mathematics

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The Newtonian Operator and Global Convergence Balls for Newton's Method

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Received: 2 June 2020; Accepted: 22 June 2020; Published: 2 July 2020

Abstract: We obtain results of restricted global convergence for Newton's method from ideas based on the Fixed-Point theorem and using the Newtonian operator and auxiliary points. The results are illustrated with a non-linear integral equation of Davis-type and improve the results previously given by the authors.

Keywords: fixed-point theorem; Newton's method; Newtonian operator; global convergence; convergence ball; Fredholm integral equation

MSC: 45B99; 47H10; 65J15

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1. Introduction

Consider a twice continuously Fréchet differentiable operator $F : \Omega \subseteq X \to Y$ defined on a non-empty open convex domain Ω of a Banach space X with values in a Banach space Y and Newton's method,

$$x_{n+1} = N_F(x_n) = x_n - [F'(x_n)]^{-1}F(x_n), \quad n \geq 0, \quad \text{with } x_0 \text{ given}, \tag{1}$$

to solve the non-linear equation $F(x) = 0$. Please note that a solution of the equation $F(x) = 0$ is a fixed point of the Newtonian operator $N_F : \Omega \subseteq X \to X$.

We can analyze the convergence of Newton's method in three different ways. The first, called *semilocal convergence*, consists of giving conditions on the starting point x_0 and on the operator F, obtaining a ball of existence of solution of the equation $F(x) = 0$, $\overline{B(x_0, R_1)}$, called existence ball, and the convergence of the method taking x_0 as the starting point. The second, called *local convergence*, assumes that there exists a solution x^* of the equation $F(x) = 0$ and, under certain conditions on the operator F, we obtain a ball $\overline{B(x^*, R_2)}$, called a convergence ball, in which the convergence of the method is guaranteed by taking any point on the ball as the starting point x_0. The third, called *global convergence*, consists of giving conditions on the operator F to obtain a domain in which a solution of the equation $F(x) = 0$ exists, so that the method converges to the solution taking any point of the domain obtained as the starting point x_0.

Taking the above as a perspective, the most interesting fact is to prove the existence of a solution of the equation $F(x) = 0$ and in turn find a domain in which any of its points, taken as the starting point of the method, provides the convergence of the method. Obviously, this case refers to a study of the global convergence that, as is known, is obtained for specific operators in certain situations.

In [1], we use a technique based on the use of auxiliary points [2,3] to present a study of the convergence of Newton's method in which we obtain global convergence balls of the form $\overline{B(\widetilde{x}, R)}$, where $\widetilde{x}$ is an auxiliary point. These balls allow us to locate a solution of the equation $F(x) = 0$ and separate it from other possible solutions. For this, the auxiliary point $\tilde{x}$ verifies that the operator $\tilde{\Gamma} = [F'(\tilde{x})]^{-1}$ exists with $\|\tilde{\Gamma}\| \leq \beta$ and $\|\tilde{\Gamma}F(\tilde{x})\| \leq \eta$, and $M\beta\eta \leq 1/6$, where M is the Lipschitz constant of the operator F', provided that F' is Lipschitz continuous in the domain involved.

There are two main aims in this work. First, we obtain results of global convergence restricted to a subset of the space in which the operator F is defined by using the Newtonian operator N_F under the usual conditions of Kantorovich [4,5]. In addition, our study is based on the following Fixed-Point Theorem [6]:

> If D is a convex and compact set of X and the operator $P : D \to D$ is a contraction, then the operator P has a unique fixed point in D and it can be approximated by the method of successive approximations, $x_{n+1} = P(x_n)$, $n \geq 0$, from any $x_0 \in D$.

Also, we remember that the operator P is a contraction if $\|P(x) - P(y)\| < K\|x - y\|$ with $K \in [0, 1)$, for all $x, y \in D$. If the operator P is derivable, it is enough that the condition $\|P'(x)\| < 1$, for all $x \in D$, is satisfied to see that P is a contraction.

The second aim is to improve the study presented in [1]. For this, we find, from an auxiliary point $\tilde{x}$ such that $M\beta\eta \leq 1/2$, a new point $\tilde{y}$ and obtain global convergence balls of the form $\overline{B(\tilde{y}, R)}$ that locate a solution of the equation $F(x) = 0$ and separate it from other possible solutions, so that the condition $M\beta\eta \leq 1/6$ is relaxed to $M\beta\eta \leq 1/2$ and the study given in [1] is then expanded.

Throughout the paper, we denote $\overline{B(x,r)} = \{y \in X; \|y - x\| \leq r\}$, $B(x,r) = \{y \in X; \|y - x\| < r\}$ and the set of bounded linear operators from X to Y by $\mathcal{L}(X, Y)$, and use the infinity norm in X.

2. The Newtonian Operator

We start by locating a domain that contains a fixed point of the operator N_F. For this, we suppose the following conditions:

(A1) For some $\tilde{x} \in \Omega$, there exists $\tilde{\Gamma} = [F'(\tilde{x})]^{-1} \in \mathcal{L}(Y, X)$ with $\|\tilde{\Gamma}\| \leq \beta$ and $\|\tilde{\Gamma}F(\tilde{x})\| \leq \eta$.
(A2) There exists a constant $M \geq 0$ such that $\|F''(x)\| \leq M$ for all $x \in \Omega$.
(A3) $h = M\beta\eta \leq 1/2$.

Under the conditions (A1)-(A2)-(A3), it is well known that the Newton–Kantorovich theorem holds with $\tilde{x} = x_0$ ([5,7]).

Theorem 1 (The Newton–Kantorovich theorem)**.** *Let $F : \Omega \subseteq X \to Y$ be a twice continuously Fréchet differentiable operator defined on a non-empty open convex domain Ω of a Banach space X with values in a Banach space Y. Suppose that the conditions (A1)-(A2)-(A3) are satisfied and $B(x_0, \rho^*) \subset \Omega$ with $\rho^* = \frac{1-\sqrt{1-2h}}{h}\eta$. Then, Newton's sequence defined in (1) and starting at x_0 converges to a solution x^* of the equation $F(x) = 0$ and the solution x^* and the iterates x_n belong to $\overline{B(x_0, \rho^*)}$, for all $n \geq 0$ Moreover, if $h < \frac{1}{2}$, the solution x^* is unique in $B(x_0, \rho^{**}) \cap \Omega$, where $\rho^{**} = \frac{1+\sqrt{1-2h}}{h}\eta$, and, if $h = \frac{1}{2}$, x^* is unique in $\overline{B(x_0, \rho^*)}$. Furthermore, we have the following error estimates:*

$$\|x^* - x_n\| \leq \frac{1}{2^{n-1}}(2h)^{2^n-1}\eta, \quad n = 0, 1, 2, \ldots$$

In addition, in the proof of Theorem 1, the following estimates are also deduced [7], and will be used later:

$$h_n = M\beta_n\eta_n \leq \frac{1}{2}(2h)^{2^n}, \tag{2}$$

where $\|\Gamma_n\| \leq \beta_n$ and $\|\Gamma_n F(x_n)\| \leq \eta_n$ with $\Gamma_n = [F'(x_n)]^{-1}$.

The first step in locating domains that contain possible fixed points is to describe these domains. In this work, we consider balls of the form $\overline{B(\tilde{y}, R)}$. Thus, the first two aims are to locate the point $\tilde{y} \in \Omega$ and calculate the value of R, so that $N_F : \overline{B(\tilde{y}, R)} \to \overline{B(\tilde{y}, R)}$. Then we will be able to apply the Fixed-Point theorem. For this, we establish the following theorem.

Theorem 2. *Let $\widetilde{x} \in \Omega$ and suppose that the conditions (A1)-(A2)-(A3) hold. Then, there exist $\widetilde{y} \in \overline{B(\widetilde{x}, \rho^*)}$, where $\rho^* = \frac{1-\sqrt{1-2h}}{h}\eta$, and $R > 0$ such that $N_F : \overline{B(\widetilde{y}, R)} \to \overline{B(\widetilde{y}, R)}$ with $R \in [R_-, R_+]$, where $R_- = \frac{1-\sqrt{1-6Mba}}{3Mb}$, $R_+ = \frac{1+\sqrt{1-6Mba}}{3Mb}$, $\|[F'(\widetilde{y})]^{-1}\| \leq b$, and $\|[F'(\widetilde{y})]^{-1}F(\widetilde{y})\| \leq a$.*

Proof. If $M\beta\eta \leq 1/6$, then we take $\widetilde{y} = \widetilde{x}$, so that $Mba \leq 1/6$. Otherwise, we observe from (2) that if we take $x_0 = \widetilde{x}$ for Newton's method, then we can consider

$$N_0 = 1 + \left[\frac{\log\left(-\frac{\log 3}{\log 2h}\right)}{\log 2} \right], \tag{3}$$

where $[z]$ denotes the integer part of any positive real number z, so that the N_0-th iterate of the method is such that $h_{N_0} \leq 1/6$. Therefore, if we choose $\widetilde{y} = x_{N_0}$, it is clear that $Mba \leq 1/6$. Consequently, $[R_-, R_+] \neq \emptyset$ and we can then consider R in this interval.

Second, we see that the operator $\Gamma = [F'(x)]^{-1}$ exists for all $x \in \overline{B(\widetilde{y}, R)}$. Indeed, from

$$\|I - [F'(\widetilde{y})]^{-1}F'(x)\| \leq \|[F'(\widetilde{y})]^{-1}\|\|[F'(\widetilde{y}) - F'(x)\| \leq bM\|x - \widetilde{y}\| \leq MbR < 1, \tag{4}$$

since $R \in [R_-, R_+]$, it follows, by the Banach lemma on invertible operators that the operator Γ exists with $\|\Gamma\| \leq \frac{\beta}{1-MbR}$ and $\|\Gamma F'(\widetilde{y})\| \leq \frac{1}{1-MbR}$. Consequently, the operator N_F is well defined in the domain indicated.

Third, we see that $N_F(x) \in \overline{B(\widetilde{y}, R)}$ for all $x \in \overline{B(\widetilde{y}, R)}$. Indeed, from

$$\begin{aligned}
\|N_F(x) - \widetilde{y}\| &= \|-\Gamma\left(F(x) + F'(x)(\widetilde{y} - x)\right)\| \\
&= \left\| -\Gamma F(\widetilde{y}) + \Gamma \int_0^1 \left(F'(x + t(\widetilde{y} - x)) - F'(x)\right) dt(\widetilde{y} - x) \right\| \\
&\leq \|\Gamma F'(\widetilde{y})\|\|[F'(\widetilde{y})]^{-1}F(\widetilde{y})\| + \frac{M}{2}\|\Gamma\|\|\widetilde{y} - x\|^2 \\
&\leq \frac{MbR^2 + 2a}{2(1 - MbR)} \leq R,
\end{aligned}$$

it follows that $N_F(x) \in \overline{B(\widetilde{y}, R)}$, provided that $Mba \leq 1/6$ and $R \in [R_-, R_+]$. □

To apply the Fixed-Point theorem to the Newtonian operator N_F we need to ensure that this operator is a contraction in the domain considered. One way to check that N_F is a contraction is to directly check the condition $\|N_F(x) - N_F(y)\| < K\|x - y\|$ with $K \in [0, 1)$, for all $x, y \in \overline{B(\widetilde{x}, R)}$. Other variant that we can consider to prove that the operator N_F is a contraction comes from the fact that N_F is a derivable operator such that $\|N_F'(x)\| < 1$. For this, we consider the operator L_F, called degree of logarithmic convexity [5] that is defined as follows

$$L_F(x) : \Omega \xrightarrow{F''(x)[F'(x)]^{-1}F(x)} Y \xrightarrow{\quad [F'(x)]^{-1} \quad} \Omega.$$

Therefore, $L_F(x) = [F'(x)]^{-1}F''(x)\left([F'(x)]^{-1}F(x)\right) \in \mathcal{L}(X, X)$, with $[F'(x)]^{-1} \in \mathcal{L}(Y, X)$. This operator satisfies $L_F(x) = N_F'(x)$, as we can see in [5].

3. Restricted Global Convergence

First, we see that the operator N_F is a contraction from the conditions (A1)-(A2)-(A3).

Lemma 1. *Let $\widetilde{x} \in \Omega$ and suppose that the conditions (A1)-(A2)-(A3) hold. Then, there exist $\widetilde{y} \in \overline{B(\widetilde{x}, \rho^*)}$, where $\rho^* = \frac{1-\sqrt{1-2h}}{h}\eta$, and $R > 0$ such that $R \leq \delta_- = \frac{4-\sqrt{10+6Mba}}{3Mb}$ and the operator $N_F : \overline{B(\widetilde{y}, R)} \to \overline{B(\widetilde{y}, R)}$ is a contraction.*

Proof. Take $\widetilde{y} = x_0 = \widetilde{x}$. Then $Mba \leq 1/2$, $\delta_- = \frac{4-\sqrt{10+6Mba}}{3Mb} > 0$ and there exists R such that $0 < R \leq \delta_-$.

If $x \in \overline{B(\widetilde{y}, R)}$, then

$$[F'(\widetilde{y})]^{-1}F(x) = [F'(\widetilde{y})]^{-1}F(\widetilde{y}) + (x-\widetilde{y}) + [F'(\widetilde{y})]^{-1}\int_0^1 \left(F'(\widetilde{y}+t(x-\widetilde{y})) - F'(\widetilde{y})\right)(x-\widetilde{y})\,dt,$$

so that

$$\|[F'(\widetilde{y})]^{-1}F(x)\| \leq \|[F'(\widetilde{y})]^{-1}F(\widetilde{x})\| + \|x-\widetilde{y}\| + \frac{M}{2}\|[F'(\widetilde{y})]^{-1}\|\|x-\widetilde{y}\|^2 \leq \frac{M}{2}bR^2 + R + a.$$

Now, if $x, y \in \overline{B(\widetilde{x}, R)}$, as $MbR < 1$, from (4), there exists $\Gamma_x = [F'(x)]^{-1}$ and $\Gamma_y = [F'(y)]^{-1}$, and

$$\begin{aligned}
N_F(x) - N_F(y) &= x - \Gamma_x F(x) - y + \Gamma_y F(y) \\
&= \Gamma_y \left(F'(y)(x-y) - F'(y)\Gamma_x F(x) + F(y)\right) \\
&= \Gamma_y \left(F(x) - \int_y^x (F'(z) - F'(y))\,dz - F'(y)\Gamma_x F(x)\right) \\
&= \Gamma_y \left(\left(\int_0^1 F''(y+t(x-y))\,dt(x-y)\right)\Gamma_x F(x) - \int_y^x (F'(z)-F'(y))\,dz\right), \\
\|N_F(x) - N_F(y)\| &= \|\Gamma_y\| \left(M\|x-y\|\|\Gamma_x F(x)\| + \frac{M}{2}\|x-y\|^2\right) \\
&= \|\Gamma_y\| M \left(\|\Gamma_x F(x)\| + \frac{1}{2}\|x-y\|\right)\|x-y\| \\
&\leq \|\Gamma_y\| M \left(\|\Gamma_x F(x)\| + R\right)\|x-y\| \\
&= \alpha\|x-y\|.
\end{aligned}$$

Taking now into account that $\|[F'(x)]^{-1}\| \leq \frac{b}{1-MbR}$, for all $x \in \overline{B(\widetilde{y}, R)}$, we have $\|\Gamma_y\| \leq \frac{b}{1-MbR}$. Moreover,

$$\|\Gamma_x F(x)\| \leq \|\Gamma_x F'(\widetilde{y})\|\|[F'(\widetilde{y})]^{-1}F(x)\| \leq \frac{\frac{M}{2}bR^2 + R + a}{1-MbR}.$$

Thus, we can consider $\alpha = \dfrac{Mb(2a+4R-MbR^2)}{2(1-MbR)^2}$. Therefore, $\alpha < 1$ if $0 < R \leq \delta_-$ and the operator N_F is then a contraction. □

Second, we locate the value of R.

Lemma 2. *If $Mba \leq \frac{-3+\sqrt{13}}{4} = 0.1513\ldots$, there always exists $R > 0$ such that $R \in [R_-, \delta_-]$.*

Proof. If $Mba \leq 0.1513\ldots$, we observe that $R_- \leq \delta_-$, since $4(Mba)^2 + 6(Mba) - 1 \leq 0$, so that $[R_-, \delta_-] \neq \emptyset$. Moreover, as $R_+ > \delta_-$, the proof is complete. □

Third, from the Fixed-Point theorem given above, we obtain the existence of a unique fixed point of N_F in $\overline{B(\widetilde{y}, R)}$ and its approximation by Newton's method.

Theorem 3. *Let $\widetilde{x} \in \Omega$ and suppose that the conditions (A1)-(A2)-(A3) hold. Then, there exist $\widetilde{y} \in \overline{B(\widetilde{x}, \rho^*)}$, where $\rho^* = \frac{1-\sqrt{1-2h}}{h}\eta$, and $R > 0$ such that $R \in [R_-, \delta_-]$. Moreover, the operator $N_F : \overline{B(\widetilde{y}, R)} \to \overline{B(\widetilde{y}, R)}$ has a unique fixed point x^* and Newton's method converges quadratically to x^* from any starting point in $\overline{B(\widetilde{y}, R)}$.*

Proof. If $M\beta\eta \leq 0.1513\ldots$, then we take $\widetilde{y} = \widetilde{x}$, so that $Mba \leq 0.1513\ldots$ Otherwise, we observe from (2) that if we take $x_0 = \widetilde{x}$ for Newton's method, then we can consider

$$N_0 = 1 + \left[\frac{\log\left(\frac{\log(0.3026\ldots)}{\log 2h}\right)}{\log 2} \right],$$

so that the N_0-th iterate of the method is such that $h_{N_0} \leq 0.1513\ldots$ Therefore, if we choose $\widetilde{y} = x_{N_0}$, it is clear that $Mba \leq 0.1513\ldots$

Now, from Theorem 2, it follows that $N_F(x) \in \overline{B(\widetilde{y}, R)}$, for all $x \in \overline{B(\widetilde{y}, R)}$, and, from Lemmas 1 and 2, the operator $N_F : \overline{B(\widetilde{y}, R)} \to \overline{B(\widetilde{y}, R)}$ is a contraction. Then, the proof is concluded from the application of the Fixed-Point theorem given in Section 1 to the operator N_F in $\overline{B(\widetilde{y}, R)}$. The fact that Newton's method has quadratic convergence follows easily from the known Newton–Kantorovich theorem [5]. □

Since R_- is the smallest value that R can take and δ_- the largest, we can consider $\overline{B(\widetilde{y}, R_-)}$ as ball of location of the fixed point and $\overline{B(\widetilde{y}, \delta_-)}$ as ball of uniqueness, and thus achieve a greater separation of other possible fixed points of the Newtonian operator. Moreover, we observe that we have global convergence in any ball $\overline{B(\widetilde{y}, R)}$ with $R \in [R_-, \delta_-]$.

On the other hand, we prove that operator N_F is a contraction from the condition on $N_F' = L_F$ and consequently has a unique fixed point.

Theorem 4. *Let $\widetilde{x} \in \Omega$ and suppose that the conditions (A1)-(A2)-(A3) hold. Then, there exists $\widetilde{y} \in \overline{B(\widetilde{x}, \rho^*)}$, where $\rho^* = \frac{1-\sqrt{1-2h}}{h}\eta$, such that, if $\|L_F(x)\| \leq K < 1$, for all $x \in B(\widetilde{y}, R)$, the operator $N_F : \overline{B(\widetilde{y}, R)} \to \overline{B(\widetilde{y}, R)}$ has a unique fixed point x^* for all $R \in [R_-, R_+]$ and Newton's method converges quadratically to x^* from any starting point in $\overline{B(\widetilde{x}, R)}$.*

Proof. If $M\beta\eta \leq 1/6$, then we take $\widetilde{y} = \widetilde{x}$, so that $Mba \leq 1/6$. Otherwise, we consider $x_0 = \widetilde{x}$ for Newton's method and take $\widetilde{y} = x_{N_0}$ with N_0 given in (3), so that $Mba \leq 1/6$.

Under the conditions required, we have seen in theorem 2 that $N_F(x) \in \overline{B(\widetilde{y}, R)}$ for all $x \in \overline{B(\widetilde{y}, R)}$, where $R \in [R_-, R_+]$. Moreover, as $\|N_F'(x)\| \leq \|L_F(x)\| \leq K < 1$, for all $x \in \overline{B(\widetilde{y}, R)}$, the operator N_F is a contraction. Therefore, from the application of the Fixed-Point theorem mentioned above to the operator N_F in $\overline{B(\widetilde{y}, R)}$, the proof is complete. □

In this case, we have $\overline{B(\widetilde{y}, R_-)}$ as ball of location of the fixed point and $\overline{B(\widetilde{y}, R_+)}$ as ball of uniqueness. We note that the most favorable ball of global convergence in this case is the last.

The existence condition of the operator L_F in a certain set seems very strong, since it implies the existence of the inverse operator $[F'(x)]^{-1}$ at each point in the set. However, by requiring the conditions (A1)-(A2)-(A3), it is guaranteed the existence of L_F in a domain, as we can see in the following result.

Theorem 5. *Let $\widetilde{x} \in \Omega$ and suppose that the conditions (A1)-(A2)-(A3) hold. Then, there exists $L_F(x)$ for all $x \in B\left(\widetilde{x}, \frac{1}{M\beta}\right)$ and $\|L_F(x)\| \leq \dfrac{M\beta\,\|\widetilde{\Gamma}F(x)\|}{(1 - M\beta\|x - \widetilde{x}\|)^2}$.*

Proof. Observe that

$$\widetilde{\Gamma} \int_{\widetilde{x}}^{x} F''(v)\,dv = \widetilde{\Gamma}F'(x) - I$$

and

$$\|\widetilde{\Gamma}F'(x) - I\| = \left\| \widetilde{\Gamma} \int_{\widetilde{x}}^{x} F''(v)\,dv \right\| \leq M\beta\|x - \widetilde{x}\| < 1.$$

Then, from the Banach lemma on invertible operators, there exists $[\widetilde{\Gamma}F'(x)]^{-1}$ for all $x \in B\left(\widetilde{x}, \frac{1}{M\beta}\right)$ and

$$\|[\widetilde{\Gamma}F'(x)]^{-1}\| \leq \frac{1}{1 - M\beta\|x - \widetilde{x}\|}.$$

Now, from

$$L_F(x) = [\widetilde{\Gamma}F'(x)]^{-1}\widetilde{\Gamma}F''(x)[\widetilde{\Gamma}F'(x)]^{-1}\widetilde{\Gamma}F(x), \tag{5}$$

the proof is complete. □

Moreover, we observe that $B(\widetilde{x}, R) \subset B\left(\widetilde{x}, \frac{1}{M\beta}\right)$ if $R \in [R_-, R_+]$, since $R_+ < \frac{1}{M\beta}$. Consequently, we establish the following result.

Theorem 6. *Let $\widetilde{x} \in \Omega$ and suppose that the conditions (A1)-(A2)-(A3) hold. Then, the operator $L_F(x)$ exists, for all $x \in B(\widetilde{x}, R)$, with $R \in [R_-, R_+]$, and*

$$\|L_F(x)\| \leq \frac{(M\beta R)^2 + 2(M\beta R) + 2M\beta\eta}{2\,(1 - M\beta R)^2}.$$

Proof. We have seen above that $\|\Gamma F'(\widetilde{x})\| \leq \frac{1}{1-M\beta R}$ and $\|\widetilde{\Gamma}F(x)\| \leq \frac{M}{2}\beta R^2 + R + \eta$. Now, from (5), it follows that

$$\|L_F(x)\| \leq \|\Gamma F'(\widetilde{x})\|\|\widetilde{\Gamma}\|\|F''(x)\|\|\Gamma F'(\widetilde{x})\|\|\widetilde{\Gamma}F(x)\| \leq \frac{M\beta\left(\frac{M}{2}\beta R^2 + R + \eta\right)}{2\,(1 - M\beta R)^2},$$

for all $x \in B(\widetilde{x}, R)$, and the proof is complete. □

In addition, as $N_F'(x) = L_F(x)$, for $x \in B(\widetilde{x}, R)$, with $R \in [R_-, R_+]$, we obtain the next result.

Theorem 7. *Let $\widetilde{x} \in \Omega$ and suppose that the conditions (A1)-(A2)-(A3) hold. Then, there exist $\widetilde{y} \in \overline{B(\widetilde{x}, \rho^*)}$, where $\rho^* = \frac{1-\sqrt{1-2h}}{h}\eta$, and $R > 0$ such that $R \in [R_-, \sigma_-]$, where $\sigma_- = \frac{3-\sqrt{7+2Mba}}{Mb}$, and the operator $N_F : \overline{B(\widetilde{y}, R)} \to \overline{B(\widetilde{y}, R)}$ has a unique fixed point x^* and Newton's method converges quadratically to x^* from any starting point in $\overline{B(\widetilde{y}, R)}$.*

Proof. If $M\beta\eta \leq 0.1642\ldots$, then we take $\widetilde{y} = \widetilde{x}$, so that $Mba \leq 0.1642\ldots$ In another case, we observe from (2) that if we take $x_0 = \widetilde{x}$ for Newton's method, then we can consider

$$N_0 = 1 + \left[\frac{\log\left(\frac{\log(0.3284\ldots)}{\log 2h}\right)}{\log 2}\right],$$

so that the N_0-th iterate of the method is such that $h_{N_0} \leq 0.1642\ldots$ Therefore, if we choose $\widetilde{y} = x_{N_0}$, it is clear that $Mba \leq 0.1642\ldots$

Then, from Theorem 2, we know that $N_F(x) \in \overline{B(\widetilde{y}, R)}$, for all $x \in \overline{B(\widetilde{y}, R)}$. Thus, from Theorem 6, we have

$$\|N_F'(x)\| \leq \frac{(MbR)^2 + 2(MbR) + 2Mba}{2\,(1 - MbR)^2},$$

since $N_F'(x) = L_F(x)$. Thus, $N_F(x)$ is a contraction if

$$\frac{(MbR)^2 + 2(MbR) + 2Mba}{2\,(1 - MbR)^2} < 1.$$

As $Mba \leq 1/6$, the last inequality is true if $R \leq \sigma_-$.

After that, we observe that $R_- \leq \sigma_-$ if $Mba \leq 1/6$, so that $[R_-, \sigma_-] \neq \emptyset$. Also, if $Mba \leq \frac{-5+4\sqrt{2}}{4} = 0.1642\ldots$, we have $16(Mba)^2 + 40(Mba) - 7 \leq 0$ and, as a consequence, $R_+ > \sigma_-$.

Finally, because of the above, the operator N_F is a contraction. Therefore, from the application of the Fixed-Point theorem given in Section 1 to the operator N_F in $\overline{B(\widetilde{y}, R)}$, the proof is complete. $\square$

Now, the ball of location of the fixed point is $\overline{B(\widetilde{y}, R_-)}$ and the ball of uniqueness is $\overline{B(\widetilde{y}, \sigma_-)}$, which in turn is the most favorable ball of global convergence.

4. Example

We illustrate the above study with the following particular Davis-type integral equation that is used in [1].

We consider the following integral equation:

$$x(s) = s + \frac{7}{5}\int_0^1 G(s,t)x(t)^2 dt, \quad s \in [0,1], \tag{6}$$

where the kernel $G(s,t)$ is the Green function

$$G(s,t) = \begin{cases} (1-s)t, & t \leq s, \\ s(1-t), & s \leq t. \end{cases}$$

As we cannot apply Newton's method directly, since we do not know the inverse operator that is involved in the algorithm of the method, we transform the integral equation into a finite-dimensional problem by a process of discretization, where the Gauss-Legendre quadrature formula

$$\int_0^1 \phi(t)\, dt \simeq \sum_{j=1}^{8} w_j \phi(t_j),$$

with the nodes t_j and weights w_j known, is used to approximate the integral that appears in the integral equation.

By denoting the $x(t_i)$ by x_i, for $i = 1, 2, \ldots, 8$, the integral equation is equivalent to the following system of non-linear equations:

$$x_j = t_j + \frac{7}{5}\sum_{k=1}^{8} a_{jk}\, x_k^2, \quad k = 1, 2, \ldots, 8, \tag{7}$$

where

$$a_{jk} = \begin{cases} w_k\,(1-t_j)t_k, & k \leq j, \\ w_k\,(1-t_k)t_j, & k > j. \end{cases}$$

Now, we write the last system compactly in matrix form as

$$F(\mathbf{x}) \equiv \mathbf{x} - \mathbf{v} - \frac{7}{5}A\,\mathbf{y} = 0, \qquad F : \mathbb{R}^8 \longrightarrow \mathbb{R}^8, \tag{8}$$

where

$$\mathbf{x} = (x_1, x_2, \ldots, x_8)^t, \quad \mathbf{v} = (t_1, t_2, \ldots, t_8)^t, \quad A = \left(a_{jk}\right)_{j,k=1}^{8}, \quad \mathbf{y} = \left(x_1^2, x_2^2, \ldots, x_8^2\right)^t.$$

In [1], we take $\widetilde{\mathbf{x}} = \mathbf{v}$ as auxiliary point to illustrate the study developed there and obtain the best ball of location of solution and the biggest ball of convergence. However, if for example, we choose $\widetilde{\mathbf{x}}$ as the vector $(1, 1, \ldots, 1)^t$, instead of the vector $\mathbf{v}$, then $\beta = 1.4912\ldots$, $\eta = 0.9850\ldots$ and $h = M\beta\eta = 0.3629\ldots$, where $M = 0.2471\ldots$, and choosing the max-norm. As $h > 1/6$, we cannot apply the study developed in [1] and, as a consequence, we cannot locate and separate a solution of the integral equation. However, we can apply the study presented in this work, since $h \leq 1/2$.

For Theorem 3, we have $N_0 = 2$ and $h_{N_0} = 0.000192\ldots < 0.1513\ldots$, so that $\widetilde{\mathbf{y}} = \mathbf{x}_2$ (the vector shown in Table 1), $R \in [0.0006\ldots, 0.9076\ldots]$, the best ball of location of solution is $\overline{B(\mathbf{x}_2, 0.0006\ldots)}$ and the biggest ball of convergence is $\overline{B(\mathbf{x}_2, 0.9076\ldots)}$.

Table 1. The approximation $\mathbf{x}_2$ of Newton's method for (8).

i	$\mathbf{x}_2$	i	$\mathbf{x}_2$
1	0.022629...	5	0.658303...
2	0.115872...	6	0.822537...
3	0.270122...	7	0.932596...
4	0.462132...	8	0.987941...

For Theorem 7, we have $N_0 = 2$ and $h_{N_0} = 0.000192\ldots < 0.1642\ldots$, so that $\widetilde{\mathbf{y}} = \mathbf{x}_2$, $R \in [0.0006\ldots, 1.1515\ldots]$, the best ball of location of solution is $\overline{B(\mathbf{x}_2, 0.0006\ldots)}$ and the biggest ball of convergence is $\overline{B(\mathbf{x}_2, 1.1515\ldots)}$, which is bigger than that given by Theorem 3.

From the last result, it is easy to conclude that the ball of location of solution is the same and it is very precise, and the balls of separation of solutions given by the two theorems are good enough. Observe that Theorem 7 offers a better result for the ball of convergence. In addition, we can use all the new results given in this work and the work presented in [1] is clearly improved.

Finally, we can see the convergence of Newton's method to a solution of the above non-linear system in [1], along with an approximate solution of the system.

Remark 1. *The application of the technique developed to obtain global convergence domains for Newton's method presents an initial difficulty: the location of an auxiliary point. In the Davis-type integral equation, it is known that the choice of the function s as the starting point for the application of iterative methods is a reasonable choice due to its proximity to a solution of the integral equation [8]. As we prove in this work, the choice of auxiliary points has a direct relationship with the location of starting points for Newton's method, so its choice as auxiliary points is natural. Thus, the applicability of our technique is analogous to that of starting points. It is well known that this problem is of great importance and numerous studies have been carried out to try to extend the domains of starting points [2,9–11], which can therefore also be applied to locate auxiliary points.*

5. Conclusions

From the theoretical significance of Newton's method, we prove the existence and uniqueness of solutions of a non-linear equation and obtain domains of global convergence for the method. We also obtain results for Newton's method that allow us to locate a solution of the equation involved and separate it from other possible solutions, and improve the study previously carried out in our work presented in [1]. For this, auxiliary points are used and ideas based on the Fixed-Point theorem. The improvement mentioned above is illustrated with a non-linear integral equation presented in [1].

Author Contributions: The contributions of the two authors have been similar. Both authors have worked together to develop the present manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: This research was partially supported by Ministerio de Ciencia, Innovación y Universidades under grant PGC2018-095896-B-C21.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ezquerro, J.A.; Hernández-Verón, M.A. How to obtain global convergence domains via Newton's method for nonlinear integral equations. *Mathematics* **2019**, *7*, 553. [CrossRef]
2. Ezquerro, J.A.; Hernández-Verón, M.A. Domains of global convergence for Newton's method from auxiliary points. *Appl. Math. Lett.* **2018**, *85*, 48–56. [CrossRef]
3. Ezquerro, J.A.; Hernández-Verón, M.A. Domains of global convergence for a type of nonlinear Fredholm-Nemytskii integral equations. *Appl. Numer. Math.* **2019**, *146*, 452–468. [CrossRef]
4. Argyros, I.K. On the Newton-Kantorovich hypothesis for solving equations. *J. Comput. Appl. Math.* **2004**, *169*, 315–332. [CrossRef]
5. Ezquerro, J.A.; Hernández-Verón, M.A. *Newton's Method: An Updated Approach of Kantorovich's Theory. Frontiers in Mathematics*; Birkhäuser/Springer: Cham, Switzerland, 2017.
6. Berinde, V. *Iterative Approximation of Fixed Points*; Springer: New York, NY, USA, 2007.
7. Kantorovich, L.V. On Newton's method for functional equations (Russian). *Dokl. Akad. Nauk. SSSR* **1948**, *59*, 1237–1240.
8. Keller, H.B. *Numerical Methods for Two-Point Boundary Value Problems*; Dover Publications: Mineola, NY, USA, 1992.
9. Argyros, I.K. On a theorem of L.V. Kantorovich concerning Newton's method. *J. Comput. Appl. Math.* **2003**, *155*, 223–230. [CrossRef]
10. Argyros, I.K.; Hilout, S. On the quadratic convergence of Newton's method under center-Lipschitz but not necessarily Lipschitz hypotheses. *Math. Slovaca* **2013**, *63*, 621–638. [CrossRef]
11. Gutiérrez, J. M.; Magreñán, Á.A.; Romero, N. On the semilocal convergence of Newton-Kantorovich method under center-Lipschitz conditions. *Appl. Math. Comput.* **2013**, *221*, 79–88. [CrossRef]