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Numerical Simulation for a Multidimensional Fourth-Order Nonlinear Fractional Subdiffusion Model with Time Delay

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Abstract: The purpose of this paper is to develop a numerical scheme for the two-dimensional fourth-order fractional subdiffusion equation with variable coefficients and delay. Using the $L_2 - 1_\sigma$ approximation of the time Caputo derivative, a finite difference method with second-order accuracy in the temporal direction is achieved. The novelty of this paper is to introduce a numerical scheme for the problem under consideration with variable coefficients, nonlinear source term, and delay time constant. The numerical results show that the global convergence orders for spatial and time dimensions are approximately fourth order in space and second-order in time.

Keywords: nonlinear fractional differential equation of fourth-order; $L_2 - 1_\sigma$ formula; two-dimensional; variable coefficients; delay

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1. Introduction

The theory of fractional integrals and derivatives of arbitrary real or complex orders dates back approximately three centuries. In recent years, researchers have shown that differential equations with non-integer order derivatives can better model real-world physical phenomena than integer-order derivatives. Because of their importance in science and engineering, chemistry, physics, finance, and other fields [1], fractional differential equations have received increasing attention in recent years. In some cases, the analytical solution to fractional differential equations can be obtained using Laplace and Fourier transforms, as well as the Green function [2,3]. Due to the presence of a singular kernel in fractional derivatives, it is difficult to find an analytical solution for the majority of fractional differential equations.

In this paper, we implement a difference analog of the Caputo fractional derivative known as the $L_2 - 1_\sigma$ [4] formula, which provides second-order global convergence in the time direction. Abbaszadeh and Dehghan [5] developed an accurate and robust numerical solution for solving neutral delay time-space distributed-order fractional damped diffusion-wave equation based on the Galerkin meshless method. They also obtained a meshless numerical simulation for a fractional damped diffusion-wave equation with delay [6]. Abbaszadeh et al. [7] introduced an interpolating stabilized element-free Galerkin method for neutral delay fractional damped diffusion-wave equations. In some important applications, the fourth-order derivative term plays a crucial role. Nikan et al. [8] proposed an efficient numerical procedure, the local radial basis function created by the finite difference method, for computing the approximation solution of the time-fractional fourth-order reaction-diffusion equation in terms of the Riemann–Liouville derivative. The time-fractional derivative was estimated using the second-order accurate formulation, while the spatial terms were discretized using the local radial basis function generated by the finite difference method. Huang and Stynes [9] studied α-robust error analysis of a mixed $L1$-finite element method for a time-fractional fourth-order diffusion equation. For a two-dimensional distributed-order time-fractional fourth-order partial differential equation, Fakhar-Izadi [10] investigated the space-time Petrov–Galerkin spectral approach. The problem is converted to a multi-term time-fractional equation by using an appropriate Gauss-quadrature rule to discretize the distributed integral operator. Liu et al. [11] presented and discussed a finite difference/finite element algorithm for casting about for numerical solutions to a time-fractional fourth-order reaction–diffusion problem with a nonlinear reaction term, which is based on a finite difference approximation in time and a finite element method in spatial direction. Moreover, many numerical methods have been developed for solving different classes of fractional differential equations such as spectral methods [12–19], finite difference methods [20–25], finite element methods [26,27], finite volume method [28,29], and matrix transfer technique [30,31]. The study of delay differential equations with fractional derivatives is rapidly expanding these days since they are frequently employed in modeling of elastic media and stress–strain behavior for the torsional model, control difficulties, high-speed machining communications, and so on [1].

The multidimensional fourth order nonlinear fractional subdiffusion model with time delay is obtained from the standard problem by replacing the first-order time derivative by fractional derivatives in the Caputo sense. The analytical solution of this problem is only gained for the linear case when the coefficients constants. Recently, the authors of [32–34] constructed numerical techniques for this problem with fourth-order fractional diffusion wave equations and some modifications have been considered in [35]. In [36], a Newton-linearized compact ADI scheme for Riesz space fractional nonlinear reaction-diffusion equations was constructed and analyzed. The numerical computation for a class of fourth-order linear fractional subdiffusion equations with spatially variable coefficient under the first Dirichlet boundary conditions were considered in [37]. Two finite difference schemes with second-order accuracy were derived by applying $L_2 - 1_\sigma$ formula and $FL_2 - 1_\sigma$ formula, respectively, to approximate the time Caputo derivative. Liu et al. [38] studied and analyzed a Galerkin mixed finite element method combined with time second-order discrete scheme for solving nonlinear time fractional diffusion equation with fourth-order derivative term. Most of the numerical techniques opted in the articles are based upon $L1$ approximation of Caputo time derivative which gives first-order convergence for fractional order $0 < \alpha \leq 1$, where α is the fractional order (present in time-fractional derivative). Recently, articles based on orthogonal spline collocation method are published [39] which focus on weakly singular solutions and constructing a high-order numerical scheme for fourth-order subdiffusion equations.

In the real world, parameters such as the coefficients in the problem under consideration are spatially or temporally variable. Therefore, presenting a numerical scheme and finding numerical solution becomes a tedious task. There are many difficulties to build numerical schemes for fourth-order fractional differential equations with delay due to non-locality of the problem, variable coefficients, nonlinearity, and error depends upon the history of considered problem. Therefore, the main issue we address in this paper is introducing an effective numerical approach for multi-dimensional fourth-order fractional subdiffusion equations with delay. In the studies discussed above and mainly in literature, there is a lack of studies available on delay differential equations.

In this paper, our target is to present high ordered difference scheme to solve the following two-dimensional time-fractional subdiffusion equation of fourth-order with variable coefficients and nonlinear source term having a delay constant.

$$
\begin{aligned}
{}_{0}^{C}D_{t}^{\alpha}U(x_1,x_2,t) + a(x_1,x_2)\frac{\partial^4\mathcal{U}(x_1,x_2,t)}{\partial x_1^4} + b(x_1,x_2)\frac{\partial^4\mathcal{U}(x_1,x_2,t)}{\partial x_2^4}& \\
= F(x_1,x_2,t,\mathcal{U}(x_1,x_2,t),\mathcal{U}(x_1,x_2,t-s)),& \\
s>0, \quad (x_1,x_2)\in\Omega, \quad t\in(0,T],& \qquad (1a)
\end{aligned}
$$

with the initial and the boundary conditions:

$$\mathcal{U}(x_1,x_2,t) = \phi(x_1,x_2,t), \quad (x_1,x_2)\in\Omega, \quad t\in[-s,0], \qquad (1b)$$

$$\mathcal{U}(0,x_2,t) = \alpha_1(x_2,t), \quad \mathcal{U}(L,x_2,t) = \alpha_2(x_2,t), \quad x_2\in[0,L_2], \quad t\in[0,T], \qquad (1c)$$

$$\mathcal{U}(x_1,0,t) = \beta_1(x_1,t), \quad \mathcal{U}(x_1,M,t) = \beta_2(x_1,t), \quad x_1\in[0,L_1], \quad t\in[0,T], \qquad (1d)$$

$$\frac{\partial^2\mathcal{U}(0,x_2,t)}{\partial x_1^2} = \gamma_1(x_2,t), \quad \frac{\partial^2\mathcal{U}(L,x_2,t)}{\partial x_1^2} = \gamma_2(x_2,t), \quad x_2\in[0,L_2], \quad t\in[0,T], \qquad (1e)$$

$$\frac{\partial^2\mathcal{U}(x_1,0,t)}{\partial x_2^2} = \gamma_3(x_1,t), \quad \frac{\partial^2\mathcal{U}(x_1,L,t)}{\partial x_2^2} = \gamma_4(x_1,t), \quad x_1\in[0,L_1], \quad t\in[0,T]. \qquad (1f)$$

where $\Omega = (0,L_1)\times(0,L_2)$ and $\partial\Omega$ is its boundary. Here, ${}_{0}^{C}D_{t}^{\alpha}\mathcal{U}(x_1,x_2,t)$ is the temporal Caputo fractional operator of order $\alpha\in(0,1)$, $s>0$ is the constant delay parameter and $F(x_1,x_2,t,\mathcal{U}(x_1,x_2,t),\mathcal{U}(x_1,x_2,t-s))$ is the nonlinear source function with delay. $\phi(x_1,x_2,t)$, $\alpha_1(x_2,t)$, $\alpha_2(x_2,t)$, $\beta_1(x_1,t)$, $\beta_2(x_1,t)$, $\gamma_1(x_2,t)$, $\gamma_2(x_2,t)$, $\gamma_3(x_1,t)$, and $\gamma_4(x_1,t)$ are all given sufficiently smooth functions. Now, we present the definitions of fractional integral, and fractional derivatives which would be used later.

Definition 1. *The Caputo fractional derivative [40] of order $\alpha\in[0,1]$ is defined by*

$$
{}_{0}^{C}D_{t}^{\alpha}\mathcal{U}(x_1,x_2,t) := \begin{cases} \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\xi)^{-\alpha}\frac{\partial\mathcal{U}}{\partial\xi}(x_1,x_2,\xi)d\xi, & 0\le\alpha<1; \\ \mathcal{U}_t(x_1,x_2,t), & \alpha=1. \end{cases}
$$

Next, for discretization, we partition the mesh $\Omega_{h_{x_1}h_{x_2}t} = \Omega_{h_{x_1}}\times\Omega_{h_{x_2}}\times\Omega_t$, where $\Omega_{h_{x_1}} = \{x_{1,i}|0\le i\le M_1\}$, $\Omega_{h_{x_2}} = \{x_{2,j}|0\le j\le M_2\}$, $\Omega_t = \{t_k|-n\le k\le N\}$. Introducing the positive integers M_1, M_2, and N, let $h_x = \frac{L_1}{M_1}$, $h_y = \frac{L_2}{M_2}$, $\tau = \frac{s}{n}$ ($n>0$ is a positive constant) are the spatial and temporal steps respectively. Define $x_{1,i} = i\,h_{x_1}$, $0\le i\le M_1$, $x_{2,j} = j\,h_{x_2}$, $0\le j\le M_2$, $t_k = k\,\tau$, $-n\le k\le N$.

1.1. The L_2-1_σ Discretization of the Time Fractional Operator

Alikhanov [4] constructed a discrete approximation for Caputo fractional derivative which will be mentioned firstly. Next, some Lemmas important in the later context are introduced to aid in constructing the numerical scheme for the system (1).

Definition 1 ([4]). *Let $\sigma = 1-\frac{\alpha}{2}$, $0<\alpha<1$, the approximated formula for $\mathcal{U}(t)\in C^3[0,T]$ at the fixed point $t_{k-1+\sigma}$, $k\in\{1,2,\ldots,N\}$, is called L_2-1_σ formula of second order temporal convergence for $\alpha\in(0,1)$, given as follows:*

Let $a_0 = \sigma^{1-\alpha}$, $a_l = (l+\sigma)^{1-\alpha} - (l-1+\sigma)^{1-\alpha}$, $l\ge 1$,

$b_l = \frac{1}{2-\alpha}[(l+\sigma)^{2-\alpha} - (l-1+\sigma)^{2-\alpha}] - \frac{1}{2}[(l+\sigma)^{1-\alpha} + (l-1+\sigma)^{1-\alpha}]$, $l\ge 1$,

For $k=0$, $c_0^{(k)} = a_0$, and

further when $k\ge 1$,

$$
c_j^{(k)} = \begin{cases} a_0+b_1, & j=0, \\ a_j+b_{j+1}-b_j, & 1\le j\le k-1, \\ a_j-b_j, & j=k. \end{cases} \qquad (2)
$$

Defining

$$ {}_0^C D_t^{\alpha} \mathcal{U}_{ij}^{k-1+\sigma} = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \left[c_0^{(k)} \mathcal{U}^k - \sum_{p=1}^{k-1} (c_{k-p-1}^{(k)} - c_{k-i}^{(k)}) \mathcal{U}^p - c_{k-1}^{(k)} \mathcal{U}^0 \right]. $$

Below given Lemma estimates the error of the $L_2 - 1_\sigma$ formula.

Lemma 1 ([4]). *For any $\alpha \in (0,1)$ and $u \in C^3[0, t_{k+1}]$, it holds that*

$$ \left| {}_0^C D_t^{\alpha} \mathcal{U}(t)|_{t=t_{k-1+\sigma}} - {}_0^C D_t^{\alpha} \mathcal{U}^{k-1+\sigma} \right| = O(\tau^{3-\alpha}). $$

The next lemmas are giving some technical properties for Alikhanov formula.

Lemma 2 ([41,42]). *Let function $\mathcal{U}(x) \in C^6[x_{i-1}, x_{i+1}]$, $x_{i+1} = x_i + h$, $x_{i-1} = x_i - h$ and $\mathfrak{T}(q) = (1-q)^3[5 - 3(1-q)^2]$, then*

$$ \frac{\mathcal{U}''(x_{i+1}) + 10\mathcal{U}''(x_i) + \mathcal{U}''(x_{i-1})}{12} = \frac{\mathcal{U}(x_{i+1}) - 2\mathcal{U}(x_i) + \mathcal{U}(x_{i-1})}{h^2} + \frac{h^4}{360} \int_0^1 [\mathcal{U}^{(6)}(x_i - qh) + \mathcal{U}^{(6)}(x_i + qh)] \mathfrak{T}(q)\, dq. \tag{3} $$

Define

$$ \nu_h = \{\mathcal{U} = \mathcal{U}_{ij} | (i,j) \in \Omega_{h_{x_1} h_{x_2}}\}, $$
$$ \nu_h^* = \{\mathcal{U} | \mathcal{U} \in \nu_h; \mathcal{U}_{ij} = 0 | (i,j) \in \partial\Omega\}. $$

For any $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W} \in \nu_h$, we introduce the following notations:

$$ \delta_{x_1} \mathcal{U}_{i-\frac{1}{2} j} = \frac{1}{h_x}(U_{ij} - U_{i-1 j}), \quad \delta_{x_1}^2 \mathcal{U}_{ij} = \frac{1}{h_x}(\delta_{x_1} U_{i+\frac{1}{2} j} - \delta_{x_1} \mathcal{U}_{i-\frac{1}{2} j}) $$
$$ \delta_{x_2} \mathcal{U}_{i-\frac{1}{2} j} = \frac{1}{h_{x_2}}(\mathcal{U}_{ij} - \mathcal{U}_{i j-1}), \quad \delta_{x_2}^2 U_{ij} = \frac{1}{h_{x_2}}(\delta_{x_2} \mathcal{U}_{i j+\frac{1}{2}} - \delta_{x_2} \mathcal{U}_{i j-\frac{1}{2}}) $$

$$ \mathcal{H}_1 \mathcal{U}_{ij} = \frac{1}{12}(\mathcal{U}_{i-1 j} + 12\mathcal{U}_{ij} + \mathcal{U}_{i+1 j}), \tag{4} $$
$$ \mathcal{H}_2 \mathcal{U}_{ij} = \frac{1}{12}(\mathcal{U}_{i j-1} + 12\mathcal{U}_{ij} + \mathcal{U}_{i j+1}), \tag{5} $$
$$ \mathcal{H} \mathcal{U}_{ij} = \mathcal{H}_1 \mathcal{H}_2 \mathcal{U}_{ij}. \tag{6} $$

Now, we define inner products and discrete norms useful in the analysis section of the described numerical scheme (for any $\mathcal{U}, \mathcal{V} \in \nu_h^*$):

$$(\mathcal{U},\mathcal{V}) = h_{x_1}h_{x_2}\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} U_{ij}V_{ij}, \quad ||\mathcal{U}||^2 = (\mathcal{U},\mathcal{U})$$

$$||\mathcal{U}||_\infty = \max_{1\le i\le M_1-1, 1\le j\le M_2-1} |\mathcal{U}_{ij}|$$

$$(\delta_{x_1}\mathcal{U},\delta_{x_1}\mathcal{V}) = h_{x_1}h_{x_2}\sum_{i=1}^{M_1}\sum_{j=1}^{M_2-1} \delta_{x_1}U_{i-\frac{1}{2}j}\delta_{x_1}V_{i-\frac{1}{2}j},$$

$$(\delta_{x_2}\mathcal{U},\delta_{x_2}\mathcal{V}) = h_{x_1}h_{x_2}\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2} \delta_{x_2}\mathcal{U}_{i\,j-\frac{1}{2}}\delta_{x_2}\mathcal{V}_{i\,j-\frac{1}{2}},$$

$$(\delta^2_{x_1}\mathcal{U},\delta^2_{x_1}\mathcal{V}) = h_{x_1}h_{x_2}\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} \delta^2_{x_1}\mathcal{U}_{ij}\delta^2_{x_1}\mathcal{V}_{ij},$$

$$(\delta^2_{x_2}\mathcal{U},\delta^2_{x_2}V) = h_{x_1}h_{x_2}\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} \delta^2_{x_2}\mathcal{U}_{ij}\delta^2_{x_2}\mathcal{V}_{ij},$$

1.2. Numerical Scheme Construction

A numerical scheme is constructed by employing the linear operator $(\mathcal{H})$ for the spatial dimensions and the $L_2 - 1_\sigma$ approximation of Caputo time-fractional derivative.

First, consider

$$\mathcal{V}(x_1,x_2,t) = \frac{\partial^2\mathcal{U}(x_1,x_2,t)}{\partial x^2}, \quad \mathcal{W}(x_1,x_2,t) = \frac{\partial^2\mathcal{U}(x_1,x_2,t)}{\partial x_2^2}. \tag{7}$$

Substituting (7) into (1a), yields

$$\begin{aligned}{}^C_0D^\alpha_t\mathcal{U}(x_1,x_2,t)+a(x_1,x_2)\frac{\partial^2\mathcal{V}(x_1,x_2,t)}{\partial x_1^2} + b(x_1,x_2)\frac{\partial^2\mathcal{W}(x_1,x_2,t)}{\partial x_2^2}&\\ = F(x_1,x_2,t,\mathcal{U}(x,y,t),\mathcal{U}(x_1,x_2,t-s)),& \quad (8)\\ \mathcal{V}(x_1,x_2,t) = \frac{\partial^2\mathcal{U}}{\partial x_1^2}(x_1,x_2,t),& \quad (9)\\ \mathcal{W}(x_1,x_2,t) = \frac{\partial^2\mathcal{U}}{\partial x_2^2}(x_1,x_2,t).& \quad (10)\end{aligned}$$

The numerical scheme for (1) is ready after that order reduction. Considering Equations (8)–(10) at $(x_{1,i},x_{2,j},t_{k-1+\sigma})$ yields

$$\begin{aligned}{}^C_0D^\alpha_t\mathcal{U}(x_{1,i},x_{2,j},t_{k-1+\sigma}) + a(x_{1,i},x_{2,j})\frac{\partial^2\mathcal{V}}{\partial x_1^2}(x_{1,i},x_{2,j},t_{k-1+\sigma})&\\ + b(x_{1,i},x_{2,j})\frac{\partial^2\mathcal{W}}{\partial x_2^2}(x_{1,i},x_{2,j},t_{k-1+\sigma}) = F\Big(x_{1,i},x_{2,j},t_{k-1+\sigma}&\\ ,\mathcal{U}(x_{1,i},x_{2,j},t_{k-1+\sigma}),\mathcal{U}(x_{1,i},x_{2,j},t_{k-1+\sigma-n})\Big),& \quad (11)\\ \mathcal{V}(x_{1,i},x_{2,j},t_{k-1+\sigma}) = \frac{\partial^2\mathcal{U}}{\partial x_1^2}(x_{1,i},x_{2,j},t_{k-1+\sigma}),& \quad (12)\\ \mathcal{W}(x_{1,i},x_{2,j},t_{k-1+\sigma}) = \frac{\partial^2\mathcal{U}}{\partial x_2^2}(x_{1,i},x_{2,j},t_{k-1+\sigma}).& \quad (13)\end{aligned}$$

Using Lemma 1, we can write

$$ {}_{0}^{C}D_{t}^{\alpha}\mathcal{U}(x_{1,i},x_{2,j},t_{k-1+\sigma}) = {}_{0}^{C}D_{t}^{\alpha}\mathcal{U}_{ij}^{k-1+\sigma} + O(\tau^2). \tag{14} $$

Using the Taylor's expansion we can write the following equalities:

$$ \mathcal{U}(x_{1,i},x_{2,j},t_{k-1+\sigma}) = \sigma\mathcal{U}(x_{1,i},x_{2,j},t_k) + (1-\sigma)\mathcal{U}(x_{1,i},x_{2,j},t_{k-1}) \tag{15} $$

$$ \begin{aligned} \frac{\partial^2\mathcal{V}}{\partial x_1^2}(x_{1,i},x_{2,j},t_{k-1+\sigma}) &= \sigma\frac{\partial^2\mathcal{V}}{\partial x_1^2}(x_{1,i},x_{2,j},t_k) \\ &\quad + (1-\sigma)\frac{\partial^2\mathcal{V}}{\partial x_1^2}(x_{1,i},x_{2,j},t_{k-1}) + O(\tau^2), \end{aligned} \tag{16} $$

$$ \begin{aligned} \frac{\partial^2\mathcal{W}}{\partial x_2^2}(x_{1,i},x_{2,j},t_{k-1+\sigma}) &= \sigma\frac{\partial^2\mathcal{W}}{\partial x_2^2}(x_{1,i},x_{2,j},t_k) \\ &\quad + (1-\sigma)\frac{\partial^2\mathcal{W}}{\partial x_2^2}(x_{1,i},x_{2,j},t_{k-1}) + O(\tau^2). \end{aligned} \tag{17} $$

Acting the operator $\mathcal{H}_1$ on (16) gives

$$ \begin{aligned} \mathcal{H}_1\frac{\partial^2\mathcal{V}}{\partial x^2}(x_{1,i},x_{2,j},t_{k-1+\sigma}) &= \sigma\mathcal{H}_1\frac{\partial^2\mathcal{V}}{\partial x^2}(x_{1,i},x_{2,j},t_k) \\ &\quad + (1-\sigma)\mathcal{H}_1\frac{\partial^2\mathcal{V}}{\partial x^2}(x_{1,i},x_{2,j},t_{k-1}) + O(\tau^2), \\ &= \delta_{x_1}^2\mathcal{V}_{ij}^{k-1+\sigma} + O(\tau^2) + O(h_{x_1}^4). \end{aligned} \tag{18} $$

Similarly, acting the operator $\mathcal{H}_2$ on (17) gives

$$ \mathcal{H}_2\frac{\partial^2\mathcal{W}}{\partial y^2}(x_{1,i},x_{2,j},t_{k-1+\sigma}) = \delta_{x_2}^2\mathcal{W}_{ij}^{k-1+\sigma} + O(\tau^2) + O(h_{x_2}^4). \tag{19} $$

Employing Taylor's expansion for the source term (linearization) gives

$$ \begin{aligned} &F(x_{1,i},x_{2,j},t_{k-1+\sigma},U(x_{1,i},x_{2,j},t_{k-1+\sigma}),\mathcal{U}(x_{1,i},x_{2,j},t_{k-1+\sigma-n})) \\ &= F\Big(x_{1,i},x_{2,j},t_{k-1+\sigma},\sigma(2\mathcal{U}_{ij}^{k-1}-\mathcal{U}_{ij}^{k-2}) + (1-\sigma)(2U_{ij}^{k-2}-\mathcal{U}_{ij}^{k-3}) \\ &\qquad ,\sigma U_{ij}^{k-n} + (1-\sigma)U_{ij}^{k-n-1}\Big) + O(\tau^2), \quad \text{(using Equation (15))} \\ &= F\Big(x_{1,i},x_{2,j},t_{k-1+\sigma},2\sigma\mathcal{U}_{ij}^{k-1} + (2-3\sigma)\mathcal{U}_{ij}^{k-2} - (1-\sigma)\mathcal{U}_{ij}^{k-3} \\ &\qquad ,\sigma\mathcal{U}_{ij}^{k-n} + (1-\sigma)U_{ij}^{k-n-1}\Big) + O(\tau^2). \end{aligned} \tag{20} $$

Acting the operator $\mathcal{H}$ on Equations (11)–(13) and substituting above equations gives

$$ \begin{aligned} &\mathcal{H}{}_{0}^{C}D_{t}^{\alpha}\mathcal{U}_{ij}^{k-1+\sigma} + \mathcal{H}_2 a(x_{1,i},x_{2,j})\delta_{x_1}^2 V_{ij}^{k-1+\sigma} + \mathcal{H}_1 b(x_{1,i},x_{2,j})\delta_{x_2}^2 W_{ij}^{k-1+\sigma} \\ &\quad = \mathcal{H}F\Bigg(x_{1,i},x_{2,j},t_{k-1+\sigma},2\sigma\mathcal{U}_{ij}^{k-1} + (2-3\sigma)\mathcal{U}_{ij}^{k-2} \\ &\qquad\qquad - (1-\sigma)\mathcal{U}_{ij}^{k-3},\sigma\mathcal{U}_{ij}^{k-n} + (1-\sigma)\mathcal{U}_{ij}^{k-n-1}\Bigg) \\ &\quad + |R_{ij}^k|, \end{aligned} \tag{21} $$

$$ \mathcal{H}\mathcal{V}(x_{1,i},x_{2,j},t_{k-1+\sigma}) = \mathcal{H}_2\delta_{x_1}^2\mathcal{U}_{ij}^{k-1+\sigma} + |S_{ij}^k|, \tag{22} $$

$$ \mathcal{H}\mathcal{W}(x_{1,i},x_{2,j},t_{k-1+\sigma}) = \mathcal{H}_1\delta_{x_2}^2\mathcal{U}_{ij}^{k-1+\sigma} + |C_{ij}^k|. \tag{23} $$

where $|R_{ij}^k| = |O(\tau^2 + h_{x_1}^4 + h_{x_2}^4)|, 1 \le i \le M_1 - 1, 1 \le j \le M_2 - 1, 0 \le k \le N,$
$|S_{ij}^k| = |O(\tau^2) + O(h_{x_1}^4)|, 1 \le i \le M_1 - 1, 1 \le j \le M_2 - 1, 0 \le k \le N,$
$|C_{ij}^k| = |O(\tau^2) + O(h_{x_2}^4)|, 1 \le i \le M_1 - 1, 1 \le j \le M_2 - 1, 0 \le k \le N$, and there exists a positive constant C such that $|R_{ij}^k| + |S_{ij}^k| + |C_{ij}^k| \le C(\tau^2 + h_{x_1}^4 + h_{x_2}^4), 1 \le i \le M_1 - 1,$ $1 \le j \le M_2 - 1, 0 \le k \le N.$
Noticing the initial and boundary conditions at $(x_{1,i}, x_{2,j}, t_k)$

$$\begin{aligned}
&\mathcal{U}(x_{1,i}, x_{2,j}, t_k) = \phi(x_{1,i}, x_{2,j}, t_k),\\
&0 \le i \le M_1, \quad 0 \le j \le M_2, \quad -n \le k \le 0, && (24)\\
&\mathcal{U}(0, x_{2,j}, t_k) = \alpha_1(x_{2,j}, t_k), \quad \mathcal{U}(L_1, x_{2,j}, t_k) = \alpha_2(x_{2,j}, t),\\
&0 \le j \le M_2, \quad 0 \le k \le N, && (25)\\
&\mathcal{U}(x_{1,i}, 0, t_k) = \beta_1(x_{1,i}, t_k), \quad \mathcal{U}(x_{1,i}, L_2, t_k) = \beta_2(x_{1,i}, t_k),\\
&0 \le i \le M_1, \quad 0 \le k \le N, && (26)\\
&\frac{\partial^2 \mathcal{U}}{\partial x^2}(0, x_{2,j}, t_k) = \gamma_1(x_{2,j}, t_k), \quad \frac{\partial^2 \mathcal{U}}{\partial x^2}(L_1, x_{2,j}, t_k) = \gamma_2(x_{2,j}, t_k),\\
&0 \le j \le M_2, \quad 0 \le k \le N, && (27)\\
&\frac{\partial^2 \mathcal{U}}{\partial x_2^2}(x_{1,i}, 0, t_k) = \gamma_3(x_{1,i}, t_k), \quad \frac{\partial^2 \mathcal{U}}{\partial x_2^2}(x_{1,i}, L, t_k) = \gamma_4(x_{1,i}, t_k),\\
&0 \le i \le M_1, \quad 0 \le k \le N. && (28)
\end{aligned}$$

Omitting the small terms in (21)–(23), we construct the numerical scheme for (1) as follows:

$$\begin{aligned}
&\mathcal{H}_0^C D_t^\alpha \mathcal{U}_{ij}^{k-1+\sigma} + \mathcal{H}_2 a(x_{1,i}, x_{2,j}) \delta_{x_1}^2 \mathcal{V}_{ij}^{k-1+\sigma} + \mathcal{H}_1 b(x_{1,i}, x_{2,j}) \delta_{x_2}^2 \mathcal{W}_{ij}^{k-1+\sigma} = \mathcal{H}\mathcal{F}_{i,j}^\sigma,\\
&1 \le i \le M_1 - 1, \quad 1 \le j \le M_2 - 1, \quad 0 < k \le N, && (29a)\\
&\mathcal{H}\mathcal{V}(x_{1,i}, x_{2,j}, t_{k-1+\sigma}) = \mathcal{H}_2 \delta_{x_1}^2 \mathcal{U}_{ij}^{k-1+\sigma},\\
&1 \le i \le M_1 - 1, \quad 1 \le j \le M_2 - 1, \quad 0 < k \le N, && (29b)\\
&\mathcal{H}\mathcal{W}(x_{1,i}, x_{2,j}, t_{k-1+\sigma}) = \mathcal{H}_1 \delta_{x_2}^2 \mathcal{U}_{ij}^{k-1+\sigma},\\
&1 \le i \le M_1 - 1, \quad 1 \le j \le M_2 - 1, \quad 0 < k \le N, && (29c)\\
&\mathcal{U}(x_{1,i}, x_{2,j}, t_k) = \phi(x_{1,i}, x_{2,j}, t_k),\\
&0 \le i \le M_1, \quad 0 \le j \le M_2, -n \le k \le 0, && (29d)\\
&\mathcal{U}(0, x_{2,j}, t_k) = \alpha_1(x_{2,j}, t_k), \quad \mathcal{U}(L_1, x_{2,j}, t_k) = \alpha_2(x_2, t),\\
&0 \le j \le M_2, \quad 0 \le k \le N, && (29e)\\
&\mathcal{U}(x_{1,i}, 0, t_k) = \beta_1(x_{1,i}, t_k), \quad \mathcal{U}(x_{1,i}, L_2, t_k) = \beta_2(x_{1,i}, t_k),\\
&0 \le i \le M_1, \quad 0 \le k \le N, && (29f)\\
&\frac{\partial^2 \mathcal{U}}{\partial x_1^2}(0, x_{2,j}, t_k) = \gamma_1(x_{2,j}, t_k), \quad \frac{\partial^2 \mathcal{U}}{\partial x_1^2}(L_1, x_{2,j}, t_k) = \gamma_2(x_{2,j}, t_k),\\
&0 \le j \le M_2, \quad 0 \le k \le N, && (29g)\\
&\frac{\partial^2 \mathcal{U}}{\partial x_2^2}(x_{1,i}, 0, t_k) = \gamma_3(x_{1,i}, t_k), \quad \frac{\partial^2 \mathcal{U}}{\partial x_2^2}(x_{1,i}, L_2, t_k) = \gamma_4(x_{1,i}, t_k),\\
&0 \le i \le M_1, \quad 0 \le k \le N, && (29h)
\end{aligned}$$

such that

$$\mathcal{F}^{\sigma}_{i,j} = F\Bigg(x_{1,i}, x_{2,j}, t_{k-1+\sigma}, 2\sigma\mathcal{U}^{k-1}_{ij} + (2-3\sigma)\mathcal{U}^{k-2}_{ij} - (1-\sigma)\mathcal{U}^{k-3}_{ij}, \sigma\mathcal{U}^{k-n}_{ij} + (1-\sigma)\mathcal{U}^{k-n-1}_{ij}\Bigg).$$

2. Numerical Simulations

Two examples are presented here to show the efficiency of numerical scheme (29) for the problem defined in (1). Let $u^k = u^k(h_{x_1}, h_{x_2}, \tau)$ be the solution of proposed numerical scheme.

Denote $E_\infty(h_{x_1}, h_{x_2}, \tau) = \max_{-n\le k\le N} ||U^k - u^k||$. $Order(\tau) = \log_2\big(\frac{E_\infty(h_{x_1}, h_{x_2}, 2\tau)}{E_\infty(h_{x_1}, h_{x_2}, \tau)}\big)$, $Order(h_{x_1}) = \log_2\big(\frac{E_\infty(2h_{x_1}, h_{x_2}, \tau)}{E_\infty(h_{x_1}, h_{x_2}, \tau)}\big)$, $Order(h_{x_2}) = \log_2\big(\frac{E_\infty(h_{x_1}, 2h_{x_2}, \tau)}{E_\infty(h_{x_1}, h_{x_2}, \tau)}\big)$.

Example 1. *Consider the following two-dimensional fourth-order subdiffusion equation:*

$$\begin{aligned}
{}^C_0D^{\alpha}_t\mathcal{U}(x_1,x_2,t) + x_1x_2\frac{\partial^4\mathcal{U}}{\partial x_1^4} + (x_1+x_2)\frac{\partial^4\mathcal{U}}{\partial x_2^4} &= \mathcal{U}^3(x_1,x_2,t) + \mathcal{U}(x_1,x_2,t-0.3) \\
&\quad + (x_1^2+x_2^2)e^{-t}, \\
\mathcal{U}(x_1,x_2,t) &= e^{-t}x_1^5(2-x_1)^5x_2^5(2-x_2)^5 \quad (x_1,x_2)\in\Omega, \quad t\in[-0.3,0], \\
\mathcal{U}(0,x_2,t) &= 0, \quad \mathcal{U}(1,x_2,t) = e^{-t}x_1^5x_2^5(2-x_2)^5, \quad x_2\in[0,1], \quad t\in[0,1], \\
\mathcal{U}(x_1,0,t) &= 0, \quad \mathcal{U}(x_1,1,t) = e^{-t}x_1^5(2-x_1)^5x_2^5, \quad x\in[0,1], \quad t\in[0,1], \\
\frac{\partial^2\mathcal{U}(0,x_2,t)}{\partial x_1^2} &= 0, \quad \frac{\partial^2\mathcal{U}(1,x_2,t)}{\partial x_1^2} = -10e^{-t}x_2^5(2-x_2)^5, \quad x_2\in[0,1], \quad t\in[0,1], \\
\frac{\partial^2\mathcal{U}(x_1,0,t)}{\partial x_2^2} &= 0, \quad \frac{\partial^2\mathcal{U}(x_1,1,t)}{\partial x_2^2} = -10e^{-t}x_1^5(2-x_2)^5, \quad x_1\in[0,1], \quad t\in[0,1].
\end{aligned}$$

where the initial and boundary conditions are defined specially so that the exact solution of above problem is

$$\mathcal{U}(x_1,x_2,t) = e^{-t}x_1^5(2-x_1)^5x_2^5(2-x_2)^5.$$

Problem (1) is solved using our proposed method. The numerical results are listed in Tables 1 and 2. It can be monitored from the results that our proposed method gives $O(\tau^2 + h_{x_1}^4 + h_{x_2}^4)$ order of convergences. The numerical results are obtained for different values of fractional order $\alpha \in (0,1)$ with varying values of time and spatial step sizes. Figure 1 gives surface plot for different values of time $t = 0, 0.5, 0.75, 1$. The variation in numerical values can be observed from the varying range in $U(x_1, x_2, t)$ with respect to change in the value of time variable. The obtained convergence orders are approximately 4, 4, 2 for spatial directions and time direction, respectively, and same is validated in Stability and Convergence section discussed above.

Table 1. The maximum error and convergence order in time-dimension for example 1.

$h_{x_1} = h_{x_2}$	α	τ	E_∞	**Order**(τ)	**CPU(s)**
	0.3	$1/16$	2.6368×10^{-5}	-	0.5698
		$1/32$	6.8212×10^{-6}	1.9507	0.8240
		$1/64$	1.7197×10^{-6}	1.9879	2.1024
	0.6	$1/16$	3.0455×10^{-5}	-	0.5695
$1/50$		$1/32$	7.7345×10^{-6}	1.9773	0.8240
		$1/64$	1.9366×10^{-6}	1.9978	2.1026
	0.9	$1/16$	1.5525×10^{-6}	-	0.5692
		$1/32$	3.9139×10^{-7}	1.9879	0.8241
		$1/64$	9.1213×10^{-8}	2.1013	2.1022

Table 2. The maximum error and average convergence order in spatial-dimensions for Example 1.

τ	α	$h_{x_1} = h_{x_2}$	E_∞	**Spatial-Order**	**CPU(s)**
	0.2	$1/15$	1.0550×10^{-4}	-	0.5690
		$1/30$	7.2704×10^{-6}	3.8591	0.8237
		$1/60$	4.5902×10^{-7}	3.9854	2.0191
	0.5	$1/15$	5.5330×10^{-5}	-	0.5692
$1/50$		$1/30$	3.7425×10^{-6}	3.8860	0.8237
		$1/60$	2.3451×10^{-7}	3.9963	2.0189
	0.7	$1/15$	1.5221×10^{-4}	-	0.5690
		$1/30$	9.8463×10^{-6}	3.9503	0.8235
		$1/60$	6.0793×10^{-7}	4.0176	2.0193

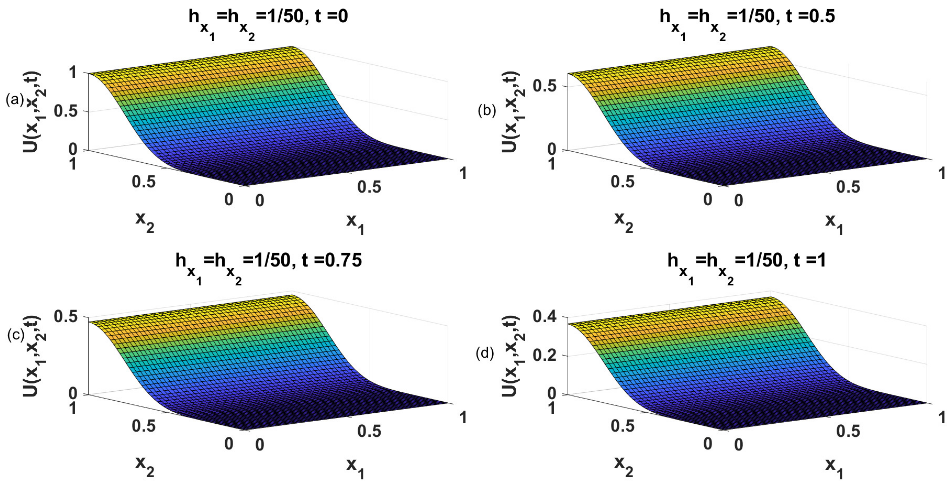

Figure 1. The numerical solutions for Example 1 for different time values (**a**: $t = 0$, **b**: $t = 0.5$, **c**: $t = 0.75$, **d**: $t = 1$).

We present a second problem solved with described numerical scheme. Maximum error was being computed for varying step sizes in space and time dimensions as given in Tables 3 and 4. The decrease in error can be observed from these tables with decrease in step size. Moreover, we can observe that approximate convergence orders for space and time are 4 and 2, respectively. Figure 2 depicts the numerical solution for different values of time variable with spatial step sizes $h_{x_1} = h_{x_2} = 1/50$. Moreover, the convergence orders for spatial and time dimensions are authenticated in Stability and convergence section as well.

Example 2. *Consider the two dimensional fourth order subdiffusion equation in the following form:*

$$
\begin{aligned}
{}_{0}^{C}D_t^{\alpha}\mathcal{U}(x_1,x_2,t)+\exp(x_1+x_2)\frac{\partial^4\mathcal{U}}{\partial x_1^4}+(x_1-x_2)\frac{\partial^4\mathcal{U}}{\partial x_2^4} &= \mathcal{U}^2(x_1,x_2,t)+\mathcal{U}(x_1,x_2,t-0.5)\\
&+\frac{10t^{3.8}}{\Gamma(0.8)}(x_1+x_2)^2\sin(t+1),\\
\mathcal{U}(x_1,x_2,t) &= x_1^4(2-x_1)^4x_2^4(2-x_2)^4\sin(t+1), \quad (x_1,x_2)\in\Omega, \quad t\in[-0.5,0],\\
\mathcal{U}(0,x_2,t) &= 0, \quad \mathcal{U}(2,x_2,t)=0, \quad x_2\in[0,2], \quad t\in[0,2],\\
\mathcal{U}(x_1,0,t) &= 0, \quad \mathcal{U}(x_1,2,t)=0, \quad x_1\in[0,2], \quad t\in[0,2],\\
\frac{\partial^2\mathcal{U}(0,x_2,t)}{\partial x_1^2} &= 0, \quad \frac{\partial^2 U(2,x_2,t)}{\partial x_1^2}=0, \quad x_2\in[0,2], \quad t\in[0,1],\\
\frac{\partial^2\mathcal{U}(x_1,0,t)}{\partial x_2^2} &= 0, \quad \frac{\partial^2\mathcal{U}(x_1,2,t)}{\partial x_2^2}=0, \quad x_1\in[0,2], \quad t\in[0,1].
\end{aligned}
$$

The exact solution of above example is

$$\mathcal{U}(x,y,t)=x_1^4(2-x_1)^4x_2^4(2-x_2)^4\sin(t+1).$$

Table 3. The maximum error and average convergence order in spatial dimensions for Example 2.

τ	α	$h_{x_1}=h_{x_2}$	E_∞	Spatial-Order
	0.2	1/15	4.0674×10^{-5}	-
		1/30	2.8291×10^{-6}	3.8457
		1/60	1.8400×10^{-7}	3.9426
1/50	0.5	1/15	1.1170×10^{-4}	-
		1/30	7.7102×10^{-6}	3.8567
		1/60	4.9571×10^{-7}	3.9592
	0.7	1/15	2.8042×10^{-5}	-
		1/30	1.7836×10^{-6}	3.9747
		1/60	1.1159×10^{-7}	3.9985

Table 4. The maximum error and convergence order in time-dimension for Example 2.

τ	α	$h_{x_1}=h_{x_2}$	E_∞	Order(τ)
	0.3	1/15	6.4458×10^{-6}	-
		1/30	1.6533×10^{-6}	1.9630
		1/60	4.1809×10^{-7}	1.9835
1/50	0.6	1/15	1.2813×10^{-5}	-
		1/30	3.2828×10^{-6}	1.9646
		1/60	8.2659×10^{-7}	1.9897
	0.9	1/15	1.5544×10^{-6}	-
		1/30	3.9067×10^{-7}	1.9923
		1/60	9.1095×10^{-8}	2.1005

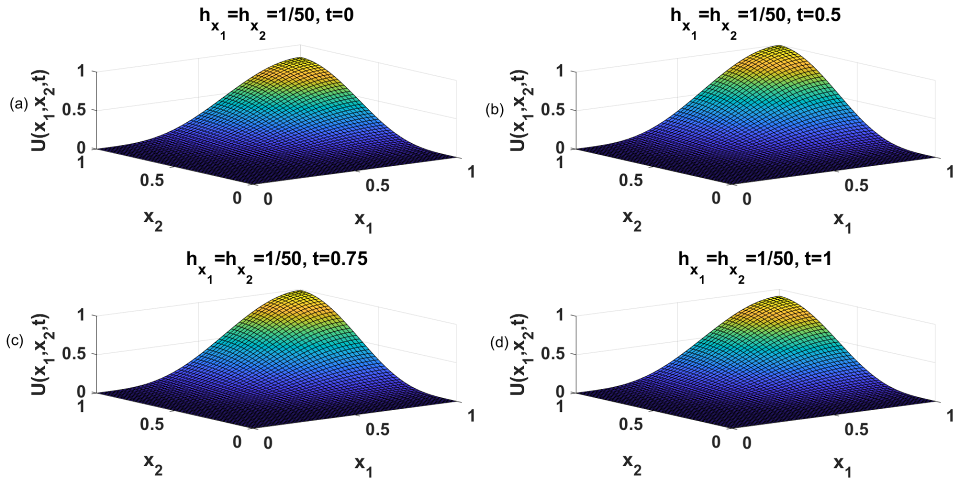

Figure 2. The numerical solutions for Example 2 for different time values (**a**: $t = 0$, **b**: $t = 0.5$, **c**: $t = 0.75$, **d**: $t = 1$).

2.1. A Final Example

Finally, we consider a problem whose exact solution is not known to us.

$$
\begin{aligned}
{}_0^C D_t^\alpha \mathcal{U}(x_1,x_2,t) + (x_1+x_2^2)\frac{\partial^4 \mathcal{U}}{\partial x_1^4} + 2\exp(x_1+x_2)\frac{\partial^4 \mathcal{U}}{\partial x_2^4} &= \mathcal{U}^3(x_1,x_2,t) + \mathcal{U}(x_1,x_2,t-0.8) \\
&\quad + \frac{(t-0.5)^2}{10\Gamma(0.8)}\exp(x_1+x_2), \\
\mathcal{U}(x_1,x_2,t) &= 2\exp(x_1+x_2)(t^3+t)(x_1-1)^4(x_2-1)^4, \quad (x_1,x_2)\in\partial\Omega, \quad t\in[-0.8,0], \\
\mathcal{U}(0,x_2,t) &= 0, \quad \mathcal{U}(1,x_2,t) = 0, \quad x_2\in[0,1], \quad t\in[0,1], \\
\mathcal{U}(x_1,0,t) &= 0, \quad \mathcal{U}(x_1,1,t) = 0, \quad x_1\in[0,1], \quad t\in[0,1], \\
\frac{\partial^2 \mathcal{U}(0,x_2,t)}{\partial x_1^2} &= 0, \quad \frac{\partial^2 \mathcal{U}(1,x_2,t)}{\partial x_1^2} = 0, \quad x_2\in[0,1], \quad t\in[0,1], \\
\frac{\partial^2 \mathcal{U}(x_1,0,t)}{\partial x_2^2} &= 0, \quad \frac{\partial^2 \mathcal{U}(x_1,1,t)}{\partial x_2^2} = 0, \quad x\in[0,1], \quad t\in[0,1].
\end{aligned}
$$

Similar to above problems, we find the numerical results using the numerical scheme given in Equation (29). The above problem has variable coefficients and the exact solution is not known to us; therefore, we check the accuracy of proposed numerical scheme by means of absolute residual error function, which is a measure of how well the approximation satisfies the original nonlinear fractional differential problem given in Example Section 2.1. The absolute residual error is defined as

$$
\begin{aligned}
R_m(x_1,x_2,t) := \Bigg| {}_0^C D_t^\alpha \mathcal{U}_m(x_1,x_2,t) + (x_1+x_2^2)\frac{\partial^4 \mathcal{U}_m}{\partial x_1^4} + 2\exp(x_1+x_2)\frac{\partial^4 \mathcal{U}_m}{\partial x_2^4} - \mathcal{U}_m^3(x_1,x_2,t) \\
- \mathcal{U}_m(x_1,x_2,t-0.8) - \frac{(t-0.5)^2}{10\Gamma(0.8)}\exp(x_1+x_2)\Bigg|, \quad (x_1,x_2)\in\Omega, t\in[0,1).
\end{aligned}
$$

The numerical results are shown in Figure 3 with different time variable values. Tables 5 and 6 show the numerical results of residual error R_n, $n = 6, 8, 10$. From our numerical results, we can conclude that numerical and theoretical results are in agreement. The numerical scheme gives the global second-order time convergence and fourth-order convergence in spatial dimensions. The accuracy of getting convergence order becomes more pronounced when step sizes are decreased. In all the calculations MATLAB is used.

Table 5. The absolute residual error and average convergence order in spatial dimensions for Example Section 2.1 with $t = 0.5$.

α	$x_1 = x_2$	m	Absolute Residual Error	Spatial-Order
0.2	0.3	6	5.500×10^{-3}	-
		8	4.7670×10^{-4}	3.5283
		10	3.6402×10^{-5}	3.7110
0.5	0.5	6	9.6002×10^{-2}	-
		8	9.0001×10^{-3}	3.4151
		10	7.0036×10^{-4}	3.6383
0.7	0.7	6	2.8001×10^{-3}	-
		8	2.6252×10^{-4}	3.4150
		10	2.0169×10^{-5}	3.7022

Table 6. The absolute residual error and average convergence order in time-dimension for Example Section 2.1 with $x_1 = x_2 = 0.5$.

α	t	m	Absolute Residual Error	Time-Order
0.2	0.3	6	6.0999×10^{-4}	-
		8	2.0118×10^{-4}	1.6003
		10	6.0908×10^{-5}	1.7238
0.5	0.5	6	7.4003×10^{-3}	-
		8	2.4001×10^{-3}	1.6245
		10	7.0036×10^{-4}	1.8020
0.7	0.7	6	4.1518×10^{-4}	-
		8	1.3862×10^{-4}	1.5826
		10	4.2205×10^{-5}	1.7156

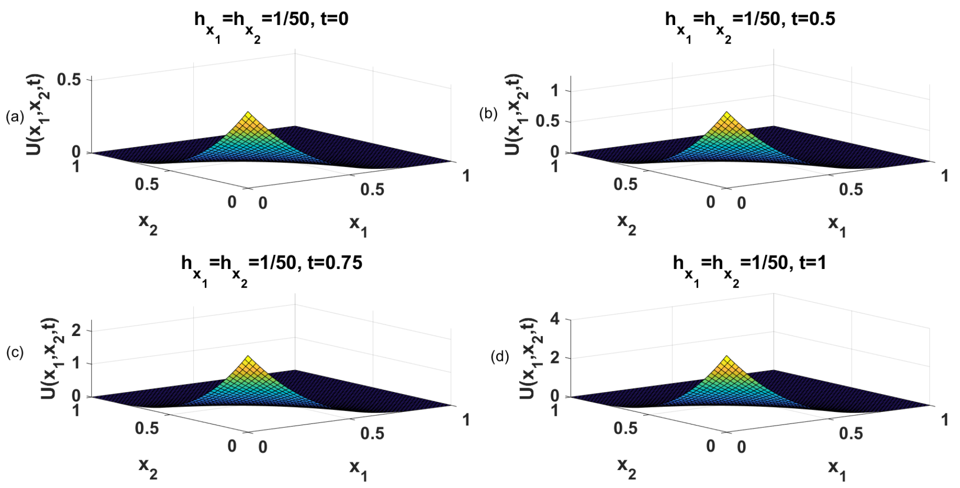

Figure 3. The numerical solutions for Example Section 2.1 for different time values (**a**: $t = 0$, **b**: $t = 0.5$, **c**: $t = 0.75$, **d**: $t = 1$).

3. Conclusions

The major contribution of this works lies in the construction of the numerical scheme and numerical results for two-dimensional fourth-order fractional subdiffusion equation with delay, nonlinear source term, and variable coefficients in case of smooth solutions. The numerical scheme presented in this paper incorporates the $L_2 - 1_\sigma$ formula for Caputo's time-fractional derivative. By invoking the works in [37,43] and due to the nonlocality of time Caputo fractional derivatives which need high computational cost and storage, we can improve our approach in the near future by presenting a high-order scheme based on the sum of exponential functions technique to speed up the evaluation. Furthermore, recalling the methodologies in [37,44] side by side to the numerical analysis in [35] and the appropriate discrete Grönwall inequality in [35,45], the unconditional convergence and stability estimates with out any constraints on time and space steps can be deduced. Assuming that $a_0 \leq a(x_1, x_2) \leq a_1$ and $b_0 \leq b(x_1, x_2) \leq b_1$, such that $a_0,\ b_0$ and $a_1,\ b_1$ are positive constants, is essential to prove the stability and convergence estimates which will be devoted to a new study in the near future. The numerical results indicate the efficiency of the numerical scheme. We obtained the fourth-order convergence in spatial dimensions and second-order convergence in the time dimension. Our numerical results validate the obtained theoretical results. The proposed numerical scheme can easily be extended and implemented for higher dimension problems.

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