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# Fractional Partial Differential Equations Associated with Lêvy Stable Process

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**Abstract:** In this study, we first present a time-fractional Lêvy diffusion equation of the exponential option pricing models of European option pricing and the risk-neutral parameter. Then, we modify a particular Lêvy-time fractional diffusion equation of European-style options. Further, we introduce a more general model based on the Lêvy-time fractional diffusion equation and review some recent findings associated with risk-neutral free European option pricing.

**Keywords:** price impact; option pricing; liquidity; Lêvy process; fractional differential equations; fractional Lêvy process

## 1. Introduction

One of the significant problems in finance is to derive value from financially traded assets that is also known as the pricing of financial instruments, for example, stocks, and it is a very interesting problem. In the literature, Merton (1990, [1]) was among the first researchers who gave the systematic solution for this problem, and proposed the Black-Scholes (BS) model where the model rests on the assumption that the natural logarithm of the stock price $S_t$ defined as follows:

$$d(\ln S_t) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dB_t \tag{1}$$

where $\mu > 0$ is the average compounded growth rate of the stock $S_t$, and $dB_t$ is the increment of Brownian motion which assumed to have the Normal or Gaussian distribution, and $\sigma \geq 0$ represents the volatility of the returns from holding $S_t$. The Equation (1) is also known as componential equation. On either side a Lêvy process is a stochastic process with independent, stationary increments that represents the motion of a point whose successive displacements are random and independent, and statistically identical over different time intervals of the same length. The mathematical theory of Lêvy process can be found in Bertoin (1996) [2] or Sato (1999) [3]. An example of a Lêvy process that is well-known from, for instance, the Black–Scholes–Merton option pricing theory is the Brownian motion (or Wiener process), where the increments are normally distributed.

Thus if we substitute the Lêvy process by the Brownian motion in componential Equation (1), the pricing partial differential equation becomes a partial integro-differential equation; further details related to the partial integro-differential equations (PIDEs) can be found in [4,5]. The partial integro-differential equations are also studied in order to understand the non-locality phenomena which are produced by the jumps in the Lêvy process.

One of the methods to solve PIDEs was the numerical method which was proposed by (Cont, [6]) and that was the finite-difference method for option pricing, having jump-diffusion as well as exponential Lêvy process models, see Lewis [7]. On the other hand, the second method was the fast

Fourier transform of European-style options (see [8]). Further similar strategies were also proposed, for example, ref. ([9]) proposed a model to use fractional calculus.

In this article, we modify European-style options under a risk-neutral probability condition for the stock-price assets, followed by liquidity market in the financial literature. We also consider to generate some partial integro-differential equation for possible application to less-studied issues such as barrier options for finite moment having log-stable (FMLS) processes in the future.

The article is based on the following: Section 2 reviews the basic concepts of Lêvy operations and applications in financial modeling. Section 3 introduces the concepts of fractional calculus and how to solve fractional differential equations, and reviews the main concepts of Lêvy process. Section 4 introduce the main result. Finally, Section 5 will conclude and discuss some applications.

## 2. Fractional Diffusion Model and Option Pricing

In a fully liquid market, regardless of the trading size, the options trader cannot influence the price of the underlying asset in the trading of the asset in order to duplicate the option. In the literature (Chen, et al. (2014), [10]) studied this model, where $L_t^{\alpha,\beta}$ is a Lêvy $\alpha$–stable process with skew parameter $\beta$. Before viewing the idea of that research we will define $\alpha$–stable distribution, that is, the distribution is said to be stable if the location and scale parameters have the same distributions of linear combination of two independent random variables with respect to this distribution. Similarly, a random variable is said to be stable if its distribution is stable. The stable dissemination family at times indicates as the Lêvy alpha-stable distribution.

**Definition 1.** *Any random variable $X$ is s-stable if for each $n \in N$ with $X_1, X_2, \ldots, X_n$ being infinitely divisible copies of $X$ $X_1 + X_2 + \ldots + X_n = bX + c$ or some constants $b = b(n) > 0$ and $c = c(n) \in R^d$. It is called strictly stable for any $n \in N$ if $c(n) = 0$ .*

For an infinitely divisible random vector $X^{*t}$ define the alpha-stable as follows.

**Definition 2.** *A stable $X$ is called alpha-stable, whenever $X^{*t} = t^{\frac{1}{\alpha}} X + c$ or some constants $c = c(r) \in \mathcal{R}^d$ , $t > 0$, and $0 < \alpha \leq 2$. When $c(t) = 0$, for $t > 0$, then $X$ is called strictly alpha-stable .*

Now, consider the following dynamic under a risk neutral probability measure for the stock price $S_t$

$$dS_t = S_t \left( (r - q)dt + \sigma dL_t^{\alpha,-1} \right) \tag{2}$$

for time $0 < t < T$, where index $\alpha$ of stability satisfies $1 < \alpha < 2$, and volatility $\sigma > 0$. When $\sigma = 0$, we will get the original BS model. Moreover where $r$ and $q$ respectively denote deterministic parameters corresponding to the risk-free rate and dividend yield. We restrict our selves to the case where $\beta = -1$ to obtain finite moments of $S_t$ and negative skewness in the return density distribution. In particular for $n > 0$, then

$$E\left[\exp\left(n\sigma L_t^{\alpha,-1}\right)\right] = \exp\left(-tn^\alpha \sigma^\alpha \sec\left(\frac{\pi\alpha}{2}\right)\right) < +\infty.$$

The model in the Equation (2) is known as Finite Moment Log Stable (FMLS) for short model. Under the risk–neutral measure the log price satisfies the following SDE:

$$d\left(\ln(S_t)\right) = (r - q - v)dt + \sigma dL_t^{\alpha,-1}, \tag{3}$$

where $v = -\frac{1}{2}\sigma^\alpha \sec\left(\frac{\pi\alpha}{2}\right)$ represents the convexity adjustment.

Let $u(t,x)$ be the price of the European call option with $x = x_t := \ln(S_t)$. (Chen et al. (2014), [10]) In order to find FPDE let $u(t,x)$ satisfy under FMLS the following fractional PDEs

$$\frac{\partial u}{\partial t}(t,x) + (r - v)\frac{\partial u}{\partial x}(t,x) + v\frac{\partial^\alpha u}{\partial x^\alpha}(t,x) - ru(t,x) = 0$$

$$u(x,T) := \begin{cases} \max(e^x - K; 0) & \text{for European call option} \\ \max(K - e^x; 0) & \text{for European put option} \end{cases}$$

where $K$ is the strike price and

$$\frac{\partial^\alpha u(t,x)}{\partial x^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^{x} \frac{u(t,u)}{(x-u)^\alpha} du$$

and $\Gamma(.)$ is the gamma function defined by:

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx.$$

## 3. The Model

In this research, we incorporate $L_t^{\alpha,\beta}$ as a Lêvy $\alpha$–stable process with skew parameter $\beta$. Consider the following dynamic under a risk-neutral probability measure for the stock price $S_t$; the goal is to consider a modified model to Equation (2) that consists on an illiquid market with impact additional term that for $0 \le t < T, 0 < \gamma < 1$ and $1 < \alpha \le 2$,with boundary condition

$$d^\gamma S_t = S_t \left( (r-q) dt^\gamma + \sigma dL_t^{\alpha,-1} \right) + \lambda(t, S_t) S_t d\beta_t^\gamma, S(0) = S_0 \tag{4}$$

where $\lambda(t, S_t) \ge 0$ is the price impact function of the trader and $\beta_t$ denotes the number of shares that the trader has in the stock at time $t$. The term $\lambda(t, S_t) d\beta_t$ represents the price impact of the investor's trading is additional term of Chen model (2) where $\gamma = 2H$ and $H$ is Hurst exponent $0 < H \le 1$. The Hurst exponent is used as a measure of long-term memory of time series. It relates to the autocorrelations of the time series and the rate at which decrease as the lag between pairs of values. $H = 0.5$ indicates a random series, and $H > 0.5$ indicates a trend reinforcing series. Similarly, if the larger the $H$ value is considered then the stronger trend. In the present study we consider the Caputo fractional integral of $f$ defined by the following expression

$$I^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_{-\infty}^{t} f(u)(t-u)^{\gamma-1} du$$

and similarly, the Caputo fractional partial derivative of $u$ defined by the expression

$$\partial_t^\gamma u(x,t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{dy}{(t-y)^{\gamma-1}} \frac{\partial u(x,y)}{\partial y}.$$

In this work, we consider trading strategies written in the following form

$$d\beta_t^\gamma = \eta_t dt^\gamma + \zeta_t dL_t^{\alpha,-1} \tag{5}$$

for some processes $(\eta_t)_{t\ge 0}$ and $(\zeta_t)_{t\ge 0}$ to be determined endogenously and $\beta_0$ is the initial number of shares in the stock. Next we consider the wealth process $(V_t)_{t\ge 0}$ corresponds to a self–financing strategy $(\theta_t, \beta_t)_{t\ge 0}$ for the trader and given by

$$V_t = \theta_t S_t^0 + \beta_t S_t = V_0 + \int_0^t \theta_u dS_u^0 + \int_0^t \beta_u dS_u.$$

To find the fractional partial differential equations that satisfy our model in Equation (4) we need a method to solve fractional equation. In this way we follow the (Demirci and Ozalp, 2012)[11] as an example where they solved the fractional differential equation for the initial value problem in the sense of Caputo type FDE given by

$$D^\gamma x(t) = f(t, x(t)), x(0) = x_0$$

which has a solution

$$x(t) = x_*\left(\frac{t^\gamma}{\Gamma(\gamma+1)}\right)$$

where $x_*(v)$ is a solution for an equation having integer order differentials.

We also need the method of the from literature (Jumarie, [12]) of the equation,

$$dx = f(t)dt^\alpha, t \geq 0, x(0) = x_0$$

where $0 < \alpha \leq 1$, has a solution defined by the equality

$$\int_0^t f(\tau)d\tau^\alpha = \alpha \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau. \tag{6}$$

Furthermore, we will use in our model the following formula

$$d^\alpha x = \Gamma(1+\alpha)dx. \tag{7}$$

The general Fourier transform is defined by

$$\hat{f}(\xi) = \mathcal{F}[f(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x} f(x)dx \tag{8}$$

and the inverse Fourier transformation is defined by

$$f(x) = \int_{-\infty}^{\infty} \exp^{i\xi x}\hat{f}(\xi)d\xi.$$

### 3.1. Lêvy Process

The distribution of Lêvy process is characterized by the Lêvy-Khintchine formula and it is considered as a modified model that characterize the Lêvy process in a very compact way. Today it is known as the Lêvy-Khintchine process. More definitely, a time-dependent random variable $X_t$ is a Lêvy process if and only if it has independent and stationary increments having log-characteristic function given by the Lêvy-Khintchine theorem:

**Theorem 1** (Lêvy-Khintchine presentation theorem)**.** *Let $(X_t)_{t\geq 0}$ Lêvy process on $R$ with characteristic triplet $(m, \sigma, w)$, then $E[e^{izX_t}] = e^{t\Psi(z)}, t \in R$, with characteristic exponent of the Lêvy process*

$$\Psi(z) = im\xi + \frac{\sigma^2}{2}(i\xi)^2 + \int_{-\infty}^{\infty}(e^{i\xi x} - 1 - i\xi I_{|x|<1})W(dx) \tag{9}$$

where, $\int_R \min[1, x^2]W(dx) < \infty$, and *$W = w(x)$ Lêvy density, $m$ in$R$, $\sigma \geq 0$.*

To accommodate how the Lêvy processes being incorporated in the derivatives pricing models, we recall the standard Black-Scholes framework and see how it was built by Gaussian shocks. To find the fair or arbitrage-free prices of a financial instrument whose value are deriven from the underlying share price $S_t$, it is also necessary to express the dynamics of $S_t$ under what is known as a neutral risk measure or the equivalent martingale scale. In the price, the European option may be expressed as the neutral condition for a risk as

$$V(t,S) = e^{-r(T-t)}E^Q[\max(S_T - k, 0)|\mathcal{F}_t]. \tag{10}$$

Fourier transform of European option can be written as (Du, [13])

$$\frac{\partial^\gamma \tilde{V}}{\partial t^\gamma} = \tilde{V}(t,S) + \Psi(\xi)\tilde{V}(t,S) - r\tilde{V}(t,S) \tag{11}$$

where

$$\Psi(\xi) = im\xi + \frac{\sigma^2}{2}(i\xi)^2 + \int_{-\infty}^{\infty}(e^{i\xi x} - 1 - i\xi I_{|x|<1})W(dx)$$

and the indicator function of set $A$ where $I_A : A \subset X \to \{0,1\}$ and defined by

$$I_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

*3.2. Lêvy Stable Processes*

Let $w(x) = w_{LS}(x)$ be Lêvy density function and given by

$$w_{LS} = \begin{cases} \frac{Dq}{|x|^{1+\alpha}} & \text{for } x < 0 \\ \frac{Dq}{x^{1+\alpha}} & \text{for } x > 0 \end{cases} \tag{12}$$

where $D > 0$, $q + p = 1$ and $\alpha \in (0, 2)$. Then by using the Equation (9) we obtain the characteristic exponent of an LS process in the parameters as follows: $\sigma, \alpha, \beta$ and $m$:

$$\Psi_{LS}(\xi) = i\xi m - \frac{1}{2}\sigma^\alpha |\xi|^\alpha + \left[1 - i\beta sign(\xi)\tan\left(\frac{\alpha\pi}{2}\right)\right]. \tag{13}$$

An equivalent form can be written as

$$\Psi_{LS}(\xi) = i\xi m - \frac{1}{4\cos(\frac{\alpha\pi}{2})}\sigma^\alpha[(1-\beta)(i\xi)^\alpha + (1+\beta)(-i\xi)^\alpha] \tag{14}$$

where $\beta = p - q$. If $\beta = -1$, then $p = 0$ and $q = 1$, that is (Alvaro Cartea et al. [9])

$$\Psi_{LS}(\xi) = i\xi m - \frac{1}{4\cos(\frac{\alpha\pi}{2})}\sigma^\alpha\left[(2)(i\xi)^\alpha\right]. \tag{15}$$

## 4. Main Results

Consider the fractional differential Lêvy equation

$$d^\gamma S_t = S_t\left((r-q)dt^\gamma + \sigma dL_t^{\alpha,-1}\right) + \lambda(t, S_t)S_t(\eta_t dt^\gamma + \zeta_t dL_t^{\alpha,-1}). \tag{16}$$

That can be rewritten as $\lambda_t = \lambda(t, S(t))$ and let $x_t = \ln S_t$

$$d^\gamma x_t = (r - q + \lambda_t\eta_t)dt^\gamma + (\sigma + \lambda_t\zeta_t)dL_t^{\alpha,-1}. \tag{17}$$

Next, we derive revised and updated FPDEs for options which are written on assets and follow the Lêvy operations that were mentioned in the previous section. In order to find the relation between the fractional price equations and LP process, [14], we make use of Fourier transform, as in

$$\hat{f}(\xi) = \mathcal{F}[f(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp^{-i\xi x} f(x)dx \tag{18}$$

of the value of European style option price $\hat{\mathcal{U}}(\xi, T)$, written on $S_t$, and satisfies

$$\frac{\partial^\gamma\hat{\mathcal{U}}}{\partial t^\gamma} = r\hat{\mathcal{U}}_{(}\xi, t) + (-q + \lambda_u\eta_u)\, i\xi\hat{\mathcal{U}}(\xi, t) - \Psi(\xi)\hat{\mathcal{U}}(\xi, t). \tag{19}$$

Let $\zeta_T(v)$ denote the Fourier transform of the time value, where

$$\zeta_T(\xi, t)(i\xi)^\gamma = \frac{\partial^\gamma\hat{\mathcal{U}}}{\partial t^\gamma}$$

Let $\mathcal{U}(\xi, T)$ denote the Fourier transform of a European-style option and defined by

$$\zeta_T(\xi,t)(i\xi)^{\gamma} = r\hat{\mathcal{U}}_(\xi,t) + (-q+\lambda_u\eta_u)\, i\xi\hat{\mathcal{U}}(\xi,t) - \Psi(\xi)\hat{\mathcal{U}}(\xi,t). \tag{20}$$

with boundary condition $\mathcal{U}(\xi, T) = \Pi(\xi, T)$.

Now substitute the Equation (15) in Equation (19) and taking the inverse Fourier transform we reach

$$\zeta_T(\xi,t)(i\xi)^{\gamma} = r\hat{\mathcal{U}}(\xi,t) + (r-q+\lambda_u\eta_u)\, i\xi\hat{\mathcal{U}}(\xi,t) - \left[-\frac{1}{4}\sec\left(\frac{\alpha\pi}{2}\right)\sigma^{\alpha}(2)(i\xi)^{\alpha}\right]\hat{\mathcal{U}}(\xi,t)$$

then taking the inverse Fourier transform delivered to

$$\frac{\partial^{\gamma}\mathcal{U}}{\partial t^{\gamma}}(x,t) + (r-q+\lambda_u\eta_u)\frac{\partial\mathcal{U}}{\partial x}(x,t) + \frac{1}{2}\sec\left(\frac{\alpha\pi}{2}\right)\sigma^{\alpha}\frac{\partial^{\alpha}\mathcal{U}}{\partial x^{\alpha}}(x,t) = r\mathcal{U}(x,t). \tag{21}$$

To prove Equation (17) satisfies the Equation (19).

First we can find the solution of Equation (17). Rewrite the Equation (17) in the form, where $x_T = \ln(S_T)$

$$dx_T = \frac{1}{\Gamma(1+\gamma)}\left[(r-q+\lambda_t\eta_t)dt^{\gamma} + (\sigma+\lambda_t\zeta_t)dL_t^{\alpha,-1}\right].$$

Take the integral for the above equation and using method (6) we get

$$x_T = \frac{\gamma}{\Gamma(1+\gamma)}\int_t^T (t-\tau)^{\gamma-1}[(r-q+\lambda\eta)d\tau + dL_u^{\alpha,-1}].$$

So

$$S_t = S_t \exp\left[\frac{(T-t)^{\gamma}}{\Gamma(1+\gamma)}((r-q+\lambda_t\eta_t) + \int_t^T dL_u^{\alpha,-1})\right]. \tag{22}$$

By the same way and using method of (Demirci and Ozalp (2012)) the Equation (19) has a solution

$$\hat{\mathcal{U}}(\xi,t) = \exp\left[r - i\xi(r-q+\lambda_t\eta_t+\psi(-\xi))\frac{(T-t)^{\gamma}}{\Gamma(1+\gamma)}\right].$$

To prove Equation (17) satisfies the Equation (19), start with

$$\mathcal{U}(x,t) = e^{\left[\frac{-r(T-t)^{\gamma}}{\Gamma(1+\gamma)}\right]} E^{Q}(\Pi(x_T,T))$$

using inverse Fourier of $\Pi(x_T, T)$, thus

$$\mathcal{U}(x,t) = \frac{1}{2\pi}e^{\left[\frac{-r(T-t)^{\gamma}}{\Gamma(1+\gamma)}\right]}\int_{i\xi+R} E^{Q}(e^{(i\xi x_T)})\hat{\Pi}(\xi,T)d\xi$$

from solution (22) we get

$$\mathcal{U}(x,t) = \frac{1}{2\pi}e^{\left[\frac{-r(T-t)^{\gamma}}{\Gamma(1+\gamma)}\right]}\int_{i\xi+R} e^{\left[i\xi(r-q+\lambda_t\eta_t+\psi(-\xi))\frac{(T-t)^{\gamma}}{\Gamma(1+\gamma)}\right]}\hat{\Pi}(\xi,T)d\xi. \tag{23}$$

That is

$$\hat{\mathcal{U}}(\xi,t) = \exp\left[-r - i\xi(r-q+\lambda_t\eta_t+\psi(-\xi))\frac{(T-t)^{\gamma}}{\Gamma(1+\gamma)}\right]\hat{\Pi}(\xi,T)$$

is a solution of the Equation (19).

## 5. Conclusions

In this paper, we modified the particular Lêvy-time fractional diffusion equation, and applied to the price of fractional financial derivatives of European-style options. A more general class of model based on the fractional diffusion equation as Lêvy process was also presented in the form of fractional partial differential equation (FPDE) then the solution was obtained and applied to the European option pricing having risk-free parameters.

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## References

1. Merton, R. *Continuous-Time Finance*, 1st ed.; Blackwell: Oxford, UK, 1990.
2. Bertoin, J. *Lêvy Processes*; Cambridge University Press: Cambridge, UK, 1996.
3. Sato, K.I. *Lêvy Processes and Infinitely Divisible Distributions*; Cambridge University Press: Cambridge, UK, 1999.
4. Schoutens, W. *Lêvy Processes in Finance: Pricing Financial Derivativese*, 1st ed.; Wiley Series in Probability and Statistics; Wiley: Chichester, UK, 2003.
5. Shokrollahi, F.; Kılıçman, A.; Ibrahim, N.A. Greeks and Partial Differential Equations for some Pricing Currency Option Models. *Malaysian J. Math. Sci.* **2015**, *9*, 417–442.
6. Cont, R.; Voltchkova, E. Finite difference methods for option pricing in jump diffusion and exponential Lêvy models. *SIAM J. Numer. Anal.* **2005**, *43*, 1596–1626. [CrossRef]
7. Lewis, A.L. *A Simple Option Formula for General Jump-Diffusion and Other Exponential Lêvy Processes*; Working paper; Envision Financial Systems and Option City: Newport Beach, CA, USA, 2001.
8. Carr, P.; Madan, D. Option valuation using the fast Fourier transform. *J. Comput. Financ.* **1999**, *2*, 61–73. [CrossRef]
9. Cartea, A.; del-Castillo-Negrete, D. Fractional Diffusion Models of Option Prices in Markets with Jumps. *Phys. A Stat. Mech. Its Appl.* **2006**, *374*, 749–763. [CrossRef]
10. Chen, W.; Xu, X.; Zhu, S.P. Analytical pricing European-style option under the modified Black-Scholes equation with a partial-fractional derivative. *Q. Appl. Math.* **2014**, *72*, 597–611. [CrossRef]
11. Demirci, E.; Ozalp, N. A method for solving differential equations of fractional order. *J. Comput. Appl. Math.* **2012**, *236*, 2754–2762. [CrossRef]
12. Jumarie, G. Derivation and solutions of some farctional Black-Scholes equations in space and time. *J. Comput. Math. Appl.* **2010**, *59*, 1142–1164. [CrossRef]
13. Du, M. *Analytically Pricing European Options under the CGMY Model*; University of Wollongong Thesis Collection 1954–2016; University of Wollongong: Dubai, United Arab Emirates.
14. De Olivera, F.; Mordecki, E. Computing Greeks for Lêvy Models: The Fourier Transform Approach. *arXiv* **2014**, arXiv:1407.1343v1.