above isomorphism Γ between $\mathcal{IFS}(X)$ and $\mathcal{R}_5^X$, instead of elements from $\mathcal{IFS}(X)$ we use elements of $\mathcal{R}_5^X$. Let $\mathcal{A} = \{p_y : y \in Y\} \subseteq \mathcal{R}_5^X$ be a $\mathcal{R}_5$-partition of X. For $x \in X, k \in K$, *let* $p_y(x)(k) = (\alpha_{yxk}, \beta_{yxk}) \in \mathcal{R}_2$. *According to Definition 10, for arbitrary* $f \in \mathcal{R}_5^X$, $f(x)(k) = (f_{1,xk}, f_{2,xk})$, $x \in X, k \in k$, *we obtain the lower F-transform* $F^{\downarrow}_{\mathcal{A},X} : \mathcal{R}_5^X \to \mathcal{R}_5^Y$ *by*

$$y \in Y, k \in K, \quad F^{\downarrow}_{\mathcal{A},X}(f)(y)(k) = (\sum_{x\in X}^{\mathcal{R}_5^*} \Phi_5(p_y(x)) \times_5^* f(x))(k) =$$

$$\sum_{x\in X}^{\mathcal{R}_2^*} \Phi_2(p_y(x)(k)) \times_2^* f(x)(k) = \sum_{x\in X}^{\mathcal{R}_2^*} \Phi_2(\alpha_{yxk}, \beta_{yxk}) \times_2^* (f_{1,xk}, f_{2,xk}) =$$

$$\sum_{x\in X}^{\mathcal{R}_2^*} (\beta_{yxk}, \alpha_{yxk}) \times_2^* (f_{1,xk}, f_{2,xk}) = \sum_{x\in X}^{\mathcal{R}_2^*} (\beta_{yxk} \oplus f_{1,xk}, \alpha_{yxk} \otimes f_{2,xk}) =$$

$$(\bigwedge_{x\in X} \beta_{yxk} \oplus f_{1,xk}, \bigvee_{x\in X} \alpha_{yxk} \otimes f_{2,xk}).$$

In the following theorem we will show how to characterize the lower and upper F-transforms for $\mathcal{R}$-fuzzy sets also without the use of a fuzzy partition.

Theorem 1. *Let* $(\mathcal{R}, \mathcal{R}^*)$ *be adjoint pair of po-semirings with adjoint isomorphism* Φ. *Let* $F : R^X \to R^Y$ *be an arbitrary mapping.*

1. *The following statements are equivalent.*
   (a) *F is the* $\mathcal{R}$*-po-semimodule homomorphism and there exists a relation* $u \subseteq X \times Y$ *with* $dom(u) = X, codom(u) = Y$ *and such that*
   $$\forall (x, y) \in u, \quad F(\eta_X(x))(y) = 1_{\mathcal{R}}.$$
   (b) *There exists a* $\mathcal{R}$*-partition* $\mathcal{A} = \{p_y : y \in Y\}$ *such that* $F = F^{\uparrow}_{\mathcal{A},X}$.
2. *The following statements are equivalent.*
   (a) *F is the* $\mathcal{R}^*$*-po-semimodule homomorphism* $(\mathcal{R}^*)^X \to (\mathcal{R}^*)^Y$ *and there exists a relation* $u \subseteq X \times Y$ *with* $dom(u) = X, codom(u) = Y$ *and such that*
   $$\forall (x, y) \in u, \quad F(\neg(\eta_X(x))(y)) = 1^*_{\mathcal{R}},$$
   where $\neg(\chi_X(x))(x') = \Phi(\chi_X(x)(x'))$ *for* $x' \in X$.
   (b) *There exists a* $\mathcal{R}$*-partition* $\mathcal{A} = \{p_y : y \in Y\}$ *such that* $F = F^{\downarrow}_{\mathcal{A},X}$.

Proof.

(1) Let the condition (b) holds. We set $(x, y) \in u \Leftrightarrow A_y(x) = 1_{\mathcal{R}}$. It is easy to that $F^{\uparrow}_{\mathcal{A},X}$ is the $\mathcal{R}$-*po*-semimodule homomorphism and that the additional condition holds.
Let the condition (a) holds. According to Lemma 2, for arbitrary element $f \in \mathcal{R}^X$ we obtain $f = \boxplus^{\mathcal{R}^X}_{x\in X} f(x) \star \eta_X(x)$. Because F is a $\mathcal{R}$-semimodule homomorphism, according to Definition 7, we obtain

$$F(f)(y) = F(\boxplus_{x\in X}^{\mathcal{R}^X} f(x) \star \eta_X(x))(y) = \sum_{x\in X}^{\mathcal{R}} f(x) \times F(\eta_X(x))(y) =$$

$$\sum_{x\in X}^{\mathcal{R}} f(x) \times p_y(x),$$

where we set $p_y(x) = F(\eta_X(x))(y)$. Therefore, $\mathcal{A} = \{p_y : y \in Y\}$ is a $\mathcal{R}$-partition and the statement 2. holds.

(2) Let the condition (b) holds. For $r \in \mathcal{R}^*, f, g \in (\mathcal{R}^*)^X$ and $y \in Y$ we obtain

$$F^{\downarrow}_{\mathcal{A},X}(r \star^* f +^* g)(y) = \sum_{x \in X}^{\mathcal{R}^*} \Phi(p_y(x)) \times^* (r \star^* f +^* g)(x) =$$
$$\sum_{x \in X}^{\mathcal{R}^*} r \times^* \Phi(p_y(x)) \times^* f(x) +^* \Phi(p_y(x)) \times^* g(x) =$$
$$r \times^* F^{\downarrow}_{\mathcal{A},X}(f)(y) +^* F^{\downarrow}_{\mathcal{A},X}(g)(y),$$

and $F^{\downarrow}_{\mathcal{A},X}$ is the $\mathcal{R}^*$-*po*-semimodule homomorphism. Moreover, for $(x, y) \in u$ we have

$$F^{\downarrow}_{\mathcal{A},X}(\neg(\eta_X(x))(y) = \sum_{z \in X}^{\mathcal{R}^*} \Phi(p_y(z)) \times^* \Phi(\eta_X(x)(z))) = \Phi(p_y(x)) \times^* \Phi(1_{\mathcal{R}}) =$$
$$\Phi(p_y(x)) \times^* 1^*_{\mathcal{R}} = \Phi(p_y(x)) = \Phi(1_{\mathcal{R}}) = 1^*_{\mathcal{R}}.$$

Let the condition (a) holds. According to Lemma 2, we have

$$F(f)(y) = F(\boxplus_{x \in X}^{(\mathcal{R}^*)^X} f(x) \star^* \neg(\eta_X(x)))(y) =$$
$$\sum_{x \in X}^{\mathcal{R}^*} f(x) \times^* F(\neg(\eta_X(x)))(y) = \sum_{x \in X}^{\mathcal{R}^*} f(x) \times^* \Phi(p_y(x)) = F^{\downarrow}_{\mathcal{A},X}(f)(y),$$

where we set $p_y(x) = \Phi(F(\neg(\eta_X(x)))(y))$ and $\mathcal{A} = \{p_y : y \in Y\}$.

□

As we mentioned in the introduction, one of the main advantages of the classical F-transform for $\mathcal{L}$-fuzzy sets is the existence of the inverse F-transform, which allows to reconstruct with some accuracy the original function from its F-transform image. It is therefore natural to try to define a similar inverse transformation for the F-transform of $\mathcal{R}$-fuzzy sets and to determine its basic properties. We introduce the inverse F-transform for $\mathcal{R}$-fuzzy sets in the following definition.

Definition 11. *Let $(\mathcal{R}, \mathcal{R}^*)$ be adjoint pair of po-semirings with adjoint isomorphism Φ and let X be a set. Let $\mathcal{A} = \{p_y : y \in Y\}$ be a $\mathcal{R}$-partition of X.*

1. *The upper inverse F-transform of $\mathcal{R}$-fuzzy sets defined by $\mathcal{A}$ is a mapping $G^{\uparrow}_{\mathcal{A},X} : \mathcal{R}^Y \to \mathcal{R}^X$, defined by*
$$g \in R^Y, x \in X, \quad G^{\uparrow}_{\mathcal{A},X}(g)(x) = \sum_{y \in Y}^{\mathcal{R}^*} \Phi(p_y(x)) \times^* g(y).$$
2. *The lower inverse F-transform of $\mathcal{R}$-fuzzy sets defined by $\mathcal{A}$ is a mapping $G^{\downarrow}_{\mathcal{A},X} : \mathcal{R}^Y \to \mathcal{R}^X$, defined by*
$$g \in R^Y, x \in X, \quad G^{\downarrow}_{\mathcal{A},X}(g)(x) = \sum_{y \in Y}^{\mathcal{R}} p_y(x) \times g(y).$$

There are simple relationships between the transformations $F^{\uparrow}, F^{\downarrow}$ and $G^{\uparrow}, G^{\downarrow}$, respectively, as can be seen from the following proposition.

Proposition 2. *Let $(\mathcal{R}, \mathcal{R}^*)$ be adjoint pair of po-semirings with adjoint isomorphism Φ and let X be a set. Let $\mathcal{A} = \{p_y : y \in Y\}$ be a $\mathcal{R}$-partition of X. The following statements hold for arbitrary $a \in \mathcal{R}, f, g \in \mathcal{R}^X, h, k \in \mathcal{R}^Y, x \in X, y \in Y$.*

1. $F^{\uparrow}_{\mathcal{A},X}(a \star f \boxplus g) = a \star F^{\uparrow}_{\mathcal{A},X}(f) \boxplus F^{\uparrow}_{\mathcal{A},X}(g),$

2. $F^{\downarrow}_{\mathcal{A},X}(a \star^* f \boxplus^* g) = a \star^* F^{\downarrow}_{\mathcal{A},X}(f) \boxplus^* F^{\downarrow}_{\mathcal{A},X}(g)$,
3. $G^{\uparrow}_{\mathcal{A},X}(a \star^* h \boxplus^* k) = a \star^* G^{\uparrow}_{\mathcal{A},X}(h) \boxplus^* G^{\uparrow}_{\mathcal{A},X}(k)$,
4. $G^{\downarrow}_{\mathcal{A},X}(a \star h \boxplus k) = a \star G^{\downarrow}_{\mathcal{A},X}(h) \boxplus G^{\downarrow}_{\mathcal{A},X}(k)$,
5. $F^{\downarrow}_{\mathcal{A},X}(f)(y) = \Phi(F^{\uparrow}_{\mathcal{A},X}(\neg f)(y))$,
6. $G^{\downarrow}_{\mathcal{A},X}(g)(x) = \Phi(G^{\uparrow}_{\mathcal{A},X}(\neg g)(x))$,
7. $G^{\downarrow}_{\mathcal{A},X}F^{\downarrow}_{\mathcal{A},X}(f)(x) = \Phi(G^{\uparrow}_{\mathcal{A},X}F^{\uparrow}_{\mathcal{A},X}(\neg f)(x))$.
8. $F^{\uparrow}_{\mathcal{A},X}(f)(y) \geq_{\mathcal{R}} F^{\downarrow}_{\mathcal{A},X}(f)(y)$
9. $G^{\uparrow}_{\mathcal{A},X}(g)(x) \leq_{\mathcal{R}} G^{\downarrow}_{\mathcal{A},X}(g)(x)$,
10. $(x,y) \in u \Rightarrow F^{\uparrow}_{\mathcal{A},X}(f)(y) \geq_{\mathcal{R}} f(x),\, G^{\uparrow}_{\mathcal{A},X}(g)(x) \leq_{\mathcal{R}} g(y)$,
11. $(x,y) \in u \Rightarrow F^{\downarrow}_{\mathcal{A},X}(f)(y) \leq_{\mathcal{R}} f(x),\, G^{\downarrow}_{\mathcal{A},X}(g)(x) \geq_{\mathcal{R}} g(y)$,

where $\neg f$ for $f \in \mathcal{R}^X$ or $\neg g$ for $g \in \mathcal{R}^Y$ are defined by $(\neg f)(x) = \Phi(f(x))$, $x \in X$, or $(\neg g)(y) = \Phi(g(y))$, $y \in Y$.

Proof. To prove (1), (2), (3), (4) is straightforward and it will be omitted.

(5) We have

$$F^{\downarrow}_{\mathcal{A},X}(f)(y) = \sum_{x\in X}^{\mathcal{R}^*} \Phi(p_y(x)) \times^* f(x) = \Phi(\sum_{x\in X}^{\mathcal{R}} p_y(x) \times \Phi(f(x))) = \Phi(F^{\uparrow}_{\mathcal{A},X}(\neg f)(y)).$$

The proof of (6) can be done analogously and it will be omitted.

(7) We have

$$G^{\downarrow}_{\mathcal{A},X}F^{\downarrow}_{\mathcal{A},X}(f)(x) = \Phi(G^{\uparrow}(\neg F^{\downarrow}_{\mathcal{A},X}(f))(x)) = \Phi(G^{\uparrow}_{\mathcal{A},X}(\neg\neg F^{\uparrow}_{\mathcal{A},X}(\neg f))(x) = \Phi(G^{\uparrow}_{\mathcal{A},X}F^{\uparrow}_{\mathcal{A},X}(\neg f)(x)).$$

(8) Let $y \in Y$ be arbitrary and let $x \in X$ be such that $(x,y) \in u$. We have

$$F^{\uparrow}_{\mathcal{A},X}(f)(y) = \sum_{z\in X}^{\mathcal{R}} p_y(z) \times_R f(z) \geq p_y(x) \times_R f(x) = f(x)$$

and, analogously,

$$F^{\downarrow}_{\mathcal{A},X}(f)(y) = \sum_{z\in X}^{\mathcal{R}^*} \Phi(p_y(z)) \times^*_R f(z) \geq^* \Phi(p_y(x)) \times^*_R f(x) = 1^*_{\mathcal{R}} \times^* f(x) = f(x).$$

Therefore, $F^{\downarrow}_{\mathcal{A},X}(f)(y) \leq f(x)$ and it follows $F^{\uparrow}_{\mathcal{A},X}(f)(y) \geq f(x) \geq F^{\downarrow}_{\mathcal{A},X}(f)(y)$.

(9) The proof is similar to the proof of (8) and it will be omitted.

The rest of the proof can be done easily and it will be omitted. □

Analogously as for direct F-transform, the inverse F-transform can be characterised without the notion of $\mathcal{R}$-partition. It is not surprising that these characterisations are, in some aspects, dual to the characterisations of direct F-transforms and that the proofs of these characterisations are analogical to the proofs for direct F-transforms. It follows from the following lemma.

Lemma 3. *Let $(\mathcal{R}, \mathcal{R}^*)$ be adjoint pair of po-semirings with adjoint isomorphism Φ and let X, Y be sets. Let $\mathcal{A} = \{p_y : y \in Y\}$ be a $\mathcal{R}$-partition of X and let $\mathcal{B} = \{q_x : x \in X\}$ be a $\mathcal{R}$-partition of Y, such that*

$$\forall x \in X, \forall y \in Y, \quad p_y(x) = q_x(y).$$

Then for any $s \in \mathcal{R}^Y$ hold

$$F^{\uparrow}_{Y,\mathcal{B}}(s) = G^{\downarrow}_{X,\mathcal{A}}(s), \quad F^{\downarrow}_{Y,\mathcal{B}}(s) = G^{\uparrow}_{X,\mathcal{A}}(s).$$

The proof of Lemma is trivial and it will be omitted.

Using Lemma 3 and Theorem 1 it is easy to see that the following characterisations of lower and upper inverse F-transforms without the notion of $\mathcal{R}$-partition hold.

Theorem 2. *Let $(\mathcal{R}, \mathcal{R}^*)$ be adjoint pair of po-semirings with adjoint isomorphism Φ and let X be a set. Let $G : R^Y \to R^X$ be an arbitrary mapping.*

1. *The following statements are equivalent.*
   (a) *G is the $\mathcal{R}^*$-po-semimodule homomorphism $(\mathcal{R}^*)^Y \to (\mathcal{R}^*)^X$ and there exists a relation $u \subseteq X \times Y$ with $dom(u) = X, codom(u) = Y$ and such that*

   $$\forall (x,y) \in u, \quad G(\neg \eta_Y(y))(x) = 1^*_{\mathcal{R}}.$$

   where $\neg(\eta_Y(y))(y') = \Phi(\eta_Y(y)(y'))$ for $y' \in Y$.
   (b) *There exists a $\mathcal{R}$-partition $\mathcal{A} = \{p_y : y \in Y\}$ such that $G = G^{\uparrow}_{\mathcal{A},X}$.*
2. *The following statements are equivalent.*
   (a) *G is the $\mathcal{R}$-po-semimodule homomorphism $\mathcal{R}^Y \to \mathcal{R}^X$ and there exists a relation $u \subseteq X \times Y$ with $dom(u) = X, codom(u) = Y$ and such that*

   $$\forall (x,y) \in u, \quad G(\eta_Y(y))(x) = 1_{\mathcal{R}}.$$

   (b) *There exists a $\mathcal{R}$-partition $\mathcal{A} = \{p_y : y \in Y\}$ such that $G = G^{\downarrow}_{\mathcal{A},X}$.*

The proof follows directly from Theorem 2 and Lemma 3 and it will be omitted.

Notation 2. *Let $\mathcal{F}(X) = (F(X), \cup, \cap, \neg, \circ)$ be a fuzzy type structure in a set X which is transformable to $\mathcal{R}$-fuzzy sets.*

1. *The mappings $F^{\uparrow}_{X,\mathcal{A}}, F^{\downarrow}_{X,\mathcal{A}}$ from Definition 10 are called upper and lower F-transform of this fuzzy type structure $\mathcal{F}(X)$.*
2. *The mapping $G^{\uparrow}_{Y,\mathcal{A}}, G^{\downarrow}_{X,\mathcal{A}}$ from Definition 11 are called the upper and lower inverse F-transform of this fuzzy type structure $\mathcal{F}(X)$.*

For the classical $\mathcal{L}$-valued F-transform and its inversion more properties hold than stated in Proposition 2. Because $\mathcal{R}$-fuzzy sets comprise a lot of fuzzy type structures and some of them could be very different from $\mathcal{L}$-fuzzy sets, we cannot expect that the F-transforms for arbitrary $\mathcal{R}$-fuzzy sets will have the same properties as F-transforms for $\mathcal{L}$-fuzzy sets. On the other hand, many $\mathcal{R}$-fuzzy sets defined for particular fuzzy type structures have additional properties that imply other important properties of F-transforms and their inversions. Let us consider the following example of these additional properties.

Definition 12. *Let $(\mathcal{R}, \mathcal{R}^*)$ be adjoint pair of po-semirings with adjoint isomorphism Φ. We say that $(\mathcal{R}, \mathcal{R}^*)$ satisfies the axiom (+), if the following condition holds.*

$$\forall a, b \in \mathcal{R}, \quad \Phi(a) \times^* (a \times b) \geq b.$$

Remark 4. *It should be observed that the following dual condition is equivalent to the axiom (+):*

$$\forall a, b \in \mathcal{R}, \quad a \times (\Phi(a) \times^* b) \geq^* b.$$

In the following examples we show that the adjoint pair $(\mathcal{R}_3, \mathcal{R}^*_3)$ satisfy the axiom (+), but $(\mathcal{R}_1, \mathcal{R}^*_1)$, $(\mathcal{R}_2, \mathcal{R}^*_2)$, $(\mathcal{R}_5, \mathcal{R}^*_5)$ and $(\mathcal{R}_4, \mathcal{R}^*_4)$ do not satisfy the axiom (+).

Example 11. *Let $(\mathcal{R}_3, \mathcal{R}_3^*)$ be the adjoint pair from Example 7. For $(E, \phi), (F, \psi) \in \mathcal{R}_3$ and $k \in K$ we have*

$$\Phi(E,\phi) \times_3^* ((E,\phi) \times_3 (F,\psi))(k) = (E, \neg\phi) \times_3^* (E \cap F, \phi \otimes \psi)(k) =$$
$$(E \cap F, \neg\phi \oplus (\phi \otimes \psi))(k) = \begin{cases} \neg\phi(k) \oplus (\phi(k) \otimes \psi(k)) \geq \psi(k), & k \in E \cap F \\ 0_L, & k \notin E \cap F \end{cases} \geq (F,\psi)(k),$$

according to the definition of the pre-order in $\mathcal{R}_3$. Therefore, $(\mathcal{R}_3, \mathcal{R}_3^*)$ satisfies the axiom (+).

Example 12. *Let $(\mathcal{R}_2, \mathcal{R}_2^*)$ be the adjoint pair from Example 6 and let $\mathcal{L}$ be the Lukasiewicz algebra. We set $(\alpha, \beta) = (0.4, 0.5), (\gamma, \delta) = (0.8, 0.1) \in \mathcal{R}_2$. We have*

$$\Phi(\alpha,\beta) \times_2^* ((\alpha,\beta) \times_2 (\gamma,\delta)) = (0.5, 0.4) \times_2^* (0.4 \otimes 0.8, 0.5 \oplus 0.1) =$$
$$(0.7, 0) \ngeq (0.8, 0.1) = (\gamma, \delta)$$

and it follows that $(\mathcal{R}_2, \mathcal{R}_2^*)$ does not satisfy the axiom (+).

Example 13. *Let $(\mathcal{R}_1, \mathcal{R}_1^*)$ be the adjoint pair from Example 5. Let $\mathcal{L}$ be the Łukasiewicz algebra and let $A = \{0.7, 0.5\}, B = \{0.4\}$. Then $0.3 = \neg 0.7 \oplus (0.5 \otimes 0.4) \in \Phi(A) \times_1^* (A \times_1 B)$, but $0.3 \ngeq_L 0.4$. Therefore, $\Phi(A) \times_1^* (A \times_1 B) \ngeq_1 B$ and (+) does not hold.*

To show that axiom (+) does not hold for other adjoint pairs can be done analogously as in Example 12.

As we can see from the following proposition, for adjoint pairs of *po*-semirigs $(\mathcal{R}, \mathcal{R}^*)$ which satisfy the axiom (+), many analogies properties of direct and inverse F-transforms well known for classical F-transform for $\mathcal{L}$-valued fuzzy sets also hold.

Proposition 3. *Let $(\mathcal{R}, \mathcal{R}^*)$ be adjoint pair of po-semirings with adjoint isomorphism Φ, which satisfies the axiom (+). Let $\mathcal{A} = \{p_y : y \in Y\}$ be a $\mathcal{R}$-partition of X. The following statements hold.*

1. $G^{\uparrow}_{\mathcal{A},X} F^{\uparrow}_{\mathcal{A},X}(f) \geq f$,
2. $G^{\downarrow}_{\mathcal{A},X} F^{\downarrow}_{\mathcal{A},X}(f) \leq f$,
3. $F^{\uparrow}_{\mathcal{A},X} G^{\uparrow}_{\mathcal{A},X}(g) \leq g$,
4. $F^{\downarrow}_{\mathcal{A},X} G^{\downarrow}_{\mathcal{A},X}(g) \geq g$,
5. $F^{\uparrow}_{\mathcal{A},X} G^{\uparrow}_{\mathcal{A},X} F^{\uparrow}_{\mathcal{A},X}(f) = F^{\uparrow}_{\mathcal{A},X}(f)$,
6. $G^{\downarrow}_{\mathcal{A},X} F^{\downarrow}_{\mathcal{A},X} G^{\downarrow}_{\mathcal{A},X}(g) = G^{\downarrow}_{\mathcal{A},X}(g)$.
7. $F^{\uparrow}_{\mathcal{A},X}(f) \leq_Y g \Rightarrow f \leq_X G^{\uparrow}_{\mathcal{A},X}(g)$,
8. $G^{\downarrow}_{\mathcal{A},X}(g) \leq_X f \Rightarrow g \leq_Y F^{\downarrow}_{\mathcal{A},X}(f)$.

Proof. For simplicity, instead of $F^{\uparrow}_{\mathcal{A},X}$ we use only $F^{\uparrow}$ and similarly for other F-transforms.

(1) According to axiom (+) we have

$$G^{\uparrow}F^{\uparrow}(f)(x) = \sum_{y \in Y}^{\mathcal{R}^*} (\Phi(p_y(x)) \times^* F^{\uparrow}(f)(y)) = \quad (3)$$

$$\sum_{y \in Y}^{\mathcal{R}^*} (\Phi(p_y(x)) \times^* (\sum_{z \in X}^{\mathcal{R}} p_y(z) \times f(z))) = \quad (4)$$

$$\sum_{y \in Y}^{\mathcal{R}^*} \sum_{z \in X}^{\mathcal{R}} \Phi(p_y(x)) \times^* (p_y(z) \times f(z)). \quad (5)$$

Because $\Phi(p_y(x)) \times^* (p_y(z) \times f(z) \in R$, according to Definition 2(v) we have $\Phi(p_y(x)) \times^* (p_y(z) \times f(z) \geq 0$ and, according to axiom (+), for arbitrary $y \in Y$ it follows

$$\sum_{z \in X}^{\mathcal{R}} (\Phi(p_y(x)) \times^* (p_y(z) \times f(z))) \geq \Phi(p_y(x)) \times^* (p_y(x) \times f(x)) \geq f(x). \quad (6)$$

According to Definition 5, for arbitrary $y \in Y$ we have

$$\sum_{z \in X}^{\mathcal{R}} (\Phi(p_y(x)) \times^* (p_y(z) \times f(z))) \leq^* \Phi(p_y(x)) \times^* (p_y(x) \times f(x)) \leq^* f(x).$$

Therefore, using inequalities (3), (4), (5) and (6), we have

$$G^{\uparrow}F^{\uparrow}(f)(x) \leq^* \sum_{y \in Y}^{\mathcal{R}^*} \Phi(p_y(x)) \times^* (p_y(x) \times f(x)) \leq^*$$
$$\sum_{y \in Y}^{\mathcal{R}^*} f(x) = f(x) \times^* (\sum_{y \in Y} 1^*) = f(x) \times^* 1^* = f(x).$$

and we obtain $G^{\uparrow}F^{\uparrow}(f)(x) \geq f(x)$.

(2) According to Proposition 2 and previous part (1), we have

$$G^{\downarrow}F^{\downarrow}(f)(x) = \Phi(G^{\uparrow}F^{\uparrow}(\neg f))(x) \geq^* \Phi(\Phi(f(x))) = f(x),$$

and it follows that $G^{\downarrow}F^{\downarrow}(f)(x) \leq f(x)$.

(3) From the axiom (+) and its dual version, analogously as in (1), it follows

$$F^{\uparrow}G^{\uparrow}(g)(y) = \sum_{x \in X}^{\mathcal{R}} p_y(x) \times G^{\uparrow}(g)(x) =$$
$$\sum_{x \in X}^{\mathcal{R}} \sum_{t \in Y}^{\mathcal{R}^*} p_y(x) \times (\Phi(p_t(x)) \times^* g(t)) \leq$$
$$\sum_{x \in X}^{\mathcal{R}} p_y(x) \times (\Phi(p_y(x)) \times^* g(y)) \leq \sum_{x \in X}^{\mathcal{R}} g(y) = g(y).$$

(4) The proof can be analogously as in (2) and it will be omitted.

(5) From (3) it follows $F^{\uparrow}G^{\uparrow}F^{\uparrow}(f) \leq F^{\uparrow}(f)$ and because $F^{\uparrow}$ is order preserving, from $G^{\uparrow}F^{\uparrow}(f) \geq f$ we obtain the other inequality.

(6) From the property (3) it follows $G^{\downarrow}F^{\downarrow}(G^{\downarrow}(g)) \leq G^{\downarrow}(g)$. Because $G^{\downarrow}$ preserves the ordering $\leq$, from the property (4) it follows $G^{\downarrow}(F^{\downarrow}G^{\downarrow}(g)) \geq G^{\downarrow}(g)$.

(7) From $F^{\uparrow}(f) \leq_Y g$ it follows $F^{\uparrow}(f) \geq^*_Y g$ and because $G^{\uparrow}$ preserves $\leq^*_Y$-ordering, from (3) we obtain $f \geq^*_X G^{\uparrow}F^{\uparrow}(f) \geq^*_X G^{\uparrow}(g)$. Hence, $G^{\uparrow}(g) \geq_X f$.

(8) The proof can be done analogously as in (7) and it will be omitted.

□

From Propositions 2 and 3 and from Example 11 the following Corollary follows, which is important for possible applications of the F-transform theory for other fuzzy type structures.

Corollary 1.

1. *For arbitrary fuzzy type structure which is transformable to $\mathcal{R}$-fuzzy sets, the F-transform and inverse F-transform of this fuzzy type structure satisfy all properties from Proposition 2.*
2. *The F-transform and inverse F-transform of $\mathcal{L}$-fuzzy soft sets satisfy all properties of Proposition 3.*

4. Discussion and Conclusions

In lattice-valued fuzzy set theory, there are a number of fuzzy type structures that represent either generalisations of classical lattice-valued fuzzy sets or are built on these fuzzy sets using special constructions. A common feature of these new fuzzy type structures is especially their usability in specific applications, as evidenced by a number of research publications related to these structures. Although these new fuzzy type structures are in some way based on the classical theory of lattice-valued fuzzy sets, for the theory and methods used in these fuzzy type structures, own procedures are developed that often represent procedures modified in one or another way from the classical fuzzy set theory. It is therefore natural to ask the following question. *Is it possible to develop a new general theory based on methods or theories from classical lattice-valued fuzzy sets, which could be directly applied to a large part of new fuzzy type structures and at the same time had as many properties similar to the original theory as possible?* In this paper, we have tried to answer this question at least in part and to present one possible variant of such a general theory that could be successfully used to create new methods in new lattice-valued fuzzy type structures in a unified way. This method is based on the use of the so-called $\mathcal{R}$-fuzzy sets, i.e., fuzzy sets with commutative pre-ordered semirings $\mathcal{R}$ as value sets and with operations defined by operations of these semirings. In the paper, we have shown that many of new $\mathcal{L}$-fuzzy type structures, where $\mathcal{L}$ is a complete MV-algebra, including hesitant, intuitionistic, neutrosophic or fuzzy soft sets, including their mutual combinations *are transformable to $\mathcal{R}$-fuzzy sets for suitable semirings* $\mathcal{R}$. This allows us to "copy" the classical methods used in L-fuzzy sets and simply transfer these methods to these new fuzzy type structures in a unified way. In addition, if a given fuzzy type structure is transformable to R-fuzzy sets, *we can determine in advance what properties the possible application of a new method to this fuzzy type structure will have*.

As an illustrative example of such a procedure, we chose the F-transform method, which is often used in $\mathcal{L}$-fuzzy sets and their applications but so far was not used in these new fuzzy type structures. For this purpose, we defined the F-transform method for general $\mathcal{R}$-fuzzy sets in the way formally similar to the classical F-transform, and using the transformation of new $\mathcal{L}$-fuzzy type structures to $\mathcal{R}$-fuzzy sets, we introduced the F-transform for these new fuzzy type structures. The advantage of this procedure is, among other things, that the properties of these F-transforms in new fuzzy type structures are known in advance, because these properties are proven for any $\mathcal{R}$-fuzzy set.

Like any method, the use of $\mathcal{R}$-fuzzy sets has its limitations. One limitation is that to transform a given L-fuzzy type structure to $\mathcal{R}$-fuzzy sets, L is required to be a MV-algebra. However, since a large part of applications using L-fuzzy type structures is based on Łukasiewicz algebra L, which is the MV-algebra, this is not a fundamental limitation. A certain limitation of this method results from its ability to cover a number of fuzzy type structures. Due to the differences of individual structures, it is not expected that all these structures will have the same properties. As a result, general $\mathcal{R}$-fuzzy sets have only properties *that apply to all L-fuzzy type structures that can be transformed into $\mathcal{R}$-fuzzy sets*. On the other hand, *$\mathcal{R}$-fuzzy sets have not properties which are special properties only for some transformable fuzzy type structures*. An example of a property that does not apply to general $\mathcal{R}$-fuzzy sets is the axiom (+) from Definition 12. This axiom applies only to some types of $\mathcal{R}$-fuzzy sets and therefore only to some types of new L-fuzzy type structures.

We must emphasise that this paper is intended to be a theoretical basis for the possible transformation of methods which are standardly used in classical fuzzy sets to applications in various fuzzy type structures. For further development of methods based on the theory of $\mathcal{R}$-fuzzy sets, we will deal with, among other things, *rough $\mathcal{R}$-fuzzy sets* and their applications to various new fuzzy type structures. Although fuzzy type structures are often used in both theory and some applications, most real applications of these structures are based on the use of $[0,1]$ value set with specific operations instead of MV-algebra $\mathcal{L}$. It is therefore appropriate to focus on the transformation of these fuzzy type structures into $\mathcal{R}$-fuzzy set, where $\mathcal{R}$ will be appropriate semirings, or their generalisations with specific operations and properties.

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