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Discrete Hypergeometric Legendre Polynomials

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Abstract: A discrete analog of the Legendre polynomials defined by discrete hypergeometric series is investigated. The resulting polynomials have qualitatively similar properties to classical Legendre polynomials. We derive their difference equations, recurrence relations, and generating function.

Keywords: Legendre polynomial; hypergeometric function; special functions; recurrence relation; generating function; difference equation

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1. Introduction

We investigate the discrete Legendre polynomials

$$P_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^k(-n)_k(n+1)_k}{(k!)^2 2^k} t^{\underline{k}}, \tag{1}$$

where $t^{\underline{k}} = t(t-1)\dots(t-k+1)$ denotes the falling power. We will show that (1) forms a sequence of variable parameter hypergeometric polynomials.

The classical Legendre polynomials $\mathcal{P}_n$ form a sequence of orthogonal polynomials with many historical applications. Their use continues in recent times in applications such as beam theory [1], phone segmentation [2], neural networks [3], and signal processing [4]. The wide range of historical and recent applications of Legendre polynomials inspires us to consider their discrete analog.

The most common existing notion of discrete Legendre polynomials of order n are given by the summation $\sum_{\ell=0}^{n}(-1)^\ell \binom{n}{\ell}\binom{n+\ell}{\ell}\frac{t^{\underline{\ell}}}{N^{\underline{\ell}}}$, which is different from the series (1). These discrete Legendre polynomials have been used in various applications such as digital filters [5], the analysis of time-varying delayed systems [6], and adaptive image filtering [7]. A different approach to discrete Legendre polynomials to (1) appears in [8], which studies a sequence of discrete Legendre polynomials given by the sum (simplified here for convenience) $\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k(2n-2k)!t^{\underline{n-2k}}}{2^k k!(n-k)!(n-2k)!}$. That study's polynomials do not appear to have a representation as a discrete hypergeometric series, but many recurrences and an h-difference equation are derived.

A famous classical example of a discrete analog of classical orthogonal polynomials is the Meixner polynomials, which were realized as variable parameter hypergeometric series and were historically called "discrete Laguerre polynomials" [9]. Discrete Bessel functions studied in [10] were later shown to be useful in understanding qualitative properties of discrete wave and diffusion equations [11] and in semidiscrete diffusion equations [12]. More recently, discrete Chebyshev polynomials of the first and second kind were investigated [13] and their recurrences and interrelations were developed; however, they were shown to not form a sequence of orthogonal polynomials. In all of the above cases, these discrete special functions may be realized as instances of discrete hypergeometric series [14]. We refer readers interested in other types of analogs of special functions to the theory of time scales calculus [15] or q-calculus [16]. Here we consider the Legendre polynomials obtained by using the discrete hypergeometric series.

2. Preliminaries and Definitions

The classical ${}_2\mathcal{F}_1$ hypergeometric series is given by

$$ {}_2\mathcal{F}_1(a_1, a_1; b_1; t) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{(b_1)_k} \frac{t^k}{k!} $$

and the classical hypergeometric ${}_3\mathcal{F}_1$ function is given by

$$ {}_3\mathcal{F}_1(a_1, a_2, a_3; b_1; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k (a_3)_k}{(b_1)_k} \frac{z^k}{k!}, $$

where $(x)_k = x(x+1)\dots(x+k-1)$. The classical Legendre polynomials are given by the hypergeometric function

$$ \mathcal{P}_n(t) = {}_2\mathcal{F}_1\left(-n, n+1; 1; \frac{1-t}{2}\right), \tag{2} $$

which terminates when $n \in \mathbb{N}_0$. See [17] as a standard reference for the properties of the Legendre polynomials. They solve the differential equation

$$ (t^2-1)\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2t\frac{\mathrm{d}y}{\mathrm{d}t} - n(n+1)y = 0. \tag{3} $$

They obey the three-term recurrence

$$ (n+1)\mathcal{P}_{n+1}(t) - (2n+1)t\mathcal{P}_n(t) + n\mathcal{P}_{n-1}(t) = 0, \tag{4} $$

and the additional recurrence

$$ \frac{t^2-1}{n}\mathcal{P}_n'(t) - t\mathcal{P}_n(t) + \mathcal{P}_{n-1}(t) = 0. \tag{5} $$

Discrete hypergeometric ${}_2F_1$ series are defined in [14] by the formula

$$ {}_2F_1(a_1, a_2; b_1; t, n, \xi) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{(b_1)_k} \frac{\xi^k t^{\underline{nk}}}{k!}. $$

It is well-known ([14], Proposition 2) that the ${}_2F_1$ function with $n=1$ is a ${}_3\mathcal{F}_1$ function with variable parameters of the form

$$ {}_2F_1(a_1, a_2; b_1; t, 1, \xi) = {}_3\mathcal{F}_1(a_1, a_2, -t; b_1; -\xi). \tag{6} $$

3. Discrete Legendre Polynomials

The argument $\dfrac{1-t}{2}$ appearing in (2) is not directly amenable to the creation of a discrete analog with our method, see ([13], Example 1) and the prose after it for more details. There, it was suggested that replacing t with $t+1$ ameliorates this issue. Doing so in (2) yields

$$ \mathcal{P}_n(t+1) = {}_2\mathcal{F}_1\left(-n, n+1; 1; -\frac{t}{2}\right). \tag{7} $$

We have included a comparison between the $\mathcal{P}_n(t)$, the $\mathcal{P}_n(t+1)$, and the $P_n(t)$ below in Table 1 and visualized in Figure 1. Similarly, (3) yields

$$(t^2+2t)\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+2(t+1)\frac{\mathrm{d}y}{\mathrm{d}t}-n(n+1)y=0, \tag{8}$$

(4) yields

$$(n+1)\mathcal{P}_{n+1}(t+1)-(2n+1)(t+1)\mathcal{P}_n(t+1)+n\mathcal{P}_{n-1}(t+1)=0, \tag{9}$$

and (5) yields

$$\frac{t^2+2t}{n}\frac{\mathrm{d}\mathcal{P}_n}{\mathrm{d}t}(t+1)-(t+1)P_n(t+1)+P_{n-1}(t+1)=0. \tag{10}$$

Inspired by (7), we defined the discrete Legendre functions by (1) and this representation has discrete hypergeometric form

$$P_n(t)={}_2F_1\left(-n,n+1;1;t,1,-\frac{1}{2}\right). \tag{11}$$

Immediately from (1), we see

$$P_n(0)=1, \tag{12}$$

and due to (6) and (13), we have

$$P_n(t)={}_3\mathcal{F}_1\left(-n,n+1,-t;1;\frac{1}{2}\right). \tag{13}$$

Table 1. The Legendre polynomials $\mathcal{P}_n(t)$, the shifted Legendre polynomials $\mathcal{P}_n(t+1)$, and the discrete Legendre polynomials $P_n(t)$ for $n\in\{0,1,2,3,4\}$. The highest-order terms of all of these polynomials actually agree, so there is asymptotic equivalence among all of them in the complex plane as $|t|\to\infty$.

n	$\mathcal{P}_n(t)$	$\mathcal{P}_n(t+1)$	$P_n(t)$
0	1	1	1
1	t	$t+1$	$t+1$
2	$\frac{3t^2}{2}-\frac{1}{2}$	$\frac{3t^2}{2}+3t+1$	$\frac{3t^2}{2}+\frac{3t}{2}+1$
3	$\frac{5t^3}{2}-\frac{3t}{2}$	$\frac{5t^3}{2}+\frac{15t^2}{2}+6t+1$	$\frac{5t^3}{2}+\frac{7t}{2}+1$
4	$\frac{35t^4}{8}-\frac{15t^2}{4}+\frac{3}{8}$	$\frac{35t^4}{8}+\frac{35t^3}{2}+\frac{45t^2}{2}+10t+1$	$\frac{35t^4}{8}-\frac{35t^3}{4}+\frac{145t^2}{8}-\frac{15t}{4}+1$

We now argue that the discrete Legendre polynomials do not form an orthogonal polynomial sequence. This is because if $\{P_n\}$ is a sequence of orthogonal polynomials, then it is well-known that they must satisfy a three-term recurrence of the form $P_n=(A_nt+B_n)P_{n-1}-C_nP_{n-2}$ for constants $A_n,C_n\in(0,\infty)$ and $B_n\in\mathbb{R}$ [18]. We shall check this recurrence for $n=3$, i.e., whether there exist appropriate constants A_3, B_3, and C_3 such that $P_3(t)=(A_3t+B_3)P_2(t)-C_3P_1(t)$. When the polynomials are inserted into this equation, the following system of equations results:

$$\begin{cases}\frac{5}{2} &= \frac{3}{2}A_3\\ 0 &= \frac{3A_3}{2}+\frac{3B_3}{2}\\ \frac{7}{2} &= -C_3\\ 1 &= A_3+B_3-C_3,\end{cases}$$

for which it is straightforward to show that no solution exists. Therefore, we conclude that the discrete Legendre polynomials do not form a sequence of orthogonal polynomials. In general, this means that discrete hypergeometric series do not preserve the orthogonality of their continuous counterparts. This raises a general question of when orthogonality is preserved by such discrete analogs, and more generally, when any sequence of polynomials defined by a discrete hypergeometric series may form an orthogonal polynomial sequence.

We now analyze the series (1) for general $t, n \in \mathbb{C}$.

Theorem 1. *If $n \in \{0,1,\dots\}$ or $t \in \{0,1,\dots\}$, then the series (1) converges. Otherwise, it diverges.*

Proof. If $n \in \{0,1,\dots\}$, then $(-n)_{k+1}$ will ultimately become zero, and the series terminates. If $t \in \{0,1,\dots\}$, then $t^{\underline{k}}$ will become zero for sufficiently large k, and the series terminates. Now assume $n \in \mathbb{C} \setminus \{0,1,\dots\}$ and $t \in \mathbb{C} \setminus \{0,1,\dots\}$. By the ratio test,

$$\begin{aligned}
\lim_{k\to\infty} \frac{\alpha_{n,k+1}}{\alpha_{n,k}} &= \lim_{k\to\infty} \left| \frac{(-n)_{k+1}(n+1)_{k+1} t^{\underline{k+1}} (k!)^2 2^k}{((k+1)!)^2 2^{k+1} (-n)_k (n+1)_k t^{\underline{k}}} \right| \\
&= \lim_{k\to\infty} \frac{(-n+k)(n+1+k)(t-k)}{2(k+1)^2} = \infty,
\end{aligned}$$

so the series diverges, completing the proof. □

From the proof of Theorem 1, it follows that if $n \in \{0,1,\dots\}$, then (1) reduces to polynomials.

We now derive the delay-difference equations that these polynomials obey by finding a discrete analog of (8). We see that each term of the form $t^m \dfrac{\mathrm{d}^j}{\mathrm{d}t^j} f(t)$ appearing in (8) becomes $t^{\underline{m}} \Delta^j f(t-m)$ in the analogous difference equation.

Theorem 2. *If $n \in \{0,1,\dots\}$, then the polynomials (1) solve the difference equation*

$$t^{\underline{2}} \Delta^2 y(t-2) + 2t\Delta^2 y(t-1) + 2t\Delta y(t-1) + 2\Delta y(t) - n(n+1)y(t) = 0. \tag{14}$$

Proof. Define the operator $\Theta = t\rho\Delta$, where $(\rho f)(t) = f(t-1)$. By the hypergeometric representation (11) and ([14], Theorem 7), $y(t) = P_n(t)$ satisfies the difference equation

$$\Big[\Theta^2 - \xi t\rho(\Theta - n)(\Theta + n + 1)\Big] y(t) = 0. \tag{15}$$

First compute $\Theta^2 y(t) = t\rho\Delta\Big[t\Delta y(t-1)\Big] = t\Delta y(t-1) + t^{\underline{2}}\Delta^2 y(t-2)$. Now compute

$$\begin{aligned}
&\frac{t}{2}\rho(t\rho\Delta - n)(t\rho\Delta + n + 1)y(t) \\
&\quad= \frac{t}{2}\rho(t\rho\Delta - n)\Big[t\Delta y(t-1) + (n+1)y(t)\Big] \\
&\quad= \frac{t}{2}\rho\Bigg[t\rho\Big\{\Delta y(t) + t\Delta^2 y(t-1) + (n+1)\Delta y(t)\Big\} \\
&\qquad - nt\Delta y(t-1) - n(n+1)y(t)\Bigg] \\
&\quad= \frac{t}{2}\rho\Bigg[t\Delta y(t-1) + t^{\underline{2}}\Delta^2 y(t-2) + (n+1)t\Delta y(t-1) \\
&\qquad - nt\Delta y(t-1) - n(n+1)y(t)\Bigg] \\
&\quad= \frac{t^{\underline{2}}}{2}\Delta y(t-2) + \frac{t^{\underline{3}}}{2}\Delta^2 y(t-2) + \frac{t^{\underline{2}}}{2}\Delta y(t-2) - \frac{n(n+1)}{2} t y(t-1).
\end{aligned}$$

Therefore, (15) becomes

$$t\Delta y(t-1)+t^2\Delta^2 y(t-2)+\frac{t^2}{2}\Delta y(t-2)+\frac{t^3}{2}\Delta^2 y(t-2) \\ +\frac{t^2}{2}\Delta y(t-2)-\frac{n(n+1)}{2}ty(t-1)=0.$$

Dividing by t, replacing t with $t+1$, and multiplying by 2 completes the proof. □

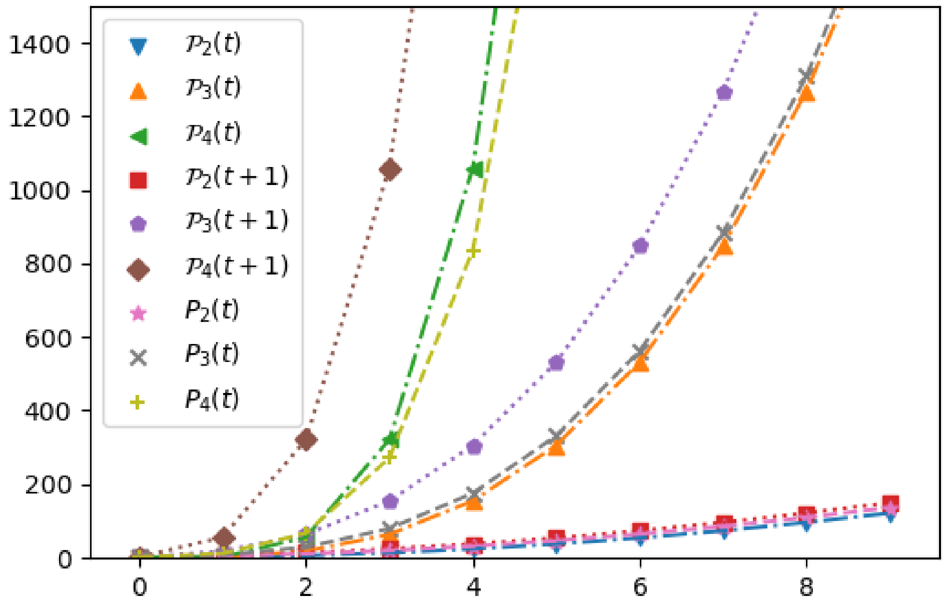

Figure 1. Plot of $\mathcal{P}_n(t)$, $\mathcal{P}_n(t+1)$, and $P_n(t)$ for $n \in \{2,3,4\}$.

The delay-difference Equation (14) can be written as a third-order linear difference equation with polynomial coefficients.

Theorem 3. *If $n \in \{0,1,\ldots\}$, then the difference Equation (14) can be written as*

$$(2t+6)\Delta^3 y(t)+(t^2+7t+14-n(n+1))\Delta^2 y(t) \\ +(10t+30-2n(n+1))\Delta y(t)+(8t+24-n(n+1))y(t)=0. \quad (16)$$

Proof. First replace t with $t+2$ in (14) to obtain

$$(t+2)^2\Delta^2 y(t)+2(t+2)\Delta^2 y(t+1) \\ +2(t+2)\Delta y(t+1)+2\Delta y(t+2)-n(n+1)y(t+2)=0. \quad (17)$$

Directly from the definition of the difference operator, we obtain the formulas $y(t+1)=\Delta y(t)+y(t)$,

$$y(t+2)=\Delta^2 y(t)+2\Delta y(t)+y(t), \quad (18)$$

and $y(t+3)=\Delta^3 y(t)+3\Delta^2 y(t)+7\Delta y(t)+5y(t)$. Therefore, we observe that

$$\Delta y(t+1)=\Delta^2 y(t)+\Delta y(t), \quad (19)$$

$$\Delta y(t+2)=\Delta^3 y(t)+2\Delta^2 y(t)+5\Delta y(t)+4y(t), \quad (20)$$

and

$$\Delta^2 y(t+1)=\Delta^3 y(t)+\Delta^2 y(t)+4\Delta y(t)+4y(t). \quad (21)$$

Substituting (18)–(21) into (17) and simplifying completes the proof. □

We now derive the discrete analog of (9).

Theorem 4. *If $n \in \{1, 2, \ldots\}$, then the polynomials (1) obey the recurrence*

$$(n+1)P_{n+1}(t) - (2n+1)tP_n(t-1) - (2n+1)P_n(t) + nP_{n-1}(t) = 0. \tag{22}$$

Proof. Let $\alpha_{n,k} = \dfrac{(-1)^k(-n)_k(n+1)_k}{(k!)^2 2^k}$, hence $\alpha_{n,0} = 1$ for all n. Then the left-hand side of (22) can be written as

$$\sum_{k=0}^{\infty} \Big[(n+1)\alpha_{n+1,k}t^{\underline{k}} - (2n+1)\alpha_{n,k}t^{\underline{k+1}} - (2n+1)\alpha_{n,k}t^{\underline{k}} + n\alpha_{n-1,k}t^{\underline{k}}\Big]. \tag{23}$$

The $k = 0$ term on the first, third, and fourth terms of (23) yield

$$(n+1)\alpha_{n+1,0} - (2n+1)\alpha_{n,0} + n\alpha_{n-1,0} = 0.$$

We reindex the second term of (23) and obtain

$$\sum_{k=1}^{\infty} \Big[(n+1)\alpha_{n+1,k} - (2n+1)\alpha_{n,k-1} - (2n+1)\alpha_{n,k} + n\alpha_{n-1,k}\Big]t^{\underline{k}} \tag{24}$$

From the definition of Pochhammer symbols, compute

$$\begin{aligned}
(-n-1)_k &= (-n-1)(-n)(-n+1)_{k-2},\\
(-n)_{k-1} &= (-n)(-n+1)_{k-2},\\
(-n)_k &= (-n)(-n+1)_{k-2}(-n+k-1),\\
(-n+1)_k &= (-n+1)_{k-2}(-n+k-1)(-n+k),\\
(n+2)_k &= (n+2)_{k-2}(n+k)(n+k+1),\\
(n+1)_{k-1} &= (n+1)(n+2)_{k-2},\\
(n+1)_k &= (n+1)(n+2)_{k-2}(n+k),\\
(n)_k &= n(n+1)(n+2)_{k-2}.
\end{aligned}$$

Therefore, if $\beta_{n,k} = \dfrac{(-1)^k(-n+1)_{k-2}(n+2)_{k-2}}{(k!)^2 2^k}$, then

$$\begin{aligned}
(n+1)\alpha_{n+1,k} &= (n+1)(-n-1)(-n)(n+k)(n+k+1)\beta_{n,k},\\
-(2n+1)\alpha_{n,k-1} &= 2k^2(2n+1)(-n)(n+1)\beta_{n,k},\\
-(2n+1)\alpha_{n,k} &= -(2n+1)(-n)(-n+k-1)(n+1)(n+k)\beta_{n,k},\\
n\alpha_{n-1,k} &= n(-n+k-1)(-n+k)n(n+1)\beta_{n,k}.
\end{aligned}$$

Thus, the coefficient of $t^{\underline{k}}$ in (24) becomes

$$\Big[(n+1)(n+k)(n+k+1) - 2k^2(2n+1) + (2n+1)(-n+k-1)(n+k) \\ + n(-n+k-1)(-n+k)\Big]n(n+1)\beta_{n,k},$$

which is identically zero, completing the proof. □

By (13), we observe that (22) encodes recurrences for the ${}_3\mathcal{F}_1$ function.

Corollary 1. *If $n \in \{1, 2, \ldots\}$, then the following relation holds:*

$$(n+1)_3\mathcal{F}_1\left(-n-1, n+2, -t; 1; \frac{1}{2}\right) - (2n+1)t_3\mathcal{F}_1\left(-n, n+1, -t+1; 1; \frac{1}{2}\right) - (2n+1)_3\mathcal{F}_1\left(-n, n+1, -t; 1; \frac{1}{2}\right) + n_3\mathcal{F}_1\left(-n+1, n, -t; 1; \frac{1}{2}\right) = 0.$$

Now we derive the discrete analog of (10).

Theorem 5. *If $n \in \{1, 2, \ldots\}$, then*

$$\frac{t^{\underline{2}}}{n}\Delta P_n(t-2) + \frac{2t}{n}\Delta P_n(t-1) - tP_n(t-1) - P_n(t) + P_{n-1}(t) = 0. \tag{25}$$

Proof. Let $\alpha_{n,k} = \dfrac{(-1)^k(-n)_k(n+1)_k}{(k!)^2 2^k}$. Compute $\Delta P_n(t) = \sum_{k=1}^{\infty} k\alpha_{n,k}t^{\underline{k-1}}$, hence

$$\frac{t^{\underline{2}}}{n}\Delta P_n(t-2) = \frac{1}{n}\sum_{k=1}^{\infty} k\alpha_{n,k}t^{\underline{k+1}} = \frac{1}{n}\sum_{k=2}^{\infty}(k-1)\alpha_{n,k-1}t^{\underline{k}},$$

$$\frac{2}{n}t\Delta P_n(t-1) = \frac{2}{n}\sum_{k=1}^{\infty} k\alpha_{n,k}t^{\underline{k}} = \frac{2}{n}\alpha_{n,1}t + \frac{2}{n}\sum_{k=2}^{\infty} k\alpha_{n,k}t^{\underline{k}},$$

$$tP_n(t-1) = \sum_{k=0}^{\infty}\alpha_{n,k}t^{\underline{k+1}} = \sum_{k=1}^{\infty}\alpha_{n,k-1}t^{\underline{k}} = \alpha_{n,0}t + \sum_{k=2}^{\infty}\alpha_{n,k-1}t^{\underline{k}}.$$

$$P_n(t) = \sum_{k=0}^{\infty}\alpha_{n,k}t^{\underline{k}} = \alpha_{n,0} + \alpha_{n,1}t + \sum_{k=2}^{\infty}\alpha_{n,k}t^{\underline{k}},$$

$$P_{n-1}(t) = \sum_{k=0}^{\infty}\alpha_{n-1,k}t^{\underline{k}} = \alpha_{n-1,0} + \alpha_{n-1,1}t + \sum_{k=2}^{\infty}\alpha_{n-1,k}t^{\underline{k}},$$

From the definition of Pochhammer symbols, we obtain

$$\begin{aligned}
(n+1)_{k-1} &= (n+1)(n+2)_{k-2} \\
(n+1)_k &= (n+1)(n+2)_{k-2}(n+k) \\
(n)_k &= n(n+1)(n+2)_{k-2} \\
(-n)_{k-1} &= (-n)(-n+1)_{k-2} \\
(-n)_k &= (-n)(-n+1)_{k-2}(-n+k-1) \\
(-n+1)_k &= (-n+1)_{k-2}(-n+k-1)(-n+k)
\end{aligned}$$

Now we compute

$$\begin{aligned}
&\frac{t^{\underline{2}}}{n}\Delta P_n(t-2) + \frac{2}{n}t\Delta P_n(t-1) - tP_n(t-1) - P_n(t) + P_{n-1}(t) \\
&\quad= \Big[-\alpha_{n,0} + \alpha_{n-1,0}\Big] + \left[\frac{2}{n}\alpha_{n,1} - \alpha_{n,0} - \alpha_{n,1} + \alpha_{n-1,1}\right] \\
&\quad+ \sum_{k=2}^{\infty}\left[\left(\frac{k-1}{n} - 1\right)\alpha_{n,k-1} + \left(\frac{2k}{n} - 1\right)\alpha_{n,k} + \alpha_{n-1,k}\right]t^{\underline{k}}.
\end{aligned} \tag{26}$$

Compute directly

$$-\alpha_{n,0} + \alpha_{n-1,0} = -1 + 1 = 0$$

and

$$\frac{2}{n}\alpha_{n,1} - \alpha_{n,0} - \alpha_{n,1} + \alpha_{n-1,1} = \frac{2n - n^2 - n + n^2 - n}{2} = 0.$$

Define $\gamma_{n,k} = \dfrac{(-1)^k(-n+1)_{k-2}(n+2)_{k-2}}{(k!)^2 2^k}$. With this notation, we write

$$\left(\frac{k-1}{n}-1\right)\alpha_{n,k-1} = -2k^2\left(\frac{k-1}{n}-1\right)(-n)(n+1)\gamma_{n,k},$$

$$\left(\frac{2k}{n}-1\right)\alpha_{n,k} = \left(\frac{2k}{n}-1\right)(-n)(-n+k-1)(n+1)(n+k)\gamma_{n,k},$$

and

$$-\alpha_{n-1,k} = (-n+k-1)(-n+k)n(n+1)\gamma_{n,k}.$$

Therefore, the coefficient of $t^{\underline{k}}$ in (26) becomes

$$\left[2k^2\left(\frac{k-1}{n}-1\right) - \left(\frac{2k}{n}-1\right)(-n+k-1)(n+k) + (-n+k-1)(-n+k)\right]n(n+1)\gamma_{n,k},$$

which is identically zero, completing the proof. □

Use (13) and expand the differences appearing in (25) to generate a recurrence for the ${}_3\mathcal{F}_1$ functions.

Corollary 2. *If $n \in \{1,2,\ldots\}$, then the following formula holds:*

$$\frac{t-t^2}{n}{}_3\mathcal{F}_1\left(-n,n+1,-t+2;1;\frac{1}{2}\right) + {}_3\mathcal{F}_1\left(-n+1,n,-t;1;\frac{1}{2}\right) + \left(\frac{t^2-(n+3)t}{n}\right){}_3\mathcal{F}_1\left(-n,n+1,-t+1;1;\frac{1}{2}\right) + \left(\frac{2t}{n}-1\right){}_3\mathcal{F}_1\left(-n,n+1,-t;1;\frac{1}{2}\right) = 0.$$

By definition, the generating function for the discrete Legendre polynomials is given by

$$T(t,x) = T_t(x) := \sum_{k=0}^{\infty} P_k(t)x^k. \tag{27}$$

It is easy to see that

$$x\frac{\partial T}{\partial x} = \sum_{k=0}^{\infty} kP_k(t)x^k. \tag{28}$$

We see that $T(t,0) = P_0(t) = 1$. If $t = 0$, then (12) shows that for $-1 < x < 1$,

$$T(0,x) = \sum_{k=0}^{\infty} P_k(0)x^k = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}. \tag{29}$$

We now derive the partial differential-difference equation that T obeys.

Theorem 6. *If $t \in \{1,2,\ldots\}$ and $-1 < x < 1$, then the generating function T satisfies*

$$\frac{\partial T_t}{\partial x} + \left(\frac{-1}{1-x}\right)T_t = \frac{2tx}{(1-x)^2}\frac{\partial T_{t-1}}{\partial x} + \frac{t}{(1-x)^2}T_{t-1}. \tag{30}$$

Proof. First notice that when $x = 0$, (30) is true by definition. Assume $x \neq 0$. Replace n with k in (22), multiply by x^{k+1}, and sum to obtain

$$\sum_{k=0}^{\infty}(k+1)P_{k+1}(t)x^{k+1} - \sum_{k=0}^{\infty}(2k+1)tP_k(t-1)x^{k+1} \\ - \sum_{k=0}^{\infty}(2k+1)P_k(t)x^{k+1} + \sum_{k=0}^{\infty}kP_{k-1}(t)x^{k+1} = 0.$$

Use the definition of T and (28) to get

$$(x-2x^2+x^3)\frac{\partial T}{\partial x}(t,x) - 2tx^2\frac{\partial T}{\partial x}(t-1,x) - txT(t-1,x) + (x^2-x)T(t,x) = 0.$$

Divide by x and perform routine algebra to complete the proof. □

Theorem 7. *If $t \in \{1,2,\dots\}$ and $-1 < x < 1$, then generating function $T_t(x)$ obeys the recurrence*

$$T_t(x) = \frac{1}{1-x} + \frac{2tx}{(1-x)^2}T_{t-1}(x) + \frac{t+1}{1-x}\int_0^x \frac{T_{t-1}(\tau)}{(1-\tau)^2}\mathrm{d}\tau, \tag{31}$$

with initial condition $T_0(x) = \dfrac{1}{1-x}$.

Proof. When $t=0$, (12) shows that (27) is the geometric series, yielding the formula for $T_0(x)$. Now multiply (32) by the integrating factor $1-x$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big((1-x)T_t(x)\Big) = \frac{2tx}{1-x}\frac{\partial T_{t-1}}{\partial x} + \frac{t}{1-x}T_{t-1}.$$

Integrating from 0 to x yields

$$(1-x)T_t(x) - 1 = 2t\int_0^x \frac{\xi}{1-\xi}\frac{\partial T_{t-1}}{\partial \xi}\mathrm{d}\xi + t\int_0^x \frac{T_{t-1}(\xi)}{1-\xi}\mathrm{d}\xi. \tag{32}$$

Apply integration by parts to the first term on the right-hand side to obtain

$$\begin{aligned}\int_0^x \frac{\xi}{1-\xi}\frac{\partial T_{t-1}}{\partial \xi}\mathrm{d}\xi &= \frac{\xi}{1-\xi}T_{t-1}(\xi)\Big|_0^x - \int_0^x \frac{T_{t-1}(\xi)}{(1-\xi)^2}\mathrm{d}\xi \\ &= \frac{x}{1-x}T_{t-1}(x) - \int_0^x \frac{T_{t-1}(\xi)}{(1-\xi)^2}\mathrm{d}\xi.\end{aligned}$$

Now (32) becomes

$$\begin{aligned}(1-x)T_t(x) - 1 &= 2t\left[\frac{x}{1-x}T_{t-1}(x) - \int_0^x \frac{T_{t-1}(\xi)}{(1-\xi)^2}\mathrm{d}\xi\right] + t\int_0^x \frac{T_{t-1}(\xi)}{1-\xi}\mathrm{d}\xi \\ &= \frac{2tx}{1-x}T_{t-1}(x) + \int_0^x\left(-\frac{2t}{(1-\xi)^2} + \frac{t}{1-\xi}\right)T_{t-1}(\xi)\mathrm{d}\xi \\ &= \frac{2tx}{1-x}T_{t-1}(x) + \int_0^x \frac{-2t+t-t\xi}{(1-\xi)^2}T_{t-1}(\xi)\mathrm{d}\xi \\ &= \frac{2tx}{1-x}T_{t-1}(x) - t\int_0^x \frac{1+\xi}{(1-\xi)^2}T_{t-1}(\xi)\mathrm{d}\xi,\end{aligned}$$

and dividing by $1-x$ completes the proof. □

4. Conclusions

By Theorem 1, we have shown that the discrete Legendre polynomials are truly discrete in that they do not converge outside of the nonnegative integers. We showed that they solve a delay difference equation whose form resembles the Legendre differential equation, and we derived the linear third-order difference equation with polynomial coefficients that they satisfy. We found discrete analogs of classical recurrence relations and noted the recurrences they induce for classical ${}_3\mathcal{F}_1$ hypergeometric series. Finally, we found a partial differential-difference equation for their generating function and derived a recurrence to compute it with elementary calculus. Further research in this area could include investigating the other two linearly independent solutions of (16), generalizing the second parameter 1 in (11) to some $\ell \in \mathbb{C}$, and the generalization to associated discrete Legendre functions, which would change the first parameter 1 in (11) to $1-\mu$ for some $\mu \in \mathbb{C}$. Another possible direction for further research may be inspired from the work in [11,12]—discrete analogs of partial differential equations (such as Laplace's equation) where discrete Legendre polynomials could be used to understand solutions.

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