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On the Approximation by Balázs–Szabados Operators

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Abstract: We present three new approximation properties of the Balázs–Szabados operators. Firstly, we prove that, in certain cases, these operators approximate some super-exponential functions on compact intervals. Next, we provide a new estimate of the error of approximation using a suitable modulus of continuity. Finally, we characterize the functions which can be uniformly approximated in the weighted norm of polynomial weight spaces.

Keywords: positive linear operators; Balázs operators; rational Bernstein type operators; weighted spaces; modulus of continuity; tail inequality; super-exponential functions

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1. Introduction

The study of positive linear operators is an important research area in approximation theory. The problem of Weierstrass [1,2] to approximate continuous functions using a sequence of polynomials initiated this research field, but the systematic study of these operators began only in the 1960s with the books of Korovkin [3,4] and Lorentz [5]. Korovkin has discovered simple conditions to verify if a sequence of positive linear operators forms an approximation process, and since then, there have been constructed many operators. One such example is the sequence of operators defined for $\beta \in (0,1)$ by

$$R_n^{[\beta]}(f,x) = \frac{1}{(1+n^{\beta-1}x)^n} \sum_{k=0}^{n} \binom{n}{k} (n^{\beta-1}x)^k \cdot f\left(\frac{k}{n^\beta}\right), \text{ for } n \geq 1,\ x \in [0,\infty). \tag{1}$$

They were introduced in 1975 by K. Balázs [6]. She proved some pointwise approximation results for the particular value $\beta = 2/3$. In 1982, Balázs and Szabados [7] extended the study for $\beta \in (0, 2/3]$ and investigated both the pointwise and the uniform approximation. In 1984, Totik [8] considered $\beta \in (0,1)$ and provided the saturation properties of these operators. Many more articles [9–43] have appeared since then, presenting different new properties and generalizing the form of the operators. The vast majority refer to the operators defined by (1) as the Balázs–Szabados operators.

In this paper, we first present a new approximation result on compact intervals, extending the space of functions to be approximated. Balázs [6] showed that operators (1) approximate, on compact intervals, continuous functions with exponential growth. We prove that for certain values of β, operators (1) approximate even functions with super-exponential growth. This fact is surprising for two reasons. Many classical operators approximate only functions with an exponential growth or some other fixed growth of exponential type. In our case, the growth can be enlarged indefinitely, in some sense, and this is correlated with the proper choice of the parameter β. Another interesting thing about the Balázs–Szabados operators is that they are built using only a finite sum and not a series and, so, the possibility of approximating functions with such a high growth comes as a surprise. This result is presented in Theorem 1 of Section 2. The key ingredient of its proof is Lemma 1, which contains an inequality similar to the tail inequalities for the probability distributions. We also present, in Remark 3, an example of a super-exponential function which cannot be approximated by the operators (1) and, thus, we correct the recent result of ([39], Theorem 1), which gave the impression that every continuous function defined on $[0,\infty)$ can be approximated on compacts by the operators $R_n^{[\beta]}$, although the author mentions at the beginning of the article, when he defines the operators, that "continuous functions defined on $\mathbb{R}_+$ satisfying a certain growth condition" are considered.

In Section 3, we present a new estimate of the rate of approximation by using a suitable modulus of continuity. In the literature, there are some estimations of this rate, but they are valid only for some values of β. We have obtained an estimation that is valid for every choice of $\beta \in (0,1)$. In addition, this estimation of the rate is uniform, uses only one modulus of continuity, and the constant in front of the modulus is explicit. We have used a modulus of continuity which is appropriate for continuous functions with a finite limit at infinity because it is known that these are precisely the functions that can be uniformly approximated by $R_n^{[\beta]}$ (see [7,8]). We must remark that better estimations can be obtained for specific values of the parameter β and the value $\beta = 1/2$ gives the best rate (see also the paper of Totik [8] which has arrived at the same conclusion).

In Section 4, we give a characterization of the functions which can be uniformly approximated in polynomial weight spaces. It is known that polynomial functions of degree m are mapped by R_n^{β} into rational functions with a growth not larger than a polynomial function of degree m (see [16]), but I could not find a result in the literature that specifies the functions that can be uniformly approximated in the polynomial weight space. Recently, by considering a generalization of the operators (1), Agratini ([39], Theorem 5) has given a negative result by presenting an example of a function which cannot be approximated in the weighted norm. He has estimated the error of approximation, too, but only pointwise. Our results from Lemma 3 and Theorem 3 complete Agratini's results, characterizing the functions that can be uniformly approximated in the weighted norm.

We present now some notations. Let $I = [0,\infty)$. A function $w : I \to (0,\infty)$ will be called weight. The space of all functions $f : I \to \mathbb{R}$ with the property that there is $M > 0$ such that

$$|f(x)| \leq M \cdot w(x), \text{ for every } x \in I$$

is called weight space or weighted space and is denoted by $B_w(I)$. Some authors prefer the "big O" notation to express the growth rate of a function. In this case, $f \in B_w(I)$ is equivalent to $f(x) = \mathcal{O}(w(x))$. The space $B_w(I)$ is a normed space, endowed with the w-norm

$$\|f\|_w = \sup_{x\in I} \frac{|f(x)|}{w(x)}.$$

We will denote by $C_w(I)$ the space of functions from $B_w(I)$ which are continuous on I.

2. Approximation on Compact Intervals of Super-Exponential Functions

The following lemma is very important in proving approximation results for the operators of Balázs and Szabados for functions with a high growth. For $y_n = 1$, this is in fact a tail inequality for the probability distribution attached to the operators (1). We improve the idea used by Cernoff [44] to prove such inequalities (see also [45]).

Lemma 1. *Let $\beta \in (0,1)$, $x \in [0,\infty)$ and $\delta > 0$ be given and consider $y_n > 0$ such that $y_n(1 - xn^{\beta-1}) \leq e^{\frac{1}{\delta n^{\beta}}}$, for all $n \geq n_0$, for some $n_0 \in \mathbb{N}$. Then, for all $n \geq n_0$, we have*

$$\sum_{k\geq(x+\delta)n^{\beta}}^{n} \binom{n}{k} \frac{(xy_n n^{\beta-1})^k}{(1+xn^{\beta-1})^n} \leq C(x,\delta) \cdot \frac{y_n^{(x+\delta)n^{\beta}}}{n^{\beta}} \cdot e^{-x\delta\cdot n^{2\beta-1}}, \tag{2}$$

where $C(x,\delta) = \frac{xe^{\frac{1-x-\delta+xe^{1/\delta}}{\delta}}}{\delta^2}\left(\frac{xe^{\frac{1}{\delta}}}{\delta^2}+1\right)$.

Proof. For $x = 0$, the inequality (2) holds true. Let $x > 0$ and denote $s_n = xn^{\beta-1}$. If $s_n \geq 1$, we have $k \geq ns_n + \delta n^\beta > n$, so the sum from (2) reduces to 0 and (2) is true. Consider now the case $s_n < 1$. We have for every $t_n \geq 0$

$$\begin{aligned}\sum_{k\geq(x+\delta)n^\beta}^{n}\binom{n}{k}\frac{(y_ns_n)^k}{(1+s_n)^n} &\leq \sum_{k\geq(x+\delta)n^\beta}^{n}\binom{n}{k}\frac{(y_ns_n)^k}{(1+s_n)^n}\cdot e^{t_n(k-xn^\beta-\delta n^\beta)}\cdot\left(\frac{k-xn^\beta}{\delta n^\beta}\right)^2\\ &\leq \frac{e^{-t_nns_n-t_n\delta n^\beta}}{\delta^2(1+s_n)^n}\sum_{k=0}^{n}\binom{n}{k}\left(y_ns_ne^{t_n}\right)^k\left(\frac{k}{n^\beta}-x\right)^2.\end{aligned}$$

Using the same reasoning as in ([6], Lemma 2.1), we have for every $a, b > 0$:

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}a^k\left(\frac{k}{b}-x\right)^2 &= \frac{1}{b^2}\sum_{k=0}^{n}k^2\binom{n}{k}a^k - \frac{2x}{b}\sum_{k=0}^{n}k\binom{n}{k}a^k + x^2\sum_{k=0}^{n}\binom{n}{k}a^k\\ &= \frac{a^2n^2+an}{b^2}(1+a)^{n-2} - \frac{2xan}{b}(1+a)^{n-1} + x^2(1+a)^n\\ &= (1+a)^n\left[\left(x-\frac{an}{b(1+a)}\right)^2 + \frac{an}{b^2(1+a)^2}\right].\end{aligned}$$

This implies that

$$\begin{aligned}&\sum_{k\geq(x+\delta)n^\beta}^{n}\binom{n}{k}\frac{(y_ns_n)^k}{(1+s_n)^n}\\ &\quad\leq \frac{e^{-t_n(ns_n+\delta n^\beta)}\left(1+y_ns_ne^{t_n}\right)^n}{\delta^2(1+s_n)^n}\left[x^2\left(\frac{y_ne^{t_n}}{1+y_ns_ne^{t_n}}-1\right)^2 + \frac{n^{1-2\beta}y_ns_ne^{t_n}}{(1+y_ns_ne^{t_n})^2}\right],\end{aligned}$$

for all $t_n \geq 0$. We choose $t_n = \frac{1}{\delta n^\beta} - \ln[y_n(1-s_n)]$. Let us observe that for all $n \geq n_0$ we have $t_n \geq 0$ and

$$\begin{aligned}e^{-t_n} &= y_n(1-s_n)e^{-\frac{1}{\delta n^\beta}}\\ e^{-t_n(ns_n+\delta n^\beta)} &= y_n^{(x+\delta)n^\beta}\cdot(1-s_n)^{ns_n+\delta n^\beta}\cdot e^{-\frac{x+\delta}{\delta}}\\ \left(1+y_ns_ne^{t_n}\right)^n &= \frac{\left(1+s_ne^{\frac{1}{\delta n^\beta}}-s_n\right)^n}{(1-s_n)^n} \leq \frac{e^{ns_n\left(e^{\frac{1}{\delta n^\beta}}-1\right)}}{(1-s_n)^n} \leq \frac{e^{\frac{xe^{\frac{1}{\delta}}}{\delta}}}{(1-s_n)^n}\\ \frac{n^{1-2\beta}y_ns_ne^{t_n}}{(1+y_ns_ne^{t_n})^2} &= \frac{n^{1-2\beta}s_ne^{\frac{1}{\delta n^\beta}}(1-s_n)}{\left(1+s_n\left(e^{\frac{1}{\delta n^\beta}}-1\right)\right)^2} \leq n^{1-2\beta}s_ne^{\frac{1}{\delta n^\beta}} \leq \frac{xe^{\frac{1}{\delta}}}{n^\beta}\end{aligned}$$

$$x^2\left(\frac{y_ne^{t_n}}{1+y_ns_ne^{t_n}}-1\right)^2 \leq x^2\left(y_n(1-s_n)e^{t_n}-1\right)^2 = x^2\left(e^{\frac{1}{\delta n^\beta}}-1\right)^2 \leq \frac{x^2e^{\frac{2}{\delta}}}{\delta^2n^{2\beta}}.$$

We deduce that

$$\sum_{k\geq(x+\delta)n^\beta}^{n}\binom{n}{k}\frac{(y_ns_n)^k}{(1+s_n)^n} \leq C(x,\delta)\cdot\frac{y_n^{(x+\delta)n^\beta}}{n^\beta}\cdot\frac{(1-s_n)^{ns_n+\delta n^\beta-n}}{(1+s_n)^n}.$$

With the notation $c = \delta/x > 0$, it remains to prove that

$$\frac{(1-s_n)^{\delta n^\beta+ns_n-n}}{(1+s_n)^n} = e^{n(cs_n+s_n-1)\ln(1-s_n)-n\ln(1+s_n)} < e^{-ncs_n^2}.$$

Consider the function $H(u) = (cu + u - 1)\ln(1-u) - \ln(1+u) + cu^2$ and let us show that $H(u) < 0$, for every $u \in (0,1)$. Indeed,

$$H'(u) = (c+1)\ln(1-u) + \frac{cu+u-1}{u-1} - \frac{1}{u+1} + 2cu,$$
$$H''(u) = -\frac{c+1}{1-u} - \frac{c}{(1-u)^2} + \frac{1}{(u+1)^2} + 2c$$
$$H^{(3)}(u) = -\frac{c+1}{(1-u)^2} - \frac{2c}{(1-u)^3} - \frac{2}{(u+1)^3} < 0.$$

Because $H''(0) = H'(0) = H(0) = 0$, we deduce $H(u) < 0$, for every $u \in (0,1)$. □

Lemma 2. *Consider $w(x) = e^{\alpha x^a}$ for $\alpha \geq 0$, $a \geq 1$ and $1 > \beta \geq \frac{a}{a+1}$. For every compact interval $K \subset [0,\infty)$, we have*

$$\lim_{n\to\infty} R_n^{[\beta]}(w,x) = w(x), \quad \text{uniformly on } K.$$

Proof. Let $K = [m, M]$ be a compact interval included in $[0,\infty)$. Consider $\delta > 0$ such that $\alpha \leq 1/\delta$ and define

$$p(x) = \begin{cases} w(x), & x \leq M+\delta \\ w(M+\delta), & x > M+\delta \end{cases} \quad \text{and } r(x) = \begin{cases} 0, & x \leq M+\delta \\ w(x) - w(M+\delta), & x > M+\delta \end{cases}.$$

The function w can be decomposed into $w(x) = p(x) + r(x)$ and so

$$\left|R_n^{[\beta]}(w,x) - w(x)\right| \leq \left|R_n^{[\beta]}(p,x) - p(x)\right| + \left|R_n^{[\beta]}(r,x) - r(x)\right|.$$

We prove that both p and r can be uniformly approximated on K. Formula ([6], (2.2)) proves that $R_n^{[\beta]}(1,x) = 1$, and using an idea of Shisha-Mond [46], the error of approximation of p can be estimated using the modulus of continuity by

$$|R_n^{[\beta]}(p,x) - p(x)| \leq 2\cdot\omega\left(p, \sqrt{R_n^{[\beta]}(|t-x|^2,x)}\right).$$

Formula (2.4) from [6] gives

$$R_n^{[\beta]}(|t-x|^2,x) = \frac{x^4 n^{2\beta-2} + xn^{-\beta}}{(1+xn^{\beta-1})^2} \leq M^4 n^{2\beta-2} + Mn^{-\beta},$$

for every $x \in K$ and $\beta \in (0,1)$ and proves that $R_n^{[\beta]}(|t-x|^2,x)$ converges to 0 uniformly on K. Because p has finite limit at infinity, it is uniformly continuous on $[0,\infty)$ and, so $\omega\left(p, \sqrt{R_n^{[\beta]}(|t-x|^2,x)}\right)$ tends uniformly to 0 on K. This proves the uniform convergence of $R_n^{[\beta]}p$ toward p on K.

Now, for every $x \in K$

$$|R_n^{[\beta]}(r,x) - r(x)| = R_n^{[\beta]}(r,x) \leq \sum_{\frac{k}{n^\beta} > M+\delta}^{n} \binom{n}{k} \frac{(n^{\beta-1}x)^k}{(1+n^{\beta-1}x)^n} \cdot e^{\alpha\left(\frac{k}{n^\beta}\right)^a}$$
$$\leq \sum_{\frac{k}{n^\beta} \geq x+\delta}^{n} \binom{n}{k} \frac{(n^{\beta-1}x)^k}{(1+n^{\beta-1}x)^n} \cdot e^{\alpha k n^{a-1-a\beta}}.$$

We apply Lemma 1 for $y_n = e^{\alpha n^{a-1-a\beta}}$. The inequality $y_n(1 - xn^{\beta-1}) \le e^{\frac{1}{\delta n^\beta}}$ is equivalent to $n^{1-\beta}\left(1 - e^{\frac{1}{\delta n^\beta} - \alpha n^{a-1-a\beta}}\right) \le x$. However, this is true for sufficiently large n, since

$$\lim_{n\to\infty} n^{1-\beta}\left(1 - e^{\frac{1}{\delta n^\beta} - \alpha n^{a-1-a\beta}}\right) = \lim_{n\to\infty} n^{1-\beta} \cdot \left(\alpha n^{a-1-a\beta} - \frac{n^{-\beta}}{\delta}\right)$$
$$= \lim_{n\to\infty}\left(\alpha \cdot n^{a-(a+1)\beta} - \frac{1}{\delta} \cdot n^{1-2\beta}\right) \le 0.$$

The error of approximation of the function r is bounded by

$$|R_n^{[\beta]}(r,x) - r(x)| \le \frac{x}{\delta^2 n^\beta} \cdot y_n^{(x+\delta)n^\beta} \cdot e^{-x\delta \cdot n^{2\beta-1}} \le \frac{C(M,\delta)}{n^\beta} \cdot e^{\alpha(M+\delta)n^{\beta+a-1-a\beta} - m\delta \cdot n^{2\beta-1}}.$$

Because $\beta \ge 1/2$ and $\beta + a - 1 - a\beta \le 2\beta - 1$ we obtain that $R_n^{[\beta]} r$ converges uniformly to r on K, and the proof of the lemma is complete. $\square$

Remark 1. *For $a = 1$ the result of Lemma 2 is true for every $\beta \in (0,1)$. Indeed, we have*

$$R_n^{[\beta]}(e^{\alpha t}, x) = \left(\frac{1 + xn^{\beta-1}e^{\frac{\alpha}{n^\beta}}}{1 + xn^{\beta-1}}\right)^n.$$

For every $x \in [m, M]$, using $(1+u)^n \le e^{nu}$ and $e^u - 1 \le ue^u$, we get

$$\left(\frac{1 + xn^{\beta-1}e^{\frac{\alpha}{n^\beta}}}{1 + xn^{\beta-1}}\right)^n \le \left(\frac{1 + Mn^{\beta-1}e^{\frac{\alpha}{n^\beta}}}{1 + Mn^{\beta-1}}\right)^n \le e^{\frac{Mn^\beta(e^{\alpha/n^\beta} - 1)}{1 + Mn^{\beta-1}}} \le e^{\alpha M e^\alpha}.$$

Now, using $|u - v| \le |\ln u - \ln v| \cdot \max(u, v)$ we evaluate the error in approximating the exponential function

$$\left|R_n^{[\beta]}(e^{\alpha t}, x) - e^{\alpha x}\right| \le \left|n \ln\left(\frac{1 + xn^{\beta-1}e^{\frac{\alpha}{n^\beta}}}{1 + xn^{\beta-1}}\right) - \alpha x\right| \cdot \max\left(\left(\frac{1 + xn^{\beta-1}e^{\frac{\alpha}{n^\beta}}}{1 + xn^{\beta-1}}\right)^n, e^{\alpha x}\right)$$
$$\le \left|n \ln\left(\frac{1 + xn^{\beta-1}e^{\frac{\alpha}{n^\beta}}}{1 + xn^{\beta-1}}\right) - \alpha x\right| \cdot e^{\alpha M e^\alpha}$$

For the logarithm we use $\frac{u}{1+u} \le \ln(1+u) \le u$ and we obtain

$$\frac{xn^\beta\left(e^{\frac{\alpha}{n^\beta}} - 1\right)}{1 + xn^{\beta-1}e^{\frac{\alpha}{n^\beta}}} \le n \ln\left(\frac{1 + xn^{\beta-1}e^{\frac{\alpha}{n^\beta}}}{1 + xn^{\beta-1}}\right) \le \frac{xn^\beta\left(e^{\frac{\alpha}{n^\beta}} - 1\right)}{1 + xn^{\beta-1}}.$$

Using the inequalities $u \le e^u - 1 \le ue^u$ we have

$$\frac{xn^\beta\left(e^{\frac{\alpha}{n^\beta}} - 1\right)}{1 + xn^{\beta-1}} - \alpha x \le \frac{\alpha x e^{\frac{\alpha}{n^\beta}}}{1 + xn^{\beta-1}} - \alpha x \le \frac{\alpha x\left(e^{\frac{\alpha}{n^\beta}} - 1\right)}{1 + xn^{\beta-1}} \le \frac{\alpha^2 x}{n^\beta} \le \frac{\alpha^2 M}{n^\beta}$$

and

$$\frac{xn^\beta\left(e^{\frac{\alpha}{n^\beta}} - 1\right)}{1 + xn^{\beta-1}e^{\frac{\alpha}{n^\beta}}} - \alpha x \ge \frac{\alpha x}{1 + xn^{\beta-1}e^{\frac{\alpha}{n^\beta}}} - \alpha x \ge -\frac{\alpha x^2 n^{\beta-1}e^{\frac{\alpha}{n^\beta}}}{1 + xn^{\beta-1}e^{\frac{\alpha}{n^\beta}}} \ge -\alpha M^2 n^{\beta-1} e^\alpha.$$

Finally, for every $x \in [m, M]$ we obtain

$$\left| R_n^{[\beta]}(e^{\alpha t}, x) - e^{\alpha x} \right| \le \alpha M \max\left(\alpha n^{-\beta}, M n^{\beta-1} e^{\alpha} \right) \cdot e^{\alpha M e^{\alpha}}.$$

Theorem 1. *Consider $w(x) = e^{\alpha x^a}$ with $\alpha \ge 0$, $a \ge 1$ and $1 > \beta \ge \frac{a}{a+1}$. For every compact interval $K \subset [0, \infty)$ and for every $f \in C_w(I)$ we have*

$$\lim_{n\to\infty} R_n^{[\beta]}(f, x) = f(x), \quad \text{uniformly on } K.$$

Proof. Let $\varepsilon > 0$. Consider the compact interval $J = [\min K - \varepsilon, \max K + \varepsilon] \cap [0, \infty)$. Because f/w and w are continuous on the compact J, there is $\eta > 0$ such that for every $t, x \in J$ with $|t - x| < \eta$ we have $|(f/w)(t) - (f/w)(x)| < \varepsilon$ and $|w(t) - w(x)| < \varepsilon$. Let us define $\delta = \min(\eta, \varepsilon)$. For every $x \in K$ and $t \in I$ such that $|t - x| < \delta$, we have $t, x \in J$. So

$$\left| \frac{f(t)}{w(t)} - \frac{f(x)}{w(x)} \right| < \varepsilon \quad \text{and} \quad |w(t) - w(x)| < \varepsilon.$$

For every $x \in K$ and $t \in I$ such that $|t - x| \ge \delta$, we can write

$$\left| \frac{f(t)}{w(t)} - \frac{f(x)}{w(x)} \right| \le 2\|f\|_w \le 2\|f\|_w \cdot \frac{|t - x|^2}{\delta^2}$$

and

$$|w(t) - w(x)| \le w(t) + w(x) \le w(t) - w(x) + 2w(x) \le w(t) - w(x) + 2w(x) \cdot \frac{|t - x|^2}{\delta^2}.$$

We have proved that, for every $t \in I$ and $x \in K$, we have

$$\left| \frac{f(t)}{w(t)} - \frac{f(x)}{w(x)} \right| < \varepsilon + 2\|f\|_w \cdot \frac{|t - x|^2}{\delta^2}$$

and

$$|w(t) - w(x)| < \varepsilon + w(t) - w(x) + 2w(x) \cdot \frac{|t - x|^2}{\delta^2}.$$

Using the above inequalities and

$$|f(t) - f(x)| \le \frac{|f(t)|}{w(t)} \cdot |w(t) - w(x)| + w(x) \cdot \left| \frac{f(t)}{w(t)} - \frac{f(x)}{w(x)} \right|$$

and applying the operators $R_n^{[\beta]}$ we deduce that

$$\begin{aligned} \left| R_n^{[\beta]}(f, x) - f(x) \right| &\le \|f\|_w \left(\varepsilon + \left| R_n^{[\beta]}(w, x) - w(x) \right| + \frac{2w(x)}{\delta^2} \cdot R_n^{[\beta]}(|t - x|^2, x) \right) \\ &\quad + w(x) \cdot \left(\varepsilon + \frac{2\|f\|_w}{\delta^2} R_n^{[\beta]}(|t - x|^2, x) \right). \end{aligned}$$

Using Lemma 2 and the inequality $w(x) \le w(\max K)$ for every $x \in K$ and the fact that $R_n^{[\beta]}(|t - x|^2, x)$ converges uniformly on K, we obtain that $R_n^{[\beta]} f$ converges uniformly to f on K. □

Remark 2. *The result of Theorem 1 is valid for $\beta \ge 1/2$. For a given $\beta \in [1/2, 1)$, the growth of the functions cannot be greater than $w(x) = e^{\alpha x^a}$ with $\alpha \ge 0$ and $a \le \frac{\beta}{1-\beta} < \frac{1}{1-\beta}$.*

For $a = 1$ the result is valid for every $\beta \in (0,1)$. For $\beta = \frac{2}{3}$ and $a = 1$ the result is known from [6]. In [7], the authors proved the approximation property of the operators (1) for $\beta \in (0, 2/3]$ and the subspace of uniformly continuous functions on I.

Remark 3. *There are super-exponential functions which cannot be approximated. Take for example $g(x) = e^{x^{\frac{1}{1-\beta}+\varepsilon}}$, with an arbitrary $\varepsilon > 0$. Considering only the last term of the sum which defines the operators $R_n^{[\beta]}$, we have*

$$R_n^{[\beta]}(g,x) \geq \frac{(n^{\beta-1}x)^n}{(1+n^{\beta-1}x)^n} \cdot g\left(n^{1-\beta}\right) = (n^{\beta-1}x)^n \cdot (1+n^{\beta-1}x)^{-n} \cdot e^{n \cdot n^{(1-\beta)\varepsilon}}.$$

Using the inequality $1 + n^{\beta-1}x \leq e^{xn^{\beta-1}}$ we deduce that

$$R_n^{[\beta]}(g,x) \geq (n^{\beta-1}x)^n \cdot e^{-xn^{\beta}} \cdot e^{n \cdot n^{(1-\beta)\varepsilon}} = e^{n \ln x - (1-\beta) n \ln n - xn^{\beta} + n \cdot n^{(1-\beta)\varepsilon}}.$$

Now, it is not difficult to see that $\lim_{n\to\infty} R_n^{[\beta]}(g,x) = +\infty$, for every $x > 0$ and $\beta \in (0,1)$.

Theorem 1 from [39] asserts that the sequence $(R_n^{[\beta]} f)$ converges to f on compact sets for every $f \in C[0,\infty)$. However, the example given above proves that an approximation result cannot be true for continuous functions with an arbitrary growth. In order to be valid, such a result must impose some limitation on the growth of the function f.

3. Estimation of the Rate of Uniform Approximation

It is known from [7], that $R_n^{[\beta]} f$ approximate uniformly the continuous function f on $I = [0,\infty)$ if, and only if, f has a finite limit at infinity. In the same paper, an estimate of the rate of convergence was given using the modulus of continuity and a modulus at infinity. We will provide an estimate of the rate of convergence using the following modulus of continuity

$$\omega^*(f,\delta) = \sup_{\substack{x,t\in I \\ |e^{-t}-e^{-x}| \leq \delta}} |f(t) - f(x)|,$$

introduced and studied in [47] (a particular case of the modulus introduced in [48–50]). This modulus is suitable for the uniform approximation of functions by the Balázs–Szabados operators, since $\omega^*(f,\cdot)$ tends to zero when its argument tends to zero and the function f has a finite limit at infinity.

Theorem 2. *Consider $\beta \in (0,1)$ and f a bounded and continuous function defined on I having a finite limit at infinity. Then,*

$$\left\| R_n^{[\beta]} f - f \right\| \leq (1+e) \cdot \omega^*\left(f, \sqrt{\max\left(n^{\beta-1}, n^{-\beta}\right)}\right), \quad \text{for every } n \in \mathbb{N}.$$

Proof. Using the properties of the modulus ω^* (see the proof of ([47], Theorem 2.1)), we obtain

$$\left| R_n^{[\beta]}(f,x) - f(x) \right| \leq \left(1 + \frac{1}{\delta_n^2} \cdot R_n^{[\beta]}\left(\left|e^{-t} - e^{-x}\right|^2, x\right)\right) \cdot \omega^*(f,\delta_n).$$

It remains to estimate in the uniform norm the following expression:

$$R_n^{[\beta]}\left(\left|e^{-t} - e^{-x}\right|^2, x\right) = R_n^{[\beta]}(e^{-2t},x) - e^{2x} - 2e^{-x}\left[R_n^{[\beta]}(e^{-t},x) - e^{x}\right].$$

Because e^{-x} is a convex function, applying Jensen inequality, we obtain

$$R_n^{[\beta]}(e^{-t},x) \geq e^{-R_n^{[\beta]}(e_1,x)} = e^{\frac{-x}{1+xn^{\beta-1}}} \geq e^{-x}.$$

As a consequence, we have

$$R_n^{[\beta]}\left(\left|e^{-t}-e^{-x}\right|^2,x\right) \leq R_n^{[\beta]}(e^{-2t},x)-e^{-2x}.$$

Let us denote

$$\Delta_n(x) = R_n^{[\beta]}(e^{-2t},x)-e^{-2x} = \left(\frac{1+xn^{\beta-1}e^{-\frac{2}{n^\beta}}}{1+xn^{\beta-1}}\right)^n - e^{-2x}, \quad x \geq 0.$$

Because the limit

$$\lim_{x\to\infty} \Delta_n(x) = e^{-2n^{1-\beta}}$$

is finite and $\Delta_n(x) \geq 0$, there is a sequence (x_n) of positive numbers such that

$$\max_{x\geq 0} \Delta_n(x) = \Delta_n(x_n).$$

This implies that $\Delta_n'(x_n) = 0$, which is equivalent to

$$\left(\frac{1+x_n n^{\beta-1}e^{-\frac{2}{n^\beta}}}{1+x_n n^{\beta-1}}\right)^n = \frac{2e^{-2x_n}\left(1+x_n n^{\beta-1}e^{-\frac{2}{n^\beta}}\right)(1+x_n n^{\beta-1})}{n^\beta\left(1-e^{-\frac{2}{n^\beta}}\right)}.$$

It follows that

$$\Delta_n(x) \leq \frac{2\left(1+x_n n^{\beta-1}e^{-\frac{2}{n^\beta}}\right)(1+x_n n^{\beta-1}) - n^\beta\left(1-e^{-\frac{2}{n^\beta}}\right)}{e^{2x_n}n^\beta\left(1-e^{-\frac{2}{n^\beta}}\right)}.$$

Using $1-e^{-u} > u - \frac{u^2}{2}$ for $u = 2n^{-\beta} > 0$, we obtain

$$\Delta_n(x) \leq \frac{2x_n n^{\beta-1}\left(e^{-\frac{2}{n^\beta}}+1\right) + 2x_n^2 n^{2\beta-2}e^{-\frac{2}{n^\beta}} + 2n^{-\beta}}{e^{2x_n}n^\beta\left(1-e^{-\frac{2}{n^\beta}}\right)}.$$

Finally, using the inequality $\frac{u}{1-e^{-u}} < e$, for $u = 2n^{-\beta} \leq 2$ and $\frac{(1+u)^2}{e^{2u}} \leq 1$, for $u = x_n > 0$, we have

$$\Delta_n(x) \leq e\max\left(n^{\beta-1},n^{-\beta}\right)\cdot\left(\frac{2x_n+x_n^2+1}{e^{2x_n}}\right) < e\max\left(n^{\beta-1},n^{-\beta}\right).$$

We choose $\delta_n = \sqrt{\max\left(n^{\beta-1},n^{-\beta}\right)}$ in the first inequality of the proof. $\square$

Remark 4. *The maximum rate of approximation is obtained for $\beta = \frac{1}{2}$.*

4. Weighted Approximation in Polynomial Weight Spaces

It is known from [16] that operators $R_n^{[\beta]}$ map a polynomial function of degree m into a function with a growth not larger than a polynomial function of degree m. In the following lemma, we extend this result.

Lemma 3. *For every $\beta \in (0,1)$, $\alpha \geq 0$, $n \in \mathbb{N}$ and every $x \geq 0$ we have*

$$R_n^{[\beta]}(1+t^\alpha, x) \leq C_\alpha(1+x^\alpha),$$

for some constant $C_\alpha > 0$ independent of n, β and x.

Proof. For $\alpha = 0$, we have equality with $C_0 = 1$. For $\alpha > 0$, let $m = \lceil \alpha \rceil \geq 1$. In ([16], Lemma 2) it was proved that

$$R_n^{[\beta]}(t^m, x) \leq C_m(1+x^m), \quad \text{for every } x \geq 0,$$

where $C_m = m \cdot \max_{1 \leq j \leq m} S(m,j)$ and $S(m,j)$ are the Stirling numbers of the second kind (see ([51], 24.1.4)).

Applying Hölder inequality we get

$$R_n^{[\beta]}(1+t^\alpha, x) = 1 + R_n^{[\beta]}(t^\alpha, x) \leq 1 + \left(R_n^{[\beta]}(t^m, x)\right)^{\frac{\alpha}{m}} \leq 1 + (C_m)^{\frac{\alpha}{m}} \cdot (1+x^m)^{\frac{\alpha}{m}}.$$

However, it is known that $u^a - v^a \leq (u-v)^a$, for $u \geq v$ and $a \in (0,1]$ (see for example ([52], Example 1.1.3)). We deduce that

$$(1+x^m)^{\frac{\alpha}{m}} - (x^m)^{\frac{\alpha}{m}} \leq 1$$

and

$$R_n^{[\beta]}(1+t^\alpha, x) \leq 1 + (C_m)^{\frac{\alpha}{m}}(1+x^\alpha) \leq [1+(C_m)^{\frac{\alpha}{m}}](1+x^\alpha).$$

□

Lemma 3 proves that $R_n^\beta w$ belongs to the space $C_w(I)$, for $w(x) = 1 + x^\alpha$ with $\alpha \geq 0$. We give now a complete characterization of the functions which can be approximated in the w-norm.

Theorem 3. *Let $\beta \in (0,1)$. Consider $\alpha > 0$ and $w(x) = 1 + x^\alpha$, $x \in I$. If $f \in C_w(I)$, then*

$$\lim_{n\to\infty} \left\| R_n^{[\beta]} f - f \right\|_w = 0$$

if and only if

$$\lim_{x\to\infty} \frac{f(x)}{1+x^\alpha} = 0.$$

Proof. Using the definition of the operators, we deduce that

$$\lim_{x\to\infty} \frac{R_n^{[\beta]}(f,x)}{(1+x)^\alpha} = 0.$$

For the "if" part, we suppose that

$$\lim_{x\to\infty} \frac{f(x)}{(1+x)^\alpha} = 0.$$

We obtain
$$\lim_{x\to\infty}\frac{\left|R_n^{[\beta]}(f,x)-f(x)\right|}{(1+x)^\alpha}=\lim_{x\to\infty}\frac{|f(x)|}{(1+x)^\alpha}=0.$$
Hence, for $\varepsilon>0$ there is $\delta>0$ such that for every $x>\delta$ and every $n\in\mathbb{N}$
$$(1+x)^{-\alpha}|R_n^{[\beta]}(f,x)-f(x)|<\varepsilon.$$

For the compact interval $K=[0,\delta]$, we apply ([16], Theorem 2) or our Theorem 1 and deduce the existence of $n_0\in\mathbb{N}$ such that
$$\left|R_n^{[\beta]}(f,x)-f(x)\right|<\varepsilon,$$
for every $n\geq n_0$ and every $x\in K$. This proves that
$$\sup_{x\geq 0}(1+x)^{-\alpha}\left|R_n^{[\beta]}(f,x)-f(x)\right|<\varepsilon.$$

For the "only if" part, let us observe that
$$\left\|R_n^{[\beta]}f-f\right\|_w=\sup_{x\geq 0}\frac{|R_n^{[\beta]}(f,x)-f(x)|}{(1+x)^\alpha}\geq\lim_{x\to\infty}\frac{|R_n^{[\beta]}(f,x)-f(x)|}{(1+x)^\alpha}=\lim_{x\to\infty}\frac{|f(x)|}{(1+x)^\alpha}.$$

Applying the limit when n tends to infinity, we obtain that $\lim_{x\to\infty}\frac{|f(x)|}{(1+x)^\alpha}=0$. □

Remark 5. *As Agratini [39] has remarked, we cannot approximate uniformly in the w-norm all the functions from the space $C_w(I)$, where $w(x)=1+x^\alpha$ and $I=[0,\infty)$, $\alpha>0$. For $\alpha=2$, Agratini gave as an example the function $f(x)=x^2$, which cannot be uniformly approximated in the weighted w-norm by $R_n^{[\beta]}f$. Our result says that only those functions for which we have*
$$\lim_{x\to\infty}\frac{f(x)}{w(x)}=0$$
can be uniformly approximated.

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