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Valuation of Cliquet-Style Guarantees with Death Benefits in Jump Diffusion Models

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Abstract: This paper aims to value the cliquet-style equity-linked insurance product with death benefits. Whether the insured dies before the contract maturity or not, a benefit payment to the beneficiary is due. The premium is invested in a financial asset, whose dynamics are assumed to follow an exponential jump diffusion. In addition, the remaining lifetime of an insured is modelled by an independent random variable whose distribution can be approximated by a linear combination of exponential distributions. We found that the valuation problem reduced to calculating certain discounted expectations. The Laplace inverse transform and techniques from existing literature were implemented to obtain analytical valuation formulae.

Keywords: equity-indexed annuity; cliquet-style guarantee; life insurance; death benefits; jump diffusion

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1. Introduction

Since being introduced in Keyport Life Insurance in 1995, see [1], equity-indexed annuities (EIAs) have been continuously attractive to both industry and academia. EIAs usually consist of two phases. During the first accumulation phase, collected premiums are allowed to be invested in financial instruments, for example, the S&P 500 Index. As an equity-linked insurance product, the EIA benefit is linked to the performance of the reference asset through crediting mechanisms. When the payout is due, a fixed annuity or a lump-sum amount will be delivered to the recipient. By offering a guaranteed minimum return, EIAs allow policyholders to participate in the potential appreciation of the linked asset while eliminating the downside risk. This feature makes the EIA a hybrid between a life insurance (or annuity) policy and a risky investment.

In one typical EIA contract, the guaranteed return and the equity-indexed return are combined to calculate the benefit payoff. Mostly, a participation rate is offered to specify the part of equity-indexed returns that contribute to the calculation. Common crediting or indexing methods include the return of premium design, the roll-up design, the cliquet-style (also known as ratchet or ratchet-type) design, and the high-water-mark design. The most popular design seems to be the cliquet-style, since the earnings are credited per annum, based on the higher of a guaranteed minimum return rate and the annual equity-linked return multiplying a participation rate. Except for guarantees on maturity, some contracts may provide death benefits. If the insured dies before the maturity, the then-current benefits will be paid.

We aim to study the valuation of a cliquet-style EIA insurance contract that also provides the death benefits. We assume that the market is frictionless and that there are no transaction fees. We consider one such EIA contract that links to one unit of stock. At any time $t \geq 0$, the stock price follows

$$S(t) = S(0)e^{X(t)},\ S(0) = 1, \tag{1}$$

where $\{X(t), X(0) = 0\}$ is a drifted Brownian motion plus an independent compound Poisson process:

$$X(t) = \mu t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i, \tag{2}$$

where real numbers $\mu, \sigma^2 > 0$, $W(t)$ is a Brownian motion, and $N(t)$ is a Poisson process with a constant rate λ. The sequence $\{Y_1, Y_2, \ldots\}$ consists of i.i.d random variables such that $Y_i \stackrel{d}{=} Y$, denoting the size of a jump.

For an insured currently at age x, we denote their remaining lifetime by a positive random variable T_x. Throughout this paper, T_x and $S(t)$ are assumed to be independent. In addition, we take a year as 1 unit and suppose the contract matures in n $(n \in \mathbb{N}^+)$ years. We formulate the valuation problem as

$$V(\delta, T_x \wedge n) = \mathbf{E}\Big[e^{-\delta(T_x \wedge n)} B(S(T_x \wedge n))\Big], \tag{3}$$

where the payment is triggered at $T_x \wedge n$, taking the minimum of T_x and n. We let $B(\cdot)$ denote the cliquet-style payoff, and detailed expressions are given in Section 2.3. Moreover, δ denotes the constant force of interest.

The presence of T_x in (3) makes the calculation equivalent to the evaluation of a life-contingent option exercised at $T_x \wedge n$. In the existing literature, a risk-neutral setting from option theory has been widely adopted to deal with such valuation problems. Early studies on fair valuation of equity-linked contracts with guarantees include [2,3]. The authors in [1] studied the valuation of EIAs with various embedded payoffs and derived closed-form valuation formulas. The authors in [4] focused on the ratchet EIAs and proposed a lattice method for the valuation problem. In the presence of stochastic settings, Ref. [5] discussed the pricing of ratchet EIAs under a stochastic rate model. The authors in [6] exploited the valuation of EIAs under stochastic mortality and interest rate (two factors are assumed to be independent) models. To analyse how economic factors might affect the valuation of EIAs, a regime-switching framework was adopted, see [7,8].

The geometric Brownian motion has been widely used to model the dynamics of the asset price $S(t)$ for the mathematical tractability. However, this model is criticized for lacking the ability to explain some empirical facts, such as leptokurtic features. An alternative is to use exponential Lévy models. Extra jumps could be added to the drifted Brownian motion in the other areas of financial mathematics, see [9–11] for example. In particular, when the distribution of jumps follows a combination (or a mixture) of exponential distributions, the jump diffusion (2) becomes highly tractable, see [12–14], therein.

Another issue arising from valuing (3) lies in the proper assumption on T_x. We briefly recapitulate two common approaches in the literature. The first approach is to use established mortality models. When valuing a variable annuity contract, [15,16] adopted the constant force of mortality assumption and De Moivre's mortality law to obtain explicit solutions. The work was extended in [17] by using Makeham's law of mortality. We adopt the other approach, which approximates the distribution of T_x in a desired form by fitting the life table data. Due to the contributions in [18,19], it is possible to approximate the distribution of T_x by a combination of exponential distributions from a life table, see [20,21].

Randomization is a technique proposed in [22], illustrating an approach that one can presume a plausible distribution for a parameter, next somehow solve the desired problem under this random setting, finally obtains a reasonable approximation to the original problem by letting the variance of the governing distribution approach to zero. Such a technique has been successfully implemented in insurance field with the name "Erlangization", serving as an approximation of a deterministic time n. The idea is helpful in obtaining explicit identities. The authors in [23] first employed this technique in studying finite-time ruin problems, and some extensions were referred to in [24,25].

In regard to risk processes with periodic decisions, [26] first studied Erlang distributed inter-dividend-decision times in the Cramér-Lundberg model and [27] assumed Erlang distributed observation times to derive explicit expressions for the discounted penalty function at ruin. Other applications involving erlangization were referred in [28–30], among others. A recent application of erlangization in life insurance is presented in [31].

Motivated by the erlangization technique, in this work, we are also interested in valuing (3) with a erlangized maturity. To this end, we suppose that the time to maturity is divided into n non-overlapping sub-periods whose endpoints are denoted by $T_0(=0) < T_1 < T_2 < \ldots < T_n$. For $j = 1, 2, \ldots, n$, we denote the length of each sub-period by $T_j - T_{j-1}$ and assume that it follows an Erlang distribution:

$$T_{i-1} - T_i \stackrel{d}{=} \theta \sim \text{Erlang}(m, \rho), \tag{4}$$

where $m \in \mathbb{N}^+$, and $\rho > 0$. We note $T_n \sim$Erlang(nm, ρ). The reason for the choice of the Erlang(m, ρ) distribution is that we can increase m to keep the mean $\mathbf{E}\theta$ fixed, then θ converges in distribution to a point mass at $\mathbf{E}\theta$, thereby, approximating the situation of a deterministic interval/maturity time. In addition, we set $\mathbf{E}\theta = 1$ in this paper, and assume those Erlang distributed sub-periods are independent from both T_x and $S(t)$. Therefore, a randomized version of Formula (3) is given by

$$V(\delta, T_x \wedge T_n) = \mathbf{E}\Big[e^{-\delta(T_x \wedge T_n)} B(S(T_x \wedge T_n))\Big]. \tag{5}$$

When letting the variance of T_n go to zero, T_n converges to its mean n and the value of (5) converges to the one calculated by n, that is, by Formula (3).

The rest of this paper is structured as follows. Section 2 introduces the recovery of the density function of $X(t)$ stopped at an independent Erlang random variable and closed-form pricing formulae when the benefit payoff follows a European style. Regarding two maturity settings, n and T_n, we present corresponding cliquet-style payoff expressions in Section 2.3. If we attempt to use a linear combination of exponential distributions to approximate the distribution of T_x, we find it sufficient to replace T_x in (3) and (5) by an independent exponentially distributed random variable. By doing so, we propose (37) and (38) and derive their analytical valuation formulae in Sections 3 and 4, respectively. Numerical examples are provided in Section 5, and concluding remarks follow in the last section.

2. Preliminaries

2.1. Erlang Stopping of $X(t)$

By the Lévy–Khintchine formula, we assume the moment generating function (m.g.f.) of $X(t)$ exists. For $t > 0, z \in \mathbb{R}$, it is given by

$$\mathbf{E}[e^{zX(t)}] = e^{t\psi(z)}, \tag{6}$$

where $\psi(z)$ denotes the Lévy exponent of $X(t)$, i.e., the cumulative generating function of $X(1)$. We assume that $\psi(z)$ is a rational function, and its coefficient of the highest order term is 1.

We investigate the process $\{X(t)\}$ stopped at an independent random variable θ with $\theta \sim$Erlang(m, ρ). The density of $X(\theta)$ and corresponding m.g.f. are denoted by $f_{X(\theta)}$ and G_θ. By the laws of total expectation, for $z \in \mathbb{R}$, we have

$$G_\theta(z) = \mathbf{E}\Big[e^{zX(\theta)}\Big] = \mathbf{E}\Big[\mathbf{E}\Big[e^{zX(\theta)}\Big|\theta\Big]\Big] = \mathbf{E}\Big[e^{\theta\psi(z)}\Big] = \frac{\rho^m}{(\rho - \psi(z))^m}. \tag{7}$$

We assume the roots of

$$\psi(z) = \rho \tag{8}$$

are distinct and let $\{\alpha_p : \alpha_p < 0\}$ and $\{\beta_q : \beta_q > 0\}$ be root sequences. It follows from Vieta's theorem that

$$\psi(z) - \rho = \prod_p (z - \alpha_p) \prod_q (z - \beta_q). \tag{9}$$

We notice that the roots α_p and β_q are poles of $G_\theta(z)$. By partial fraction decomposition, $G_\theta(z)$ is rewritten as

$$\begin{aligned} G_\theta(z) &= \sum_p \left(\frac{a_{p1}}{\alpha_p - z} + \frac{a_{p2}}{(\alpha_p - z)^2} + \ldots + \frac{a_{pm}}{(\alpha_p - z)^m} \right) \\ &\quad + \sum_q \left(\frac{b_{q1}}{\beta_q - z} + \frac{b_{q2}}{(\beta_q - z)^2} + \ldots + \frac{b_{qm}}{(\beta_q - z)^m} \right) \\ &= \sum_p \sum_{k=1}^{m} \frac{a_{pk}}{(\alpha_p - z)^k} + \sum_q \sum_{l=1}^{m} \frac{b_{ql}}{(\beta_q - z)^l}, \end{aligned} \tag{10}$$

where the expansion coefficient sequences $\{a_{pk}\}$ and $\{b_{ql}\}$ can be calculated as follows: for $k, l = 1, 2, \ldots, m$,

$$a_{pk} = \frac{(-1)^{m-k}}{(m-k)!} \frac{d^{m-k}}{dz^{m-k}} \left(G_\theta(z)(\alpha_p - z)^m \right) \Big|_{z=\alpha_p}, \tag{11}$$

$$b_{ql} = \frac{(-1)^{m-l}}{(m-l)!} \frac{d^{m-l}}{dz^{m-l}} \left(G_\theta(z)(\beta_q - z)^m \right) \Big|_{z=\beta_q}. \tag{12}$$

Therefore, by the inverse Laplace transform, $f_{X(\theta)}$ can be inverted from (10):

$$f_{X(\theta)}(x) = \begin{cases} \displaystyle\sum_p \sum_{k=1}^{m} a_{pk}(-1)^k \frac{(-x)^{k-1} e^{-\alpha_p x}}{(k-1)!}, & x \le 0, \\ \displaystyle\sum_q \sum_{l=1}^{m} b_{ql} \frac{x^{l-1} e^{-\beta_q x}}{(l-1)!}, & x > 0. \end{cases} \tag{13}$$

The density $f_{X(\theta)}$ extends Formula (2.5) in [12]. When letting $m = 1$ in (13), we obtain the density of $X(t)$ stopped at an independent exponentially distributed random variable. For this reason, we assume $\tau \sim$Exp(γ), where $\gamma > 0$ and γ is not necessarily equal to ρ. The density function $X(\tau)$ is given by

$$f_{X(\tau)}(x) = \begin{cases} \displaystyle\sum_p w_p e^{-\omega_p x}, & x \le 0, \\ \displaystyle\sum_q v_q e^{-\nu_q x}, & x > 0, \end{cases} \tag{14}$$

where $\{\omega_p : \omega_p < 0\}$ and $\{\nu_q : \nu_q > 0\}$ denote distinct roots of $\psi(z) = \gamma$. The coefficient sequences $\{w_p\}$ and $\{v_q\}$ are computed by

$$w_p = \frac{-\gamma}{\psi'(\omega_p)}, \; v_q = \frac{\gamma}{\psi'(\nu_q)}. \tag{15}$$

Remark 1. *In recovering the density of $X(\theta)$, we assume $\psi(z)$ is a rational function. Our illustrated approach includes the case where $X(t)$ has independent jumps in both directions, and densities of upward or downward jump are expressed by a linear combination of exponential densities, see (2.9) and (2.10) in [12].*

2.2. Valuing European-Style Payoffs

Let $b(\cdot)$ represent a general class of benefit functions, which provides $b(S(\theta))$ at time θ. We consider

$$v(\delta, \rho) := \mathbf{E}\left[e^{-\delta\theta} b(S(\theta)) \right] = \int_0^\infty \mathbf{E}[b(S(t))] \frac{\rho^m t^{m-1} e^{-(\rho+\delta)t}}{(m-1)!} dt. \tag{16}$$

For any $w > -\rho$, we notice that (16) can be mathematically written as

$$\begin{aligned} v(\delta,\rho) =& \frac{\rho^m}{(\rho+w)^m}\int_0^\infty \mathbf{E}[b(S(t))]\frac{(\rho+w)^m t^{m-1} e^{-(\rho+w+\delta-w)t}}{(m-1)!}dt \\ =& \frac{\rho^m}{(\rho+w)^m} v(\delta-w,\rho+w). \end{aligned} \tag{17}$$

In particular, letting $w = \delta$ in (17) gives

$$v(\delta,\rho) = \frac{\rho^m}{(\rho+\delta)^m} v(0,\rho+\delta). \tag{18}$$

Formula (18) indicates that it suffices to study $v(0,\rho)$. To obtain $v(\delta,\rho)$, we substitute $\rho+\delta$ for ρ in $v(0,\rho)$ and multiply it by $\frac{\rho^m}{(\rho+\delta)^m}$. We note that this substitution requires modifying root and coefficient sequences. In addition, Formula (18) is regarded as a *factorization* (see Remark 4.1 in [12]), which is

$$v(\delta,\rho) = \mathbf{E}\left[e^{-\delta\theta}\right]\mathbf{E}[b(S(\vartheta))], \tag{19}$$

where $\vartheta \sim$Erlang$(\rho+\delta, m)$.

We take $\delta = 0$ in (18) and consider the following European style payoff

$$b(x) = [e^{\alpha x} - e^g]_+, \tag{20}$$

where $\alpha > 0$ denotes a participation rate, usually $\alpha \le 1$, and $g > 0$ is a fixed return rate. It follows from (16) that

$$\begin{aligned} v(0,\rho) =& \int_{-\infty}^\infty [e^{\alpha x} - e^g]_+ f_{X(\theta)}(x)dx \\ =& \int_{g/\alpha}^\infty [e^{\alpha x} - e^g] f_{X(\theta)}(x)dx. \end{aligned} \tag{21}$$

To calculate (21), we present a useful formula. For any real number $y \ge 0$ and any positive integer k, we have

$$\int_y^\infty \frac{x^{k-1}}{(k-1)!}e^{-sx}dx = \sum_{j=1}^k \frac{1}{s^{k+1-j}}\frac{y^{j-1}}{(j-1)!}e^{-sy}, \quad \mathrm{Re}(s) \ge 0, \tag{22}$$

where $\mathrm{Re}(\cdot)$ takes the real part.

By (22), for a real number h and a positive integer i, when $h > \alpha$ or $h > 0$, we introduce the notation:

$$\begin{aligned} \eta(h,i) :=& \int_0^\infty [e^{\alpha x} - e^g]_+ \frac{x^{i-1}e^{-hx}}{(i-1)!}dx \\ =& \int_{g/\alpha}^\infty \frac{x^{i-1}e^{-(h-\alpha)x}}{(i-1)!}dx - e^g\int_{g/\alpha}^\infty \frac{x^{i-1}e^{-hx}}{(i-1)!}dx \\ =& \sum_{j=1}^i \left(\frac{1}{(h-\alpha)^{i+1-j}} - \frac{1}{h^{i+1-j}}\right)\frac{(g/\alpha)^{j-1}}{(j-1)!}e^{-hg/\alpha+g}. \end{aligned} \tag{23}$$

With (23), it follows from (13) that

$$v(0,\rho) = \mathbf{E}\left[e^{\alpha X(\theta)} - e^g\right]_+ = \sum_q \sum_{l=1}^m b_{ql}\eta(\beta_q, l). \tag{24}$$

For the case where $\tau \sim$Exp(γ), we similarly obtain

$$v(\delta,\lambda) = \frac{\lambda}{\lambda+\delta} v(0,\delta+\lambda). \tag{25}$$

For any real number h satisfying $h > \alpha$ or $h > 0$, letting $i = 1$ in (23) gives

$$\zeta(h) := \frac{\alpha}{h(h-\alpha)} e^{-hg/\alpha+g} = \eta(h,1), \tag{26}$$

and then it follows from (14) that

$$v(0,\tau) = \mathbf{E}\Big[e^{\alpha X(\tau)} - e^{g}\Big]_+ = \sum_q v_q \zeta(\nu_q). \tag{27}$$

Suppose $X(t)$ stops at an independent non-negative random variable T, we have the following "put-call parity":

$$\Big[e^{g} - e^{\alpha X(T)}\Big]_+ - \Big[e^{\alpha X(T)} - e^{g}\Big]_+ = e^{g} - e^{X(T)}; \tag{28}$$

thereby, taking expectation on both sides gives

$$\mathbf{E}\Big[e^{g} - e^{\alpha X(T)}\Big]_+ - \mathbf{E}\Big[e^{\alpha X(T)} - e^{g}\Big]_+ = e^{g} - \mathbf{E}\Big[e^{X(T)}\Big]. \tag{29}$$

2.3. Cliquet-Style Payoff Structure

As revealed in [18,19], linear combinations of exponential functions are weakly dense in the positive real line. Thus, we approximate the density function f_{T_x} by

$$f_{T_x}(t) \approx \sum_j c_j f_{\tau_j}(t) = \sum_j c_j \gamma_j e^{-\gamma_j t}, \quad t \geq 0, \tag{30}$$

where $j \in \mathbb{N}^+$, τ_j's are exponentially distributed random variables with rate parameter $\gamma_j > 0$. Coefficients c_j's are constant such that $\sum_j c_j = 1$. Some coefficient c_j's can be negative.

If f_{T_x} is given by (30), then we obtain

$$V(\delta, T_x \wedge n) \approx \sum_j V(\delta,\tau_j \wedge n), \quad V(\delta, T_x \wedge T_n) \approx \sum_j V(\delta,\tau_j \wedge T_n). \tag{31}$$

The proof of (31) is given in Appendix B.

Therefore, instead of focusing on (3) and (5), we pay attention to

$$V(\delta,\tau\wedge n) = \mathbf{E}\Big[e^{-\delta(\tau\wedge n)} B(S(\tau\wedge n))\Big], \quad V(\delta,\tau\wedge T_n) = \mathbf{E}\Big[e^{-\delta(\tau\wedge T_n)} B(S(\tau\wedge T_n))\Big], \tag{32}$$

where $\tau \sim$Exp(γ), and it is independent of $S(t)$ and T_n.

It now remains to give detailed expressions for cliquet-style benefit payoffs $B(S(\tau\wedge n))$ and $B(S(\tau \wedge T_n))$. We discuss them in order. When the contract matures in n years, for $j = 1,2,\ldots,n$, we denote the return rate increment obtained during year $j-1$ to year j by

$$X_j^c := X(j) - X(j-1). \tag{33}$$

Since $X(t)$ is stationary and has independent increments, $X_1^c, X_2^c, \ldots, X_n^c$ are i.i.d. with $X_j^c \overset{d}{=} X(1)$. Then, the cliquet-style return collected to the end of year i is denoted by

$$Q_i^c := \prod_{j=1}^{i} \max\Big(e^{g}, e^{\alpha X_i^c}\Big). \tag{34}$$

We notice that (34) displays a "locked-in" feature and will be immune to possible downturns of the linked asset afterwards. Here, $B(S(n)) = Q_n^c$.

As for the other case, we attempt defining the return rate increment during a period in a similar way. Since $\theta \sim$Erlang(m, ρ), then $\mathbf{E}\theta = m/\rho$ and $\mathrm{Var}(\theta) = m/\rho^2 = \frac{(\mathbf{E}\theta)^2}{m}$. The assumption $\mathbf{E}\theta = 1$ implies $m = \rho$, if $m \to \infty$, Var $\to 0$, indicating that the mass of θ is condensed to 1. Since $\mathbf{E}T_n = n\mathbf{E}\theta = n$, $\mathrm{Var}(T_n) = n\mathrm{Var}(\theta) = \frac{n(\mathbf{E}\theta)^2}{m}$, if m goes to ∞, T_n converges to a Dirac function centered at its mean n. For $j = 1, 2, \ldots, n$, we regard $[T_{j-1}, T_j]$ as the j-th period, and define the return rate increment attained within $[T_{j-1}, T_j]$ by

$$X_j^r := X(T_j) - X(T_{j-1}). \tag{35}$$

Similarly, we notice that $X_1^r, X_2^r, \ldots, X_n^r$ are i.i.d. with $X_j^r \overset{d}{=} X(\theta)$. The whole return accrued after i periods is defined by

$$Q_i^r = \prod_{j=1}^{i} \max\left(e^g, e^{\alpha X_j^r}\right). \tag{36}$$

In this case, $B(S(T_n)) = Q_n^r$.

Remark 2. *The return rate increment in one period is defined as a difference, that is, the return at the end of a period over that in the beginning. The cliquet-style design accumulates earnings from every period in a compound or multiplicative sense. Many works pay attention to compound ratchet EIAs, for example, Ref. [4,7,8], etc. In addition, there exists the simple ratchet EIA, where returns are added together in calculation.*

With (34) and (36), we are now able to discuss two cases:

1. If the contract matures in n years, we aim to value

$$V^c(\delta, \tau \wedge n) = \mathbf{E}\left[e^{-\delta(\tau\wedge n)} Q^c_{[\tau\wedge n]} \max\left(e^g, e^{\alpha(X(\tau\wedge n) - X([\tau\wedge n]))}\right)\right]. \tag{37}$$

2. If we consider the random maturity T_n, then the valuation becomes

$$V^r(\delta, \tau \wedge T_n) = \mathbf{E}\left[e^{-\delta(\tau\wedge T_n)} Q^r_{[\tau\wedge T_n]} \max\left(e^g, e^{\alpha(X(\tau\wedge T_n) - X([\tau\wedge T_n]))}\right)\right]. \tag{38}$$

In (37), $[\cdot]$ means taking the integer part of the input while it has a slightly different interpretation in (38). The nuance will be explained in Section 4. We note that when $m \to \infty$, (35) (or (36)) approximates (33) (or (34)), and $V^r(\delta, \tau \wedge T_n)$ approaches to $V^c(\delta, \tau \wedge n)$.

3. Valuing $V^c(\delta, \tau \wedge n)$

Let $\mathbf{1}_{(\cdot)}$ be the indicator function throughout this paper. In the first, we divide (37) into a sum:

$$V^c(\delta, \tau \wedge n) = \mathbf{E}\left[e^{-\delta n} Q_n^c \mathbf{1}_{(\tau > n)}\right] + \mathbf{E}\left[e^{-\delta\tau} Q^c_{[\tau]} \max\left(e^g, e^{\alpha(X(\tau) - X([\tau]))}\right)\mathbf{1}_{(\tau\le n)}\right]. \tag{39}$$

The decomposition indicates that, if the policyholder survives n years, the amount of Q_n^c will be paid out as the survival benefit; otherwise, the death benefit will be paid out immediately after the insured's death. The death benefit is written as two components: $Q^c_{[\tau]}$ denotes the return collected in first $[\tau]$ complete years, while $\max\left(e^g, e^{\alpha(X(\tau) - X([\tau]))}\right)$ denotes the return received from last non-integer year.

Two parts in (39) are handled in order. Since τ is independent of the underlying, then by the law of total expectation, we have

$$\mathbf{E}\left[e^{-\delta n} Q_n^c \mathbf{1}_{(\tau > n)}\right] = \mathbf{E}\left[\mathbf{E}\left[e^{-\delta n} Q_n^c \mathbf{1}_{(\tau > n)} \middle| \tau\right]\right]$$

$$
\begin{aligned}
&= \mathbf{E}\Big[e^{-(\delta+\gamma)n} Q_n^c\Big] \\
&= V^c(\delta+\gamma, n).
\end{aligned} \tag{40}
$$

Recall that $X_1^c, X_2^c, \ldots, X_n^c$ are i.i.d., then we obtain

$$
V^c(\delta+\gamma, n) = [V^c(\delta+\gamma, 1)]^n. \tag{41}
$$

It suffices to calculate $V^c(\delta+\gamma, 1)$. To this end, for real numbers a, b, we introduce:

$$
\Xi^+(a,b) := \mathbf{E}\Big[e^{aX(1)} \mathbf{1}_{(X(1)>b)}\Big], \tag{42}
$$

$$
\Xi^-(a,b) := \mathbf{E}\Big[e^{aX(1)} \mathbf{1}_{(X(1)\le b)}\Big]. \tag{43}
$$

Analytical expressions of $\Xi^+(a,b)$ and $\Xi^-(a,b)$ are provided in Appendix C. When $a = 0$, $\Xi^+(0,b)$ and $\Xi^-(0,b)$ represent probabilities of events $\{X(1) > b\}$ and $\{X(1) \le b\}$. We obtain

$$
\begin{aligned}
V^c(\delta+\gamma, 1) &= e^{-(\delta+\gamma)} \mathbf{E}\Big[\max\Big(e^g, e^{\alpha X(1)}\Big)\Big] \\
&= e^{g-(\delta+\gamma)} \mathbf{P}(X(1) \le g/\alpha) + e^{-(\delta+\gamma)} \mathbf{E}\Big[e^{\alpha X(1)} \mathbf{1}_{(X(1)>g/\alpha)}\Big] \\
&= e^{g-(\delta+\gamma)} \Xi^-\Big(0, \frac{g}{\alpha}\Big) + e^{-(\delta+\gamma)} \Xi^+\Big(\alpha, \frac{g}{\alpha}\Big),
\end{aligned} \tag{44}
$$

which leads to

$$
\mathbf{E}\Big[e^{-\delta n} Q_n^c \mathbf{1}_{(\tau>n)}\Big] = \Big[e^{g-(\delta+\gamma)} \Xi^-\Big(0, \frac{g}{\alpha}\Big) + e^{-(\delta+\gamma)} \Xi^+\Big(\alpha, \frac{g}{\alpha}\Big)\Big]^n. \tag{45}
$$

By writing $\mathbf{1}_{(\tau\le n)} = \sum_{i=1}^n \mathbf{1}_{(i-1<\tau\le i)}$, the second term in (39) is also a sum of:

$$
\begin{aligned}
&\mathbf{E}\Big[e^{-\delta\tau} Q_{[\tau]}^c \max\Big(e^g, e^{\alpha(X(\tau)-X([\tau]))}\Big) \mathbf{1}_{(\tau\le n)}\Big] \\
&= \sum_{i=1}^n \mathbf{E}\Big[e^{-\delta\tau} Q_{i-1}^c \max\Big(e^g, e^{\alpha(X(\tau)-X(i-1))}\Big) \mathbf{1}_{(i-1<\tau\le i)}\Big] \\
&= \sum_{i=1}^n A_i,
\end{aligned} \tag{46}
$$

where each A_i is regarded as the present value of a payment and defined by

$$
\begin{aligned}
A_i := \; &\mathbf{E}\Big[e^{-\delta\tau} Q_{i-1}^c \max\Big(e^g, e^{\alpha(X(\tau)-X(i-1))}\Big) \mathbf{1}_{(\tau>i-1)}\Big] \\
&- \mathbf{E}\Big[e^{-\delta\tau} Q_{i-1}^c \max\Big(e^g, e^{\alpha(X(\tau)-X(i-1))}\Big) \mathbf{1}_{(\tau>i)}\Big].
\end{aligned} \tag{47}
$$

We apply the memory-less property to calculate two expectations in (47). First, we obtain

$$
\begin{aligned}
&\mathbf{E}\Big[e^{-\delta\tau} Q_{i-1}^c \max\Big(e^g, e^{\alpha(X(\tau)-X(i-1))}\Big) \mathbf{1}_{(\tau>i-1)}\Big] \\
&= \mathbf{P}(\tau > i-1) \mathbf{E}\Big[e^{-\delta(\tau-(i-1)+(i-1))} Q_{i-1}^c \max\Big(e^g, e^{\alpha(X(\tau)-X(i-1))}\Big) \Big| \tau > i-1\Big] \\
&= \mathbf{E}\Big[e^{-(\delta+\gamma)(i-1)} Q_{i-1}^c\Big] \mathbf{E}\Big[e^{-\delta(\tau-(i-1))} \max\Big(e^g, e^{\alpha(X(\tau)-X(i-1))}\Big) \Big| \tau > i-1\Big] \\
&= V^c(\delta+\gamma, i-1) \mathbf{E}\Big[e^{-\delta\tau} \max\Big(e^g, e^{\alpha X(\tau)}\Big)\Big],
\end{aligned} \tag{48}
$$

where the second equation is true due to that Q_{i-1}^c is independent from τ and $X(\tau) - X(i-1)$. We obtain $\mathbf{E}\Big[e^{-\delta\tau} \max\Big(e^g, e^{\alpha X(\tau)}\Big)\Big]$ from noting $\tau - (i-1) | \tau > i-1 \overset{d}{=} \tau$ and

$X(\tau) - X(i-1)|\tau > i-1 \stackrel{d}{=} X(\tau)$. In (48), $V^c(\delta+\gamma, i-1)$ is given by (45), and it follows from Formulas (19) and (27) that

$$\begin{aligned}\mathbf{E}\Big[e^{-\delta\tau}\max\Big(e^g, e^{\alpha X(\tau)}\Big)\Big] =& \frac{\gamma}{\gamma+\delta}\Big[e^g + \mathbf{E}\Big[e^{\alpha X(\tau')} - e^g\Big]_+\Big] \\ =& \frac{\gamma}{\gamma+\delta}\Big[e^g + \sum_k v'_k \zeta(\nu'_k)\Big] \end{aligned} \tag{49}$$

where we introduce $\tau' \sim \text{Exp}(\gamma+\delta)$. We solve $\psi(z) = \gamma+\delta$ and denote the roots, as well as coefficients by ν'_k and v'_k.

Similar to (48), we obtain

$$\begin{aligned} &\mathbf{E}\Big[e^{-\delta\tau}Q^c_{i-1}\max\Big(e^g, e^{\alpha(X(\tau)-X(i-1))}\Big)\mathbf{1}_{(\tau>i)}\Big] \\ =&\mathbf{P}(\tau>i)\mathbf{E}\Big[e^{-\delta(\tau-i+i)}Q^c_{i-1}\max\Big(e^g, e^{\alpha(X(\tau)-X(i)+X(i)-X(i-1))}\Big)\Big|\tau>i\Big] \\ =&e^{-(\delta+\lambda)}V^c(\delta+\gamma, i-1)\mathbf{E}\Big[e^{-\delta\tau}\max\Big(e^g, e^{\alpha X^c_i}e^{\alpha X(\tau)}\Big)\Big] \\ =&e^{-(\delta+\lambda)}V^c(\delta+\gamma, i-1)\mathbf{E}\Big[e^{-\delta\tau}\max\Big(e^g, e^{\alpha X(1)}e^{\alpha X(\tau)}\Big)\Big]. \end{aligned} \tag{50}$$

We note that $X(\tau)$ in (50) is an independent copy of $X(\tau) - X(i)|\tau > i$, and independent of X^c_i. Next, applying Formula (19) gives

$$\mathbf{E}\Big[e^{-\delta\tau}\max\Big(e^g, e^{\alpha X(1)}e^{\alpha X(\tau)}\Big)\Big] = \frac{\gamma}{\gamma+\delta}\Big[e^g + \mathbf{E}\Big[e^{\alpha X(1)}e^{\alpha X(\tau')} - e^g\Big]_+\Big]. \tag{51}$$

To calculate $\mathbf{E}\Big[e^{\alpha X(1)}e^{\alpha X(\tau')} - e^g\Big]_+$, we use ζ function defined in (26) and discuss the following:

1. If given $X(1) \le g/\alpha$, for a real number h' satisfying Re$(h')> \alpha$, we have

$$\begin{aligned}\int_0^\infty \Big[e^{\alpha X(1)}e^{\alpha x} - e^g\Big]_+ e^{-h'x}dx =& \int_{g/\alpha - X(1)}^\infty \Big[e^{\alpha X(1)}e^{\alpha x} - e^g\Big]e^{-h'x}dx \\ =& \zeta(h')e^{h'X(1)}. \end{aligned} \tag{52}$$

2. If given $X(1) > g/\alpha$, then, for a real number h' satisfying Re$(h')< 0$,

$$\begin{aligned}\int_{-\infty}^0 \Big[e^g - e^{\alpha X(1)}e^{\alpha x}\Big]_+ e^{-h'x}dx =& \int_{-\infty}^{g/\alpha - X(1)} \Big[e^g - e^{\alpha X(1)}e^{\alpha x}\Big]e^{-h'x}dx \\ =& \zeta(h')e^{h'X(1)}. \end{aligned} \tag{53}$$

Using (52) and (53), it then follows from (27) and (29) that, given $X(1)$,

$$\begin{aligned} &\mathbf{E}\Big[\Big[e^{\alpha X(1)}e^{\alpha X(\tau')} - e^g\Big]_+\Big|X(1)\Big] \\ =&\sum_k v'_k\zeta(\nu'_k)e^{\nu'_k X(1)}\mathbf{1}_{(X(1)\le g/\alpha)} \\ &+\Big\{\sum_j w'_j\zeta(\omega'_j)e^{\omega'_j X(1)} - e^g + e^{\alpha X(1)}\mathbf{E}\Big[e^{\alpha X(\tau')}\Big]\Big\}\mathbf{1}_{(X(1)>g/\alpha)}. \end{aligned} \tag{54}$$

Therefore, the expected value of above can be expressed via Ξ^+, Ξ^- functions and equals

$$\begin{aligned} &\mathbf{E}\Big[e^{\alpha X(1)}e^{\alpha X(\tau')} - e^g\Big]_+ \\ =&\sum_k v'_k\eta_{\tau'}(\nu'_k)\mathbf{E}\Big[e^{\nu'_k X(1)}\mathbf{1}_{(X(1)\le g/\alpha)}\Big] + \sum_j w'_j\zeta(\omega'_j)\mathbf{E}\Big[e^{\omega'_j X(1)}\mathbf{1}_{(X(1)>g/\alpha)}\Big] \end{aligned}$$

$$
\begin{aligned}
&- e^g \mathbf{E}\left[\mathbf{1}_{(X(1)>g/\alpha)}\right] + \mathbf{E}\left[e^{\alpha X(1)}\mathbf{1}_{(X(1)>g/\alpha)}\right]\mathbf{E}\left[e^{\alpha X(\tau')}\right] \\
=& \sum_k v'_k \zeta(\nu'_k)\Xi^-\left(\nu'_k, \frac{g}{\alpha}\right) + \sum_j w'_j \zeta(\omega'_j)\Xi^+\left(\omega'_k, \frac{g}{\alpha}\right) \\
&- e^g \Xi^+\left(0, \frac{g}{\alpha}\right) + \Xi^+\left(\alpha, \frac{g}{\alpha}\right)\frac{\gamma+\delta}{\gamma+\delta-\psi(\alpha)},
\end{aligned}
\tag{55}
$$

where applying (7) for $m = 1$ yields

$$
\mathbf{E}\left[e^{\alpha X(\tau')}\right] = \frac{\gamma+\delta}{\gamma+\delta-\psi(\alpha)}.
\tag{56}
$$

Now, for $i = 1, 2, \ldots, n$, A_i has a closed-form expression:

$$
\begin{aligned}
A_i = \frac{\gamma}{\gamma+\delta} V^c(\delta+\gamma, i-1)\Bigg[& e^g + \sum_k v'_k \zeta(\nu'_k) - e^{-(\gamma+\delta-g)} \\
& - e^{-(\gamma+\delta)}\mathbf{E}\left[e^{\alpha X(1)}e^{\alpha X(\tau')} - e^g\right]_+ \Bigg];
\end{aligned}
\tag{57}
$$

hence, a concise valuation formula is given by

$$
V^c(\delta, \tau \wedge n) = V^c(\delta+\gamma, n) + \sum_{i=1}^n A_i.
\tag{58}
$$

4. Valuing $V^r(\delta, \tau \wedge T_n)$

In this section, we derive a closed-form valuation formula for (38). Similar to the decomposition in (39), we have

$$
V^r(\delta, \tau \wedge T_n) = \mathbf{E}\left[e^{-\delta T_n} Q^r_n \mathbf{1}_{(\tau > T_n)}\right] + \mathbf{E}\left[e^{-\delta\tau} Q^r_{[\tau]} \max\left(e^g, e^{\alpha(X(\tau)-X([\tau]))}\right)\mathbf{1}_{(\tau\le T_n)}\right].
\tag{59}
$$

However, we explain that $[\tau]$ in (59) is slightly different. It specifies the maximum integer number such that $T_{[\tau]} \le \tau \le T_{[\tau]+1}$, that is, the number that how many complete random periods that τ goes by.

We pay attention to $\mathbf{E}\left[e^{-\delta T_n} Q^r_n \mathbf{1}_{(\tau > T_n)}\right]$ first. By the laws of total expectation, we can remove the indicator function and have

$$
\begin{aligned}
\mathbf{E}\left[e^{-\delta T_n} Q^r_n \mathbf{1}_{(\tau > T_n)}\right] &= \mathbf{E}\left[\mathbf{E}\left[e^{-\delta T_n} Q^r_n \mathbf{1}_{(\tau > T_n)} \middle| T_n\right]\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[e^{-(\delta+\gamma)T_n} Q^r_n \middle| T_n\right]\right] \\
&= \mathbf{E}\left[e^{-(\delta+\gamma)T_n} Q^r_n\right] \\
&= V^r(\delta+\gamma, T_n).
\end{aligned}
\tag{60}
$$

Recalling that, for $j = 1, 2, \ldots, n$, $T_1 - T_0, T_2 - T_1, \ldots, T_n - T_{n-1}$ are independent and have the same distribution with θ. In addition, $X^r_1, X^r_2, \ldots, X^r_n$ are independent and have the same distribution with $X(\theta)$. Thus, we are able to derive

$$
\begin{aligned}
V^r(\delta+\gamma, T_n) =& \mathbf{E}\left[e^{-(\delta+\gamma)T_n}\prod_{j=1}^n \max\left(e^g, e^{\alpha X^r_j}\right)\right] \\
=& \prod_{j=1}^n \mathbf{E}\left[e^{-(\delta+\gamma)(T_j - T_{j-1})}\max\left(e^g, e^{\alpha X^r_j}\right)\right] \\
=& \left\{\mathbf{E}\left[e^{-(\delta+\gamma)\theta}\max\left(e^g, e^{\alpha X(\theta)}\right)\right]\right\}^n.
\end{aligned}
\tag{61}
$$

By (19) and (24), we obtain

$$\begin{aligned}\mathbf{E}\left[e^{-(\delta+\gamma)\theta}\max\left(e^g, e^{\alpha X(\theta)}\right)\right] =&\mathbf{E}\left[e^{-(\delta+\gamma)\theta}\right]\left[e^g+\mathbf{E}\left[e^{\alpha X(\theta')}-e^g\right]_+\right]\\ =&\left(\frac{\rho}{\delta+\gamma+\rho}\right)^m\left[e^g+\sum_q\sum_{l=1}^m b'_{ql}\eta(\beta'_q,l)\right],\end{aligned} \tag{62}$$

where a new random variable $\theta'\sim$Erlang$(m,\rho+\delta+\gamma)$ is introduced. We solve $\psi(z)=\rho+\delta+\gamma$. The roots and coefficients are denoted by β'_q and b'_{ql}, respectively. It follows from (62) that

$$V^r(\delta+\gamma,T_n)=\left(\frac{\rho}{\delta+\gamma+\rho}\right)^{nm}\left[e^g+\sum_q\sum_{l=1}^m b'_{ql}\eta(\beta'_q,l)\right]^n. \tag{63}$$

To calculate the second expectation in (59), we rewrite the indicator function as $\mathbf{1}_{(\tau\le T_n)}=\sum_{i=1}^n\mathbf{1}_{(T_{i-1}\le\tau\le T_i)}$, and then the expectation is expressed by a sum of consecutive payments as follows,

$$\begin{aligned}&\mathbf{E}\left[e^{-\delta\tau}Q^r_{[\tau]}\max\left(e^g, e^{\alpha(X(\tau)-X(T_{[\tau]}))}\right)\mathbf{1}_{(\tau\le T_n)}\right]\\ =&\sum_{i=1}^n\mathbf{E}\left[e^{-\delta\tau}Q^r_{i-1}\max\left(e^g, e^{\alpha(X(\tau)-X(T_{i-1}))}\right)\mathbf{1}_{(T_{i-1}<\tau\le T_i)}\right]\\ =&\sum_{i=1}^n B_i,\end{aligned} \tag{64}$$

where the present value of each payment is defined by

$$\begin{aligned}B_i:=&\,\mathbf{E}\left[e^{-\delta\tau}Q^r_{i-1}\max\left(e^g, e^{\alpha(X(\tau)-X(T_{i-1}))}\right)\mathbf{1}_{(\tau>T_{i-1})}\right]\\ &-\mathbf{E}\left[e^{-\delta\tau}Q^r_{i-1}\max\left(e^g, e^{\alpha(X(\tau)-X(T_{i-1}))}\right)\mathbf{1}_{(\tau>T_i)}\right].\end{aligned} \tag{65}$$

The first term in (65) conditioned on T_{i-1} equals to

$$\mathbf{E}\left[e^{-\delta\tau}Q^r_{i-1}\max\left(e^g, e^{\alpha(X(\tau)-X(T_{i-1}))}\right)\mathbf{1}_{(\tau>T_{i-1})}\right] \tag{66}$$

$$=\mathbf{E}\left[\mathbf{E}\left[e^{-\delta\tau}Q^r_{i-1}\max\left(e^g, e^{\alpha(X(\tau)-X(T_{i-1}))}\right)\mathbf{1}_{(\tau>T_{i-1})}\middle|T_{i-1}\right]\right]. \tag{67}$$

Given that T_{i-1} is fixed, Q^r_{i-1} is independent of τ and $X(\tau)-X(T_{i-1})$. Given T_{i-1}, by the memory-less property, we have $\tau-T_{i-1}|\tau>T_{i-1}\overset{d}{=}\tau$ and $X(\tau)-X(T_{i-1})|\tau>T_{i-1}\overset{d}{=}X(\tau)$. Therefore, the conditional expectation becomes

$$\begin{aligned}&\mathbf{P}(\tau>T_{i-1})\mathbf{E}\left[e^{-\delta(\tau-T_{i-1}+T_{i-1})}Q^r_{i-1}\max\left(e^g, e^{\alpha(X(\tau)-X(T_{i-1}))}\right)\middle|\tau>T_{i-1}\right]\\ =&\mathbf{E}\left[e^{-(\delta+\lambda)T_{i-1}}Q^r_{i-1}\right]\mathbf{E}\left[e^{-\delta\tau}\max\left(e^g, e^{\alpha X(\tau)}\right)\right],\end{aligned} \tag{68}$$

and it follows that

$$\begin{aligned}&\mathbf{E}\left[e^{-\delta\tau}Q^r_{i-1}\max\left(e^g, e^{\alpha(X(\tau)-X(T_{i-1}))}\right)\mathbf{1}_{(\tau>T_{i-1})}\right]\\ =&\mathbf{E}\left[\mathbf{E}\left[e^{-(\delta+\lambda)T_{i-1}}Q^r_{i-1}\right]\mathbf{E}\left[e^{-\delta\tau}\max\left(e^g, e^{\alpha X(\tau)}\right)\right]\middle|T_{i-1}\right]\\ =&V^r(\delta+\gamma,T_{i-1})\mathbf{E}\left[e^{-\delta\tau}\max\left(e^g, e^{\alpha X(\tau)}\right)\right],\end{aligned} \tag{69}$$

where $V^r(\delta+\gamma,T_{i-1})$ can be obtained by replacing n by $i-1$ in (63), while the expectation in (69) is given by (49).

The second term in (59) can be similarly handled, as in (50). We have

$$\begin{aligned}&\mathbf{E}\left[e^{-\delta\tau}Q^r_{i-1}\max\left(e^g, e^{\alpha(X(\tau)-X(T_{i-1}))}\right)\mathbf{1}_{(\tau>T_i)}\right]\\=&\mathbf{E}\left[e^{-(\delta+\gamma)T_i}Q^r_{i-1}\right]\mathbf{E}\left[e^{-\delta\tau}\max\left(e^g, e^{\alpha X(\theta)}e^{\alpha X(\tau)}\right)\right],\end{aligned}\tag{70}$$

where the first term in (70) is given by

$$\begin{aligned}\mathbf{E}\left[e^{-(\delta+\gamma)T_i}Q^r_{i-1}\right] =&\mathbf{E}\left[e^{-(\delta+\gamma)(T_i-T_{i-1})}\right]\mathbf{E}\left[e^{-(\delta+\gamma)T_{i-1}}Q^r_{i-1}\right]\\=&\mathbf{E}\left[e^{-(\delta+\gamma)\theta}\right]V^r(\delta+\gamma, T_{i-1})\\=&\left(\frac{\rho}{\rho+\delta+\gamma}\right)^m V^r(\delta+\gamma, T_{i-1}).\end{aligned}\tag{71}$$

By (19), the second expectation in (70) is factored into

$$\begin{aligned}\mathbf{E}\left[e^{-\delta\tau}\max\left(e^g, e^{\alpha X(\theta)}e^{\alpha X(\tau)}\right)\right] =&\mathbf{E}\left[e^{-\delta\tau}\right]\left[e^g+\mathbf{E}\left[e^{\alpha X(\theta)}e^{\alpha X(\tau')}-e^g\right]_+\right]\\=&\frac{\gamma}{\gamma+\delta}\left[e^g+\mathbf{E}\left[e^{\alpha X(\theta)}e^{\alpha X(\tau')}-e^g\right]_+\right].\end{aligned}\tag{72}$$

Regarding $X(\theta)\le g/\alpha$ or $X(\theta)>g/\alpha$, we discuss two cases:

1. If $X(\theta)\le g/\alpha$, for a real number h' satisfying Re(h')$>\alpha$, we have

$$\int_0^\infty \left[e^{\alpha X(\theta)}e^{\alpha x}-e^g\right]_+ e^{-h'x}dx=\zeta(h')e^{h'X(\theta)},\tag{73}$$

2. If $X(\theta)>g/\alpha$, then, for a real number h' satisfying Re(h')<0, we similarly derive

$$\int_{-\infty}^0 \left[e^g-e^{\alpha X(\theta)}e^{\alpha x}\right]_+ e^{-h'x}dx=\zeta(h')e^{h'X(\theta)}.\tag{74}$$

Similar to the derivation of (55), we apply (29), (73), and (74) and obtain

$$\begin{aligned}&\mathbf{E}\left[e^{\alpha X(\theta)}e^{\alpha X(\tau')}-e^g\right]_+\\=&\sum_k v'_k\zeta(\nu'_k)\mathbf{E}\left[e^{\nu'_k X(\theta)}\mathbf{1}_{(X(\theta)\le g/\alpha)}\right]+\sum_j w'_j\zeta(\omega'_j)\mathbf{E}\left[e^{\omega'_j X(\theta)}\mathbf{1}_{(X(\theta)>g/\alpha)}\right]\\&-e^g\mathbf{E}\left[\mathbf{1}_{(X(\theta)>g/\alpha)}\right]+\mathbf{E}\left[e^{\alpha X(\theta)}\mathbf{1}_{(X(\theta)>g/\alpha)}\right]\mathbf{E}\left[e^{\alpha X(\tau')}\right].\end{aligned}\tag{75}$$

We obtain $\mathbf{E}\left[e^{\alpha X(\tau')}\right]$ in (56).

For real numbers a, b, we introduce:

$$\Upsilon^+(a,b):=\mathbf{E}\left[e^{aX(\theta)}\mathbf{1}_{(X(\theta)>b)}\right],\tag{76}$$

$$\Upsilon^-(a,b):=\mathbf{E}\left[e^{aX(\theta)}\mathbf{1}_{(X(\theta)\le b)}\right],\tag{77}$$

where closed-form expressions of $\Upsilon^+(a,b)$ and $\Upsilon^-(a,b)$ are provided in Appendix D. When $a=0$, $\Upsilon^+(0,b)$ and $\Upsilon^-(0,b)$ represent probabilities of events $\{X(\theta)>b\}$ and $\{X(\theta)\le b\}$. Using Υ^+, Υ^- functions, (75) is further expressed by

$$\begin{aligned}&\mathbf{E}\left[e^{\alpha X(\theta)}e^{\alpha X(\tau')}-e^g\right]_+\\=&\sum_k v'_k\zeta(\nu'_k)\Upsilon^-\left(\nu'_k,\frac{g}{\alpha}\right)+\sum_j w'_j\zeta(\omega'_j)\Upsilon^+\left(\omega'_j,\frac{g}{\alpha}\right)\end{aligned}$$

$$- e^{g} \Upsilon^{+}\left(0, \frac{g}{\alpha}\right) + \frac{\gamma+\delta}{\gamma+\delta-\psi(\alpha)} \Upsilon^{+}\left(\alpha, \frac{g}{\alpha}\right). \tag{78}$$

Hence, for $i = 1, 2, \ldots, n$, B_i is explicitly given by

$$\begin{aligned} B_i = \frac{\gamma}{\gamma+\delta} V^r(\delta+\gamma, i-1) &\left[e^{g} + \sum_k v'_k \zeta(v'_k) - e^{g} \left(\frac{\rho}{\rho+\delta+\gamma}\right)^m \right. \\ &\left. - \left(\frac{\rho}{\rho+\delta+\gamma}\right)^m \mathbf{E}\left[e^{\alpha X(\theta)} e^{\alpha X(\tau')} - e^{g} \right]_+ \right], \end{aligned} \tag{79}$$

which immediately leads to a succinct expression for $V^r(\delta, \tau \wedge T_n)$:

$$V^r(\delta, \tau \wedge T_n) = V^r(\delta+\gamma, T_n) + \sum_{i=1}^{n} B_i. \tag{80}$$

5. Numerical Examples

We provide several numerical examples to illustrate our method of valuing the cliquet-style contract considered in this paper. The dynamics of the stock price process are modelled by the renowned Kou model, where the jump size Y follows a double exponential distribution. The density function of Y is given by

$$f_Y(y) = p_J \cdot \eta_1 e^{-\eta_1 y} \mathbf{1}_{(y \geq 0)} + q_J \cdot \eta_2 e^{\eta_2 y} \mathbf{1}_{(y<0)}, \; \eta_1 > 1, \; \eta_2 > 0, \tag{81}$$

where $p_J, q_J \geq 0$, such that $p_J + q_J = 1$, representing the probabilities that upward and downward jumps might occur, respectively. In the experiment, we let $p_J = q_J = 0.5$, $\lambda = 0.6$, $\eta_1 = 4$, $\eta_2 = 1$ and denote the risk-free interest rate by $r = 0.05$. Suppose that there is a risk neutral measure $\mathbb{Q}$, such that under $\mathbb{Q}$, $S(t)$ has the following dynamic:

$$S(t) = S(0) \exp(\omega t + L(t)), \tag{82}$$

where $L(t) = \sigma W(t) + \sum_{i=1}^{N(t)} Y_i$, and $\omega = r - \psi_L(1)$ denotes the risk-neutral drift under $\mathbb{Q}$. The Lévy exponent of $L(t)$, denoted by ψ_L, is given by

$$\psi_L(z) = \frac{1}{2}\sigma^2 z^2 + \lambda p \frac{z}{\eta_1 - z} - \lambda q \frac{z}{\eta_2 + z}, \quad z \in \mathbb{R}. \tag{83}$$

In Section 2.3, we mentioned that f_{T_x} can be approximated by a linear combination of exponential densities. This can be realized by fitting the life table data. There exist several methods to determine the rate parameters and corresponding expansion coefficients: Ref. [18] used a Jacobi polynomial method; Ref. [20] applied the least square method, while [21] proposed using the Hankel matrix. We follow [20] to fit the survival distribution function of T_x, denoted by $\overline{F}_{T_x}$, via the life table obtained from Appendix 2A of [32].

The fitted result was obtained by solving a system of non-linear least square equations with a constraint that the mean of approximated distribution shall be equal to the expected future lifetime calculated from the life table. We suppose that the current age of an insured is $x = 30$, and that the number of data points is 25. We fit $\overline{F}_{T_x}$ by using three and five exponential functions. The fitted densities, denoted by mortality models $\mathcal{M}_3$ and $\mathcal{M}_5$, are given below:

$$\begin{aligned} f_{T_{30}}^{\mathcal{M}_3}(t) =& 9.99796 \times 0.117051 e^{-0.117051t} \\ & - 18.9975 \times 0.0091914 e^{-0.0091914t} + 9.99953 \times 0.0602965 e^{-0.0602965t}, \end{aligned} \tag{84}$$

$$\begin{aligned} f_{T_{30}}^{\mathcal{M}_5}(t) =& 9.73457 \times 0.1058 e^{-0.1058t} + 20.2169 \times 0.1058 e^{-0.1058t} \\ & + 20.2169 \times 0.1058 e^{-0.1058t} - 59.1679 \times 0.0994942 e^{-0.0994942t} \\ & + 9.99957 \times 0.0605769 e^{-0.0605769t}. \end{aligned} \tag{85}$$

We report numerical results in Tables A1–A3, rounding at the fourth decimal. A constant maturity $n = 3$ is set. Results calculated using (58) are displayed in V^c columns. To illustrate randomized results, we set $m = 10, 20$, and 40. In all tables, we observe that, as m increases, the valuation results are closer to the ones displayed in the V^c columns.

To examine the sensitivities, we investigate the parameters g, α, σ, and *ceteris paribus*. In Table A1, we find that a higher g results in a greater outcome. A higher guaranteed rate of return raises the lower limit of overall benefit amount. Similarly, we observed that the increase of α brings a greater valuation result in Table A2. A bigger participation rate includes more returns into calculation, which yields a higher outcome after the accumulation period.

In Table A3, the result is boosted as σ increases. The larger value of σ pushes the price of the underlying upwards, making it more likely to exceed g. Although a higher σ possibly pushes the price downward as well, this negative effect on returns is eliminated by the "locked-in" feature of the contract.

6. Discussion

In this paper, we investigated the valuation problem of a cliquet-style EIA contract that also provides death benefits. The contract was equity-linked, relying on the performance of a stock. The stock price process was assumed to follow a jump diffusion model. We supposed that the remaining lifetime of an insured was a positive random variable, and its distribution was approximated by a linear combination of exponential distributions.

Our work considers two maturity scenarios, a constant and an erlangized version of it. We demonstrated the erlangization technique by assuming a generic Erlang distributed random variable for the inter-period time, thereby, approximating the deterministic valuation counterpart. In both scenarios, we explicitly derived corresponding valuation formulae. Our numerical illustrations witness a converging trend when m increased as the erlangization was in effect.

The common cap or floor policies are not covered, yet it is a feasible extension from the current framework. In addition, surrender (or lapsation) behaviour of policyholders are also of interest. Those topics are left for our future research.

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Appendix A. Supplementary Results

We summarize numerical results in the following tables.

Table A1. Valuation results w.r.t. g, $\alpha = 90\%$, $\sigma = 0.25$, and $n = 3$.

$g(\%)$	$\mathcal{M}_3$				$\mathcal{M}_5$			
	$m = 10$	$m = 20$	$m = 40$	V^c	$m = 10$	$m = 20$	$m = 40$	V^c
0.05	1.4246	1.4291	1.4314	1.4340	1.4248	1.4293	1.4315	1.4342
1.00	1.4332	1.4377	1.4400	1.4426	1.4334	1.4379	1.4402	1.4428
1.50	1.4420	1.4465	1.4488	1.4514	1.4422	1.4467	1.4490	1.4516
2.00	1.4510	1.4555	1.4578	1.4604	1.4512	1.4557	1.4580	1.4606
2.50	1.4603	1.4647	1.4670	1.4696	1.4604	1.4649	1.4672	1.4698
3.00	1.4697	1.4742	1.4764	1.4790	1.4699	1.4744	1.4766	1.4792

Table A2. Valuation results w.r.t. α, $g = 2.50\%$, $\sigma = 0.25$, and $n = 3$.

$\alpha(\%)$	$\mathcal{M}_3$				$\mathcal{M}_5$			
	$m = 10$	$m = 20$	$m = 40$	V^c	$m = 10$	$m = 20$	$m = 40$	V^c
75	1.3279	1.3314	1.3332	1.3352	1.3280	1.3315	1.3333	1.3353
80	1.3696	1.3734	1.3754	1.3776	1.3698	1.3736	1.3755	1.3777
85	1.4137	1.4178	1.4199	1.4223	1.4139	1.4180	1.4201	1.4225
90	1.4603	1.4647	1.4670	1.4696	1.4604	1.4649	1.4672	1.4698
95	1.5095	1.5144	1.5168	1.5197	1.5097	1.5146	1.5170	1.5199

Table A3. Valuation results w.r.t. σ, $g = 2.50\%$, $\alpha = 90\%$, and $n = 3$.

σ	$\mathcal{M}_3$				$\mathcal{M}_5$			
	$m = 10$	$m = 20$	$m = 40$	V^c	$m = 10$	$m = 20$	$m = 40$	V^c
0.15	1.3787	1.3823	1.3841	1.3859	1.3789	1.3824	1.3842	1.3861
0.20	1.4178	1.4218	1.4238	1.4260	1.4180	1.4220	1.4240	1.4262
0.25	1.4603	1.4647	1.4670	1.4696	1.4604	1.4649	1.4672	1.4698
0.30	1.5054	1.5104	1.5129	1.5158	1.5056	1.5106	1.5131	1.5160
0.35	1.5526	1.5582	1.5610	1.5643	1.5529	1.5584	1.5612	1.5645

Appendix B. Proof of (31)

Proof. We only address the case for a random variable T. Assume that T is independent of both T_x and $S(t)$. For notation convenience, we denote the benefit payoff function by $B(S(t))$ at time $t > 0$. First,

$$\begin{aligned} V(\delta, T_x \wedge T) &= \mathbf{E}\Big[e^{-\delta(T_x \wedge T)} B(S(T_x \wedge T))\Big] \\ &= \mathbf{E}\Big[e^{-\delta T} B(S(T)) \mathbf{1}_{(T_x > T)}\Big] + \mathbf{E}\Big[e^{-\delta T_x} B(S(T_x)) \mathbf{1}_{(T_x \le T)}\Big]. \end{aligned} \tag{A1}$$

We apply the laws of total expectation and have

$$\mathbf{E}\Big[e^{-\delta T} B(S(T)) \mathbf{1}_{(T_x > T)}\Big] = \mathbf{E}\Big[\mathbf{E}\Big[e^{-\delta T} B(S(T)) \mathbf{1}_{(T_x > T)} \Big| T\Big]\Big]. \tag{A2}$$

Since f_{T_x} is given in (30), then the inside conditional expectation equals

$$\begin{aligned} \mathbf{P}(T_x > T)\mathbf{E}\Big[e^{-\delta T} B(S(T))\Big] &\approx \sum_j c_j e^{-\gamma_j T} \mathbf{E}\Big[e^{-\delta T} B(S(T))\Big] \\ &= \sum_j c_j \mathbf{E}\Big[e^{-\delta \tau_j} B(S(T)) \mathbf{1}_{(\tau_j > T)}\Big]; \end{aligned} \tag{A3}$$

thus, we obtain

$$\begin{aligned}\mathbf{E}\left[e^{-\delta T}B(S(T))\mathbf{1}_{(T_x>T)}\right] &= \sum_j c_j\mathbf{E}\left[\mathbf{E}\left[e^{-\delta T}B(S(T))\mathbf{1}_{(\tau_j>T)}\middle|T\right]\right] \\ &= \sum_j c_j\mathbf{E}\left[e^{-\delta T}B(S(T))\mathbf{1}_{(\tau_j>T)}\right]. \end{aligned} \tag{A4}$$

Similarly, we obtain

$$\begin{aligned}\mathbf{E}\left[e^{-\delta T_x}B(S(T_x))\mathbf{1}_{(T_x\le T)}\right] &= \mathbf{E}\left[\mathbf{E}\left[e^{-\delta T_x}B(S(T_x))\mathbf{1}_{(T_x\le T)}\middle|T\right]\right] \\ &= \mathbf{E}\left[\int_0^T e^{-\delta t}B(S(t))f_{T_x}(t)dt\right] \\ &\approx \sum_j c_j\mathbf{E}\left[\int_0^T e^{-\delta t}B(S(t))f_{\tau_j}(t)dt\right] \\ &= \sum_j c_j\mathbf{E}\left[e^{-\delta\tau_j}B(S(\tau_j))\mathbf{1}_{(\tau_j\le T)}\right]. \end{aligned} \tag{A5}$$

To sum up, we obtain

$$V(\delta, T_x\wedge T) \approx \sum_j c_j\mathbf{E}\left[e^{-\delta T}B(S(T))\mathbf{1}_{(\tau_j>T)}\right] + \sum_j c_j\mathbf{E}\left[e^{-\delta\tau_j}B(S(\tau_j))\mathbf{1}_{(\tau_j\le T)}\right], \tag{A6}$$

that is,

$$V(\delta, T_x\wedge T) \approx \sum_j c_j V(\delta, \tau_j\wedge T). \tag{A7}$$

□

Appendix C. Explicit Formulas for Ξ^+ and Ξ^-

Following [33], we derive closed-form expressions for Ξ^+ and Ξ^- in terms of Hh functions. Hh function is a special function in mathematical physics. For $n\ge 0$, the Hh function is non-increasing and defined by

$$\mathrm{Hh}_n(x) = \int_x^\infty \mathrm{Hh}_{n-1}(y)dy = \frac{1}{n!}\int_x^\infty (t-x)^n e^{-t^2/2}dt \ge 0. \tag{A8}$$

When $n=-1$ and $n=0$, two special cases are

$$\mathrm{Hh}_{-1} = e^{-x^2/2} = \sqrt{2\pi}\phi(x), \quad \mathrm{Hh}_0(x) = \sqrt{2\pi}\Psi(-x), \tag{A9}$$

where $\phi(x)$ and $\Psi(x)$ denote the p.d.f. and c.d.f. of a standard normal random variable, respectively.

In addition, the Hh function can be also defined by

$$\mathrm{Hh}_n(x) = 2^{-n/2}\sqrt{\pi}e^{-\frac{x^2}{2}} \times \left\{\frac{{}_1F_1\left(\frac{1}{2}n+\frac{1}{2},\frac{1}{2},\frac{1}{2}x^2\right)}{\sqrt{2}\Gamma\left(1+\frac{1}{2}n\right)} - x\frac{{}_1F_1\left(\frac{1}{2}n+1,\frac{3}{2},\frac{1}{2}x^2\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}n\right)}\right\}, \tag{A10}$$

where ${}_1F_1$ is the confluent hypergeometric function, also known as the Kummer's function.

To calculate the expectation Ξ^+ and Ξ^-, we evaluate the following integrals first. For $n\ge 0$ and arbitrary constants α,β,ϱ, and c, we define

$$I_n^+(c;\alpha,\beta,\varrho) := \int_c^\infty e^{\alpha x}\mathrm{Hh}_n(\beta x-\varrho)dx, \tag{A11}$$

$$I_n^-(c;\alpha,\beta,\varrho) := \int_{-\infty}^c e^{\alpha x}\mathrm{Hh}_n(\beta x-\varrho)dx. \tag{A12}$$

Proposition A1. *The I_n^+ function is analytically calculated as follows,*

1. *If $\alpha \neq 0$ and $\beta > 0$, then, for all $n \geq -1$,*

$$\begin{aligned} I_n^+(c;\alpha,\beta,\varrho) = &-\frac{e^{\alpha c}}{\alpha}\sum_{i=0}^{n}\left(\frac{\beta}{\alpha}\right)^{n-i}\mathrm{Hh}_i(\beta c-\varrho) \\ &+\left(\frac{\beta}{\alpha}\right)^{n+1}\frac{\sqrt{2\pi}}{\beta}e^{\frac{\alpha\varrho}{\beta}+\frac{\alpha^2}{2\beta^2}}\Psi\left(-\beta c+\varrho+\frac{\alpha}{\beta}\right). \end{aligned} \tag{A13}$$

2. *If $\alpha < 0$ and $\beta < 0$, then, for all $n \geq -1$,*

$$\begin{aligned} I_n^+(c;\alpha,\beta,\varrho) = &-\frac{e^{\alpha c}}{\alpha}\sum_{i=0}^{n}\left(\frac{\beta}{\alpha}\right)^{n-i}\mathrm{Hh}_i(\beta c-\varrho) \\ &-\left(\frac{\beta}{\alpha}\right)^{n+1}\frac{\sqrt{2\pi}}{\beta}e^{\frac{\alpha\varrho}{\beta}+\frac{\alpha^2}{2\beta^2}}\Psi\left(\beta c-\varrho-\frac{\alpha}{\beta}\right). \end{aligned} \tag{A14}$$

3. *If $\alpha = 0$ and $\beta > 0$, for all $n \geq 0$, $I_n^+(c;\alpha,\beta,\varrho) = \frac{1}{\beta}\mathrm{Hh}_{n+1}(\beta c-\varrho)$. If $\beta \leq 0$ and $\alpha \geq 0$, then, for all $n \geq 0$, $I_n^+(c;\alpha,\beta,\varrho) = \infty$. If $\beta = 0$ and $\alpha < 0$, then for all $n \geq 0$, $I_n^+(c;\alpha,\beta,\varrho) = \frac{1}{\alpha}e^{\alpha c}\mathrm{Hh}_n(-\varrho)$.*

Proposition A2. *The I_n^- function is analytically calculated as follows,*

1. *if $\alpha \neq 0$ and $\beta < 0$, then for all $n \geq -1$,*

$$\begin{aligned} I_n^-(c;\alpha,\beta,\varrho) = &\frac{1}{\alpha}e^{\alpha c}\sum_{i=0}^{n}\left(\frac{\beta}{\alpha}\right)^{n-i}\mathrm{Hh}_i(\beta c-\varrho) \\ &-\left(\frac{\beta}{\alpha}\right)^{n+1}\frac{\sqrt{2\pi}}{\beta}e^{\frac{\varrho\alpha}{\beta}+\frac{\alpha^2}{2\beta^2}}\Psi\left(-\beta c+\varrho+\frac{\alpha}{\beta}\right). \end{aligned} \tag{A15}$$

2. *If $\alpha > 0, \beta > 0$, then, for all $n \geq -1$,*

$$\begin{aligned} I_n^-(c;\alpha,\beta,\varrho) = &\frac{1}{\alpha}e^{\alpha c}\sum_{i=0}^{n}\left(\frac{\beta}{\alpha}\right)^{n-i}\mathrm{Hh}_i(\beta c-\varrho) \\ &+\left(\frac{\beta}{\alpha}\right)^{n+1}\frac{\sqrt{2\pi}}{\beta}e^{\frac{\varrho\alpha}{\beta}+\frac{\alpha^2}{2\beta^2}}\Psi\left(\beta c-\varrho-\frac{\alpha}{\beta}\right). \end{aligned} \tag{A16}$$

3. *If $\alpha = 0$ and $\beta < 0$, then, for $n \geq 0$, $I_n^-(c;\alpha,\beta,\varrho) = -\frac{1}{\beta}\mathrm{Hh}_{n+1}(\beta c-\varrho)$. If $\alpha \leq 0$ and $\beta \geq 0$, then, for all $n \geq 0$, $I_n^-(c;\alpha,\beta,\varrho) = \infty$. If $\alpha > 0$ and $\beta = 0$, then, for all $n \geq 0$, $I_n^-(c;\alpha,\beta,\varrho) = \frac{1}{\alpha}e^{\alpha c}\mathrm{Hh}_n(-\varrho)$.*

The proofs of Proposition A1 are given in Appendix B in [33], and Proposition A2 can be derived analogously.

Since the density of Y is given in (81), it is equivalent to

$$Y \stackrel{d}{=} \begin{cases} \xi^+, & \text{w.p. } p_J, \\ \xi^-, & \text{w.p. } q_J, \end{cases} \tag{A17}$$

where ξ^+ and ξ^- are exponential r.v.s with rate parameters $\eta_1 > 1$ and $\eta_2 > 0$, respectively.

For a fixed integer $n \geq 1$, [33] derived a decomposition of the sum of double exponential random variables. Assuming that ξ_i^+ and ξ_i^- are i.i.d. exponential r.v.s with rates η_1 and η_2, [33] obtained

$$\sum_{i=1}^{n} Y_i \stackrel{d}{=} \begin{cases} \sum_{i=1}^{k} \xi_i^+, \text{ w.p. } P_{n,k},\ k = 1,2,\ldots,n, \\ \sum_{i=1}^{k} \xi_i^-, \text{ w.p. } Q_{n,k},\ k = 1,2,\ldots,n. \end{cases} \tag{A18}$$

where, for $1 \leq k \leq n-1$, $P_{n,k}$ and $Q_{n,k}$ are given by

$$P_{n,k} = \sum_{i=k}^{n-1} \binom{n-k-1}{i-k}\binom{n}{i}\left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{i-k}\left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{n-i} p_J^i q_J^{n-i}, \tag{A19}$$

$$Q_{n,k} = \sum_{i=k}^{n-1} \binom{n-k-1}{i-k}\binom{n}{i}\left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{n-i}\left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{i-k} p_J^{n-i} q_J^{i}, \tag{A20}$$

in which we denote $\binom{0}{0} = 1$ by convention.

Suppose $\{\xi_1, \xi_2, \ldots\}$ denotes a sequence of i.i.d. exponential r.v. with the same rate parameter $\eta > 0$ and Z denotes an independent normally distributed random variable with mean 0 and variance σ^2. For $n \geq 1$, [33] presents the density functions of $Z + \sum_{i=1}^{n} \xi_i$ and $Z - \sum_{i=1}^{n} \xi_i$ as follows,

$$f_{Z+\sum_{i=1}^{n}\xi_i}(x) = (\sigma\eta)^n \frac{e^{(\sigma\eta)^2/2}}{\sigma\sqrt{2\pi}} e^{-\eta x} \mathrm{Hh}_{n-1}\Big(-\frac{x}{\sigma} + \sigma\eta\Big), \tag{A21}$$

$$f_{Z-\sum_{i=1}^{n}\xi_i}(x) = (\sigma\eta)^n \frac{e^{(\sigma\eta)^2/2}}{\sigma\sqrt{2\pi}} e^{\eta x} \mathrm{Hh}_{n-1}\Big(\frac{x}{\sigma} + \sigma\eta\Big). \tag{A22}$$

Therefore, in terms of I^+ and I^-, we obtain

$$\mathbf{E}\Big[e^{a(Z+\sum_{k=1}^{n}\xi_i)}\mathbf{1}_{(Z+\sum_{i=1}^{n}\xi_n > b)}\Big] = \frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}} e^{(\sigma\eta)^2/2} I_{n-1}^{+}\Big(b, a-\eta, -\frac{1}{\sigma}, -\sigma\eta\Big), \tag{A23}$$

$$\mathbf{E}\Big[e^{a(Z-\sum_{k=1}^{n}\xi_i)}\mathbf{1}_{(Z-\sum_{i=1}^{n}\xi_n > b)}\Big] = \frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}} e^{(\sigma\eta)^2/2} I_{n-1}^{+}\Big(b, a+\eta, \frac{1}{\sigma}, -\sigma\eta\Big), \tag{A24}$$

as well as

$$\mathbf{E}\Big[e^{a(Z+\sum_{k=1}^{n}\xi_i)}\mathbf{1}_{(Z+\sum_{i=1}^{n}\xi_n \leq b)}\Big] = \frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}} e^{(\sigma\eta)^2/2} I_{n-1}^{-}\Big(b, a-\eta, -\frac{1}{\sigma}, -\sigma\eta\Big), \tag{A25}$$

$$\mathbf{E}\Big[e^{a(Z-\sum_{k=1}^{n}\xi_i)}\mathbf{1}_{(Z-\sum_{i=1}^{n}\xi_n \leq b)}\Big] = \frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}} e^{(\sigma\eta)^2/2} I_{n-1}^{-}\Big(b, a+\eta, \frac{1}{\sigma}, -\sigma\eta\Big). \tag{A26}$$

Proposition A3. *We denote $Z \sim N(0, \sigma^2)$, then, for $a, b \in \mathbb{R}$,*

$$\mathbf{E}[e^{aZ}\mathbf{1}_{(Z>b)}] = e^{\frac{1}{2}\sigma^2 a^2}\Psi\left(-\frac{b-\sigma^2 a}{\sigma}\right), \tag{A27}$$

$$\mathbf{E}[e^{aZ}\mathbf{1}_{(Z\leq b)}] = e^{\frac{1}{2}\sigma^2 a^2}\Psi\left(\frac{b-\sigma^2 a}{\sigma}\right). \tag{A28}$$

Proof. It suffices to prove (A28). By calculus, we have

$$\begin{aligned} \mathbf{E}[e^{aZ}\mathbf{1}_{(Z\leq b)}] &= \int_{-\infty}^{b} e^{ay} f_Z(y)dy \\ &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{b} \exp\left(ay - \frac{y^2}{2\sigma^2}\right)dy \end{aligned}$$

$$
\begin{aligned}
&= e^{\frac{1}{2}\sigma^2 a^2} \int_{-\infty}^{\frac{b-\sigma^2 a}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= e^{\frac{1}{2}\sigma^2 a^2} \Psi\left(\frac{b-\sigma^2 a}{\sigma}\right).
\end{aligned} \tag{A29}
$$

□

Theorem A1. *With the probability of the event $\{N(1) = n\}$ given by*

$$
\pi_n = \mathbf{P}(N(1) = n) = \frac{e^{-\lambda}\lambda^n}{n!},\ n = 0, 1, 2, \ldots, \tag{A30}
$$

we have

$$
\begin{aligned}
\Xi^+(a,b) =& \pi_0 e^{a\mu + \frac{1}{2}\sigma^2 a^2} \Psi\left(-\frac{b-\mu-\sigma^2 a}{\sigma}\right) \\
&+ \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} P_{n,k} e^{a\mu} \frac{(\sigma\eta_1)^k}{\sigma\sqrt{2\pi}} e^{(\sigma\eta_1)^2/2} I_{k-1}^{+}\left(b-\mu, a-\eta_1, -\frac{1}{\sigma}, -\sigma\eta_1\right) \\
&+ \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} Q_{n,k} e^{a\mu} \frac{(\sigma\eta_2)^k}{\sigma\sqrt{2\pi}} e^{(\sigma\eta_2)^2/2} I_{k-1}^{+}\left(b-\mu, a+\eta_2, \frac{1}{\sigma}, -\sigma\eta_2\right),
\end{aligned} \tag{A31}
$$

and

$$
\begin{aligned}
\Xi^-(a,b) =& \pi_0 e^{a\mu + \frac{1}{2}\sigma^2 a^2} \Psi\left(\frac{b-\mu-\sigma^2 a}{\sigma}\right) \\
&+ \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} P_{n,k} e^{a\mu} \frac{(\sigma\eta_1)^k}{\sigma\sqrt{2\pi t}} e^{(\sigma\eta_1)^2/2} I_{k-1}^{-}\left(b-\mu, a-\eta_1, -\frac{1}{\sigma}, -\sigma\eta_1\right) \\
&+ \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} Q_{n,k} e^{a\mu} \frac{(\sigma\eta_2)^k}{\sigma\sqrt{2\pi}} e^{(\sigma\eta_2)^2/2} I_{k-1}^{-}\left(b-\mu, a+\eta_2, \frac{1}{\sigma}, -\sigma\eta_2\right).
\end{aligned} \tag{A32}
$$

Proof. We briefly illustrate the proof for the first case. By the decomposition (A18) and laws of total expectation,

$$
\begin{aligned}
\mathbf{E}\left[e^{aX(1)}\mathbf{1}_{(X(1)>b)}\right] =& \mathbf{E}\left[\mathbf{E}\left[e^{a\left(\mu+Z+\sum_{i=1}^{N(1)} Y_i\right)}\mathbf{1}_{\left(\mu+Z+\sum_{i=1}^{N(1)} Y_i>b\right)}\middle|N(1)=n\right]\right] \\
=& \pi_0 e^{a\mu}\mathbf{E}\left[e^{aZ}\mathbf{1}_{(Z>b-\mu)}\right] \\
&+ \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} P_{n,k} e^{a\mu}\mathbf{E}\left[e^{a\left(Z+\sum_{k=1}^{n}\xi_i\right)}\mathbf{1}_{\left(Z+\sum_{i=1}^{n}\xi_n>b-\mu\right)}\right] \\
&+ \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} Q_{n,k} e^{a\mu}\mathbf{E}\left[e^{a\left(Z-\sum_{k=1}^{n}\xi_i\right)}\mathbf{1}_{\left(Z-\sum_{i=1}^{k}\xi_n>b-\mu\right)}\right].
\end{aligned} \tag{A33}
$$

The result follows from (A23) and (A24) for η_1 and η_2, together with (A28). The value of n can be chosen accordingly while Kou (2002) suggests that 10–15 are sufficient for calculation purpose. We take $n = 10$ in our numerical experiments. □

Appendix D. Explicit Formulas for Y^+ and Y^-

Explicit expressions for Y^+ and Y^- are obtained by calculus. We denote the c.d.f. of a Gamma(α, β) variable by $F(x; \alpha, \beta)$ with $\alpha, \beta > 0$. To ensure that the results of Y^+ and Y^- exist, we discuss several cases below.

Theorem A2. *For real numbers a, b, when $Re(a) < \beta_k$ for all β_k, we obtain*

$$\Upsilon^{+}(a,b) = \sum_{k}\sum_{l=1}^{m} \frac{b_{kl}}{(\beta_k - a)^l}(1 - F(b; l, \beta_k - a)). \tag{A34}$$

Proof. It follows from (13) that

$$\begin{aligned}\Upsilon^{+}(a,b) =& \mathbf{E}\left[e^{aX(\theta)}\mathbf{1}_{(X(\theta)>b)}\right] \\ =& \int_b^{\infty} e^{ax}\sum_k\sum_{l=1}^{m} b_{kl}\frac{x^{l-1}e^{-\beta_k x}}{(l-1)!}dx \\ =& \sum_k\sum_{l=1}^{m}\frac{b_{kl}}{(\beta_k - a)^l}(1 - F(b; l, \beta_k - a)).\end{aligned} \tag{A35}$$

□

We provide a useful formula here. For $l \in \mathbb{N}^{+}$, $\beta > 0$ and $c > 0$, we have

$$\int_0^a x^{l-1}e^{\beta x}dx = \sum_{i=1}^{l}\frac{(-1)^{i-1}}{\beta^i}\frac{((l-1)!}{(l-i)!}e^{a\beta}c^{l-i}. \tag{A36}$$

Theorem A3. *Two situations arise: for real numbers a, b,*

1. *when $0 < Re(a) < \beta_k$ for all β_k, we have*

$$\Upsilon^{-}(a,b) = \sum_j\sum_{i=1}^{m}\frac{a_{ji}}{(\alpha_j - a)^i} + \sum_k\sum_{l=1}^{m}\frac{b_{kl}}{(\beta_k - a)^l}F(b; l, \beta_k - a). \tag{A37}$$

2. *when $0 < \beta_k < Re(a)$, by (A36), we obtain*

$$\Upsilon^{-}(a,b) = \sum_j\sum_{i=1}^{m}\frac{a_{ji}}{(\alpha_j - a)^i} + \sum_k\sum_{l=1}^{m} b_{kl}\sum_{i=1}^{l}\frac{(-1)^{i-1}}{(a - \beta_k)^i}\frac{1}{(l-i)!}e^{(a-\beta_k)b}b^{l-i}. \tag{A38}$$

Proof.

1. When $0 < \mathrm{Re}(a) < \beta_k$ for all β_k, it follows from (13) that

$$\begin{aligned}\Upsilon^{-}(a,b) &= \int_{-\infty}^{0} e^{ax}f_{X(\theta)}(x)dx + \int_0^b e^{ax}f_{X(\theta)}(x)dx \\ &= \int_{-\infty}^{0} e^{ax}\sum_j\sum_{i=1}^{m} a_{ji}(-1)^i\frac{(-x)^{i-1}e^{-\alpha_j x}}{(i-1)!}dx + \int_0^b e^{ax}\sum_k\sum_{l=1}^{m} b_{kl}\frac{x^{l-1}e^{-\beta_k x}}{(l-1)!}dx \\ &= \sum_j\sum_{i=1}^{m}\frac{a_{ji}}{(\alpha_j - a)^i} + \sum_k\sum_{l=1}^{m}\frac{b_{kl}}{(\beta_k - a)^l}F(b; l, \beta_k - a).\end{aligned} \tag{A39}$$

2. When $0 < \beta_k < \mathrm{Re}(a)$, it follows from (13) and (A36) that

$$\begin{aligned}\Upsilon^{-}(a,b) &= \int_{-\infty}^{0} e^{ax}f_{X(\theta)}(x)dx + \int_0^b e^{ax}f_{X(\theta)}(x)dx \\ &= \sum_j\sum_{i=1}^{m}\frac{a_{ji}}{(\alpha_j - a)^i} + \sum_k\sum_{l=1}^{m} b_{kl}\sum_{i=1}^{l}\frac{(-1)^{i-1}}{(a - \beta_k)^i}\frac{1}{(l-i)!}e^{(a-\beta_k)b}b^{l-i}.\end{aligned} \tag{A40}$$

□

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