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Article

Argument and Coefficient Estimates for Certain Analytic Functions

Davood Alimohammadi [1], Nak Eun Cho [2,*], Ebrahim Analouei Adegani [3] and Ahmad Motamednezhad [3]

[1] Department of Mathematics, Faculty of Science, Arak University, Arak 38156-8-8349, Iran; d-alimohammadi@araku.ac.ir
[2] Department of Applied Mathematics, College of Natural Sciences, Pukyong National University, Busan 608-737, Korea
[3] Faculty of Mathematical Sciences, Shahrood University of Technology, P.O. Box 316-36155 Shahrood, Iran; analoey.ebrahim@gmail.com (E.A.A.); a.motamedne@gmail.com (A.M.)
* Correspondence: necho@pknu.ac.kr

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Abstract: The aim of the present paper is to introduce a new class $\mathcal{G}(\alpha, \delta)$ of analytic functions in the open unit disk and to study some properties associated with strong starlikeness and close-to-convexity for the class $\mathcal{G}(\alpha, \delta)$. We also consider sharp bounds of logarithmic coefficients and Fekete-Szegö functionals belonging to the class $\mathcal{G}(\alpha, \delta)$. Moreover, we provide some topics related to the results reported here that are relevant to outcomes presented in earlier research.

Keywords: starlike function; subordinate; univalent function

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1. Introduction and Preliminaries

Let $\mathbb{U}$ denote the open unit dick in the complex plane $\mathbb{C}$. A function $\omega : \mathbb{U} \to \mathbb{C}$ is called a *Schwarz function* if ω is a analytic function in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$. Clearly, a Schwarz function ω is the form

$$\omega(z) = w_1 z + w_2 z^2 + \cdots .$$

We denote by Ω the set of all Schwarz functions on $\mathbb{U}$.

Let $\mathcal{A}$ be consisting of all analytic functions of the following normalized form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

in the open unit disk $\mathbb{U}$. An analytic function f is said to be *univalent* in a domain if it provides a one-to-one mapping onto its image: $f(z_1) = f(z_2) \Rightarrow z_1 = z_2$. Geometrically, this means that different points in the domain will be mapped into different points on the image domain. Also, let $\mathcal{S}$ be the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$. A domain D in the complex plane $\mathbb{C}$ is called *starlike* with respect to a point $w_0 \in D$, if the line segment joining w_0 to every other point $w \in D$ lies in the interior of D. In other words, for any $w \in D$ and $0 \le t \le 1$, $tw_0 + (1-t)w \in D$. A function $f \in \mathcal{A}$ is starlike if the image $f(D)$ is starlike with respect to the origin.

For two analytic functions f and F in $\mathbb{U}$, we say that the function f *is subordinate to* the function F in $\mathbb{U}$ and we write $f(z) \prec F(z)$, if there exists a Schwarz function ω such that $f(z) = F\big(\omega(z)\big)$ for all $z \in \mathbb{U}$. Specifically, if the function F is univalent in $\mathbb{U}$, then we have the next equivalence:

$$f(z) \prec F(z) \iff f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

The logarithmic coefficients γ_n of $f \in \mathcal{S}$ are defined with the following series expansion:

$$\log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty} \gamma_n(f) z^n, \; z \in \mathbb{U}. \tag{2}$$

These coefficients are an important factor in studying diverse estimates in the theory of univalent functions. Note that we use γ_n instead of $\gamma_n(f)$. The concept of logarithmic coefficients inspired Kayumov [1] to solve Brennan's conjecture for conformal mappings. The importance of the logarithmic coefficients follows from Lebedev-Milin inequalities [2] (Chapter 2), see also [3,4], where estimates of the logarithmic coefficients were used to find bounds on the coefficients of f. Milin [2] conjectured the inequality

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\gamma_k|^2 - \frac{1}{k}\right) \le 0 \quad (n = 1, 2, 3, \cdots),$$

which implies Robertson's conjecture [5], and hence, Bieberbach's conjecture [6]. This is the famous coefficient problem in univalent function theory. L. de Branges [7] established Bieberbach's conjecture by proving Milin's conjecture.

Definition 1. *Let $q, n \in \mathbb{N}$. The q^{th} Hankel determinant is denote by $H_q(n)$ and defined by*

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \tag{3}$$

where a_k $(k = 1, 2, \ldots)$ are the coefficients of the Taylor series expansion of a function f of the form (1). *Note that $a_1 = 1$.*

The Hankel determinant $H_q(n)$ was defined by Pommerenke [8,9] and for fixed q, n the bounds of $|H_q(n)|$ have been studied for several subfamilies of univalent functions. Different properties of these determinants can be observed in [10] (Chapter 4). The Hankel determinants $H_2(1) = a_3 - a_2^2$ and $H_2(2) = a_2a_4 - a_3^2$, are well-known as *Fekete-Szegö* and *second Hankel determinant functionals*, respectively. In addition, Fekete and Szegö [11] introduced the generalized functional $a_3 - \lambda a_2^2$, where λ is a real number. Recently, Hankel determinants and other problems for various classes of bi-univalent functions have been studied, see [12–16].

For $\alpha \in [0, 1)$, we denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ including of all $f \in \mathcal{A}$ for which f is a *starlike function of order* α in $\mathbb{U}$, with

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha \quad (z \in \mathbb{U}).$$

Also, for $\alpha \in (0, 1]$, we denote by $\widetilde{\mathcal{S}}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of all $f \in \mathcal{A}$ for which f is a *strongly starlike function of order* α in $\mathbb{U}$, with

$$\left|\operatorname{Arg}\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{U}).$$

Note that $\widetilde{\mathcal{S}}^*(1) = \mathcal{S}^*(0) = \mathcal{S}^*$, the class of *starlike functions* in $\mathbb{U}$.

For $\alpha \in (0,1]$, we denote by $\widetilde{\mathcal{C}}(\alpha)$ the subclass of $\mathcal{A}$ including all of $f \in \mathcal{A}$ for which

$$\left|\mathrm{Arg}\left(f'(z)\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U}).$$

Note that $\widetilde{\mathcal{C}}(1) = \mathcal{C}$, the subclass of *close-to-convex functions* in $\mathbb{U}$. Here we understand that $\mathrm{Arg}\, w$ is a number in $(-\pi, \pi]$.

For $\alpha \in (0,1]$, Nunokawa and Saitoh in [17] defined the more general class $\mathcal{G}(\alpha)$ consisting of all $f \in \mathcal{A}$ satisfying

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{\alpha}{2} \qquad (z \in \mathbb{U}).$$

They proved that $\mathcal{G}(\alpha)$ is a subclass of $\mathcal{S}^*$. Ozaki in [18] showed that every function $\mathcal{G}(1)$ is univalent in the unit disk $\mathbb{U}$. In the following, Umezawa [19], Sakaguchi [20] and Singh and Singh [21] obtained some geometric properties of $\mathcal{G}(1)$ including, convex in one direction, close-to-convex and starlike, respectively. Obradović et al. in [22] proved the sharp coefficient bounds for the moduli of the Taylor coefficients a_n of $f \in \mathcal{G}(\alpha)$ and determined the sharp bound for the Fekete-Szegö functional for functions in $\mathcal{G}(\alpha)$ with complex parameter λ. Also, Ponnusamy et al. [22,23] studied bounds for the logarithmic coefficients for functions in $\mathcal{G}(\alpha)$.

Here, we introduce a class as follows:

Definition 2. *For $\alpha, \delta \in (0,1]$, we define the subclass $\mathcal{G}(\alpha,\delta)$ of $\mathcal{A}$ as the following:*

$$\mathcal{G}(\alpha,\delta) := \left\{ f \in \mathcal{A} : \left| \mathrm{Arg}\left(\frac{2+\alpha}{\alpha} - \frac{2}{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right)\right| < \frac{\delta\pi}{2} \quad (z \in \mathbb{U}) \right\}.$$

It is clear that $\mathcal{G}(\alpha,1) = \mathcal{G}(\alpha)$ for $\alpha \in (0,1]$. Let $\alpha, \delta \in (0,1]$, identity function on $\mathbb{U}$ belongs to $\mathcal{G}(\alpha,\delta)$ which implies that $\mathcal{G}(\alpha,\delta) \neq \emptyset$. By means of the principle of subordination between analytic functions, we deduce

$$\mathcal{G}(\alpha,\delta) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec -\frac{\alpha}{2}\left(\frac{1+z}{1-z}\right)^{\delta} + \frac{2+\alpha}{2} := \phi(z) \quad (z \in \mathbb{U}) \right\}. \tag{4}$$

Since the function f defined by

$$f(z) = \int_0^z \exp\left(\int_0^x \frac{-\frac{\alpha}{2}\left(\frac{1+t}{1-t}\right)^{\delta} + \frac{\alpha}{2}}{t}\, \mathrm{d}t \right) \mathrm{d}x \qquad (z \in \mathbb{U})$$

satisfies

$$1 + \frac{zf''(z)}{f'(z)} = \phi(z) \prec \phi(z),$$

we deduce $f \in \mathcal{G}(\alpha,\delta)$.

The aim of the present paper is to study some geometric properties for the class $\mathcal{G}(\alpha,\delta)$ such as strongly starlikeness and close-to-convexity. Also we investigate sharp bounds on logarithmic coefficients and Fekete-Szegö functionals for functions belonging to the class $\mathcal{G}(\alpha,\delta)$, which incorporate some known results as the special cases.

2. Some Properties of the Class $\mathcal{G}(\alpha, \delta)$

We denote by Q the class of all complex-valued functions q for which q is univalent at each $\overline{\mathbb{U}} \setminus \mathrm{E}(q)$ and $q'(\xi) \neq 0$ for all $\xi \in \partial\mathbb{U} \setminus \mathrm{E}(q)$ where

$$\mathrm{E}(q) = \left\{ \xi \in \partial\mathbb{U} : \lim_{z \to \xi} q(z) = \infty \right\}.$$

The following lemmas will be required to establish our main results.

Lemma 1 ([24] (Lemma 2.2d (i))). *Let $q \in Q$ with $q(0) = a$ and let $p(z) = a + p_n z^n + \ldots$ be analytic in $\mathbb{U}$ with $p(z) \not\equiv 1$ and $n \geq 1$. If p is not subordinate to q in $\mathbb{U}$ then there exist $z_0 \in \mathbb{U}$ and $\xi_0 \in \partial\mathbb{U} \setminus E(q)$ such that $\{p(z) : z \in \mathbb{U},\ |z| < |z_0|\} \subset q(\mathbb{U})$,*

$$p(z_0) = q(\xi_0).$$

Lemma 2. (see [25,26]) *Let the function p given by*

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p(z) \neq 0$ for all $z \in \mathbb{U}$. If there exists a point $z_0 \in \mathbb{U}$ with

$$|\arg(p(z))| < \frac{\beta\pi}{2} \qquad (|z| < |z_0|)$$

and

$$|\arg(p(z_0))| = \frac{\beta\pi}{2},$$

for some $\beta > 0$, then

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta \qquad (i = \sqrt{-1}),$$

where

$$k \geq \frac{a + a^{-1}}{2} \geq 1 \quad \text{when} \quad \arg(p(z_0)) = \frac{\beta\pi}{2} \tag{5}$$

and

$$k \leq -\frac{a + a^{-1}}{2} \leq -1 \quad \text{when} \quad \arg(p(z_0)) = -\frac{\beta\pi}{2}, \tag{6}$$

where

$$[p(z_0)]^{1/\beta} = \pm ia \quad \text{and} \quad a > 0.$$

Theorem 1. *Let $\alpha, \beta \in (0, 1]$. If $f \in \mathcal{A}$ satisfies the condition*

$$\left| \mathrm{Arg}\left(\frac{2+\alpha}{\alpha} - \frac{2}{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right)\right| < \mathrm{Arctan}\left(\frac{4\beta}{2+\alpha}\right), \tag{7}$$

then

$$\left|\mathrm{Arg}\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\beta\pi}{2} \quad (z \in \mathbb{U}).$$

Proof. Let $f \in \mathcal{A}$ and define the function $p : \mathbb{U} \to \mathbb{C}$ by

$$p(z) = \frac{zf'(z)}{f(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{U}).$$

Then it follows that p is analytic in $\mathbb{U}$, $p(0) = 1$,

$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)} \quad (z \in \mathbb{U})$$

and $p(z) \neq 0$ for all $z \in \mathbb{U}$. In fact, if p has a zero $z_0 \in \mathbb{U}$ of order m, then we may write

$$p(z) = (z - z_0)^m p_1(z) \quad (m \in \mathbb{N} = 1, 2, 3, \cdots),$$

where p_1 is analytic in $\mathbb{U}$ with $p_1(z_0) \neq 0$. Then

$$\frac{2+\alpha}{\alpha} - \frac{2}{\alpha}\left(p(z) + \frac{zp'(z)}{p(z)}\right) = \frac{2+\alpha}{\alpha} - \frac{2}{\alpha}\left(p(z) + \frac{zp_1'(z)}{p_1(z)} + \frac{mz}{z - z_0}\right).$$

Thus, choosing $z \to z_0$, suitably the argument of the right-hand of the above equality can take any value between $-\pi$ and π, which contradicts (7).

Define the function $q : \overline{\mathbb{U}} \setminus \{1\} \to \mathbb{C}$ by

$$q(z) = \left(\frac{1+z}{1-z}\right)^{\beta} \quad (z \in \overline{\mathbb{U}} \setminus \{1\}).$$

Then $q \in Q$, $q(0) = 1$ and $\mathrm{E}(q) = \{1\}$. It is clear that $\left|\operatorname{Arg}\left(p(z)\right)\right| < \frac{\beta\pi}{2}$ for all $z \in \mathbb{U}$ if and only if $p \prec q$ on $\mathbb{U}$. Let $\left|\operatorname{Arg}\left(p(z_1)\right)\right| \geq \frac{\beta\pi}{2}$ for some $z_1 \in \mathbb{U}$. Then p is not subordinate to q. By Lemma 1 there exists $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U} \setminus \{1\}$ such that $\{p\,(z) : z \in \mathbb{U}, |z| < |z_0|\} \subset q(\mathbb{U})$ and $p(z_0) = q(\zeta_0)$. Therefore,

$$\left|\operatorname{Arg}\left(p(z)\right)\right| < \frac{\beta\pi}{2},$$

for all $z \in \mathbb{U}$ with $|z| < |z_0|$ and

$$\left|\operatorname{Arg}\left(p(z_0)\right)\right| = \frac{\beta\pi}{2}.$$

Then, Lemma 2, gives us that

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,$$

where $[p(z_0)]^{\frac{1}{\beta}} = \pm ia$ $(a > 0)$ and k is given by (5) or (6).

Define the function $g : (0, a) \to \mathbb{R}$ by

$$g(t) = \frac{\frac{2}{2+\alpha}\left(t^{\beta}\sin(\frac{\beta\pi}{2}) + \beta\right)}{1 - \frac{2}{2+\alpha}t^{\beta}\cos\frac{\beta\pi}{2}} \quad t \in (0, a).$$

Then g is a differentiable function on $(0, a)$ and $g'(t) > 0$ for all $t \in (0, a)$. This implies that the function $h : (0, a) \to \mathbb{R}$ defined by

$$h(t) = \operatorname{Arctan}\left(g(t)\right) \quad t \in (0, a),$$

is a non-decreasing function on $(0, a)$. Thus

$$h(a) \geq \lim_{t \to 0^+} h(t) = \operatorname{Arctan}\left(\frac{2\beta}{2+\alpha}\right).$$

Therefore, we have

$$\operatorname{Arctan}\left(\frac{\frac{2}{2+\alpha}\left(a^{\beta}\sin\frac{\beta\pi}{2}+\beta\right)}{1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}}\right)\geq \operatorname{Arctan}\left(\frac{2\beta}{2+\alpha}\right). \tag{8}$$

Now we consider six cases for estimation of $\operatorname{Arg}\left(p(z_0)\right)$ as follows:

Case 1. $\operatorname{Arg}\left(p(z_0)\right)=\dfrac{\beta\pi}{2}$ and $1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}>0$. In this case we have $[p(z_0)]^{\frac{1}{\beta}}=ia\ (a>0)$, and $k\geq 1$. Therefore,

$$\begin{aligned}
\operatorname{Arg}\left(\frac{2+\alpha}{\alpha}\left(1-\frac{2}{2+\alpha}\left(p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}\right)\right)\right)&=\operatorname{Arg}\left(1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}-i\frac{2}{2+\alpha}\left(a^{\beta}\sin\frac{\beta\pi}{2}+k\beta\right)\right)\\
&=\operatorname{Arctan}\left(\frac{-\frac{2}{2+\alpha}\left(a^{\beta}\sin\frac{\beta\pi}{2}+k\beta\right)}{1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}}\right)\\
&\leq \operatorname{Arctan}\left(\frac{-\frac{2}{2+\alpha}\left(a^{\beta}\sin\frac{\beta\pi}{2}+\beta\right)}{1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}}\right)\\
&=-\operatorname{Arctan}\left(\frac{\frac{2}{2+\alpha}\left(a^{\beta}\sin\frac{\beta\pi}{2}+\beta\right)}{1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}}\right)\\
&=-h(a)\\
&\leq -\operatorname{Arctan}\left(\frac{2\beta}{2+\alpha}\right).
\end{aligned} \tag{9}$$

Now applying (8) and (9) we get

$$\begin{aligned}
\operatorname{Arg}\left(\frac{2+\alpha}{\alpha}\left(1-\frac{2}{2+\alpha}\left(p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}\right)\right)\right)&=\operatorname{Arg}\left(1-\frac{2}{2+\alpha}\left(p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}\right)\right)\\
&=\operatorname{Arg}\left(1-\frac{2}{2+\alpha}\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right)\right)\\
&\leq -\operatorname{Arctan}\left(\frac{\frac{2}{2+\alpha}\left(a^{\beta}\sin\frac{\beta\pi}{2}+\beta\right)}{1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}}\right)\\
&\leq -\operatorname{Arctan}\left(\frac{2\beta}{2+\alpha}\right),
\end{aligned}$$

which contradicts (7).

Case 2. $\operatorname{Arg}\left(p(z_0)\right)=\dfrac{\beta\pi}{2}$ and $1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}=0$. In this case, we have $p(z_0)=a^{\beta}(\cos\frac{\beta\pi}{2}+i\sin\frac{\beta\pi}{2})$ and $k\geq 1$. Thus $-\frac{2}{2+\alpha}\left(a^{\beta}\sin\frac{\beta\pi}{2}+k\beta\right)<0$ and so

$$\begin{aligned}
\operatorname{Arg}\left(\frac{2+\alpha}{\alpha}\left(1-\frac{2}{2+\alpha}\left(p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}\right)\right)\right)&=\operatorname{Arg}\left(-i\frac{2}{2+\alpha}\left(a^{\beta}\sin\frac{\beta\pi}{2}+k\beta\right)\right)\\
&=-\frac{\pi}{2}<-\operatorname{Arctan}\left(\frac{2\beta}{2+\alpha}\right),
\end{aligned}$$

which contradicts (7).

Case 3. $\operatorname{Arg}\left(p(z_0)\right)=\dfrac{\beta\pi}{2}$ and $1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}<0$. In this case, we have $p(z_0)=a^{\beta}(\cos\frac{\beta\pi}{2}+i\sin\frac{\beta\pi}{2})$ and $k\geq 1$. Thus

$$\frac{-\frac{2}{2+\alpha}\left(a^{\beta}\sin\frac{\beta\pi}{2}+k\beta\right)}{1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}}>0.$$

Therefore,

$$\begin{aligned}
\operatorname{Arg}\left(\frac{2+\alpha}{\alpha}\left(1-\frac{2}{2+\alpha}\left(p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}\right)\right)\right) &= \operatorname{Arg}\left(1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}-i\frac{2}{2+\alpha}\left(a^{\beta}\sin\frac{\beta\pi}{2}+k\beta\right)\right)\\
&= -\pi+\operatorname{Arctan}\left(\frac{-\frac{2}{2+\alpha}\left(a^{\beta}\sin\frac{\beta\pi}{2}+k\beta\right)}{1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}}\right)\\
&< -\pi+\frac{\pi}{2}\\
&= -\frac{\pi}{2}\\
&< -\operatorname{Arctan}\left(\frac{2\beta}{2+\alpha}\right),
\end{aligned}$$

which contradicts (7).

Case 4. $\operatorname{Arg}\left(p(z_0)\right)=-\dfrac{\beta\pi}{2}$ and $1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}>0$. In this case we have $p(z_0)=a^{\beta}(\cos\frac{\beta\pi}{2}-i\sin\frac{\beta\pi}{2})$ and $k\leq -1$. Thus $-\frac{2}{2+\alpha}\left(-a^{\beta}\sin\frac{\beta\pi}{2}+k\beta\right)<0$. Now , applying (8) we get

$$\begin{aligned}
\operatorname{Arg}\left(\frac{2+\alpha}{\alpha}\left(1-\frac{\alpha}{2+\alpha}\left(p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}\right)\right)\right) &= \operatorname{Arg}\left(1-\frac{2}{2+\alpha}\left(a^{\beta}e^{\frac{-i\beta\pi}{2}}+ik\beta\right)\right)\\
&= \operatorname{Arctan}\left(\frac{-\frac{2}{2+\alpha}\left(-a^{\beta}\sin\frac{\beta\pi}{2}+k\beta\right)}{1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}}\right)\\
&\geq \operatorname{Arctan}\left(\frac{-\frac{2}{2+\alpha}\left(-a^{\beta}\sin\frac{\beta\pi}{2}-\beta\right)}{1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}}\right)\\
&= \operatorname{Arctan}\left(\frac{\frac{2}{2+\alpha}\left(a^{\beta}\sin\frac{\beta\pi}{2}+\beta\right)}{1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}}\right)\\
&\geq \operatorname{Arctan}\left(\frac{2\beta}{2+\alpha}\right),
\end{aligned}$$

which contradicts (7).

For other cases applying the same method in Case 2. and Case 3. with $k\leq -1$ we obtain

$$\operatorname{Arg}\left(\frac{2+\alpha}{\alpha}\left(1-\frac{2}{2+\alpha}\left(p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}\right)\right)\right)\geq \operatorname{Arctan}\left(\frac{2\beta}{2+\alpha}\right),$$

which contradicts (7). Hence the proof is completed. □

Corollary 1. *Let* $\alpha,\beta\in(0,1]$ *and* $\delta=\frac{2}{\pi}\operatorname{Arctan}\left(\frac{2\beta}{2+\alpha}\right)$. *If* $f\in\mathcal{G}(\alpha,\delta)$, *then* $f\in\widetilde{\mathcal{S}}^{*}(\beta)$.

Theorem 2. *Let* $\alpha,\beta\in(0,1]$. *If* $f\in\mathcal{A}$ *and*

$$\left|\operatorname{Arg}\left(\frac{2+\alpha}{\alpha}-\frac{2}{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)\right)\right|<\operatorname{Arctan}\left(\frac{2\beta}{\alpha}\right), \tag{10}$$

then

$$\left|\operatorname{Arg}\left(f'(z)\right)\right|<\frac{\beta\pi}{2}\quad(z\in\mathbb{U}).$$

Proof. Define the function $p : \mathbb{U} \to \mathbb{C}$ by

$$p(z) = f'(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{U}).$$

Then p is analytic in $\mathbb{U}$, $p(0) = 1$,

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{zp'(z)}{p(z)}.$$

and $p(z) \neq 0$ for all $z \in \mathbb{U}$. If there exists a point $z_0 \in \mathbb{U}$ such that

$$|\operatorname{Arg}(p(z))| < \frac{\beta\pi}{2},$$

for all $z \in \mathbb{U}$ with $|z| < |z_0|$ and

$$|\operatorname{Arg}(p(z_0))| = \frac{\beta\pi}{2}.$$

Then, Lemma 2, gives us that

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,$$

where $[p(z_0)]^{\frac{1}{\beta}} = \pm ia \ (a > 0)$ and k is given by (5) or (6).

For the case $\operatorname{Arg}(p(z_0)) = \dfrac{\alpha\pi}{2}$ when

$$p(z_0)]^{\frac{1}{\beta}} = ia \qquad (a > 0)$$

and $k \geq 1$, we have

$$\begin{aligned}
\operatorname{Arg}\left(\frac{2+\alpha}{\alpha}\left(1 - \frac{2}{2+\alpha}\left(1 + \frac{z_0 p'(z_0)}{p(z_0)}\right)\right)\right) &= \operatorname{Arg}\left(1 - \frac{2}{2+\alpha}\left(1 + \frac{z_0 p'(z_0)}{p(z_0)}\right)\right) \\
&= \operatorname{Arg}\left(1 - \frac{2}{2+\alpha}(1 + ik\beta)\right) \\
&= \operatorname{Arctan}\left(\frac{-2k\beta}{\alpha}\right) \\
&\leq -\operatorname{Arctan}\left(\frac{2\beta}{\alpha}\right),
\end{aligned}$$

which contradicts (10).

Next, for the case $\operatorname{Arg}(p(z_0)) = -\dfrac{\alpha\pi}{2}$ when

$$p(z_0) = -ia \qquad (a > 0)$$

and $k \leq -1$, using the same method as before, we can obtain

$$\begin{aligned}
\operatorname{Arg}\left(\frac{2+\alpha}{\alpha}\left(1 - \frac{2}{2+\alpha}\left(1 + \frac{z_0 p'(z_0)}{p(z_0)}\right)\right)\right) &= \operatorname{Arg}\left(1 - \frac{2}{2+\alpha}\left(1 + \frac{z_0 p'(z_0)}{p(z_0)}\right)\right) \\
&= \operatorname{Arg}\left(1 - \frac{2}{2+\alpha}(1 + ik\beta)\right) \\
&= \operatorname{Arctan}\left(\frac{-2k\beta}{\alpha}\right) \\
&\geq \operatorname{Arctan}\left(\frac{2\beta}{\alpha}\right),
\end{aligned}$$

which is a contradicts (10).

Consequently, from the two above-discussed contradictions, it follows that

$$\left|\operatorname{Arg}\left(f'(z)\right)\right| < \frac{\beta\pi}{2} \quad (z \in \mathbb{U}).$$

and hence the proof is completed. □

Corollary 2. *Let $\alpha, \beta \in (0,1]$ and $\delta = \frac{2}{\pi}\operatorname{Arctan}\left(\frac{2\beta}{\alpha}\right)$. If $f \in \mathcal{G}(\alpha,\delta)$, then $f \in \widetilde{\mathcal{C}}(\beta)$. In other words, if $f \in \mathcal{G}(\alpha,\delta)$, then $f(z)$ is close-to-convex (univalent) in $\mathbb{U}$.*

3. Coefficient Bounds

In this section, we give a the general problem of coefficients in the class $\mathcal{G}(\alpha,\delta)$ like the estimates of coefficients for membership of this, bounds of logarithmic coefficients and the Fekete-Szegö problem with sharp inequalities. In order to achieve our aim we need to establish some knowledge.

Lemma 3 ([27] (p. 172)). *Let $\omega \in \Omega$ with $\omega(z) = \sum_{n=1}^{\infty} w_n z^n$ for all $z \in \mathbb{U}$. Then $|w_1| \leq 1$ and*

$$|w_n| \leq 1 - |w_1|^2 \quad \text{for all } n \in \mathbb{N} \text{ with } n \geq 2.$$

Lemma 4 ([28] (Inequality 7, p. 10)). *Let $\omega \in \Omega$ with $\omega(z) = \sum_{n=1}^{\infty} w_n z^n$ for all $z \in \mathbb{U}$. Then*

$$|w_2 - t w_1^2| \leq \max\{1, |t|\} \quad \text{for all } t \in \mathbb{C}.$$

The inequality is sharp for the functions $\omega(z) = z^2$ or $\omega(z) = z$.

Lemma 5 ([29]). *If $\omega \in \Omega$ with $\omega(z) = \sum_{n=1}^{\infty} w_n z^n$ $(z \in \mathbb{U})$, then for any real numbers q_1 and q_2, we have the following sharp estimate:*

$$|p_3 + q_1 w_1 w_2 + q_2 w_1^3| \leq H(q_1;q_2),$$

where

$$H(q_1;q_2) = \begin{cases} 1 & \text{if} \quad (q_1,q_2) \in D_1 \cup D_2 \cup \{(2,1)\}, \\ |q_2| & \text{if} \quad (q_1,q_2) \in \cup_{k=3}^{7} D_k, \\ \frac{2}{3}(|q_1|+1)\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}} & \text{if} \quad (q_1,q_2) \in D_8 \cup D_9, \\ \frac{q_2}{3}\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_2-1)}\right)^{\frac{1}{2}} & \text{if} \quad (q_1,q_2) \in D_{10} \cup D_{11} \setminus \{(2,1)\}, \\ \frac{2}{3}(|q_1|-1)\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}} & \text{if} \quad (q_1,q_2) \in D_{12}, \end{cases}$$

and the sets D_k, $k = 1,2,\ldots,12$ are stated as given below:

$$D_1 = \left\{(q_1,q_2) : |q_1| \leq \frac{1}{2},\ |q_2| \leq 1\right\},$$
$$D_2 = \left\{(q_1,q_2) : \frac{1}{2} \leq |q_1| \leq 2,\ \frac{4}{27}\left((|q_1|+1)^3\right) - (|q_1|+1) \leq q_2 \leq 1\right\},$$
$$D_3 = \left\{(q_1,q_2) : |q_1| \leq \frac{1}{2},\ q_2 \leq -1\right\},$$

$$D_4 = \left\{ (q_1, q_2) : |q_1| \geq \frac{1}{2},\ |q_2| \leq -\frac{2}{3}(|q_1|+1) \right\},$$
$$D_5 = \{ (q_1, q_2) : |q_1| \leq 2,\ q_2 \geq 1 \},$$
$$D_6 = \left\{ (q_1, q_2) : 2 \leq |q_1| \leq 4,\ q_2 \geq \frac{1}{12}(q_1^2+8) \right\},$$
$$D_7 = \left\{ (q_1, q_2) : |q_1| \geq 4,\ q_2 \geq \frac{2}{3}(|q_1|-1) \right\},$$
$$D_8 = \left\{ (q_1, q_2) : \frac{1}{2} \leq |q_1| \leq 2,\ -\frac{2}{3}(|q_1|+1) \leq q_2 \leq \frac{4}{27}\left((|q_1|+1)^3\right) - (|q_1|+1) \right\},$$
$$D_9 = \left\{ (q_1, q_2) : |q_1| \geq 2,\ -\frac{2}{3}(|q_1|+1) \leq q_2 \leq \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \right\},$$
$$D_{10} = \left\{ (q_1, q_2) : 2 \leq |q_1| \leq 4,\ \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \leq q_2 \leq \frac{1}{12}(q_1^2+8) \right\},$$
$$D_{11} = \left\{ (q_1, q_2) : |q_1| \geq 4,\ \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \leq q_2 \leq \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4} \right\},$$
$$D_{12} = \left\{ (q_1, q_2) : |q_1| \geq 4,\ \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4} \leq q_2 \leq \frac{2}{3}(|q_1|-1) \right\}.$$

We assume that φ is a univalent function in the unit disk $\mathbb{U}$ satisfying $\varphi(0) = 1$ such that it has the power series expansion of the following form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots,\ z \in \mathbb{U}, \quad \text{with} \quad B_1 \neq 0. \tag{11}$$

Lemma 6 ([30] (Theorem 2))**.** *Let the function $f \in \mathcal{K}(\varphi)$. Then the logarithmic coefficients of f satisfy the inequalities*

$$|\gamma_1| \leq \frac{|B_1|}{4}, \tag{12}$$

$$|\gamma_2| \leq \begin{cases} \dfrac{|B_1|}{12} & \text{if} \quad |4B_2 + B_1^2| \leq 4|B_1|, \\[2ex] \dfrac{|4B_2 + B_1^2|}{48} & \text{if} \quad |4B_2 + B_1^2| > 4|B_1|, \end{cases} \tag{13}$$

and if B_1, B_2, and B_3 are real values,

$$|\gamma_3| \leq \frac{|B_1|}{24} H(q_1; q_2), \tag{14}$$

where $H(q_1; q_2)$ is given by Lemma 5, $q_1 = \frac{B_1 + \frac{4B_2}{B_1}}{2}$ and $q_2 = \frac{B_2 + \frac{2B_3}{B_1}}{2}$. The bounds (12) and (13) are sharp.

Theorem 3. *Let $f \in \mathcal{G}(\alpha, \delta)$. Then*

$$|a_2| \leq \frac{\alpha\delta}{2}, \quad |a_3| \leq \frac{\alpha\delta}{6}, \quad |a_4| \leq \frac{\alpha\delta}{12} H(q_1; q_2),$$

where $H(q_1; q_2)$ is given by Lemma 5,

$$q_1 = \frac{-3\alpha\delta}{2} + 2\delta \quad \text{and} \quad q_2 = \delta^2\left(\frac{-3\alpha}{2} + \frac{\alpha^2}{2} + \frac{2}{3}\right) + \frac{1}{3}.$$

The first two bounds are sharp.

Proof. Set $g(z) =: zf'(z)$, where $f \in \mathcal{G}(\alpha, \delta)$ and suppose that $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Hence $b_n = na_n$ for $n \geq 1$. Then from (4), it follows that

$$\begin{aligned}\frac{zg'(z)}{g(z)} &\prec -\frac{\alpha}{2}\left(\frac{1+z}{1-z}\right)^{\delta} + \frac{2+\alpha}{2} =: \phi(z)\\ &= 1 - \alpha\delta z - \alpha\delta^2 z^2 - \frac{1}{3}\alpha\delta(2\delta^2+1)z^3 + \cdots\\ &:= 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots .\end{aligned}$$

Now, by the definition of the subordination, there is a $\omega \in \Omega$ with $\omega(z) = \sum_{n=1}^{\infty} w_n z^n$ so that

$$\begin{aligned}\frac{zg'(z)}{g(z)} =&\phi(\omega(z))\\ =&1 + B_1 w_1 z + (B_1 w_2 + B_2 w_1^2)z^2 + (B_1 w_3 + 2w_1 w_2 B_2 + B_3 w_1^3)z^3 + \cdots .\end{aligned}$$

From the above equality, it concludes that

$$\begin{cases} b_2 = B_1 w_1 \\ 2b_3 - b_2^2 = B_1 w_2 + B_2 w_1^2 \\ 3b_4 - 3b_2 b_3 + b_2^3 = B_1 w_3 + 2w_1 w_2 B_2 + B_3 w_1^3. \end{cases}$$

First, for b_2, from Lemma 3 we get $|b_2| \leq \alpha\delta$, and so $|a_2| \leq \frac{\alpha\delta}{2}$. Next, utilizing Lemma 3 for b_3 and using $|B_2 + B_1^2| \leq |B_1|$, we have

$$\begin{aligned}|b_3| \leq &\frac{|B_1|(1-|w_1|^2) + |B_2 + B_1^2||w_1|^2}{2}\\ =&\frac{|B_1| + \left[|B_2 + B_1^2| - |B_1|\right]|w_1|^2}{2}\\ \leq&\frac{|B_1|}{2} = \frac{\alpha\delta}{2}.\end{aligned}$$

Ultimately, utilizing Lemma 5 for a_4, we have

$$\begin{aligned}|b_4| \leq &\frac{B_1}{3}\left|c_3 + \left(\frac{3}{2}B_1 + \frac{2B_2}{B_1}\right) w_1 w_2 + \left(\frac{3}{2}B_2 + \frac{1}{2}B_1^2 + \frac{B_3}{B_1}\right) w_1^3\right|\\ \leq&\frac{B_1}{3}H(q_1; q_2),\end{aligned}$$

where

$$q_1 = \frac{3}{2}B_1 + \frac{2B_2}{B_1} = \frac{-3\alpha\delta}{2} + 2\delta \quad \text{and} \quad q_2 = \frac{3}{2}B_2 + \frac{1}{2}B_1^2 + \frac{B_3}{B_1} = \delta^2\left(\frac{-3\alpha}{2} + \frac{\alpha^2}{2} + \frac{2}{3}\right) + \frac{1}{3}.$$

The extremal functions for the initial coefficients a_n $(n = 2, 3)$ are of the form:

$$f_n(z) = \int_0^z \exp\left(\int_0^x \frac{\phi(t^n) - 1}{t}\,\mathrm{d}t\right)\mathrm{d}x = z - \frac{\alpha\beta}{n(n+1)}z^{n+1} + \frac{\alpha\beta^2(\alpha/n - 1)}{2n(2n+1)}z^{2n+1} + \cdots,$$

obtained by taking $\omega(z) = z^n$ in (4). Therefore, this completes the proof. □

Theorem 4. *Let $f \in \mathcal{G}(\alpha, \delta)$. Then*

$$|\gamma_1| \leq \frac{\alpha\delta}{4}, \quad |\gamma_2| \leq \frac{\alpha\delta}{12}, \quad |\gamma_3| \leq \frac{\alpha\delta}{24}H(q_1; q_2),$$

where $H(q_1; q_2)$ is given by Lemma 5, $q_1 = \frac{-\alpha\delta+4\delta}{2}$, and $q_2 = \frac{-\alpha\delta^2+\frac{2(2\delta^2+1)}{3}}{2}$. The first two bounds are sharp.

Proof. The results are concluded from Theorem 6 by setting $\varphi := \phi$. Also, two first bounds are sharp for $f_n(z)$ for $n = 1, 2$, respectively. Therefore, this completes the proof. □

Theorem 5. *Let $f \in \mathcal{G}(\alpha, \delta)$. Then we have sharp inequalities for complex parameter μ*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{\alpha\delta^2}{6}\left|1 - \alpha + \frac{3\mu}{2}\alpha\right| & for \quad \left|\mu + \frac{2}{3\alpha}(1-\alpha)\right| \geq \frac{2}{3\alpha\delta}, \\ \frac{\alpha\delta}{6} & for \quad \left|\mu + \frac{2}{3\alpha}(1-\alpha)\right| < \frac{2}{3\alpha\delta}. \end{cases}$$

Proof. Let $f \in \mathcal{G}(\alpha, \delta)$, then from (4), by the definition of the subordination, there is a $\omega \in \Omega$ with $\omega(z) = \sum_{n=1}^{\infty} w_n z^n$ so that

$$1 + \frac{zf''(z)}{f'(z)} = \phi(\omega(z)) = 1 + B_1 w_1 z + (B_1 w_2 + B_2 w_1^2)z^2 + \cdots .$$

Therefore, we get that

$$2a_2 = B_1 w_1 \qquad \text{and} \qquad 6a_3 - 4a_2^2 = B_1 w_2 + B_2 w_1^2.$$

Form the above equalities, we have

$$\left|a_3 - \mu a_2^2\right| = \frac{1}{6}|B_1|\left|w_2 + \nu w_1^2\right|.$$

The results are obtained by the application of Lemma 4 with $\nu = \left[\frac{B_2}{B_1} + B_1(1 - \frac{3\mu}{2})\right]$, where $B_1 = -\alpha\delta$ and $B_2 = -\alpha\delta^2$. Equality is attained in the first inequality by the function $f = f_1$ and in the second inequality for $f = f_2$. □

Remark 1.

(i) *Taking into account $\delta = 1$ in Theorem 3, we get the result obtained in [31] (Theorem 1) for $n = 2, 3, 4$.*
(ii) *Setting $\delta = 1$ in Theorem 3, we have the result obtained in [23] (Theorem 2.10).*
(iii) *Letting $\delta = 1$ in Theorem 4, we obtain a correction of the result presented in [31] (Theorem 2).*

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References

1. Kayumov, I.R. On Brennan's conjecture for a special class of functions. *Math. Notes* **2005**, *78*, 498–502. [CrossRef]
2. Milin, I.M. *Univalent Functions and Orthonormal Systems*; Amer Mathematical Society, Translations of Mathematical Monographs: Proovidence, RI, USA, 1977; Volume 49.

3. Milin, I.M. On a property of the logarithmic coefficients of univalent functions. In *Metric Questions in the Theory of Functions*; Naukova Dumka: Kiev, Ukraine, 1980; pp. 86–90. (In Russian)
4. Milin, I.M. On a conjecture for the logarithmic coefficients of univalent functions. *Zap. Nauch. Semin. Leningr. Otd. Mat. Inst. Steklova* **1983**, *125*, 135–143. (In Russian) [CrossRef]
5. Robertson, M.S. A remark on the odd-schlicht functions. *Bull. Am. Math. Soc.* **1936**, *42*, 366–370. [CrossRef]
6. Bieberbach, L. Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitkreises vermitteln. *Sitz. Preuss. Akad. Wiss.* **1916**, *138*, 940–955.
7. De Branges, L. A proof of Bieberbach conjecture. *Acta Math.* **1985**, *154*, 137–152. [CrossRef]
8. Pommerenke, C. On the coefficients and Hankel determinant of univalent functions. *J. Lond. Math. Soc.* **1966**, *41*, 111–112. [CrossRef]
9. Pommerenke, C. On the Hankel determinants of univalent functions. *Mathematika* **1967**, *14*, 108–112. [CrossRef]
10. Vein, R.; Dale, P. Determinants and Their Applications in Mathematical Physics. In *Applied Mathematical Sciences*; Springer: New York, NY, USA, 1999; Volume 134.
11. Fekete, M.; Szegö, G. Eine Bemerkung Über Ungerade Schlichte Funktionen. *J. Lond. Math. Soc.* **1933**, *8*, 85–89. [CrossRef]
12. Analouei Adegani, E.; Cho, N.E.; Motamednezhad, A.; Jafari, M. Bi-univalent functions associated with Wright hypergeometric functions. *J. Comput. Anal. Appl.* **2020**, *28*, 261–271.
13. Cho, N.E.; Analouei Adegani, E.; Bulut, S.; Motamednezhad, A. The second Hankel determinant problem for a class of bi-close-to-convex functions. *Mathematics* **2019**, 7, 986. [CrossRef]
14. Deniz, E.; Çağlar, M.; Orhan, H. Second Hankel determinant for bi-starlike and bi-convex functions of order β. *Appl. Math. Comput.* **2015**, 271, 301–307. [CrossRef]
15. Kanas, S.; Analouei Adegani, E.; Zireh, A. An unified approach to second Hankel determinant of bi-subordinate functions. *Mediterr. J. Math.* **2017**, *14*, 233. [CrossRef]
16. Motamednezhad, A.; Bulboacă, T.; Adegani, E.A.; Dibagar, N. Second Hankel determinant for a subclass of analytic bi-univalent functions defined by subordination. *Turk. J. Math.* **2018**, *42*, 2798–2808. [CrossRef]
17. Nunokawa, M.; Saitoh, H. On certain starlike functions. *Srikaisekikenkysho Kkyroku* **1996**, *963*, 74–77.
18. Ozaki, S. On the theory of multivalent functions, II. *Sci. Rep. Tokyo Bunrika Daigaku. Sect. A* **1941**, *4*, 45–87.
19. Umezawa, T. Analytic functions convex in one direction. *J. Math. Soc. Jpn.* **1952**, *4*, 194–202. [CrossRef]
20. Sakaguchi, K. A property of convex functions and an application to criteria for univalence. *Bull. Nara Univ. Ed. Nat. Sci.* **1973**, *22*, 1–5.
21. Singh, R.; Singh, S. Some sufficient conditions for univalence and starlikeness. *Colloq. Math.* **1982**, *47*, 309–314. [CrossRef]
22. Obradović, M.; Ponnusamy, S.; Wirths, K.-J. Logarithmic coeffcients and a coefficient conjecture for univalent functions. *Monatshefte Math.* **2017**. [CrossRef]
23. Ponnusamy, S.; Sharma, N.L.; Wirths. K.-J. Logarithmic coefficients problems in families related to starlike and convex functions. *J. Aust. Math. Soc.* **2019**, in press. [CrossRef]
24. Miller, S.S.; Mocanu, P.T. *Differential Subordinations. Theory and Applications*; Marcel Dekker Inc.: New York, NY, USA, 2000; ISBN 0-8247-0029-5.
25. Nunokawa, M. On properties of non-Carathéodory functions. *Proc. Jpn. Acad. Ser. A Math. Sci.* **1992**, *68*, 152–153. [CrossRef]
26. Nunokawa, M. On the order of strongly starlikeness of strongly convex functions. *Proc. Jpn. Acad. Ser. A Math. Sci.* **1993** *69*, 234–237. [CrossRef]
27. Nehari, Z. *Conformal Mapping*; McGraw-Hill: New York, NY, USA, 1952.
28. Keogh, F.R.; Merkes, E.P. A coefficient inequality for certain classes of analytic functions. *Proc. Am. Math. Soc.* **1969**, *20*, 8–12. [CrossRef]
29. Prokhorov, D.V.; Szynal, J. Inverse coefficients for $(\alpha;\beta)$-convex functions. *Ann. Univ. Mariae Curie-Sklodowska Sect. A* **1984**, *35*, 125–143.

30. Analouei Adegani, E.; Cho, N.E.; Jafari M. Logarithmic coefficients for univalent functions defined by subordination. *Mathematics* **2019**, *7*, 408. [CrossRef]
31. Obradović, M.; Ponnusamy, S.; Wirths, K.-J. Coefficient characterizations and sections for some univalent functions. *Sib. Math. J.* **2013**, *54*, 679–696. [CrossRef]