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Article

Janowski Type q-Convex and q- Close-to-Convex Functions Associated with q-Conic Domain

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Abstract: Certain new classes of q-convex and q-close to convex functions that involve the q-Janowski type functions have been defined by using the concepts of quantum (or q-) calculus as well as q-conic domain $\left(\Omega_{k,q}[\lambda,\alpha]\right)$. This study explores some important geometric properties such as coefficient estimates, sufficiency criteria and convolution properties of these classes. A distinction of new findings with those obtained in earlier investigations is also provided, where appropriate.

Keywords: analytic functions; Janowski functions; conic domain; q-convex functions; q-close-to-convex functions

1. Introduction

The mathematical study of q-calculus, particularly q-fractional calculus and q-integral calculus, q-transform analysis has been a topic of great interest for researchers due to its wide applications in different fields (see [1,2]). Some of the earlier work on the applications of the q-calculus was introduced by Jackson [3,4]. Later, q-analysis with geometrical interpretation was turned into identified through quantum groups. Due to the applications of q-analysis in mathematics and other fields, numerous researchers [3,5–14] did some significant work on q-calculus and studied its several other applications. Recently, Srivastava [15] in his survey-cum-expository article, explored the mathematical application of q-calculus, fractional q- calculus and fractional q-differential operators in geometric function theory. Keeping in view the significance of q-operators instead of ordinary operators and due to the wide range of applications of q-calculus, many researchers comprehensively studied q-calculus such as Srivastava et al. [16], Muhammad and Darus [17], Kanas and Reducanu [18] and Muhammad and Sokol [19]. Motivated by [15–21], we consider subfamilies of q-convex functions and q-close to convex functions with respect to Janowski functions connected with q-conic domain.

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, z \in U, \tag{1}$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C}, |z| < 1\}$. Let $\mathcal{A} \supseteq \mathcal{S}$, where $\mathcal{S}$ represents the set of all univalent functions in U. The classes of starlike (S^*) and convex (C) functions in U are the well known subclasses of S. Moreover, the class K of close to convex functions in U consists of normalized functions $f \in \mathcal{A}$ that satisfy the following conditions:

$$f \in \mathcal{A} \quad \text{and} \quad Re\left(\frac{zf^{'}(z)}{g(z)}\right) > 0, \quad \text{where} \quad g(z) \in S^{*}.$$

Now, for $\kappa \geq 0$, the classes κ-uniformly convex mappings ($\kappa - UCV$) and κ-starlike mappings ($\kappa - UST$), explored by Kanas and Wiśniowska, see [22–28]. Kanas and Wiśniowska [22,23] also initiated the study of analytic functions on conic domain Ω_κ, $\kappa \geq 0$ as:

$$\Omega_\kappa = \left\{u + iv : u > \kappa\sqrt{(u-1)^2 + v^2}\right\}.$$

See [22,23] for geometric interpretation of Ω_κ. These conic regions are images of the unit disk under the extremal functions $h_\kappa(z)$ given by:

$$h_\kappa(z) = \begin{cases} \frac{1+z}{1-z} & \kappa = 0, \\ 1 + \left(\log \frac{\sqrt{z}+1}{1-\sqrt{z}}\right)^2 \frac{2}{\pi^2} & \kappa = 1, \\ 1 + \sinh^2\left\{\arctan h\sqrt{z}\left(\frac{2}{\pi}\arccos\kappa\right)\right\}\frac{2}{1-\kappa^2} & 0 < \kappa < 1, \\ 1 + \frac{1}{\kappa^2-1}\sin\left(\frac{\pi}{2R(y)}\int_0^{\frac{u(z)}{\sqrt{y}}} \frac{dx}{\sqrt{1-x^2}\sqrt{1-y^2x^2}}\right) + \frac{1}{\kappa^2-1} & \kappa > 1, \end{cases} \tag{2}$$

where

$$u(z) = \frac{z - \sqrt{y}}{1 - \sqrt{yz}}, \quad z \in U.$$

Here, $\kappa = \cosh(\pi R'(y)/(4R(y))) \in (0,1)$, where $R(y)$ is Legendre's complete elliptic integral of first kind and $R'(y) = R(\sqrt{1-y^2})$ is its complementary integral, see [22,23,29–34]. If $h_\kappa(z) = 1 + \delta(\kappa)z + \delta_1(\kappa)z^2 + \cdots$ is taken from [23] for (2), then

$$\delta(\kappa) = \begin{cases} \frac{8(\arccos\kappa)^2}{\pi^2(1-\kappa^2)} & 0 \leq \kappa < 1, \\ \frac{8}{\pi^2} & \kappa = 1, \\ \frac{\pi^2}{4\sqrt{y}(\kappa^2-1)R^2(y)(1+y)} & \kappa > 1, \end{cases} \tag{3}$$

$$\delta_1(\kappa) = \delta_2(\kappa)\,\delta(\kappa),$$

where

$$\delta_2(\kappa) = \begin{cases} \frac{\mathcal{T}_1^2+2}{3} & 0 \leq \kappa < 1, \\ \frac{2}{3} & \kappa = 1, \\ \frac{4R^2(y)\left(y^2+6y+1\right)-\pi^2}{24R^2(y)(1+y)\sqrt{y}} & \kappa > 1, \end{cases} \tag{4}$$

where $\mathcal{T}_1 = \frac{2}{\pi}\arccos\kappa$, and $y \in (0,1)$.

Definition 1. *([35]) Let $p \in \mathcal{A}$ and $p(0) = 1$ be in the class $\mathcal{P}(\lambda, \alpha)$ if and only if*

$$p(z) \prec \frac{1+\lambda z}{1+\alpha z}, \quad (-1 \leq \alpha < \lambda \leq 1),$$

where $\prec$ stands for subordination.

Janowski [35] initiated the class $\mathcal{P}(\lambda,\alpha)$ by showing that $p \in \mathcal{P}(\lambda,\alpha)$ if and only if there exists a mapping $p \in \mathcal{P}$ such that

$$\frac{p(z)(\lambda+1)-(\lambda-1)}{p(z)(\alpha+1)-(\alpha-1)} \prec \frac{1+\lambda z}{1+\alpha z},$$

where $\mathcal{P}$ is class the of mappings with non-negative real parts.

Definition 2. *([36]) Let function $f \in \mathcal{A}$ be in the class $\mathcal{S}^*(\lambda,\alpha)$ if and only if*

$$\frac{zf'(z)}{f(z)} = \frac{p(z)(\lambda+1)-(\lambda-1)}{p(z)(\alpha+1)-(\alpha-1)}, \quad (-1 \le \alpha < \lambda \le 1).$$

Definition 3. *([36]) Let function $f \in \mathcal{A}$ is in the class $\mathcal{C}(\lambda,\alpha)$ if and only if*

$$\frac{\left(zf'(z)\right)'}{(f(z))'} = \frac{p(z)(\lambda+1)-(\lambda-1)}{p(z)(\alpha+1)-(\alpha-1)}, \quad (-1 \le \alpha < \lambda \le 1).$$

Definition 4. *([7]) Let function $f \in \mathcal{A}$, $n \in \mathbb{N}_0$ and $q \in (0,1)$, the q-difference (or $q-$derivative) operator D_q is defined as:*

$$D_q f(z) = -\frac{f(z)-f(qz)}{(q-1)z}.$$

Note that

$$D_q z^n = [n]_q z^{n-1}, \; D_q\left\{\sum_{n=1}^{\infty} a_n z^n\right\} = \sum_{n=1}^{\infty} [n]_q a_n z^{n-1},$$

where

$$[n]_q = \frac{1-q^n}{1-q}.$$

Definition 5. *([37]) Let function $f \in \mathcal{A}$ is in the class $\mathcal{S}_q^*(\lambda,\alpha)$ if and only if*

$$\frac{zD_q f(z)}{f(z)} = \frac{(\lambda+1)\widetilde{p}(z)-(\lambda-1)}{(\alpha+1)\widetilde{p}(z)-(\alpha-1)}, \quad (-1 \le \alpha < \lambda \le 1), \; q \in (0,1).$$

By principle of subordination we can be written as follows:

$$\frac{zD_q f(z)}{f(z)} \prec \frac{(\lambda+1)z+2+(\lambda-1)qz}{(\alpha+1)z+2+(\alpha-1)qz},$$

where

$$\widetilde{p}(z) = \frac{1+z}{1-qz}.$$

Definition 6. *([37]) Let function $f \in \mathcal{A}$ is in class $\mathcal{C}_q(\lambda,\alpha)$ if and only if*

$$\frac{D_q\left(zD_q f(z)\right)}{D_q f(z)} = \frac{\widetilde{p}(z)(\lambda+1)-(\lambda-1)}{\widetilde{p}(z)(\alpha+1)-(\alpha-1)}, \quad (-1 \le \alpha < \lambda \le 1), \quad q \in (0,1).$$

Similarly, by principle of subordination, we can be written as follows:

$$\frac{D_q\left(zD_q f(z)\right)}{D_q f(z)} \prec \frac{z(\lambda+1)+(\lambda-1)qz+2}{z(\alpha+1)+(\alpha-1)qz+2}.$$

Mahmood et al. [38] introduced the class $k-\mathcal{P}_q(\lambda,\alpha)$ as:

Definition 7. *([38]) A function $h \in k - \mathcal{P}_q(\lambda, \alpha)$, if and only if*

$$h(z) \prec \frac{(\lambda O_1 + O_3) h_k(z) - (\lambda O_1 - O_3)}{(\alpha O_1 + O_3) h_k(z) - (\alpha O_1 - O_3)}, \quad k \geq 0, \ q \in (0,1),$$

where

$$O_1 = 1 + q \ \text{ and } \ O_3 = 3 - q.$$

In addition, $h_k(z)$ is defined in Label (2). Geometrically, the mapping $h \in k - \mathcal{P}_q(\lambda, \alpha)$ takes all domain values $\Omega_{k,q}(\lambda, \alpha)$, $1 \leq \alpha < \lambda \leq 1$, $k \geq 0$, which is definable as:

$$\Omega_{k,q}(\lambda, \alpha) = \{r = u + iv : \Re(\Psi) > k|\Psi - 1|\},$$

where

$$\Psi = \frac{(\alpha O_1 - O_3) r(z) - (\lambda O_1 - O_3)}{(\alpha O_1 + O_3) r(z) - (\lambda O_1 + O_3)}.$$

This domain describes the conic type domain; for details, see [38].

Note that

(i) When $q \to 1$, then domain $\Omega_{\kappa,q}(\lambda, \alpha)$ reduces to the domain $\Omega_\kappa(\lambda, \alpha)$ (see [39]).
(ii) When $q \to 1$, then the class $\kappa - \mathcal{P}_q(\lambda, \alpha)$ reduces to the class $\kappa - \mathcal{P}(\lambda, \alpha)$ (see [39]).
(iii) When $q \to 1$, and $\kappa = 0$, then $\kappa - \mathcal{P}_q(\lambda, \alpha) = \mathcal{P}(\lambda, \alpha)$ also $\kappa - \mathcal{P}(1, -1) = \mathcal{P}(h_\kappa)$ (see ([35]).

Definition 8. *([38]) Let $f \in \mathcal{A}$ be in the class $k - \mathcal{ST}_q(\beta, \gamma)$, if and only if*

$$\Re\left(\frac{(\gamma O_1 - O_3)\frac{zD_q f(z)}{f(z)} - (\beta O_1 - O_3)}{(\gamma O_1 + O_3)\frac{zD_q f(z)}{f(z)} - (\beta O_1 + O_3)}\right)$$
$$> k\left|\frac{(\gamma O_1 - O_3)\frac{zD_q f(z)}{f(z)} - (\beta O_1 - O_3)}{(\gamma O_1 + O_3)\frac{zD_q f(z)}{f(z)} - (\beta O_1 + O_3)} - 1\right|,$$

or, equivalently,

$$\frac{zD_q f(z)}{f(z)} \in k - \mathcal{P}_q(\beta, \gamma),$$

where $k \geq 0$, $-1 \leq \gamma < \beta \leq 1$.

We can see that, when $q \to 1$, then $\kappa - \mathcal{ST}_q(\beta, \gamma)$ diminishes to the renowned class which is stated in [39].

Motivated by the definition above, we introduced new classes $\kappa - \mathcal{UCV}_q(\beta, \gamma)$, $\kappa - \mathcal{UK}_q(\lambda, \alpha, \beta, \gamma)$ and $\kappa - \mathcal{UQ}_q(\lambda, \alpha, \beta, \gamma)$ of analytic functions.

Definition 9. *Let $f \in \mathcal{A}$, be in the class $k - \mathcal{UCV}_q(\beta, \gamma)$ if and only if*

$$\Re\left(\frac{(\gamma O_1 - O_3)\frac{D_q(zD_q f(z))}{D_q f(z)} - (\beta O_1 - (O_3)}{(\gamma O_1 + O_3)\frac{D_q(zD_q f(z))}{D_q f(z)} - (\beta O_1 + O_3)}\right)$$
$$> k\left|\frac{(\gamma O_1 - O_3)\frac{D_q(zD_q f(z))}{D_q f(z)} - (\beta O_1 - O_3)}{(\gamma O_1 + O_3)\frac{D_q(zD_q f(z))}{D_q f(z)} - (\beta O_1 + O_3)} - 1\right|,$$

or, equivalently,

$$\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)} \in k-\mathcal{P}_q(\beta,\gamma),$$

where $k \geq 0$, $-1 \leq \gamma < \beta \leq 1$.

One can clearly see that

$$f \in \kappa - \mathcal{UCV}_q(\beta,\gamma) \Leftrightarrow zD_q(z) \in \kappa - \mathcal{ST}_q(\beta,\gamma). \tag{5}$$

Note that, when $q \rightarrow 1$, then the class $\kappa - \mathcal{UCV}_q(\beta,\gamma)$ reduces to a well-known class defined in [39].

Definition 10. *Let $f \in \mathcal{A}$, be in the class $k-\mathcal{UK}_q(\lambda,\alpha,\beta,\gamma)$ if and only if there exists $g \in k-\mathcal{ST}_q(\beta,\gamma)$, such that*

$$\Re\left(\frac{(\alpha O_1 - O_3)\frac{zD_qf(z)}{g(z)} - (\lambda O_1 - O_3)}{(\alpha O_1 + O_3)\frac{zD_qf(z)}{g(z)} - (\lambda O_1 + O_3)}\right)$$

$$> k\left|\frac{(\alpha O_1 - O_3)\frac{zD_qf(z)}{g(z)} - (\lambda O_1 - O_3)}{(\alpha O_1 + O_3)\frac{zD_qf(z)}{g(z)} - (\lambda O_1 + O_3)} - 1\right|.$$

We can write equivalently

$$\frac{zD_qf(z)}{g(z)} \in k-\mathcal{P}_q(\lambda,\alpha),$$

where $k \geq 0$, $-1 \leq \gamma < \beta \leq 1$, $-1 \leq \alpha < \lambda \leq 1$.

Note that, when $q \rightarrow 1$, then, the class $k-\mathcal{UK}_q(\lambda,\alpha,\beta,\gamma)$ reduces into the well-known class that is defined in (see [40]).

Definition 11. *Let $f \in \mathcal{A}$, belong to the class $k-\mathcal{UQ}_q(\lambda,\alpha,\beta,\gamma)$ if and only if there exist $g \in k-\mathcal{CV}_q(\beta,\gamma)$, such that*

$$\Re\left(\frac{(\alpha O_1 - O_3)\frac{D_q\left(zD_qf(z)\right)}{D_qg(z)} - (\lambda O_1 - O_3)}{(\alpha O_1 + O_3)\frac{D_q\left(zD_qf(z)\right)}{D_qg(z)} - (\lambda O_1 + O_3)}\right)$$

$$> k\left|\frac{(\alpha O_1 - O_3)\frac{D_q\left(zD_qf(z)\right)}{D_qg(z)} - (\lambda O_1 - O_3)}{(\alpha O_1 + O_3)\frac{D_q\left(zD_qf(z)\right)}{D_qg(z)} - (\lambda O_1 + O_3)} - 1\right|,$$

or, equivalently,

$$\frac{D_q\left(zD_qf(z)\right)}{D_qg(z)} \in k-\mathcal{P}_q(\lambda,\alpha),$$

where, for $k \geq 0$, $-1 \leq \gamma < \beta \leq 1$, $-1 \leq \alpha < \lambda \leq 1$.

It is simple to verify this

$$f \in \kappa - \mathcal{UQ}_q(\lambda,\alpha,\beta,\gamma) \Leftrightarrow zD_qf \in \kappa - \mathcal{UK}_q(\lambda,\alpha,\beta,\gamma). \tag{6}$$

A special case arises when $q \to 1$, then the class $\kappa - \mathcal{UQ}_q(\lambda, \alpha, \beta, \gamma)$ reduces to a well known class defined in [40].

2. Set of Lemmas

Lemma 1. *([41]) Suppose* $1 + \sum_{n=1}^{\infty} c_n z^n = d(z) \prec H(z) = 1 + \sum_{n=1}^{\infty} C_n z^n$. *If* $H(U)$ *is convex and* $H(z) \in \mathcal{A}$, *then*

$$|c_n| \leq |C_1|, \quad n \geq 1.$$

Lemma 2. *([38]) Suppose* $d(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in k - \mathcal{P}_q(\lambda, \alpha)$, *then*

$$|c_n| \leq |\delta\,(k, \lambda, \alpha)| = \frac{O_1(\lambda - \alpha)}{4}\delta(k),$$

where $\delta\,(k)$ *is given by (3).*

Lemma 3. *([38]) Suppose* $d \in k - \mathcal{ST}_q(\beta, \gamma)$, $k \geq 0$ *is given by*

$$d(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in U,$$

then

$$|b_n| \leq \prod_{m=0}^{n-2} \left(\frac{\left|\delta(k) O_1(\beta - \gamma) - 4q\,[m]_q\,\gamma\right|}{4q\,[m+1]_q} \right),$$

where $\delta(k)$ *is given by (3).*

Lemma 4. *([42]) Suppose* $d \in \mathcal{S}^*$, $f \in \mathcal{C}$ *and* $G \in \mathcal{S}$, *then we have*

$$\frac{f(z) * d(z)G(z)}{f(z) * d(z)} \in \overline{co}(G(U)), \quad z \in U.$$

Here, "" means convolution and* $\overline{co}(G(U)$ *means the closed convex hull* $G(U)$.

Lemma 5. *([38]) The function* $f \in \mathcal{A}$ *will belong to the class* $k - \mathcal{ST}_q(\beta, \gamma)$, *if the following inequality holds:*

$$\sum_{n=2}^{\infty} \left\{ 2O_3(1+k)q\,[n-1]_q + \left| (\gamma O_1 + O_3)\,[n]_q - (\beta O_1 + (O_3) \right| \right\} |a_n| \leq O_1\,|\gamma - \beta|.$$

Throughout this paper, we assume that $k \geq 0$, $-1 \leq \gamma < \beta \leq 1$, $-1 \leq \alpha < \lambda \leq 1$, *and* $q \in (0,1)$, *unless otherwise specified.*

3. Main Results

Theorem 1. *Let* $f \in \mathcal{A}$; *then,* f *is in the class* $k - \mathcal{UCV}_q(\beta, \gamma)$, *if the following inequality holds:*

$$\sum_{n=2}^{\infty} [n]_q \left\{ 2O_3(k+1)q\,[n-1]_q + \left| (\gamma O_1 + O_3)\,[n]_q - (\beta O_1 + O_3) \right| \right\} |a_n| \leq O_1\,|\gamma - \beta|.$$

Proof. By Lemma 5 and relation (5), the proof is straightforward. □

For $q \to 1^-$, in Theorem 1, then we obtained following corollary, proved by Malik and Noor [39].

Corollary 1. *Let $f \in \mathcal{A}$; then, f belongs to $k - \mathcal{UCV}(\beta,\gamma)$, if the following inequality holds*

$$\sum_{n=2}^{\infty} n \left\{2(k+1)(n-1) + \left|n(\gamma+1) - (\beta+1)\right|\right\} |a_n| \leq |\gamma - \beta| .$$

Theorem 2. *Let $f \in \mathcal{A}$, then f is in the class $k - \mathcal{UK}_q(\lambda,\alpha,\beta,\gamma)$, if the condition (7) holds*

$$\sum_{n=2}^{\infty} \left\{2O_3(k+1)\left|b_n - [n]_q a_n\right| + \left|(\alpha O_1 + O_3)[n]_q a_n - (\lambda O_1 + O_3) b_n\right|\right\}$$

$$\leq O_1 |\alpha - \lambda| . \tag{7}$$

Proof. Presuming that (7) holds, then it is enough to show that

$$k\left|\frac{(\alpha O_1 - O_3)\frac{zD_q f(z)}{g(z)} - (\lambda O_1 - O_3)}{(\alpha O_1 + O_3)\frac{zD_q f(z)}{g(z)} - (\lambda O_1 + O_3)} - 1\right|$$

$$- \operatorname{Re}\left\{\frac{(\alpha O_1 - O_3)\frac{zD_q f(z)}{g(z)} - (\lambda O_1 - O_3)}{(\alpha O_1 + O_3)\frac{zD_q f(z)}{g(z)} - (\lambda O_1 + O_3)} - 1\right\}$$

$$< 1.$$

We have

$$k\left|\frac{(\alpha O_1 - O_3)\frac{zD_q f(z)}{g(z)} - (\lambda O_1 - O_3)}{(\alpha O_1 + O_3)\frac{zD_q f(z)}{g(z)} - (\lambda O_1 + O_3)} - 1\right|$$

$$- \operatorname{Re}\left\{\frac{(\alpha O_1 - O_3)\frac{zD_q f(z)}{g(z)} - (\lambda O_1 - O_3)}{(\alpha O_1 + O_3)\frac{zD_q f(z)}{g(z)} - (\lambda O_1 + O_3)} - 1\right\},$$

$$\leq (k+1)\left|\frac{(\alpha O_1 - O_3)\frac{zD_q f(z)}{g(z)} - (\lambda O_1 - O_3)}{(\alpha O_1 + O_3)\frac{zD_q f(z)}{g(z)} - (\lambda O_1 + O_3)} - 1\right|, \tag{8}$$

$$= 2O_3(k+1)\left|\frac{g(z) - zD_q f(z)}{(\alpha O_1 + O_3)\, zD_q f(z) - (\lambda O_1 + O_3)\, g(z)}\right|,$$

$$= 2O_3(k+1)\left|\frac{\sum_{n=2}^{\infty}\left\{b_n - [n]_q a_n\right\} z^n}{O_1(\alpha - \lambda) z + \sum_{n=2}^{\infty}\left\{(\alpha O_1 + O_3)[n]_q a_n - (\lambda O_1 + O_3) b_n\right\} z^n}\right|,$$

$$\leq \frac{2O_3(k+1)\sum_{n=2}^{\infty}\left\{\left|b_n - [n]_q a_n\right|\right\}}{O_1|\alpha - \lambda| - \sum_{n=2}^{\infty}\left|(\alpha O_1 + O_3)[n]_q a_n - (\lambda O_1 + O_3) b_n\right|}.$$

The expression (8) is bounded above by 1 if

$$\sum_{n=2}^{\infty}\left[2O_3(k+1)\left|b_n-[n]_q\,a_n\right|+\left|(\alpha O_1+O_3)\,[n]_q\,a_n-(\lambda O_1+O_3)b_n\right|\right]$$
$$\leq (O_1)\,|\alpha-\lambda|\,.$$

□

Corollary 2. *([40]) Let $f\in\mathcal{A}$. Then, f is in the class $k-\mathcal{UK}_{q\to 1}(\lambda,\alpha,\beta,\gamma)=k-\mathcal{UK}(\lambda,\alpha,\beta,\gamma)$, if the following condition holds:*

$$\sum_{n=2}^{\infty}\left\{2(k+1)\,|b_n-na_n|+|(\alpha+1)na_n-(\lambda+1)b_n|\right\}\leq|\alpha-\lambda|\,.$$

Here, $q\to 1$ represents the limiting value of q as it approaches 1.

Theorem 3. *Let $f\in\mathcal{A}$. Then, f is in the class $k-\mathcal{UQ}_q(\lambda,\alpha,\beta,\gamma)$, if the following condition holds:*

$$\sum_{n=2}^{\infty}[n]_q\left[2O_3(k+1)\left|b_n-[n]_q\,a_n\right|+\left|(\alpha O_1+O_3)\,[n]_q\,a_n-(\lambda O_1+O_3)b_n\right|\right]$$
$$\leq O_1\,|\alpha-\lambda|\,.$$

Proof. By Theorem 2 and relation (6), the proof is straightforward. □

Corollary 3. *([40]) Let $f\in\mathcal{A}$. Then, f is in the class $k-\mathcal{UK}_{q\to 1}(\lambda,\alpha,\beta,\gamma)=k-\mathcal{UQ}(\lambda,\alpha,\beta,\gamma)$, if*

$$\sum_{n=2}^{\infty}n\left\{2(k+1)\,|b_n-na_n|+|(\alpha+1)na_n-(\lambda+1)b_n|\right\}\leq|\alpha-\lambda|\,.$$

Corollary 4. *([43]) Let $f\in\mathcal{A}$. Then, f is in the class $1-\mathcal{UK}_{q\to 1}(1-2\tau,-1,1,-1)=\mathcal{UK}(\tau)$ if*

$$\sum_{n=2}^{\infty}n^2\,|a_n|\leq\frac{1-\tau}{2}.$$

Theorem 4. *Let $f\in k-\mathcal{UCV}_q(\beta,\gamma)$, is of the form (1). Then,*

$$|a_n|\leq\frac{1}{[n]_q}\prod_{m=0}^{n-2}\left(\frac{\left|\delta(k)O_1(\beta-\gamma)-4q\,[m]_q\,\gamma\right|}{4q\,[m+1]_q}\right),$$

where $\delta(k)$ is given by (3).

Proof. By Lemma 3 and relation (5), the proof is straightforward. □

For $q\to 1^-$, Theorem 4 brings to the following corollary, proved by Noor [39].

Corollary 5. *Let $f\in k-\mathcal{UCV}(\beta,\gamma)$. Then,*

$$|a_n|\leq\frac{1}{n}\prod_{m=0}^{n-2}\left(\frac{|\delta(k)(\beta-\gamma)-2m\gamma|}{2\,(m+1)}\right),$$

where $\delta(k)$ is given by (3).

Theorem 5. *If $f \in k - \mathcal{UK}_q(\lambda, \alpha, \beta, \gamma)$ and $g \in k - \mathcal{ST}_q(\beta, \gamma)$, then,*

$$|a_n| \leq \begin{cases} \frac{1}{[n]_q} \prod_{m=0}^{n-2} \frac{\left|\delta(k)O_1(\beta-\gamma)-4q[m]_q\gamma\right|}{4q[m+1]_q} \\ +\frac{\delta(k)O_1(\lambda-\alpha)}{4[n]_q} \sum_{j=1}^{n-1} \prod_{m=0}^{j-2} \frac{\left|\delta(k)O_1(\beta-\gamma)-4q[j]_q\gamma\right|}{4q[j+1]_q}, \quad n \geq 2, \end{cases}$$

where $\delta(k)$ is given in (3).

Proof. Let us take

$$\frac{zD_q f(z)}{g(z)} = h(z), \tag{9}$$

where

$$h \in k - \mathcal{P}_q(\lambda, \alpha) \text{ and } g \in k - \mathcal{ST}_q(\beta, \gamma).$$

Now, from (9), we have

$$zD_q f(z) = g(z)h(z),$$

which implies that

$$z + \sum_{n=2}^{\infty} [n]_q \, a_n z^n = \left(1 + \sum_{n=1}^{\infty} c_n z^n\right) \left(z + \sum_{n=2}^{\infty} b_n z^n\right).$$

By equating z^n coefficients

$$[n]_q \, a_n = b_n + \sum_{j=1}^{n-1} b_j c_{n-j}, \quad a = 1, \; b_1 = 1.$$

This implies that

$$[n]_q \, |a_n| \leq |b_n| + \sum_{j=1}^{n-1} \left|b_j\right| \left|c_{n-j}\right|. \tag{10}$$

Since $h \in k - \mathcal{P}_q(\lambda, \alpha)$, therefore, by using Lemma 2 on (10), we have

$$[n]_q \, |a_n| \leq |b_n| + \frac{\delta(k)O_1(\lambda-\alpha)}{4} \sum_{j=1}^{n-1} \left|b_j\right|. \tag{11}$$

Again $g \in k - \mathcal{ST}_q(\beta, \gamma)$, therefore, by using Lemma 3 on (11), we have

$$|a_n| \leq \begin{cases} \frac{1}{[n]_q} \prod_{m=0}^{n-2} \left(\frac{\left|\delta(k)O_1(\beta-\gamma)-4q[m]_q\gamma\right|}{4q[m+1]_q} \right) \\ +\frac{\delta(k)O_1(\lambda-\alpha)}{4[n]_q} \sum_{j=1}^{n-1} \prod_{m=0}^{j-2} \left(\frac{\left|\delta(k)O_1(\beta-\gamma)-4q[m]_q\gamma\right|}{4q[m+1]_q} \right). \end{cases}$$

□

Corollary 6. *([40]) If $f \in k - \mathcal{UK}_{q\to 1}(\lambda, \alpha, \beta, \gamma) = k - \mathcal{UK}(\lambda, \alpha, \beta, \gamma)$, then*

$$|a_n| \leq \begin{cases} \frac{1}{n} \prod_{m=0}^{n-2} \left(\frac{|\delta(k)(\beta-\gamma)-2m\gamma|}{2(m+1)} \right) \\ +\frac{\delta(k)(\lambda-\alpha)}{2n} \sum_{j=1}^{n-1} \prod_{m=0}^{j-2} \left(\frac{|\delta(k)(\beta-\gamma)-2m\gamma|}{2(m+1)} \right), \; n \geq 2, \end{cases}$$

where $\delta(k)$ is defined by (3).

Corollary 7. *([26]) If $f \in k-\mathcal{UK}_{q\to 1}(1,-1,1,-1) = k-\mathcal{UK}$, then*

$$|a_n| \leq \frac{(\delta(k))_{n-1}}{n!} + \frac{\delta(k)}{n}\sum_{j=0}^{n-1}\frac{(\delta(k))_{j-1}}{(j-1)!}, \quad n \geq 2.$$

Corollary 8. *([44]) If $f \in 0-\mathcal{UK}_{q\to 1}(1,-1,1,-1) = \mathcal{K}$, then*

$$|a_n| \leq n, \quad n \geq 2.$$

Theorem 6. *If $f \in k-\mathcal{UQ}_q(\lambda,\alpha,\beta,\gamma)$, then*

$$|a_n| \leq \begin{cases} \frac{1}{\left([n]_q\right)^2}\prod_{m=0}^{n-2}\frac{\left|\delta(k)O_1(\beta-\gamma)-4q[m]_q\gamma\right|}{4q[m+1]_q} \\ +\frac{\delta(k)O_1(\lambda-\alpha)}{4\left([n]_q\right)^2}\sum_{j=1}^{n-1}\prod_{m=0}^{j-2}\frac{\left|\delta(k)O_1(\beta-\gamma)-4q[j]_q\gamma\right|}{4q[j+1]_q}, \quad n \geq 2, \end{cases}$$

where $\delta(k)$ is defined by (3).

Proof. By Theorem 5 and relation (6), the proof is straightforward. □

Corollary 9. *([40]) If $f \in k-\mathcal{UQ}_{q\to 1}(\lambda,\alpha,\beta,\gamma) = \mathcal{UQ}(\lambda,\alpha,\beta,\gamma)$ and is of the form (1), then*

$$|a_n| \leq \begin{cases} \frac{1}{n^2}\prod_{m=0}^{n-2}\left(\frac{|\delta(k)(\beta-\gamma)-2m\gamma|}{2(m+1)}\right) \\ +\frac{\delta(k)(\lambda-\alpha)}{2n^2}\sum_{j=1}^{n-1}\prod_{m=0}^{j-2}\left(\frac{|\delta(k)(\beta-\gamma)-2m\gamma|}{2(m+1)}\right), \quad n \geq 2. \end{cases}$$

Theorem 7. *If $f \in k-\mathcal{P}_q(\beta,\gamma)$ and $\chi \in \mathcal{C}$, then $f * \chi \in k-\mathcal{P}_q(\beta,\gamma)$.*

Proof. Here, we prove that

$$\frac{zD_q\left(\chi(z) * f(z)\right)}{\left(\chi(z) * f(z)\right)} \in k-\mathcal{P}_q(\beta,\gamma).$$

Consider

$$\begin{aligned} \frac{zD_q\left(\chi(z) * f(z)\right)}{\left(\chi(z) * f(z)\right)} &= \frac{\chi(z) * f(z)\left(\frac{zD_qf(z)}{f(z)}\right)}{\chi(z) * f(z)}, \\ &= \frac{\chi(z) * f(z)\Psi(z)}{\chi(z) * f(z)}, \end{aligned}$$

where $\frac{zD_qf(z)}{f(z)} = \Psi(z) \in \mathcal{P}_q(\beta,\gamma)$. By using Lemma 4, we obtain the required result. □

Theorem 8. *If $f \in k-\mathcal{UK}_q(\lambda,\alpha,\beta,\gamma)$ and $\chi \in \mathcal{C}$, then $f * \chi \in k-\mathcal{UK}_q(\lambda,\alpha,\beta,\gamma)$.*

Proof. Since $f \in k-\mathcal{UK}_q(\lambda,\alpha,\beta,\gamma)$, there exist $g \in k-\mathcal{ST}_q(\beta,\gamma)$, such that $\frac{zD_qf(z)}{g(z)} \in k-\mathcal{P}_q(\lambda,\alpha)$.
It follows from Lemma 4 that $\chi * g \in k-\mathcal{ST}_q(\beta,\gamma)$.

Consider

$$\frac{zD_q\left(\chi(z)*f(z)\right)}{\left(\chi(z)*g(z)\right)}=\frac{\chi(z)*\left(zD_qf(z)\right)}{\left(\chi(z)*g(z)\right)},$$

$$=\frac{\chi(z)*\left(\frac{zD_qf(z)}{g(z)}\right)g(z)}{\chi(z)*f(z)},$$

$$=\frac{\chi(z)*F(z)g(z)}{\chi(z)*g(z)},$$

where $F\in k-\mathcal{ST}_q(\lambda,\alpha)$. By using Lemma 4, we obtain the required result. □

4. Conclusions

In this paper, we use Quantum Calculus to define new subclasses $k-\mathcal{CV}_q(\beta,\gamma)$, $k-\mathcal{UK}_q(\lambda,\alpha,\beta,\gamma)$ and $k-\mathcal{UQ}_{q\to 1}(\lambda,\alpha,\beta,\gamma)$ of analytic functions involving conic domain and associated with Janowski type function. We then investigate many geometric properties and characteristics of each of these families such as coefficient inequalities, sufficient condition, necessary condition, and convolution properties. For verification and validity of our main results, we have also pointed out relevant connections of our main results with those in several earlier related works on this subject.

For further investigation, we can make connections between the q-analysis and (p,q)-analysis, and the results for q-analogues which we have included in this article for $0<q<1$ can be possibly be translated into the relevant findings for the (p,q)-analogues with $(0<q<p\le 1)$ by adding some parameter.

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