Derivative-Free King's Scheme for Multiple Zeros of Nonlinear Functions

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Abstract: There is no doubt that the fourth-order King's family is one of the important ones among its counterparts. However, it has two major problems: the first one is the calculation of the first-order derivative; secondly, it has a linear order of convergence in the case of multiple roots. In order to improve these complications, we suggested a new King's family of iterative methods. The main features of our scheme are the optimal convergence order, being free from derivatives, and working for multiple roots ($m \geq 2$). In addition, we proposed a main theorem that illustrated the fourth order of convergence. It also satisfied the optimal Kung–Traub conjecture of iterative methods without memory. We compared our scheme with the latest iterative methods of the same order of convergence on several real-life problems. In accordance with the computational results, we concluded that our method showed superior behavior compared to the existing methods.

Keywords: King's method; nonlinear equations; optimal iterative methods; multiple roots; Kung–Traub conjecture

MSC: 65G99; 65H10

1. Introduction

The most demanding task of science and engineering problems [1–3] is to find the solutions of nonlinear equations. Most of the time, these are complicated or unsolvable when using analytical methods. Therefore, this leads to the construction of iterative techniques that provide an approximate solution. Many higher order schemes [4,5] with certain conditions have been proposed by researchers to solve a nonlinear function $f(x) = 0$. Along with the simple roots of nonlinear equations, multiple roots of nonlinear equations play a significant role in many areas such as the Ideal Gas Law [6], which describes the nature of a real gas and the relation between molecular size and attraction forces. Taking this into consideration, we focused on the study of multiple roots of function $f : \mathbb{D} \subset \mathbb{C} \to \mathbb{C}$ such that $f(x_r) = f'(x_r) = f''(x_r) = f''(x_r) = \cdots = 0$ and $f^m(x_r) \neq 0$, where x_r is the exact root of function f with multiplicity m.

The most popular and classical method for multiple roots is Newton's modified method [7,8], defined as:

$$x_{p+1} = x_p - m\frac{f(x_p)}{f'(x_p)},$$ (1)

where $f'(x_p)$ is the derivative of the function $f(x)$. With multiplicity known in advance, its order of convergence is quadratic. Later, some authors [9–16] constructed higher order methods to solve the multiple root scalar equation taking Newton's modified method as a base step and using some weight functions. The major drawback of these schemes is

the computation of the first-order derivative at each step, which consumes much time. To reduce this complexity, researchers [17–22] have worked on derivative-free schemes of multiple roots of scalar equations with the concept of the divided difference introduced by Traub–Steffensen [4]:

$$x_{p+1} = x_p - m\frac{f(x_p)}{f[\nu_p, x_p]}, \tag{2}$$

where $\nu_p = x_p + \alpha f(x_p)$, $\alpha \neq 0 \in \mathbb{C}$ is any finite real number. Hueso et al. [17] developed a fourth-order derivative-free technique to find the multiple roots of nonlinear equations. Here, the authors approximated the first-order derivative with the divided difference operator as:

$$f'(x) \approx f[\nu_p, x_p], \tag{3}$$

where $\nu_p = x_p + f(x_p)^q$, $q \in \mathbb{R}$.

In 2020, Kumar et al. [18] and Sharma et al. [19–21] constructed derivative-free methods of second-order, fourth-order, and eighth-order convergence, respectively. Recently, Behl et al. [6] proposed an optimal derivative-free Chebyshev–Halley family for multiple roots of a nonlinear equation. This scheme utilized three functional evaluations, one weighted function $H(\tau)$, and one parameter α to obtain the fourth-order convergence. Besides, the mentioned methods are optimal, which means the order of convergence is 2^{n-1}, where n represents the number of function evaluations per iteration, known as the Kung–Traub conjecture [23]. Keeping all the above-mentioned facts in mind, we focused on the construction of the well-known King's optimal higher order family. The main significance of King's family is that it is completely derivative free and preserves the same fourth order of convergence as in the case of simple roots. The present work is organized as follows: The second section explains the convergence of the iterative formula. The third section demonstrates the numerical work. Lastly, Section 4 presents the concluding remarks.

2. Construction of the Higher Order Scheme

Here, we constructed an optimal fourth-order family of King's method [7,8] for multiple zeros ($m \geq 2$) with a simple and compact body structure, which is defined by:

$$\begin{aligned} w_p &= x_p - m\frac{f(x_p)}{f[\nu_p, x_p]}, \\ x_{p+1} &= w_p + (w_p - x_p)\frac{1 + \beta t_p}{1 + (\beta - 2)t_p}(\eta y_p + Q(t_p)), \end{aligned} \tag{4}$$

where $\nu_p = x_p + \alpha f(x_p)$, $\alpha \neq 0 \in \mathbb{C}$ is any finite real number and $m \geq 2$ is the known multiplicity of the required zero. In addition, $f[\nu_p, x_p]$ is the finite difference of order one $f[\nu_p, x_p] = \frac{f(\nu_p) - f(x_p)}{\nu_p - x_p}$. Moreover, $Q(t_p)$ is a single variable function with two multi-valued functions $t_p = \left(\frac{f(w_p)}{f(x_p)}\right)^{\frac{1}{m}}$ and $y_p = \left(\frac{f(w_p)}{f(\nu_p)}\right)^{\frac{1}{m}}$. Suppose their principal analytic branches (see [24]) t_p as a principal root given by $t_p = \exp\left[\frac{1}{m}\log\left(\frac{f(w_p)}{f(x_p)}\right)\right]$, with $\log\left(\frac{f(w_p)}{f(x_p)}\right) = \log\left|\frac{f(w_p)}{f(x_p)}\right| + i\arg\left(\frac{f(w_p)}{f(x_p)}\right)$ for $-\pi < \arg\left(\frac{f(w_p)}{f(x_p)}\right) \leq \pi$. The choice of $\arg(z)$ for $z \in \mathbb{C}$ agrees with that of $\log(z)$ to be employed later in the numerical experiments of the section. We have in an analogous way $t_p = \left|\frac{f(w_p)}{f(x_p)}\right|^{\frac{1}{m}} \cdot \exp\left[\frac{1}{m}\arg\left(\frac{f(w_p)}{f(x_p)}\right)\right] = O(e_p)$. It is important to note that the proposed scheme (4) can be reducedto the well-known King's method for $m = 1$, $\eta = 0$ and $\alpha \to 0$.

In Theorems 1–3, we illustrate that the constructed scheme (4) attains the maximum fourth-order convergence for all $\alpha \neq 0 \in \mathbb{C}$, without adopting any supplementary evaluation of the function or its derivative.

Theorem 1. *Suppose $x = x_*$ is multiple solutions of multiplicity $m = 2$ of function f. Consider that function $f : \mathbb{D} \subset \mathbb{C} \to \mathbb{C}$ is analytic in $\mathbb{D}$ surrounding the required zero x_*. Then, the presented scheme (4) has fourth-order convergence, when:*

$$Q(0) = 0, \ Q'(0) = 1 - \eta, \ Q''(0) = 0, \ \eta = \frac{1}{2}, \ Q'''(0) = \tau \in \mathbb{R}, \tag{5}$$

and satisfies the following error equation:

$$e_{p+1} = \frac{(\lambda_2 + 2B_1)}{384}\left[\lambda_2^2(12\beta - \tau - 12) + 4A_1\lambda_2(12\beta - \tau - 9) + 4A_1^2(12\beta - \tau + 9) - 48A_2\right]e_p^4 + O(e_p^5),$$

where $\lambda_2 = \alpha f''(x_*)$.

Proof. Let us consider that $e_p = x_p - x_*$ and $B_k = \frac{2!}{(2+k)!}\frac{f^{(2+k)}(x_*)}{f^{(2)}(x_*)}$, $k = 1, 2, 3, 4$ are the error in the pth iteration and the asymptotic error constant numbers, respectively. Now, we adopted Taylor's series expansions for the functions $f(x_p)$ and $f(\nu_p)$ around $x = x_*$ with the assumption $f(x_*) = f'(x_*) = 0$ and $f''(x_*) \neq 0$, which are given by:

$$f(x_p) = \frac{f''(x_*)}{2!}e_p^2\left(1 + B_1 e_p + B_2 e_p^2 + B_3 e_p^3 + B_4 e_p^4 + O(e_p^5)\right) \tag{6}$$

and:

$$\begin{aligned}
f(\nu_p) = &\frac{f''(x_*)}{2!}e_p^2\bigg[1 + (\lambda_2 + B_1)e_p + \frac{1}{4}\left(\lambda_2^2 + 10\lambda_2 B_1 + 4B_2\right)e_p^2 + \frac{1}{4}\bigg(5\lambda_2^2 B_1 \\
&+ 6\lambda_2 B_1^2 + 12\lambda_2 B_2 + 4B_3\bigg)e_p^3 + \frac{1}{8}\bigg(\lambda_2^3 B_1 + 14\lambda_2^2 B_1^2 + 16\lambda_2^2 B_2 \\
&+ 28\lambda_2 B_1 B_2 + 28\lambda_2 B_3 + 8B_4\bigg)e_p^4 + O(e_p^5)\bigg],
\end{aligned} \tag{7}$$

respectively.

By using Expressions (6) and (7) in the first substep of Scheme (4), we obtained:

$$\begin{aligned}
w_p - x_* = &\frac{1}{4}(\lambda_2 + 2B_1)e_p^2 - \left[\frac{1}{16}\lambda_2^2 - \frac{1}{2}\lambda_2 B_1 - B_2 + \frac{3B_1^2}{4}\right]e_p^3 + \frac{1}{64}\bigg[\lambda_2^3 \\
&- 10B_1\left(\lambda_2^2 + 16B_2\right) - 20\lambda_2 B_1^2 + 64\lambda_2 B_2 + 72B_1^3 + 96B_3\bigg]e_p^4 + O(e_p^5).
\end{aligned} \tag{8}$$

Expression (8) and Taylor series expansion led us to:

$$\begin{aligned}
f(w_p) = &\frac{f''(x_*)}{2!}e_p^2\bigg[\frac{1}{16}(\lambda_2 + 2B_1)^2 e_p^2 - \frac{1}{32}(\lambda_2 + 2B_1)\left(\lambda_2^2 - 8\lambda_2 B_1 + 12B_1^2 - 16B_2\right)e_p^3 \\
&+ \frac{1}{256}\bigg(3\lambda_2^4 - 4B_1\left(7\lambda_2^3 - 48\lambda_2 B_2 - 96B_3\right) + 96\lambda_2^2 B_2 + 32B_1^2(\lambda_2^2 - 32B_2) \\
&- 80\lambda_2 B_1^3 + 192\lambda_2 B_3 + 464B_1^4 + 256B_2^2\bigg)e_p^4 + O(e_p^5)\bigg].
\end{aligned} \tag{9}$$

By adopting Expressions (6), (7), and (9), we obtained:

$$\begin{aligned}
t_p = \left(\frac{f(w_p)}{f(x_p)}\right)^{\frac{1}{m}} = &\frac{1}{4}(\lambda_2 + 2B_1)e_p + \left(\frac{-\lambda_2^3 + 4\lambda_2^2 B_1 - 4\lambda_2 B_1^2 + 16\lambda_2 B_2 - 32B_1^3 + 32B_1 B_2}{16(\lambda_2 + 2B_1)}\right)e_p^2 \\
&+ \frac{1}{64}\left(\lambda_2^3 - 6\lambda_2^2 B_1 - 22\lambda_2 B_1^2 + 56\lambda_2 B_2 + 116B_1^3 - 208B_1 B_2 + 96B_3\right)e_p^3 + O(e_p^4)
\end{aligned} \tag{10}$$

and:

$$y_p = \left(\frac{f(w_p)}{f(v_p)}\right)^{\frac{1}{m}} = \frac{1}{4}(\lambda_2 + 2B_1)e_p + \left(\frac{-3\lambda_2^3 - 4\lambda_2^2 B_1 - 12\lambda_2 B_1^2 + 16\lambda_2 B_2 - 32B_1^3 + 32B_1 B_2}{16(\lambda_2 + 2B_1)}\right)e_p^2$$
$$+ \frac{1}{64}\left(7\lambda_2^3 - 22\lambda_2^2 B_1 - 14\lambda_2 B_1^2 + 24\lambda_2 B_2 + 116B_1^3 - 208B_1 B_2 + 96B_3\right)e_p^3 + O(e_p^4). \tag{11}$$

Expression (10) demonstrates that t_p is of order one ($t_p = O(e_p)$). Then, we have:

$$Q(t_p) = Q(0) + Q'(0)t_p + \frac{1}{2!}Q''(0)t_p^2 + \frac{1}{3!}Q'''(0)t_p^3. \tag{12}$$

We obtained the following expression by inserting Equations (6)–(12) in Scheme (4):

$$e_{p+1} = -Q(0)e_p + \sum_{i=1}^{3} \Lambda_i e_p^{i+1} + O(e_p^5), \tag{13}$$

where $\Lambda_i = \Lambda_i\left(f''(x_*), \alpha, \beta, \eta, B_1, B_2, B_3, B_4, Q(0), Q'(0), Q''(0), Q'''(0)\right)$. For example, $\Lambda_1 = -Q(0)$, $\Lambda_2 = -\frac{1}{4}\left(\eta + Q(0) + Q'(0) - 1\right)(\lambda_2 + 2B_1)$, etc.

From Expression (13), we deduced that Scheme (4) reaches at least second-order convergence, if:

$$Q(0) = 0. \tag{14}$$

Adopting Expression (14) and $\Lambda_1 = 0$, we obtained:

$$-\frac{1}{4}\left(\eta + Q'(0) - 1\right)(\lambda_2 + 2B_1) = 0, \tag{15}$$

which further gives:

$$Q'(0) = 1 - \eta. \tag{16}$$

By inserting Expressions (14) and (16) in $\Lambda_2 = 0$, we have:

$$-\frac{1}{32}(\lambda_2 + 2B_1)\left(\lambda_2\left(-4\eta + Q''(0) + 2\right) + 2Q''(0)B_1\right) = 0, \tag{17}$$

which further leads to:

$$Q''(0) = 0, \quad \eta = \frac{1}{2}. \tag{18}$$

Next, by inserting (14), (16), and (18) in (13), we obtained:

$$e_{p+1} = \frac{(\lambda_2 + 2B_1)}{384}\left[\lambda_2^2(12\beta - \tau - 12) + 4A_1\lambda_2(12\beta - \tau - 9) + 4A_1^2(12\beta - \tau + 9) - 48A_2\right]e_p^4 + O(e_p^5), \tag{19}$$

where $\tau = Q'''(0) \in \mathbb{R}$. Hence, the scheme (4) has fourth-order convergence for $m = 2$. $\square$

Theorem 2. *Adopting the same hypotheses of Theorem 1, then the proposed scheme (4) has fourth-order convergence for $m = 3$ when:*

$$Q(0) = 0, \quad Q'(0) = 1 - \eta, \quad Q''(0) = 0, \quad Q'''(0) = \tau \in \mathbb{R},$$

It satisfies the following error equation:

$$e_{p+1} = \frac{C_1}{162}\left[C_1^2(12\beta - \tau + 12) - 9\left(2C_2 - (\eta - 1)\lambda_3\right)\right]e_p^4 + O(e_p^5), \quad \text{where } \lambda_3 = \alpha f'''(x_*).$$

Proof. We adopted Taylor's series expansions for the functions $f(x_p)$ and $f(\nu_p)$ around $x = x_*$ with the assumptions $f(x_*) = f'(x_*) = f''(x_*) = 0$ and $f'''(x_*) \neq 0$, which are defined as follows:

$$f(x_p) = \frac{f'''(x_*)}{3!} e_p^3 \left(1 + C_1 e_p + C_2 e_p^2 + C_3 e_p^3 + C_4 e_p^4 + O(e_p^5) \right) \tag{20}$$

and:

$$f(\nu_p) = \frac{f'''(x_*)}{3!} e_p^3 \left[1 + C_1 e_p + \frac{1}{2}(\lambda_3 + 2C_2) e_p^2 + \left(\frac{7}{6} \lambda_3 C_1 + C_3 \right) e_p^3 + O(e_p^4) \right], \tag{21}$$

where $C_j = \frac{3!}{(3+j)!} \frac{f^{(3+j)}(x_*)}{f^{(3)}(x_*)}$, $j = 1, 2, 3, 4$ are asymptotic error constant numbers.

By using Expressions (20) and (21) in the first substep of Scheme (4), we have:

$$w_p - x_* = \frac{C_1}{3} e_p^2 + \frac{1}{18} \left(3\lambda_3 - 8C_1^2 + 12C_2 \right) e_p^3 + \left(\frac{1}{9} C_1(2\lambda_3 - 13C_2) + \frac{16C_1^3}{27} + C_3 \right) e_p^4 + O(e_p^5). \tag{22}$$

Expression (22) and Taylor series expansion lead to:

$$f(w_p) = \frac{f'''(x_*)}{3!} e_p^3 \left[\frac{C_1^3}{27} e_p^3 + \frac{1}{54} C_1^2 \left(3\lambda_3 - 8C_1^2 + 12C_2 \right) e_p^4 + O(e_p^5) \right]. \tag{23}$$

By adopting Expressions (20), (21), and (23), we obtained

$$y_p = \left(\frac{f(w_p)}{f(\nu_p)} \right)^{\frac{1}{m}} = \frac{C_1}{3} e_p + \frac{1}{18C_1^2} \left(3\lambda_3 C_1^2 - 10C_1^4 + 12C_2 C_1^2 \right) e_p^2 + \frac{1}{27} \left(3\lambda_3 C_1 + 23C_1^3 - 48C_2 C_1 \right.$$
$$\left. + 27C_3 \right) e_p^3 + O(e_p^4) \tag{24}$$

and:

$$t_p = \left(\frac{f(w_p)}{f(x_p)} \right)^{\frac{1}{m}} = \frac{C_1}{3} e_p + \frac{1}{18C_1^2} \left(3\lambda_3 C_1^2 - 10C_1^4 + 12C_2 C_1^2 \right) e_p^2 + \frac{1}{54} \left(9\lambda_3 C_1 + 46C_1^3 \right.$$
$$\left. - 96C_2 C_1 + 54C_3 \right) e_p^3 + O(e_p^4). \tag{25}$$

Expression (25) demonstrates that t_p is of order one ($t_p = O(e_p)$). Then, we have:

$$Q(t_p) = Q(0) + Q'(0)t_p + \frac{1}{2!} Q''(0)t_p^2 + \frac{1}{3!} Q'''(0)t_p^3. \tag{26}$$

We have the following expression by inserting Equations (20)–(26) in Scheme (4):

$$e_{p+1} = -Q(0)e_p + \sum_{i=1}^{3} \bar{\Lambda}_i e_p^{i+1} + O(e_p^5), \tag{27}$$

where $\bar{\Lambda}_i = \bar{\Lambda}_i(f'''(x_*), \alpha, \beta, \eta, C_1, C_2, C_3, C_4, Q(0), Q'(0), Q''(0), Q'''(0))$. For example, $\bar{\Lambda}_1 = -Q(0)$, $\bar{\Lambda}_2 = \frac{C_1}{3}(\eta + Q(0) + Q'(0) - 1)$, etc.

From Expression (27), we deduced that Scheme (4) reaches at least second-order convergence, if:

$$Q(0) = 0. \tag{28}$$

Adopting Expression (28) and $\bar{\Lambda}_1 = 0$, we obtained:

$$-\frac{C_1}{3} \left(\eta + Q'(0) - 1 \right) = 0, \tag{29}$$

which further gives:

$$Q'(0) = 1 - \eta. \tag{30}$$

By inserting Expressions (28) and (30) in $\bar{\Lambda}_2 = 0$, we have:

$$-\frac{1}{18}Q''(0)C_1^2 = 0, \tag{31}$$

which further leads to:

$$Q''(0) = 0. \tag{32}$$

Next, by inserting (28), (30), and (32) in (27), we obtained:

$$e_{p+1} = \frac{C_1}{162}\left[C_1^2(12\beta - \tau + 12) - 9\left(2C_2 - (\eta - 1)\lambda_3\right)\right]e_p^4 + O(e_p^5). \tag{33}$$

Hence, Scheme (4) has fourth-order convergence for $m = 3$. □

Theorem 3. *Adopting the same hypotheses of Theorem 1, then the proposed scheme (4) has fourth-order convergence for $m \geq 4$ provided:*

$$Q(0) = 0, \quad Q'(0) = 1 - \eta, \quad Q''(0) = 0, \quad Q'''(0) = \tau \in \mathbb{R},$$

It satisfies the following error equation:

$$e_{n+1} = \frac{D_1^3\left(12\beta - \tau + 3(m+1)\right) - 6mD_1D_2}{6m^3}e_p^4 + O(e_p^5).$$

Proof. Let us consider that $e_p = x_p - x_*$ and $D_j = \frac{m!}{(m+j)!}\frac{f^{(m+j)}(x_*)}{f^{(m)}(x_*)}$, $j = 1, 2, 3, 4$ are the error in the pth iteration and asymptotic error constant numbers, respectively. Now, we adopted Taylor's series expansions for the functions $f(x_p)$ and $f(\nu_p)$ around $x = x_*$ with the assumption $f(x_*) = f'(x_*) = f^{(m-1)}(x_*) = 0$ and $f^{(m)}(x_*) \neq 0$, which are given respectively by:

$$f(x_p) = \frac{f^{(m)}(x_*)}{m!}e_p^m\left(1 + D_1e_p + D_2e_p^2 + D_3e_p^3 + D_4e_p^4 + O(e_p^5)\right) \tag{34}$$

and:

$$f(\nu_p) = \frac{f^m(x_*)}{m!}e_p^m\left[1 + \sum_{i=0}^{2}\Gamma_ie_p^{i+1} + O(e_p^4)\right], \tag{35}$$

where $\Gamma_i = \Gamma_i(m, f^{(m)}(x_*), \alpha, D_1, D_2, D_3, D_4)$. For example, the first coefficient is explicitly written as $\Gamma_0 = D_1$, $\Gamma_1 = D_2, \Gamma_2 = \begin{cases} \frac{1}{6}\left(\alpha f^{(4)}(x_*) + 6D_3\right), & m = 4 \\ D_3, & m \geq 5 \end{cases}$, etc.

By adopting Expressions (34) and (35) in the first substep of Scheme (4), we obtained:

$$e_{w_p} = w_p - x_* = \frac{D_1}{m}e_p^2 + \frac{1}{m^2}\left(2mD_2 - (1+m)D_1^2\right)e_p^3 + \frac{1}{m^3}\left(3m^2D_3 + (m+1)^2D_1^3 - m(3m+4)D_2D_1\right)e_p^4 + O(e_p^5). \tag{36}$$

Expression (36) and Taylor series expansion lead to:

$$f(w_p) = \frac{f^{(m)}(x_*)}{m!}e_{w_p}^m\left[1 + D_1e_{w_p} + D_2e_{w_p}^2 + D_3e_{w_p}^3 + D_4e_{w_p}^4 + O(e_p^5)\right]. \tag{37}$$

From Expressions (34), (35), and (37), we obtained:

$$y_p = \left(\frac{f(w_p)}{f(v_p)}\right)^{\frac{1}{m}} = \frac{D_1}{m}e_p + \frac{1}{m^2}\left(2mD_2 - (m+2)D_1^2\right)e_p^2 + \frac{1}{2m^3}\left((2m^2 + 7m + 7)D_1^3 - 2m(3m+7)D_1D_2\right. \tag{38}$$
$$\left. + 6m^2D_3\right)e_p^3 + O(e_p^4)$$

and:

$$t_p = \left(\frac{f(w_p)}{f(x_p)}\right)^{\frac{1}{m}} = \frac{D_1}{m}e_p + \left(\frac{2}{m}D_2 - \frac{(m+2)}{m^2}D_1^2\right)e_p^2 + \frac{1}{2m^3}\left[(2m^2 + 7m + 7)D_1^3 - 2m(3m+7)D_1D_2\right.$$
$$\left. + 6m^2D_3\right]e_p^3 + \frac{1}{3m^4}\left[-\left(\frac{11m^2}{2} + \frac{33m}{2} + 14\right)D_1^4 - 12m^2D_3D_1 + 3m(6m+11)D_2D_1^2 \right. \tag{39}$$
$$\left. - 6m^2D_2^2\right]e_p^4 + O(e_p^5).$$

Expression (39) demonstrates that t_p is of order one ($t_p = O(e_p)$). Then, we have:

$$Q(t_p) = Q(0) + Q'(0)t_p + \frac{1}{2!}Q''(0)t_p^2 + \frac{1}{3!}Q'''(0)t_p^3. \tag{40}$$

We have the following expression by inserting Equations (34)–(40) in Scheme (4):

$$e_{p+1} = -Q(0)e_p + \sum_{i=1}^{3} \bar{\bar{\Lambda}}_i e_p^{i+1} + O(e_p^5), \tag{41}$$

where $\bar{\bar{\Lambda}}_i = \bar{\bar{\Lambda}}_i(f^{(m)}(x_*), m, \alpha, \beta, \eta, D_1, D_2, D_3, D_4, Q(0), Q'(0), Q''(0), Q'''(0))$. For example, $\bar{\bar{\Lambda}}_1 = -Q(0)$, $\bar{\bar{\Lambda}}_2 = -\frac{D_1}{m}\left(\eta + Q(0) + Q'(0) - 1\right)$, etc.

From Expression (41), we deduced that Scheme (4) reaches at least second-order convergence, if:

$$Q(0) = 0. \tag{42}$$

Adopting Expression (42) and $\bar{\Lambda}_1 = 0$, we obtained:

$$-\frac{D_1}{m}\left(\eta + Q'(0) - 1\right) = 0, \tag{43}$$

which further gives:

$$Q'(0) = 1 - \eta. \tag{44}$$

By inserting Expressions (42) and (44) in $\bar{\Lambda}_2 = 0$, we have:

$$-\frac{1}{2m^2}Q''(0)D_1^2 = 0, \tag{45}$$

which further leads to:

$$Q''(0) = 0. \tag{46}$$

Next, by inserting (42), (44), and (46) in (41), we obtained:

$$e_{n+1} = \frac{D_1^3\left(12\beta - \tau + 3(m+1)\right) - 6mD_1D_2}{6m^3}e_p^4 + O(e_p^5). \tag{47}$$

Hence, Scheme (4) has fourth-order convergence for $m \geq 4$. □

Remark 1. *The final body structure of our proposed Scheme (4), is given by:*

$$
\begin{aligned}
w_p &= x_p - m \frac{f(x_p)}{f[\nu_p, x_p]}, \\
x_{p+1} &= w_p + (w_p - x_p) \frac{1 + \beta t_p}{1 + (\beta - 2)t_p} \left(\frac{1}{2} y_p + Q(t_p) \right).
\end{aligned}
\tag{48}
$$

The above Scheme (48) satisfies all the hypothesis that are mentioned in Theorems 1–3.

3. Numerical Experimentation

Here, we present the results of the proposed derivative-free iterative methods named as $M_1(\alpha = \frac{1}{4}, \beta = \frac{1}{2}, Q'''(0) = 2)$, $M_2(\alpha = \frac{1}{4}, \beta = \frac{1}{2}, Q'''(0) = \frac{1}{3})$, $M_3(\alpha = \frac{1}{4}, \beta = 1, Q'''(0) = -1)$, and $M_4(\alpha = \frac{1}{4}, \beta = \frac{1}{2}, Q'''(0) = 0)$, respectively, in comparison to existing derivative-free methods. Several numerical examples were tested to verify the results. We considered the following derivative-free techniques developed by Hueso et al. [17], Sharma et al. [19], and Sharma et al. [20], respectively, for the comparisons.

Hueso et al.'s scheme (HM):

$$
\begin{aligned}
y_p &= x_p - \frac{2m}{m+2} \frac{f(x_p)}{f[x_p, \mu_p]}, \\
x_{p+1} &= x_p - \left(s_1 + s_2 H(x_p, y_p) + s_3 H(y_p, x_p) + s_4 \left(H(x_p, y_p)^2 \right) \right) \frac{f(x_p)}{f[x_p, \mu_p]},
\end{aligned}
$$

where:

$$
\begin{aligned}
\mu_p &= x_p + (f(x_p))^q, \quad q \in \mathbb{R}, \\
H(x_p, y_p) &= \frac{f[x_p, y_p]}{f[x_p, \mu_p]}, \\
s_1 &= -\frac{1}{4} m \left(m^3 + 3m^2 + 2m - 4 \right) \\
s_2 &= \frac{1}{8} m \left(\frac{m}{m+2} \right)^m (m+2)^3 \\
s_3 &= \frac{1}{8} m^4 \left(\frac{m}{m+2} \right)^{-m} \\
s_4 &= 0
\end{aligned}
$$

Note that for $m \geq 4$, the value of q is considered as one, and for other values of m, q is taken as two.

Sharma et al.'s scheme (S_1):

$$
\begin{aligned}
y_p &= x_p - m \frac{f(x_p)}{f[x_p, \mu_p]}, \\
x_{p+1} &= y_p - \left(\frac{m}{2} h_p (1 + 3h_p) \right) \left(1 + \frac{1}{v_p} \right) \frac{f(x_p)}{f[x_p, \mu_p]},
\end{aligned}
$$

where:

$$\mu_p = x_p + \gamma f(x_p), \quad \gamma \in \mathbb{R},$$

$$u_p = \left(\frac{f(y_p)}{f(x_p)} \right)^{\frac{1}{m}},$$

$$v_p = \left(\frac{f(\mu_p)}{f(x_p)} \right)^{\frac{1}{m}},$$

$$h_p = \frac{u_p}{1 + u_p}.$$

Sharma et al.'s scheme (S_2):

$$y_p = x_p - m \frac{f(x_p)}{f[x_p, \mu_p]},$$

$$x_{p+1} = x_p - h_p \frac{f(x_p)}{f[x_p, \mu_p]},$$

where:

$$\mu_p = x_p + \gamma g(x_p), \quad \gamma \in \mathbb{R},$$

$$r_p = \left(\frac{f(y_p)}{f(x_p)} \right)^{\frac{1}{m}},$$

$$s_p = \left(\frac{f(y_p)}{f(\mu_p)} \right)^{\frac{1}{m}},$$

$$h_p = r_p + m(r_p)^2 + (m - 1)s_p + mr_p s_p.$$

All the numerical results were performed with the software Mathematica 10 using 3000 multiple precision digits of the mantissa after the first four iterative procedures. To check the better performance of the proposed method, we displayed the errors between two consecutive iterations $e_p = |x_p - x_{p-1}|$, the absolute residual error $|f(x_p)|$ at the (p)th iteration, and the approximate computational order of convergence (ACOC), denoted as ρ, in Tables 1–5. The starting point is indicated in each example, and the final results depended on this initial guess, but were similar when we were in a neighborhood of the exact solution, such as the local convergence studies in iterative methods have confirmed. The following formula was used to evaluate the ACOC:

$$\rho = \frac{ln \frac{|x_{p+1} - x_p|}{|x_p - x_{p-1}|}}{ln \frac{|x_p - x_{p-1}|}{|x_{p-1} - x_{p-2}|}}, \quad \text{for each } p = 2, 3, \ldots \tag{49}$$

Note that the expression $b(\pm a)$ represents $b \times 10^{\pm a}$ in all the tables.

Example 1. *Here, we considered the famous van der Waals equation of an ideal gas [6] that represents the nature of a real gas on the basis of parameters a, b of a particular gas.*

$$\left(P + \frac{an^2}{V^2} \right)(V - nb) = nRT.$$

The parameters $n, R,$ and T were also evaluated with terms a and b. Therefore, we have the following nonlinear equations in terms of the volume of gas (V), which is represented as x by:

$$f_1(x) = x^3 - 5.22x^2 + 9.0825x - 5.2675.$$

The desired zero is $r = 1.75$ of multiplicity $m = 2$. Table 1 depicts the performance of different iterative schemes with initial guess $x_0 = 1.9$. The proposed methods M_1, M_2, M_3, and M_4 converged to the root much faster than the other derivative-free methods HM, S_1, and S_2.

Table 1. Numerical results of function f_1.

Schemes	$|e_2|$	$|e_3|$	$|e_4|$	$|f(e_4)|$	ρ
HM	$1.5(-2)$	$1.8(-4)$	$1.8(-11)$	$1.1(-79)$	3.998
S_1	$1.9(-2)$	$5.6(-4)$	$4.3(-9)$	$7.6(-60)$	3.990
S_2	$1.6(-2)$	$2.8(-4)$	$1.4(-10)$	$2.9(-72)$	3.996
M_1	$8.8(-5)$	$5.9(-13)$	$1.2(-45)$	$4.3(-92)$	3.999
M_2	$1.0(-4)$	$1.1(-12)$	$1.9(-44)$	$1.1(-89)$	3.999
M_3	$1.8(-4)$	$1.6(-11)$	$1.1(-39)$	$3.4(-80)$	3.998
M_4	$1.0(-4)$	$1.3(-12)$	$3.3(-44)$	$3.2(-89)$	3.999

Example 2. *Next, we tested the proposed scheme on Planck's radiation equation, which determines the spectral density of electromagnetic radiation released by a black body at a given temperature and at thermal equilibrium [25] as:*

$$F(y) = \frac{8\pi chy^{-5}}{e^{\frac{ch}{ykT}} - 1},$$

where T, y, k, h, and c denote the absolute temperature of the black-body, the wavelength of the radiation, Boltzmann's constant, Planck's constant, and the speed of light in a medium (vacuum), respectively. The wavelength y was determined by the maximum energy density $F(y)$, that is $F'(y) = 0$, resulting in the following equation.

$$\frac{(\frac{ch}{ykT})e^{\frac{ch}{ykT}}}{e^{\frac{ch}{ykT}} - 1} = 5.$$

The nonlinear equation was formulated by taking $x = \frac{ch}{ykT}$ as:

$$f_2(x) = \left(e^{-x} - 1 + \frac{x}{5}\right)^m.$$

In order to obtain the approximated zero $r = 4.96511423174427630369$ of multiplicity $m = 4$, one can easily find the wavelength y from the relation $x = \frac{ch}{ykT}$. Planck's problem was tested with initial guess $x_0 = 5.5$, and the results are demonstrated in Table 2.

Table 2. Numerical results of function f_2.

Schemes	$|e_2|$	$|e_3|$	$|e_4|$	$|f(e_4)|$	ρ
HM	$1.1(-3)$	$1.7(-14)$	$1.1(-57)$	$2.3(-922)$	4.000
S_1	$6.4(-6)$	$2.8(-25)$	$1.0(-102)$	$1.4(-1630)$	4.000
S_2	$5.6(-6)$	$1.4(-25)$	$4.7(-104)$	$2.8(-1672)$	4.000
M_1	$6.8(-26)$	$2.6(-105)$	$5.4(-423)$	$1.2(-1692)$	4.000
M_2	$7.2(-26)$	$3.2(-105)$	$1.3(-422)$	$4.1(-1691)$	4.000
M_3	$9.0(-26)$	$8.2(-105)$	$5.8(-421)$	$1.6(-1684)$	4.000
M_4	$7.3(-26)$	$3.4(-105)$	$1.6(-422)$	$8.2(-1699)$	4.000

Example 3. *Another application of nonlinear equations is the characteristic equations of a large matrix [6] to find the eigenvalues. The determination of the roots/eigenvalues of such higher order characteristic equations is a difficult task if we apply the linear algebra approach. Therefore, one of the best ways is to use the numerical techniques. Here, we considered the following square matrix of order nine:*

$$A = \frac{1}{8} \begin{pmatrix} -12 & 0 & 0 & 19 & -19 & 76 & -19 & 18 & 437 \\ -64 & 24 & 0 & -24 & 24 & 64 & -8 & 32 & 376 \\ -16 & 0 & 24 & 4 & -4 & 16 & -4 & 8 & 92 \\ -40 & 0 & 0 & -10 & 50 & 40 & 2 & 20 & 242 \\ -4 & 0 & 0 & -1 & 41 & 4 & 1 & 2 & 25 \\ -40 & 0 & 0 & 18 & -18 & 104 & -18 & 20 & 462 \\ -84 & 0 & 0 & -29 & 29 & 84 & 21 & 42 & 501 \\ 16 & 0 & 0 & -4 & 4 & -16 & 4 & 16 & -92 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 \end{pmatrix},$$

then the characteristic equation was formulated into a nonlinear equation as:

$$f_3(x) = x^9 - 29x^8 + 349x^7 - 2261x^6 + 8455x^5 - 17663x^4 + 15927x^3 + 6993x^2 - 24732x + 12960.$$

The zero of f_3 is $r = 3$ with multiplicity $m = 4$. Table 3 depicts the better performance of the proposed schemes in comparison with existing techniques by taking initial guess $x_0 = 3.1$.

Table 3. Numerical results of function f_3.

Schemes	$|e_2|$	$|e_3|$	$|e_4|$	$|f(e_4)|$	ρ
HM	5.4(−2)	3.4(−2)	1.1(−3)	5.9(−35)	4.041
S_1	3.8(−3)	3.7(−11)	3.4(−43)	2.8(−681)	4.001
S_2	3.8(−3)	2.3(−11)	3.2(−44)	1.5(−698)	4.001
M_1	4.7(−11)	3.0(−43)	5.3(−172)	6.6(−684)	4.000
M_2	1.3(−10)	2.2(−41)	1.6(−164)	5.6(−654)	4.000
M_3	5.2(−10)	6.0(−39)	1.1(−154)	1.2(−614)	4.000
M_4	1.5(−10)	3.7(−41)	1.2(−163)	1.8(−650)	4.000

Example 4. *Now, we considered the problem of a continuous stirred-tank reactor [26] shown in Figure 1.*
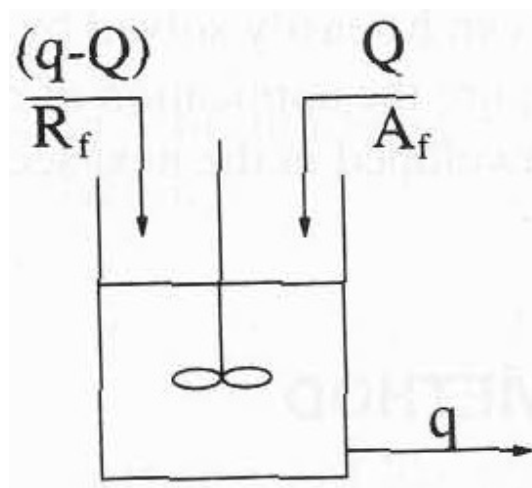
Figure 1. Continuous stirred-tank reactor.

Here, components A and R are fed to the reactor at rates Q and $q - Q$, respectively. The reaction schemes developed in this reactor are:

$$A + R \rightarrow B;$$
$$B + R \rightarrow C;$$
$$C + R \rightarrow D;$$
$$D + R \rightarrow E;$$
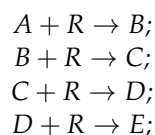
Douglas [27] converted this problem into a mathematical expression:

$$K_c \frac{2.98(t + 2.25)}{(t + 1.45)(t + 2.85)^2(t + 4.35)} = -1, \tag{50}$$

where K_c is the gain of the proportional controller. This control system is stable for the values of K_c. If we take $K_c = 0$, we obtain the poles of the open-loop transferred function as the solutions of the following univariate equation:

$$f_4(x) = x^4 + 11.50x^3 + 47.49x^2 + 83.06325x + 51.23266875 = 0. \tag{51}$$

The function $f_4(x)$ has three zeros, -1.45, -2.85, and -4.35, where the root -2.85 has multiplicity $m = 2$. The computation results are demonstrated in Table 4 on initial guess -3.2. It can be concluded that the results were better for the suggested methods in terms of less residual and functional error with a similar utilization of the number of iterations for the methods developed by HM, S_1, and S_2.

Table 4. Numerical results of function f_4.

Schemes	$|e_2|$	$|e_3|$	$|e_4|$	$|f(e_4)|$	ρ
HM_1	4.3(−2)	9.0(−6)	1.8(−20)	1.6(−157)	4.000
S_1	6.9(−3)	2.7(−7)	1.2(−28)	3.6(−227)	4.000
S_2	7.0(−3)	2.7(−7)	1.1(−28)	1.5(−227)	4.000
M_1	9.1(−18)	2.7(−70)	2.1(−280)	9.5(−560)	4.000
M_2	1.1(−17)	5.9(−70)	5.5(−279)	6.4(−557)	4.000
M_3	1.7(−17)	6.9(−69)	1.7(−274)	6.3(−548)	4.000
M_4	1.1(−17)	6.8(−70)	1.0(−278)	2.1(−556)	4.000

Example 5. *Lastly, we have the standard academic problem [6] as follows:*

$$f_5(x) = \left(\tan^{-1}\left(\frac{\sqrt{5}}{2}\right) - \tan^{-1}\left(\sqrt{x^2 - 1}\right) + \sqrt{6}\left(\tan^{-1}\left(\sqrt{\frac{x^2-1}{6}}\right) - \tan^{-1}\left(\frac{1}{2}\sqrt{\frac{5}{6}}\right)\right) - \frac{11}{63}\right)^4,$$

which has approximated zero $r = 1.8411027704926161$ of multiplicity four. The results were obtained on initial guess $x_0 = 1.5$ and shown in Table 5.

Table 5. Numerical results of function f_5.

Schemes	$|e_2|$	$|e_3|$	$|e_4|$	$|f(e_4)|$	ρ
HM_1	9.3(−3)	4.1(−9)	1.6(−34)	1.4(−543)	4.000
S_1	2.9(−5)	1.4(−20)	8.0(−82)	3.0(−1306)	4.000
S_2	2.9(−5)	9.9(−21)	1.4(−82)	4.2(−1319)	4.000
M_1	2.0(−21)	1.7(−85)	9.1(−342)	4.3(−1366)	4.000
M_2	2.1(−21)	2.0(−85)	1.8(−341)	6.2(−1365)	4.000
M_3	2.4(−21)	3.9(−85)	2.7(−340)	3.2(−1360)	4.000
M_4	2.1(−21)	2.1(−85)	2.0(−341)	1.1(−1364)	4.000

Clearly, the proposed methods M_1, M_2, M_3, M_4 depicted better results in Tables 1–5 for every function considered in Examples 1–5. The suggested techniques showed less residual error and less functional error in comparison with the existing methods HM, S_1, and S_2.

4. Concluding Remarks

In this paper, we sorted out the two main issues (that related to multiple roots $m \geq 2$ and the calculation of the derivative at each step) of the well-known King's family of fourth order. In simple words, we suggested a new family of King's method that has the key features of optimal order of convergence for multiple roots and being free from derivatives, unlike the classical King's method. In addition, we established a main theorem that confirmed the theoretical fourth-order convergence. Each member of our scheme

satisfied the famous Kung–Traub conjecture that deals with the optimality of iterative methods (without memory). In accordance with the obtained computational consequences, we deduced that our method *M*1 performed far better than the latest iterative methods of the same generation. The computational Tables 1–5 also confirmed and supported this conclusion.

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