mathematics

MDPI

Article

Approximation by Generalized Lupaş Operators Based on q-Integers

Mohd Qasim [1], **M. Mursaleen** [2,3,4,*], **Asif Khan** [3] **and Zaheer Abbas** [1]

[1] Department of Mathematical Sciences, Baba Ghulam Shah Badshah University, Rajouri 185234, Jammu and Kashmir, India; bgsbuqasim@gmail.com (M.Q.); az1129200@yahoo.co.in (Z.A.)
[2] Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung 40402, Taiwan
[3] Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India; asifjnu07@gmail.com
[4] Department of Computer Science and Information Engineering, Asia University, Taichung 41354, Taiwan
* Correspondence: mursaleenm@gmail.com; Tel.: +91-9411491600

Received: 6 December 2019; Accepted: 27 December 2019; Published: 2 January 2020

Abstract: The purpose of this paper is to introduce q-analogues of generalized Lupaş operators, whose construction depends on a continuously differentiable, increasing, and unbounded function ρ. Depending on the selection of q, these operators provide more flexibility in approximation and the convergence is at least as fast as the generalized Lupaş operators, while retaining their approximation properties. For these operators, we give weighted approximations, Voronovskaja-type theorems, and quantitative estimates for the local approximation.

Keywords: generalized Lupaș operators; q analogue; Korovkin's type theorem; convergence theorems; Voronovskaya type theorem

1. Introduction

Approximation theory rudimentarily deals with the approximation of functions by simpler functions or more easily calculated functions. Broadly, it is divided into theoretical and constructive approximation. In 1912, S.N. Bernstein [1] was the first to construct a sequence of positive linear operators to provide a constructive proof of the prominent Weierstrass approximation theorem [2] using a probabilistic approach. One can find a detailed monograph about the Bernstein polynomials in [3,4]. Cárdenas et al. [5] in 2011, defined the Bernstein type operators by $B_n(f o\tau^{-1})o\tau$ and showed that its Korovkin set is $\{e_0, \tau, \tau^2\}$ instead of $\{e_0, e_1, e_2\}$. These operators present an interesting byproduct sequence of positive linear operators of polynomial type with nice geometric shape preserving properties, which converge to the identity, which in a certain sense improves B_n in approximating a number of increasing functions, and which, apart from the constant functions, fixes suitable polynomials of a prescribed degree. The notion of convexity with respect to τ plays an important role. Recently, Aral et al. [6] in 2014 defined a similar modification of Szász-Mirakyan type operators obtaining approximation properties of these operators on the interval $[0,\infty)$.

Very recently motivated by the above work, İlarslan et al. [7] introduced a new modification of Lupaş operators [8] using a suitable function ρ, which satisfies the following properties:

(ρ_1) ρ be a continuously differentiable function on $[0,\infty)$,
(ρ_2) $\rho(0) = 0$ and $\inf\limits_{u\in[0,\infty)} \rho^{'}(u) \geq 1$.

The generalized Lupaş operators are defined as

$$\mathcal{L}_m^{\rho}(f;u) = 2^{-m\rho(u)} \sum_{l=0}^{\infty} \frac{(m\rho(u))_l}{2^l l!} \left(f o \rho^{-1}\right)\left(\frac{l}{m}\right), \tag{1}$$

for $m \geq 1$, $u \geq 0$, and suitable functions f are defined on $[0, \infty)$. If $\rho(u) = u$ then (1) reduces to the Lupaş operators defined in [8].

İlarslan et al. [7] gave uniform convergence results on a weighted space, where the weight function is $\phi(u) = 1 + \rho^2(u)$ satisfying the conditions (ρ_1) and and (ρ_2) given above, in the sense of Gadjiev's results [9,10]. For the rate of convergence, the authors used a weighted modulus of continuity stated by Holhoş in [11] using the weight function. They obtained a Voronovskaya-type result and monotonicity of the sequence of operators $\mathcal{L}_m^{\rho}(f; u)$. Moreover, they obtained some quantitative type theorems on weighted spaces.

The purpose of this paper is to define the q-analogue of operators (1) which depend on ρ.

Before proceeding further, let us recall some basic definitions and notations of quantum calculus [12]. For any fixed real number $q > 0$, the q-integer $[l]_q$, for $l \in \mathbb{N}$ (set of natural numbers) are defined as

$$[l]_q := \begin{cases} \frac{(1-q^l)}{(1-q)}, & q \neq 1 \\ l, & q = 1, \end{cases}$$

and the q-factorial by

$$[l]_q! := \begin{cases} [l]_q [l-1]_q ... [1]_q, & l \geq 1 \\ l, & l = 0. \end{cases}$$

The q-Binomial expansion is

$$(u + y)_q^m := (u + y)(u + qy)(u + q^2 y) \cdots (u + q^{m-1} y),$$

and the q-binomial coefficients are as follows:

$$\left[\begin{array}{c} m \\ l \end{array} \right]_q := \frac{[m]_q!}{[l]_q! [m-l]_q!}.$$

The Gauss-formula is defined as:

$$(u + y)_q^m = \sum_{j=0}^{m} \left[\begin{array}{c} m \\ l \end{array} \right]_q q^{j(j-1)/2} y^j u^{m-j}.$$

After development of q-calculus, Lupaş [13] introduced the q-Lupaş operator (rational) as follows:

$$L_{m,q}(f; u) = \sum_{l=0}^{m} \frac{f\left(\frac{[l]_q}{[m]_q}\right) \left[\begin{array}{c} m \\ l \end{array} \right]_q q^{\frac{l(l-1)}{2}} u^l \, (1-u)^{m-l}}{\prod_{j=1}^{m} \{(1-u) + q^{j-1} u\}}, \tag{2}$$

and studied its approximation properties.

Similarly, Phillips [14] constructed another q-analogue of Bernstein operators (polynomials) as follows:

$$B_{m,q}(f; u) = \sum_{l=0}^{m} \left[\begin{array}{c} m \\ l \end{array} \right]_q u^l \prod_{s=0}^{m-l-1} (1 - q^s u) \; f\left(\frac{[l]_q}{[m]_q}\right), \quad u \in [0, 1] \tag{3}$$

where $B_{m,q} : C[0, 1] \to C[0, 1]$ defined for any $m \in \mathbb{N}$ and any function $f \in C[0, 1]$, where $C[0, 1]$ denotes the set of all continuous functions on $[0, 1]$.

The basis of these operators have been used in Computer Aided Geometric Design (CAGD) to study curves and surfaces. Then it became an active area of research in approximation theory as well as CAGD [15–17]. In the recent past, q-analogues of various operators were investigated by several researchers (see [18–23]).

The q-analogue is a very interesting idea. It can also be used in statistical and biological physics, multi- type directed scalefree percolation, and modeling epidemic spread with awareness and heterogeneous transmission rates in networks.

Persuaded by the above mentioned work, we introduce the q-analogue of operators (1) which depends on a suitable function ρ, as follows:

Definition 1. *Let $0 < q < 1$ and $m \in \mathbb{N}$. For $f : [0, \infty) \to \mathbb{R}$, we define q-analogue of generalized Lupaş operators as*

$$\mathcal{L}^{\rho}_{m,q}(f;u) = 2^{-[m]_q\rho(u)} \sum_{l=0}^{\infty} \frac{([m]_q\rho(u))_l}{2^l [l]_q!} \left(fo\rho^{-1}\right)\left(\frac{[l]_q}{[m]_q}\right), \tag{4}$$

where $([m]_q\rho(u))_l$ is the rising factorial defined as:

$$\begin{aligned} ([m]_q\rho(u))_0 &= 1, \\ ([m]_q\rho(u))_l &= ([m]_q\rho(u))([m]_q\rho(u)+1)([m]_q\rho(u)+2)\cdots([m]_q\rho(u)+l-1), \quad l \geq 0. \end{aligned}$$

The operators (4) are linear and positive. For $q = 1$, the operators (4) turn out to be generalized Lupaş operators defined in (1). Next, we prove some auxiliary results for (4).

Lemma 1. *Let $\mathcal{L}^{\rho}_{m,q}$ be given by (4). Then for each $u \geq 0$ and $m \in \mathbb{N}$ we have*

(i) $\mathcal{L}^{\rho}_{m,q}(1;u) = 1$,
(ii) $\mathcal{L}^{\rho}_{m,q}(\rho;u) = \rho(u)$,
(iii) $\mathcal{L}^{\rho}_{m,q}(\rho^2;u) = \rho^2(u) + \frac{(1+q)}{[m]_q}\rho(u)$,
(iv) $\mathcal{L}^{\rho}_{m,q}(\rho^3;u) = \rho^3(u)q^3 + \frac{(3q^3+q^2+2q)}{[m]_q}\rho^2(u) + \frac{(2q^3+q^2+2q+1)}{[m]_q^2}\rho(u)$,
(v) $\mathcal{L}^{\rho}_{m,q}(\rho^4;u) = \rho^4(u)q^6 + \frac{(6q^6+q^5+2q^4+3q^3)}{[m]_q}\rho^3(u) + \frac{(11q^6+3q^5+6q^4+1oq^3+3q^2+3q)}{[m]_q^2}\rho^2(u)$
$+ \frac{(6q^6+2q^5+4q^4+7q^3+3q^2+3q+1)}{[m]_q^3}\rho(u)$.

Proof. By taking into account the recurrence relation $([m]_q\rho(u))_0 = 1, ([m]_q\rho(u))_l = ([m]_q\rho(u))$ $([m]_q\rho(u)+1)_{l-1}, l \geq 1$, we have

(i)

$$\mathcal{L}^{\rho}_{m,q}(1;u) = 2^{-[m]_q\rho(u)} \sum_{l=0}^{\infty} \frac{([m]_q\rho(u))_l}{2^l [l]_q!} = 1.$$

(ii)

$$\begin{aligned} \mathcal{L}^{\rho}_{m,q}(\rho;u) &= 2^{-[m]_q\rho(u)} \sum_{l=0}^{\infty} \frac{([m]_q\rho(u))_l}{[l]_q! 2^l} \frac{[l]_q}{[m]_q} \\ &= \frac{2^{-([m]_q\rho(u)+1)}[m]_q\rho(u)}{[m]_q} \sum_{l=0}^{\infty} \frac{([m]_q\rho(u)+1)_l}{[l]_q! 2^l} \\ &= \rho(u). \end{aligned}$$

(iii)

$$\begin{aligned}
\mathcal{L}^{\rho}_{m,q}(\rho^2;u) &= 2^{-[m]_q\rho(u)}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u))_l}{[l]_q!2^l}\frac{[l]_q^2}{[m]_q^2}\\
&= \frac{2^{-[m]_q\rho(u)}}{[m]_q^2}\sum_{l=0}^{\infty}\frac{([m]_q u)_l}{[l]_q!2^l}[l]_q^2\\
&= \frac{2^{-([m]_q\rho(u)+1)}[m]_q\rho(u)}{[m]_q^2}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+1)_l}{[l]_q!2^l}[l]_q\\
&= \rho^2(u)+\frac{(1+q)}{[m]_q}\rho(u).
\end{aligned}$$

(iv)

$$\mathcal{L}^{\rho}_{m,q}(\rho^3;u) = 2^{-[m]_q\rho(u)}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u))_l}{[l]_q!2^l}\frac{[l]_q^3}{[m]_q^3}$$

Now by using $[l+1]_q=(1+q[l]_q)$ and shifting l to $l+1$, we have

$$\begin{aligned}
\mathcal{L}^{\rho}_{m,q}(\rho^3;u) &= \frac{2^{-([m]_q\rho(u)+1)}[m]_q\rho(u)}{[m]_q^3}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+1)_l}{[l]_q!2^l}[l+1]_q^2\\
&= \frac{2^{-([m]_q\rho(u)+1)}[m]_q\rho(u)}{[m]_q^3}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+1)_l}{[l]_q!2^l}(1+q[l]_q)^2\\
&= \frac{2^{-([m]_q\rho(u)+1)}[m]_q\rho(u)}{[m]_q^3}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+1)_l}{[l]_q!2^l},\\
&+ \frac{2^{-([m]_q\rho(u)+1)}[m]_q\rho(u)}{[m]_q^3}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+1)_l}{[l]_q!2^l}q^2[l]_q^2\\
&+ \frac{2^{-([m]_q\rho(u)+1)}[m]_q\rho(u)}{[m]_q^3}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+1)_l}{[l]_q!2^l}2q[l]_q,\\
&= A+B+C(Say).
\end{aligned}$$

Now, let us calculate the values of A, B, and C

$$\begin{aligned}
A &= \frac{2^{-([m]_q\rho(u)+1)}[m]_q\rho(u)}{[m]_q^3}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+1)_l}{[l]_q!2^l},\\
&= \frac{\rho(u)}{[m]_q^2}.
\end{aligned}$$

$$\begin{aligned}
B &= \frac{2^{-([m]_q\rho(u)+1)}[m]_q\rho(u)}{[m]_q^3}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+1)_l}{[l]_q!2^l}q^2[l]_q^2\\
&= \frac{2^{-([m]_q\rho(u)+2)}([m]_q\rho(u)+1)\rho(u)}{[m]_q^2}q^2\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+2)_l}{[l]_q!2^l}[l+1]_q\\
&= \frac{2^{-([m]_q\rho(u)+2)}([m]_q\rho(u)+1)\rho(u)}{[m]_q^2}q^2\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+2)_l}{[l]_q!2^l}(1+q[l]_q)\\
&= \rho^3(u)q^3+\frac{3\rho^2(u)q^3}{[m]_q}+\frac{2\rho(u)q^3}{[m]_q^2}+\frac{\rho^2(u)q^2}{[m]_q}+\frac{\rho(u)q^2}{[m]_q^2}.
\end{aligned}$$

Also,

$$\begin{aligned} C &= \frac{2^{-([m]_q\rho(u)+1)}[m]_q\rho(u)}{[m]_q^3}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+1)_l}{[l]_q!2^l}2q[l]_q, \\ &= \frac{2\rho^2(u)q}{[m]_q}+\frac{2\rho(u)q}{[m]_q^2}. \end{aligned}$$

On adding A, B, and C we have,

$$\mathcal{L}_{m,q}^{\rho}(\rho^3;u)=\rho^3(u)q^3+\frac{(3q^3+q^2+2q)}{[m]_q}\rho^2(u)+\frac{(2q^3+q^2+2q+1)}{[m]_q^2}\rho(u).$$

(v)

$$\mathcal{L}_{m,q}^{\rho}(\rho^4;u) = 2^{-[m]_q\rho(u)}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u))_l}{[l]_q!2^l}\frac{[l]_q^4}{[m]_q^4}.$$

Now, by using $[l+1]_q=(1+q[l]_q)$ and shifting l to $l+1$, we have

$$\begin{aligned} \mathcal{L}_{m,q}^{\rho}(\rho^4;u) &= \frac{2^{-([m]_q\rho(u)+1)}[m]_q\rho(u)}{[m]_q^4}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+1)_l}{[l]_q!2^l}[l+1]_q^3 \\ &= \frac{2^{-([m]_q\rho(u)+1)}\rho(u)}{[m]_q^3}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+1)_l}{[l]_q!2^l}(1+q[l]_q)^3 \\ &= \frac{2^{-([m]_q\rho(u)+1)}\rho(u)}{[m]_q^3}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+1)_l}{[l]_q!2^l} \\ &+ \frac{2^{-([m]_q\rho(u)+1)}\rho(u)}{[m]_q^3}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+1)_l}{[l]_q!2^l}q^3[l]_q^3 \\ &+ \frac{2^{-([m]_q\rho(u)+1)}\rho(u)}{[m]_q^3}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+1)_l}{[l]_q!2^l}3q^2[l]_q^2 \\ &+ \frac{2^{-([m]_q\rho(u)+1)}\rho(u)}{[m]_q^3}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+1)_l}{[l]_q!2^l}3q[l]_q, \\ &= D+E+F+G(Say). \end{aligned}$$

Now, let us calculate the values of D, E, F, and G

$$\begin{aligned} D &= \frac{2^{-([m]_q\rho(u)+1)}\rho(u)}{[m]_q^3}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+1)_l}{[l]_q!2^l} \\ &= \frac{\rho(u)}{[m]_q^3}. \end{aligned}$$

$$
\begin{aligned}
E &= \frac{2^{-([m]_q\rho(u)+1)}\rho(u)}{[m]_q^3}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+1)_l}{[l]_q!2^l}q^3[l]_q^3 \\
&= \frac{2^{-([m]_q\rho(u)+2)}\rho(u)([m]_q\rho(u)+1)}{[m]_q^3}q^3\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+2)_l}{[l]_q!2^l}[l+1]_q^2 \\
&= \frac{2^{-([m]_q\rho(u)+2)}\rho(u)([m]_q\rho(u)+1)}{[m]_q^3}q^3\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+2)_l}{[l]_q!2^l}(1+q[l]_q)^2 \\
&= \frac{2^{-([m]_q\rho(u)+2)}\rho(u)([m]_q\rho(u)+1)}{[m]_q^3}q^3\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+2)_l}{[l]_q!2^l}(1+q^2[l]_q^2+2q[l]_q) \\
&= \left(\rho^4(u)+\frac{6\rho^3(u)}{[m]_q}+\frac{11\rho^2(u)}{[m]_q^2}+\frac{6\rho(u)}{[m]_q^3}\right)q^6+\left(\frac{\rho^3(u)}{[m]_q}+\frac{3\rho^2(u)}{[m]_q^2}+\frac{2\rho(u)}{[m]_q^3}\right)q^5 \\
&+ \left(\frac{2\rho^3(u)}{[m]_q}+\frac{6\rho^2(u)}{[m]_q^2}+\frac{4\rho(u)}{[m]_q^3}\right)q^4+\left(\frac{\rho^2(u)}{[m]_q^2}+\frac{\rho(u)}{[m]_q^3}\right)q^3.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
F &= \frac{2^{-([m]_q\rho(u)+1)}\rho(u)}{[m]_q^3}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+1)_l}{[l]_q!2^l}3q^2[l]_q^2 \\
&= \left(\frac{3\rho^3(u)}{[m]_q}+\frac{9\rho^2(u)}{[m]_q^2}+\frac{6\rho(u)}{[m]_q^3}\right)q^3+\left(\frac{3\rho^2(u)}{[m]_q^2}+\frac{3\rho(u)}{[m]_q^3}\right)q^2.
\end{aligned}
$$

Also,

$$
\begin{aligned}
G &= \frac{2^{-([m]_q\rho(u)+1)}\rho(u)}{[m]_q^3}\sum_{l=0}^{\infty}\frac{([m]_q\rho(u)+1)_l}{[l]_q!2^l}3q[l]_q \\
&= \left(\frac{3\rho^2(u)}{[m]_q^2}+\frac{3\rho(u)}{[m]_q^3}\right)q.
\end{aligned}
$$

On adding D, E, F, and G we have,

$$
\begin{aligned}
\mathcal{L}_{m,q}^{\rho}(\rho^4;u) &= \rho^4(u)q^6+\frac{(6q^6+q^5+2q^4+3q^3)}{[m]_q}\rho^3(u) \\
&+ \frac{(11q^6+3q^5+6q^4+1oq^3+3q^2+3q)}{[m]_q^2}\rho^2(u) \\
&+ \frac{(6q^6+2q^5+4q^4+7q^3+3q^2+3q+1)}{[m]_q^3}\rho(u).
\end{aligned}
$$

□

Corollary 1. *For $n=1,2,3,4$ the nth order central moments of $\mathcal{L}_{m,q}^{\rho}$ defined as $\mu_{n,m}^{\rho}(q;u)=\mathcal{L}_{m,q}^{\rho}((\rho(t)-\rho(u))_q^n;u)$, by using linearity of operators (4) and by Lemma 1 we have*

(i) $\mathcal{L}_{m,q}^{\rho}(\rho(t)-\rho(u);u) = \mathcal{L}_{m,q}^{\rho}(\rho(t);u)-\rho(u)\mathcal{L}_{m,q}^{\rho}(1;u)=0,$

(ii) $\mathcal{L}_{m,q}^{\rho}((\rho(t)-\rho(u))^2;u)=\mathcal{L}_{m,q}^{\rho}(\rho^2(t);u)+\rho^2(u)\mathcal{L}_{m,q}^{\rho}(1;u)-2\rho(u)\mathcal{L}_{m,q}^{\rho}(\rho(t);u)=\frac{(1+q)}{[m]_q}\rho(u),$

(iii) $\mathcal{L}_{m,q}^{\rho}((\rho(t)-\rho(u))^3;u) = \mathcal{L}_{m,q}^{\rho}(\rho^3(t);u)-\rho^3(u)\mathcal{L}_{m,q}^{\rho}(1;u)-3\rho(u)\mathcal{L}_{m,q}^{\rho}(\rho^2(t);u)-3\rho^2(u)\mathcal{L}_{m,q}^{\rho}(\rho(t);u)=(q^3-1)\rho^3(u)+\frac{(3q^3+q^2-q-3)}{[m]_q}\rho^2(u)+\frac{(2q^3+q^2+2q+1)}{[m]_q^2}\rho(u),$

(iv) $\mathcal{L}_{m,q}^{\rho}((\rho(t)-\rho(u))^4;u) = \mathcal{L}_{m,q}^{\rho}(\rho^4(t);u)+\rho^4(u)\mathcal{L}_{m,q}^{\rho}(1;u)+6\rho^2(u)\mathcal{L}_{m,q}^{\rho}(\rho^2(t);u)-4\rho^3(u)\mathcal{L}_{m,q}^{\rho}(\rho(t);u)-4\rho(u)\mathcal{L}_{m,q}^{\rho}(\rho^3(t);u) = (q^6-4q^3+3)\rho^4(u)+\frac{(6q^6+q^5+2q^4-9q^3-4q^2-2q+6)}{[m]_q}\rho^3(u)+\frac{(11q^6+3q^5+6q^4+2q^3-q^2-5q-4)}{[m]_q^2}\rho^2(u)+\frac{(6q^6+2q^5+4q^4+7q^3+3q^2+3q+1)}{[m]_q^3}\rho(u).$

Remark 1. *We observe from Lemma 1 and Corollary 1, that for $q = 1$, we get the moments and central moments of generalized Lupaş operators [7].*

2. Weighted Approximation

We start by noting that ρ not only defines a Korovkin-type set $\{1, \rho, \rho^2\}$ but also characterizes growth of the functions that are approximated.

Let $\phi(u) = 1 + \rho^2(u)$ be a weight function satisfying the conditions (ρ_1) and and (ρ_2) given above let $\mathcal{B}_\phi[0, \infty)$ be the weighted space defined by

$$\mathcal{B}_\phi[0, \infty) = \{f : [0, \infty) \to \mathbb{R} | |f(u)| \leq \mathcal{K}_f \phi(u), u \geq 0\},$$

where $\mathcal{K}_f$ is a constant which depends only on f. $\mathcal{B}_\phi[0, \infty)$ is a normed linear space equipped with the norm

$$\| f \|_\phi = \sup_{u \in [0,\infty)} \frac{|f(u)|}{\phi(u)}.$$

Also, we define the following subspaces of $\mathcal{B}_\phi[0, \infty)$ as

$$\mathcal{C}_\phi[0, \infty) = \{f \in \mathcal{B}_\phi[0, \infty) : \quad f \ \ is \ \ continuous \ \ on \quad [0, \infty)\},$$

$$\mathcal{C}^*_\phi[0, \infty) = \left\{ f \in \mathcal{C}_\phi[0, \infty) : \lim_{u \to \infty} \frac{f(u)}{\phi(u)} = \mathcal{K}_f \right\},$$

where $\mathcal{K}_f$ is a constant depending on f and

$$U_\phi[0, \infty) = \{f \in \mathcal{C}_\phi[0, \infty) : \frac{f(u)}{\phi(u)} \ \text{ is uniformly continuous on } \ [0, \infty)\}.$$

Obviously,

$$\mathcal{C}^*_\phi[0, \infty) \subset U_\phi[0, \infty) \subset \mathcal{C}_\phi[0, \infty) \subset \mathcal{B}_\phi[0, \infty).$$

For the weighted uniform approximation by linear positive operators acting from $\mathcal{C}_\phi[0, \infty)$ to $\mathcal{B}_\phi[0, \infty)$, we state the following results due to Gadjiev in [9,10].

Lemma 2 ([9]). *Let $(\mathcal{A}_m)_{m \geq 1}$ be a sequence of positive linear operators which acts from $\mathcal{C}_\phi[0, \infty)$ to $\mathcal{B}_\phi[0, \infty)$ if and only if the inequality*

$$|\mathcal{A}_m(\phi; u)| \leq \mathcal{K}_m \phi(u), \ \ u \geq 0,$$

holds, where $\mathcal{K}_m > 0$ is a constant depending on m.

Theorem 1 ([10]). *Let $(\mathcal{A}_m)_{m \geq 1}$ be a sequence of positive linear operators, acting from $\mathcal{C}_\phi[0, \infty)$ to $\mathcal{B}_\phi[0, \infty)$ and satisfying*

$$\lim_{m \to \infty} \| \mathcal{L}_m \rho^i - \rho^i \|_\phi = 0, \quad i = 0, 1, 2.$$

Then we have

$$\lim_{m \to \infty} \| \mathcal{L}_m(f) - f \|_\phi = 0, \quad for \quad any \quad f \in C^*_\phi[0, \infty).$$

Remark 2. *It is clear from Lemma 1 and Lemma 2 that the operators $\mathcal{L}^\rho_{m,q}$ act from $\mathcal{C}_\phi[0, \infty)$ to $\mathcal{B}_\phi[0, \infty)$. Also the convergence of these operators are applicable in studying switched linear systems, see: Subspace confinement for switched linear systems, also see in [24,25].*

Theorem 2. *Let q_m be a sequence in $(0,1)$, such that $q_m \to 1$ as $m \to \infty$. Then for each function $f \in C^*_\phi[0,\infty)$ we have*

$$\lim_{m\to\infty} \| \mathcal{L}^{\rho}_{m,q_m}(f) - f \|_{\phi} = 0.$$

Proof. By Lemma 1 (i) and (ii), it is clear that

$$\| \mathcal{L}^{\rho}_{m,q_m}(1;u) - 1 \|_{\phi} = 0.$$
$$\| \mathcal{L}^{\rho}_{m,q_m}(\rho;u) - \rho \|_{\phi} = 0.$$

and by Lemma 1 (iii), we have

$$\| \mathcal{L}^{\rho}_{m,q_m}(\rho^2;u) - \rho^2 \|_{\phi} = \sup_{u\in[0,\infty)} \frac{(1+q)\rho(u)}{[m]_q(1+\rho^2(u))} \leq \frac{1+q}{[m]_q}. \tag{5}$$

Then from Lemma 1 and (5) we get $\lim_{m\to\infty} \| \mathcal{L}^{\rho}_{m,q_m}(\rho^i) - \rho^i \|_{\phi} = 0, i = 0,1,2$. Hence, the proof is completed. □

3. Rate of Convergence or Order of Approximation

In this section, we determine the rate of convergence for $\mathcal{L}^{\rho}_{m,q}$ by weighted modulus of continuity $\omega_\rho(f;\delta)$ which was recently considered by Holhoş [11] as follows:

$$\omega_\rho(f;\delta) = \sup_{u,z\in[0,\infty), |\rho(z)-\rho(u)|\leq\delta} \frac{|f(z)-f(u)|}{\phi(z)+\phi(u)}, \quad \delta > 0, \tag{6}$$

where $f \in \mathcal{C}_\phi[0,\infty)$, with the following properties:

(i) $\omega_\rho(f;0) = 0$,

(ii) $\omega_\rho(f;\delta) \geq 0$, $\delta \geq 0$ for $f \in \mathcal{C}_\phi[0,\infty)$,

(iii) $\lim_{\delta\to 0} \omega_\rho(f;\delta) = 0$, for each $f \in U_\phi[0,\infty)$.

Theorem 3 ([11]). *Let $\mathcal{A}_m : \mathcal{C}_\phi[0,\infty) \to \mathcal{B}_\phi[0,\infty)$ be a sequence of positive linear operators with*

$$\| \mathcal{A}_m(\rho^0) - \rho^0 \|_{\phi^0} = a_m, \tag{7}$$
$$\| \mathcal{A}_m(\rho) - \rho \|_{\phi^{\frac{1}{2}}} = b_m, \tag{8}$$
$$\| \mathcal{A}_m(\rho^2) - \rho^2 \|_{\phi} = c_m, \tag{9}$$
$$\| \mathcal{A}_m(\rho^3) - \rho^3 \|_{\phi^{\frac{3}{2}}} = d_m, \tag{10}$$

where the sequences a_m, b_m, c_m, and d_m converge to zero as $m \to \infty$. Then

$$\| \mathcal{A}_m(f) - f \|_{\phi^{\frac{3}{2}}} \leq (7 + 4a_m + 2c_m)\omega_\rho(f;\delta_m) + \| f \|_{\phi} a_m, \tag{11}$$

for all $f \in \mathcal{C}_\phi[0,\infty)$, where

$$\delta_m = 2\sqrt{(a_m + 2b_m + c_m)(1+a_m)} + a_m + 3b_m + 3c_m + d_m.$$

Theorem 4. *Let for each $f \in \mathcal{C}_\phi[0,\infty)$ with $0 < q < 1$. Then we have*

$$\| \mathcal{L}^{\rho}_{m,q}(f) - f \|_{\phi^{\frac{3}{2}}} \leq \left(7 + \frac{2(1+q)}{[m]_q}\right)\omega_\rho(f;\delta_{m,q}),$$

where ω_ρ is the weighted modulus of continuity defined in (6) and

$$\delta_{m,q} = 2\sqrt{\frac{(1+q)}{[m]_q}} + \frac{3(1+q)}{[m]_q} + \left((q^3-1) + \frac{(3q^3+q^2+2q)}{[m]_q} + \frac{(2q^3+q^2+2q+1)}{[m]_q^2}\right).$$

Proof. By using Lemma 1, we have

$$\| \mathcal{L}_{m,q}^{\rho}(\rho^0) - \rho^0 \|_{\phi^0} = a_{m,q} = 0,$$

$$\| \mathcal{L}_{m,q}^{\rho}(\rho) - \rho \|_{\phi^{\frac{1}{2}}} = b_{m,q} = 0,$$

and

$$\| \mathcal{L}_{m,q}^{\rho}(\rho^2) - \rho^2 \|_{\phi} \leq \frac{(1+q)}{[m]_q} = c_{m,q}.$$

Finally,

$$\| \mathcal{L}_{m,q}^{\rho}(\rho^3) - \rho^3 \|_{\phi^{\frac{3}{2}}} \leq (q^3-1) + \frac{(3q^3+q^2+2q)}{[m]_q} + \frac{(2q^3+q^2+2q+1)}{[m]_q^2} = d_{m,q}.$$

Thus, the sequences $a_{m,q}$, $b_{m,q}$, $c_{m,q}$, and $d_{m,q}$ are calculated. The sequences a_m, b_m, c_m, and d_m converge to zero as $m \to \infty$. Then

$$\| \mathcal{L}_{m,q}^{\rho}(f) - f \|_{\phi^{\frac{3}{2}}} \leq (7 + 4a_{m,q} + 2c_{m,q})\omega_\rho(f;\delta_{m,q}) + \| f \|_{\phi} a_{m,q}, \tag{12}$$

for all $f \in C_\phi[0,\infty)$, where

$$\delta_{m,q} = 2\sqrt{(a_{m,q} + 2b_{m,q} + c_{m,q})(1 + a_{m,q})} + a_{m,q} + 3b_{m,q} + 3c_{m,q} + d_{m,q}.$$

Hence, by substituting the values of $a_{m,q}$, $b_{m,q}$, $c_{m,q}$ and $d_{m,q}$ we obtain the desired result. □

Remark 3. *For* $\lim_{\delta\to 0} \omega_\rho(f;\delta) = 0$ *in Theorem 4, we find*

$$\lim_{m\to\infty} \| \mathcal{L}_{m,q}^{\rho}(f) - f \|_{\phi^{\frac{3}{2}}} = 0, \quad \text{for} \quad f \in U_\phi[0,\infty).$$

4. Voronovskaya-Type Theorem

In this section, using a technique developed in [5] by Cardenas-Morales, Garrancho and Raşa, we prove pointwise convergence of $\mathcal{L}_{m,q}^{\rho}$ by obtaining Voronovskaya-type theorems.

Theorem 5. *Let $f \in C_\phi[0,\infty)$, $u \in [0,\infty)$ with $0 < q_m < 1, q_m \to 1$ as $m \to \infty$. Suppose that $\left(fo\rho^{-1}\right)'$ and $\left(fo\rho^{-1}\right)''$ exist at $\rho(u)$. If $\left(fo\rho^{-1}\right)''$ is bounded on $[0,\infty)$, then we have*

$$\lim_{m\to\infty} [m]_{q_m} \left[\mathcal{L}_{m,q_m}^{\rho}(f;u) - f(u)\right] = \rho(u) \left(fo\rho^{-1}\right)''(\rho(u)).$$

Proof. By using the q-Taylor expansion of $\left(fo\rho^{-1}\right)$ at $\rho(u) \in [0,\infty)$, there exist a point w lying between u and z, then we have

$$\begin{aligned} f(w) = \left(fo\rho^{-1}\right)(\rho(w)) &= \left(fo\rho^{-1}\right)(\rho(u)) + \left(fo\rho^{-1}\right)'(\rho(u))\,(\rho(w)-\rho(u)) \\ &+ \frac{\left(fo\rho^{-1}\right)''(\rho(u))\,(\rho(w)-\rho(u))^2}{[2]_q} + \lambda_u^q(w)\,(\rho(w)-\rho(u))^2, \end{aligned} \tag{13}$$

where

$$\lambda_u^q(w) = \frac{\left(fo\rho^{-1}\right)''(\rho(w)) - \left(fo\rho^{-1}\right)''(\rho(u))}{[2]_q}. \tag{14}$$

Therefore, by (14) together with the assumption on f ensures that

$$|\lambda_u^q(w)| \leq \mathcal{K}, \quad \text{for} \quad \text{all} \quad w \in [0,\infty)$$

and is convergent to zero as $w \to u$. Now applying the operators (4) to the equality (13), we obtain

$$\begin{aligned} \left[\mathcal{L}_{m,q_m}^{\rho}(f;u) - f(u)\right] &= \left(fo\rho^{-1}\right)'(\rho(u))\mathcal{L}_{m,q_m}^{\rho}\left((\rho(w)-\rho(u));u\right) \\ &+ \frac{\left(fo\rho^{-1}\right)''(\rho(u))\mathcal{L}_{m,q_m}^{\rho}\left((\rho(w)-\rho(y))^2;u\right)}{[2]_q} \\ &+ \mathcal{L}_{m,q_m}^{\rho}\left(\lambda_u^q(w)\left((\rho(w)-\rho(u))^2;u\right)\right). \end{aligned} \tag{15}$$

By Lemma 1 and Corollary 1, we get

$$\lim_{m\to\infty}[m]_{q_m}\mathcal{L}_{m,q_m}^{\rho}\left((\rho(w)-\rho(u));u\right) = 0, \tag{16}$$

and

$$\lim_{m\to\infty}[m]_{q_m}\mathcal{L}_{m,q_m}^{\rho}\left((\rho(w)-\rho(u))^2;u\right) = [2]_q\rho(u). \tag{17}$$

By estimating the last term on the right hand side of equality (15), we will get the proof.

Since from (14), for every $\epsilon > 0$, $\lim_{w\to u}\lambda_u^{q_m}(w) = 0$. Let $\delta > 0$ such that $|\lambda_u^{q_m}(w)| < \epsilon$ for every $w \geq 0$. Using a Cauchy-Schwartz inequality, we have

$$\begin{aligned} \lim_{m\to\infty}[m]_{q_m}\mathcal{L}_{m,q_m}^{\rho}\left(|\lambda_u^{q_m}(w)|\left(\rho(w)-\rho(u)\right)^2;u\right) &\leq \epsilon\lim_{m\to\infty}[m]_{q_m}\mathcal{L}_{m,q_m}^{\rho}\left((\rho(w)-\rho(u))_{q_m}^2;u\right) \\ &+ \frac{\mathcal{K}}{\delta^2}\lim_{m\to\infty}[m]_{q_m}\mathcal{L}_{m,q_m}^{\rho}\left((\rho(w)-\rho(u))_{q_m}^4;u\right). \end{aligned}$$

Since

$$\lim_{m\to\infty}[m]_{q_m}\mathcal{L}_{m,q_m}^{\rho}\left((\rho(w)-\rho(u))_{q_m}^4;u\right) = 0, \tag{18}$$

we obtain

$$\lim_{m\to\infty}[m]_{q_m}\mathcal{L}_{m,q_m}^{\rho}\left(|\lambda_u^{q_m}(w)|\left(\rho(w)-\rho(y)\right)_{q_m}^2;y\right) = 0. \tag{19}$$

Thus, by using Equations (16), (17) and (19) to Equation (15) the proof is completed. □

5. Local Approximation

In this section, we present local approximation theorems for the operators $\mathcal{L}_{m,q}^{\rho}$. By $C_B[0,\infty)$, we denote the space of real-valued continuous and bounded functions f defined on the interval $[0,\infty)$. The norm $\|\cdot\|$ on the space $C_B[0,\infty)$ is given by

$$\| f \| = \sup_{0\leq u<\infty} | f(x) | .$$

For $\delta > 0$ and $W^2 = \{s \in C_B[0,\infty) : s', s'' \in C_B[0,\infty)\}$. The $\mathcal{K}$-functional is defined as

$$\mathcal{K}_2(f,\delta) = \inf_{s\in W^2}\{\| f - s \| + \delta \| g'' \|\},$$

By Devore and Lorentz ([26], p. 177, Theorem 2.4), there exists an absolute constant $\mathcal{C} > 0$ such that

$$\mathcal{K}(f,\delta) \leq \mathcal{C}\omega_2(f,\sqrt{\delta}). \tag{20}$$

The second order modulus of smoothness is as follows,

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{u\in[0,\infty)} \mid f(u+2h) - 2f(u+h) + f(u) \mid$$

where $f \in C_B[0,\infty)$. The usual modulus of continuity of $f \in C_B[0,\infty)$ is defined by

$$\omega(f,\delta) = \sup_{0<h\leq\delta} \sup_{u\in[0,\infty)} \mid f(u+h) - f(u) \mid .$$

Theorem 6. *Let $f \in \mathcal{C}_B[0,\infty)$ with $0 < q < 1$. Let ρ be a function satisfying the conditions (ρ_1), (ρ_2), and $||\rho''||$ is finite. Then, there exists an absolute constant $\mathcal{C} > 0$ such that*

$$|\mathcal{L}^{\rho}_{m,q}(f;u) - f(u)| \leq \mathcal{C}\mathcal{K}\left(f, \frac{(1+q)}{[m]_q}\rho(u)\right).$$

Proof. Let $s \in W^2$ and $u, z \in [0,\infty)$. Using Taylor's formula we have

$$s(z) = s(u) + \left(so\rho^{-1}\right)'(\rho(u))(\rho(z)-\rho(u)) + \int_{\rho(u)}^{\rho(z)} (\rho(z)-v)\left(so\rho^{-1}\right)''(v)dv. \tag{21}$$

Using the equality

$$\left(so\rho^{-1}\right)''(\rho(u)) = \frac{s''(u)}{(\rho'(u))^2} - s''(u)\frac{\rho''(u)}{(\rho'(u))^3}. \tag{22}$$

Now, put $v = \rho(y)$ in the last term in equality (21), we get

$$\begin{aligned}
\int_{\rho(u)}^{\rho(z)} (\rho(z)-v)\left(so\rho^{-1}\right)''(v)dv &= \int_u^z (\rho(z)-\rho(y))\left[\frac{s''(y)\rho'(y) - s'(y)\rho''(v)}{(\rho'(y))^2}\right]dy \\
&= \int_{\rho(u)}^{\rho(z)} (\rho(z)-v)\frac{s''(\rho^{-1}(v))}{(\rho'(\rho^{-1}(v))^2}dv \\
&- \int_{\rho(u)}^{\rho(z)} (\rho(z)-v)\frac{s'(\rho^{-1}(v))\rho''(\rho^{-1}(v))}{(\rho'(\rho^{-1}(v))^3}dv.
\end{aligned} \tag{23}$$

By using Lemma 1 and (23) and applying the operator (4) to the both sides of equality (21), we deduce

$$\begin{aligned}
\mathcal{L}^{\rho}_{m,q}(s;u) &= s(u) + \mathcal{L}^{\rho}_{m,q}\left(\int_{\rho(u)}^{\rho(z)} (\rho(z)-v)\frac{s''(\rho^{-1}(v))}{(\rho'(\rho^{-1}(v))^2}dv;u\right) \\
&- \mathcal{L}^{\rho}_{m,q}\left(\int_{\rho(u)}^{\rho(z)} (\rho(z)-v)\frac{s'(\rho^{-1}(v))\rho''(\rho^{-1}(v))}{(\rho'(\rho^{-1}(v))^3}dv;u\right).
\end{aligned}$$

As we know ρ is strictly increasing on $[0,\infty)$ and with condition (ρ_2), we have

$$|\mathcal{L}^{\rho}_{m,q}(s;u) - s(u)| \leq \mathcal{M}^{\rho}_{m,2}(u)(\|s''\| + \|s'\|\|\rho''\|),$$

where

$$\mathcal{M}^{\rho}_{m,2}(u) = \mathcal{L}^{\rho}_{m,q}((\rho(t)-\rho(u))^2;u).$$

For $f \in \mathcal{C}_B[0,\infty)$, we have

$$\begin{aligned} \left|\mathcal{L}^{\rho}_{m,q}(s;u)\right| &\leq \|f o\rho^{-1}\| 2^{-[m]_q\rho(u)} \sum_{l=0}^{\infty} \frac{([m]_q\rho(u))_l}{2^l [l]_q!} \\ &\leq \|f\| \mathcal{L}^{\rho}_{m,q}(1;u) = \|f\|. \end{aligned} \tag{24}$$

Hence we have

$$\begin{aligned} \left|\mathcal{L}^{\rho}_{m,q}(f;u) - f(u)\right| &\leq \left|\mathcal{L}^{\rho}_{m,q}(f-s;u)\right| + \left|\mathcal{L}^{\rho}_{m,q}(s;u) - s(u)\right| + \left|s(u) - f(u)\right| \\ &\leq 2\|f-g\| + \frac{(1+q)}{[m]_q}\rho(u)\left(\|s''\| + \|s'\|\|\rho''\|\right). \end{aligned}$$

If we choose $\mathcal{C} = \max\{2, \|\rho''\|\}$, then

$$\left|\mathcal{L}^{\rho}_{m,q}(f;u) - f(u)\right| \leq \mathcal{C}\left(2\|f-g\| + \frac{(1+q)}{[m]_q}\rho(u)\|s''\|_{W^2}\right).$$

Taking infimum over all $s \in W^2$ we obtain

$$\left|\mathcal{L}^{\rho}_{m,q}(f;u) - f(u)\right| \leq \mathcal{C}\mathcal{K}\left(f, \frac{(1+q)}{[m]_q}\rho(u)\right).$$

□

Now, we recall local approximation in terms of α order Lipschitz-type maximal functions given in [27]. Let ρ be a function satisfying the conditions (ρ_1), (ρ_2), $0 < \alpha \leq 1$ and $Lip_{\mathcal{M}}(\rho(u);\alpha)$, $\mathcal{M} \geq 0$ is the set of functions$f$ satisfying the inequality

$$\left|f(z) - f(u)\right| \leq \mathcal{M}\left|\rho(z) - \rho(u)\right|^{\alpha}, u, z \geq 0.$$

Moreover, for a bounded subset $\mathcal{E} \subset [0,\infty)$, we say that the function $f \in \mathcal{C}_B[0,\infty)$ belongs to $Lip_{\mathcal{M}}(\rho(u);\alpha)$, $0 < \alpha \leq 1$ on $\mathcal{E}$ if

$$\left|f(z) - f(u)\right| \leq \mathcal{M}_{\alpha,f}\left|\rho(z) - \rho(u)\right|^{\alpha}, u \in \mathcal{E} \quad and \quad z \geq 0,$$

where $\mathcal{M}_{\alpha,f}$ is a constant depending on α and f.

Theorem 7. *Let ρ be a function satisfying the conditions (ρ_1), $(\rho_2.)$ Then for any $f \in Lip_{\mathcal{M}}(\rho(u);\alpha)$, $0 < \alpha \leq 1$ with $0 < q < 1$ and for every $u \in (0,\infty)$, $m \in \mathbb{N}$, we have*

$$\left|\mathcal{L}^{\rho}_{m,q}(f;u) - f(u)\right| \leq \mathcal{M}\left(\frac{(1+q)}{[m]_q}\rho(u)\right)^{\frac{\alpha}{2}}, \tag{25}$$

Proof. Assume that $\alpha = 1$. Then, for $f \in Lip_{\mathcal{M}}(\alpha;1)$ and $u \in (0,\infty)$, we have

$$\begin{aligned} |\mathcal{L}^{\rho}_{m,q}(f;u) - f(u)| &\leq \mathcal{L}^{\rho}_{m,q}(|f(z) - f(u)|;u) \\ &\leq \mathcal{M}\mathcal{L}^{\rho}_{m,q}(|\rho(z) - f(u)|;u). \end{aligned}$$

By applying the Cauchy–Schwartz inequality, we find

$$\begin{aligned} |\mathcal{L}^{\rho}_{m,q}(f;u)-f(u)| &\leq \mathcal{M}\left[\mathcal{L}^{\rho}_{m,q}((\rho(t)-\rho(u))^2;u)\right]^{\frac{1}{2}} \\ &\leq \mathcal{M}\sqrt{\frac{(1+q)\rho(u)}{[m]_q}}. \end{aligned}$$

Let us assume that $\alpha \in (0,1)$. Then, for $f \in Lip_{\mathcal{M}}(\alpha;1)$ and $u \in (0,\infty)$, we have

$$\begin{aligned} |\mathcal{L}^{\rho}_{m,q}(f;u)-f(u)| &\leq \mathcal{L}^{\rho}_{m,q}(|f(z)-f(u)|;u) \\ &\leq \mathcal{M}\mathcal{L}^{\rho}_{m,q}(|\rho(z)-f(u)|^{\alpha};u). \end{aligned}$$

From Hölder's inequality with $p=\frac{1}{\alpha}$ and $q=\frac{1}{1-\alpha}$, for $f \in Lip_{\mathcal{M}}(\rho(u);\alpha)$, we have

$$|\mathcal{L}^{\rho}_{m,q}(f;u)-f(u)| \leq \mathcal{M}\left[\mathcal{L}^{\rho}_{m,q}(|(\rho(t)-\rho(u)|;u)\right]^{\alpha}.$$

Finally by the Cauchy–Schwartz inequality, we get

$$|\mathcal{L}^{\rho}_{m,q}(f;u)-f(u)| \leq \mathcal{M}\left(\frac{(1+q)}{[m]_q}\rho(u)\right)^{\frac{\alpha}{2}}.$$

□

A relationship between local smoothness of functions and the local approximation was given by Agratini in [28]. Here we will prove the similar result for operators $\mathcal{L}^{\rho}_{m,q}$ $(m \in \mathbb{N})$ for functions from $Lip_{\mathcal{M}}(\rho(u))$ on a bounded subset.

Theorem 8. *Let $\mathcal{E}$ be a bounded subset of $[0,\infty)$ and ρ be a function satisfying the conditions (ρ_1), (ρ_2). Then for any $f \in Lip_{\mathcal{M}}(\rho(u);\alpha)$, $0<\alpha\leq 1$ on $\mathcal{E}$ $\alpha \in (0,1]$, we have*

$$|\mathcal{L}^{\rho}_{m,q}(f;u)-f(u)| \leq \mathcal{M}_{\alpha,f}\left\{\left[\frac{(1+q)\rho(u)}{[m]_q}\right]^{\frac{\alpha}{2}} + 2[\rho'(u)]^{\alpha}d^{\alpha}(u,\mathcal{E})\right\}, u \in [0,\infty), m \in \mathbb{N},$$

where $d(u,\mathcal{E}) = inf\{\|u-y\| : y \in \mathcal{E}\}$ and $\mathcal{M}_{\alpha,f}$ is a constant depending on α and f.

Proof. Let $\overline{\mathcal{E}}$ be the closure of $\mathcal{E}$ in $[0,\infty)$. Then, there exists a point $u_0 \in \overline{\mathcal{E}}$ such that $d(u,\mathcal{E}) = |u-u_0|$. Using the monotonicity of $\mathcal{L}^{\rho}_{m,q}$ and the hypothesis of f, we obtain

$$\begin{aligned} |\mathcal{L}^{\rho}_{m,q}(f;u)-f(u)| &\leq \mathcal{L}^{\rho}_{m,q}\left(|f(z)-f(u_0)|;u\right) + \mathcal{L}^{\rho}_{m,q}\left(|f(u)-f(u_0)|;u\right) \\ &\leq \mathcal{M}_{\alpha,f}\left\{\mathcal{L}^{\rho}_{m,q}\left(|\rho(z)-\rho(u_0)|^{\alpha};u\right) + |\rho(u)-\rho(u_0)|^{\alpha}\right\} \\ &\leq \mathcal{M}_{\alpha,f}\left\{\mathcal{L}^{\rho}_{m,q}\left(|\rho(z)-\rho(u)|^{\alpha};u\right) + 2|\rho(u)-\rho(u_0)|^{\alpha}\right\}. \end{aligned}$$

By choosing $p=\frac{2}{\alpha}$ and $q=\frac{2}{2-\alpha}$, as well as the fact $|\rho(u)-\rho(u_0)| = \rho'(u)|\rho(u)-\rho(u_0)|$ in the last inequality. Then by using Hölder's inequality we easily conclude

$$|\mathcal{L}^{\rho}_{m,q}(f;u)-f(u)| \leq \mathcal{M}_{\alpha,f}\left\{\left[\mathcal{L}^{\rho}_{m,q}((\rho(z)-\rho(u))^2;u)\right]^{\frac{1}{2}} + 2[\rho'(u)|\rho(u)-\rho(u_0)|]^{\alpha}\right\}.$$

Hence, by Corollary 1 we get the proof. □

Now, for $f \in C_B[0,\infty)$, we recall local approximation in terms of α order generalized Lipschitz-type maximal function given by Lenze [29] as

$$\widetilde{\omega}^{\rho}_{\alpha}(f;u) = \sup_{z\neq u, z\in(0,\infty)} \frac{|f(z)-f(u)|}{|z-u|^{\alpha}}, \ u \in [0,\infty) \text{ and } \alpha \in (0,1]. \tag{26}$$

Then we get the next result

Theorem 9. *Let $f \in C_B[0,\infty)$ and $\alpha \in (0,1]$ with $0<q<1$. Then, for all $u \in [0,\infty)$, we have*

$$|\mathcal{L}^{\rho}_{m,q}(f;u) - f(u)| \leq \widetilde{\omega}^{\rho}_{\alpha}(f;u) \left(\frac{(1+q)}{[m]_q} \rho(u) \right)^{\frac{\alpha}{2}}.$$

Proof. We know that

$$|\mathcal{L}^{\rho}_{m,q}(f;u) - f(u)| \leq \mathcal{L}^{\rho}_{m,q}(|f(t)-f(u)|;u).$$

From Equation (26), we have

$$|\mathcal{L}^{\rho}_{m,q}(f;u) - f(u)| \leq \widetilde{\omega}^{\rho}_{\alpha}(f;u) \mathcal{L}^{\rho}_{m,q}(|\rho(z)-\rho(u)|^{\alpha};u).$$

From Hölder's inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we have

$$\begin{aligned} |\mathcal{L}^{\rho}_{m,q}(f;u) - f(u)| &\leq \widetilde{\omega}^{\rho}_{\alpha}(f;u) \left[\mathcal{L}^{\rho}_{m,q}((\rho(t)-\rho(u))^2;u)\right]^{\frac{\alpha}{2}} \\ &\leq \widetilde{\omega}^{\rho}_{\alpha}(f;u) \left(\frac{(1+q)}{[m]_q} \rho(u) \right)^{\frac{\alpha}{2}}. \end{aligned}$$

which proves the desired result. □

6. Conclusions

Here, the q-analogue of the generalized Lupaş operators are constructed. We have investigated convergence properties, order of approximation, Voronovskaja-type results and also quantitative estimates for the local approximation. The constructed operators provide better flexibility in approximating functions and rate of convergence which are dependent on the selection of the function ρ and extra parameter q. These operators also possess interesting properties and depending on the selection of q, can obtain better approximation while $q \neq 1$. The basis of these operators can be used to draw curves and surfaces in Computer Aided Geometric Design (CAGD).

Author Contributions: Supervision, A.K. and Z.A.; Writing—original draft, M.Q.; Writing—review and editing, M.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: The authors are thankful to the referees for their valuable comments and suggestions. The first author is also grateful to the Council of Scientific and Industrial Research (CSIR), India, for providing the Senior Research Fellowship (09/1172(0001)/2017-EMR-I).

Conflicts of Interest: We declare that there is no conflict of interest.

References

1. Bernstein, S.N. Démonstation du théorème de Weierstrass fondée sur le calcul de probabilités. *Commun. Soc. Math. Kharkow* **1912**, *13*, 1–2.
2. Weierstrass, K. *Über die Analytische Darstellbarkeit Sogenannter Willkürlicher Functionen Einer Reellen Veränderlichen*; Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin: Berlin, Gemany, 1885; pp. 633–639, 789–805.

3. Korovkin, P.P. On convergence of linear operators in the space of continuous functions (Russian). *Dokl. Akad. Nauk SSSR (N.S.)* **1953**, *90*, 961–964.
4. Lorentz, G.G. *Bernstein Polynomials*; University of Toronto Press: Toronto, ON, Canada, 1953.
5. Cárdenas-Morales, D.; Garrancho, P.; Rasa, I. Bernstein-type operators which preserve polynomials. *Comput. Math. Appl.* **2011**, *62*, 158–163. [CrossRef]
6. Aral, A.; Inoan, D.; Rasa, I. On the generalized Szász-Mirakyan operators. *Results Math.* **2014**, *65*, 441–452. [CrossRef]
7. İlarslan, H.G.I.; Aral, A.; Başcanbaz-Tunca, G. Generalized Lupaş operators. *AIP Conf. Proc.* **2018**, *1926*, 020019.
8. Lupaş, A. The approximation by some positive linear operators. In *Approximation Theory, Proceedings of the International Dortmund Meeting, IDoMAT 95, Witten, Germany, 13–17 March 1995*; Müller, M.W., Felten, M., Mache, D.H., Eds.; Akademie Verlag: Berlin, Germany, 1995; pp. 201–229.
9. Gadzhiev, A.D. A problem on the convergence of a sequence of positive linear operators on unbounded sets, and theorems that are analogous to P. P. Korovkin's theorem. *Dokl. Akad. Nauk SSSR* **1974**, *218*, 1001–1004. (In Russian)
10. Gadjiev, A.D. Theorems of the type of P. P. Korovkin's theorems. *Math. Zamet* **1976**, *20*, 781–786.
11. Holhoş, A. Quantitative estimates for positive linear operators in weighted spaces. *Gen. Math.* **2008**, *16*, 99–110.
12. Kac, V.; Cheung, P. *Quantum Calculus*; Springer: New York, NY, USA, 2002.
13. Lupaş, A. A q-analogue of the Bernstein operator. In *Seminar on Numerical and Statistical Calculus*; University of Cluj-Napoca: Cluj-Napoca, Rominia, 1987; Volume 9; pp. 85–92.
14. Phillips, G.M. Bernstein polynomials based on the q-integers. *Ann. Numer. Math.* **1997** *4*, 511–518.
15. Khan, K.; Lobiyal, D.K. Bézier curves based on Lupaş (p,q)-analogue of Bernstein functions in CAGD. *J. Comput. Appl. Math.* **2017**, *317*, 458–477. [CrossRef]
16. Khan, K.; Lobiyal, D.K.; Kilicman, A. A de Casteljau Algorithm for Bernstein type Polynomials based on (p,q)-integers. *Appl. Appl. Math.* **2018**, *13*, 997–1017.
17. Khan, K.; Lobiyal, D.K.; Kilicman, A. Bézier curves and surfaces based on modified Bernstein polynomials. *Azerbaijan J. Math.* **2019**, *9*, 3–21.
18. Aral, A.; Dŏgru, O. Bleimann-Butzer and Hahn operators based on q-integers. *J. Inequalities Appl.* **2008**, *2007*, 079410. [CrossRef]
19. Aral, A.; Gupta, V.; Agarwal, R.P. *Applications of q-Calculus in Operator Theory*; Springer: New York, NY, USA, 2013.
20. Derriennic, M.M. Modified Bernstein polynomials and Jacobi polynomials in q-calculus. *Rend. Circ. Mat. Palermo Ser. II* **2005**, *76*, 269–290.
21. Mahmudov, N.I. Approximation by the q-Szász-Mirakjan Operators. *Abstr. Appl. Anal.* **2012**, *2012*, 754217. [CrossRef]
22. Mishra, V.N.; Patel, P. On generalized integral Bernstein operators based on q-integers. *Appl. Math. Comput.* **2014**, *242*, 931–944. [CrossRef]
23. Mursaleen, M.; Ahasan, M. The Dunkl generalization of Stancu type q-Szász-Mirakjan-Kantrovich operators and some approximation results. *Carpathian J. Math.* **2018**, *34*, 363–370.
24. Shang, Y. Subspace confinement for switched linear systems. *Forum Math.* **2017**, *29*, 693–699. [CrossRef]
25. Wang, N.; Egerstedt, M.; Martin, C. Stability of switched linear systems and the convergence of random products. In Proceedings of the 48h IEEE Conference on Decision and Control (CDC) Held Jointly with 2009 28th Chinese Control Conference, Shanghai, China, 15–18 December 2009; pp. 3721–3726.
26. DeVore, R.A.; Lorentz, G.G. Constructive Approximation, Grundlehren Math. Wiss. In *Fundamental Principales of Mathematical Sciences*; Springer: Berlin, Germany, 1993.
27. Gadzhiev, A.D.; Aral, A. The estimates of approximation by using a new type of weighted modulus of continuity. *Comput. Math. Appl.* **2007**, *54*, 127–135.
28. Agratini, O. On the rate of convergence of a positive approximation process. *Nihonkai Math. J.* **2000**, *11*, 47–56.
29. Lenze, B. On Lipschitz type maximal functions and their smoothness spaces. *Indag. Math.* **1988**, *91*, 53–63. [CrossRef]