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Picture Fuzzy Interaction Partitioned Heronian Aggregation Operators for Hotel Selection

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Abstract: Picture fuzzy numbers (PFNs), as the generalization of fuzzy sets, are good at fully expressing decision makers' opinions with four membership degrees. Since aggregation operators are simple but powerful tools, this study aims to explore some aggregation operators with PFNs to solve practical decision-making problems. First, new operational rules, the interaction operations of PFNs, are defined to overcome the drawbacks of existing operations. Considering that interrelationships may exist only in part of criteria, rather than all of the criteria in reality, the partitioned Heronian aggregation operator is modified with PFNs to deal with this condition. Then, desirable properties are proved and several special cases are discussed. New decision-making methods with these presented aggregation operators are suggested to process hotel selection issues. Last, their practicability and merits are certified by sensitivity analyses and comparison analyses with other existing approaches. The results indicate that our methods are feasible to address such situations where criteria interact in the same part, but are independent from each other at different parts.

Keywords: picture fuzzy numbers; interaction operations; partitioned Heronian; aggregation operators; hotel selection

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1. Introduction

Decision making refers to selecting the optimal alternative according to some rules [1–3]. Generally, many uncertain or fuzzy factors may exist in the real decision-making process, which are not able to be described by crisp numbers [4,5]. In this case, fuzzy set theory [6], which was presented by Zadeh, can be used to express these uncertainty and vagueness. Nevertheless, because fuzzy set just has a membership degree, it cannot depict information in some circumstances, specifically where decision makers (DMs) disagree with each other. Then, Atanassov [7] put forward the concept of intuitionistic fuzzy sets (IFSs) with membership and non-membership functions, so that both people's consistency and inconsistency can be conveyed by IFSs. However, different DMs may have different attitudes for a certain decision-making issue, not just 'support' or 'opposition'. Hence, there are two inherent flaws in IFSs. (1) Except for consistent and inconsistent degrees, other possibilities, such as hesitant or refusal degrees, are not contained in the IFSs. (2) In addition, another shortcoming of IFSs is in their traditional operations: the interaction between membership degree and non-membership degree is not considered.

To overcome the first drawback, Cuong [8] first proposed picture fuzzy numbers (PFNs). PFNs can sufficiently depict DMs' diverse behaviors using four membership functions, including positive, neutral, negative, and refusal membership degrees [9]. Since then, the distance/similarity measure [10,11], cross entropy [12], and projection [13,14] of PFNs have been defined. Moreover, various decision-making methods based on them have been studied [15,16], such as the multi-attributive border approximation area comparison (MABAC) method [17] and the Vlsekriterijumska Optimizacija I Kompromisno

Resenje (VIKOR) method [13]. Clearly, PFNs show greater performances than fuzzy sets and IFSs in conveying complicated evaluation information. However, there is also an imperfection in the operations of PFNs: (3) The interactions among four different membership functions of PFNs are not taken into account fully. Thus, improper results may occur in certain cases, specifically when the neutral membership degree in a picture fuzzy number (PFN) is zero. When this happens, regardless of the value(s) of neutral membership degree(s) in other PFN(s), their aggregated neutral membership degree using the traditional operations is still zero.

To conquer disadvantage (2), He et al. [18] redefined the operational rules of IFSs through making an interaction between membership function and non-membership function. Likewise, for overcoming limitation (3), some interaction operational laws of PFNs are proposed.

When it comes to decision-making methods, information aggregation operators are basic and powerful at dealing with decision-making problems [19,20]. To date, several aggregation operators have been extended with PFNs. For example, Garg [21] defined the arithmetic weighted averaging, ordered weighted averaging, and hybrid averaging operators of PFNs; Wang et al. [22] adopted the geometric aggregation operators to aggregate picture fuzzy information; and Wei [23] integrated the Hamacher aggregation operators with PFNs to process practical problems. Nevertheless, a common defect of these operators is that (4) all of these operators presume that the parameters are standalone, and the interrelations among them are not considered at all. In reality, some inputs may be dependent on each other, but the above-mentioned operators are unable to deal with this situation. Lately, Xu [24] discussed the Muirhead mean operator in a picture fuzzy environment. The Muirhead mean operator is a useful method to capture the interrelationships among inputs. However, it may be difficult for DMs to determine a vector of parameters in the Muirhead mean operator.

In this case, the Bonferroni averaging [25,26] or Heronian averaging (HA) [27,28] operator may be a good choice to overcome limitation (4). Both of them can establish the relationships between two arguments, and only two parameter values need to be assigned by DMs. Compared with the Bonferroni averaging operator, the great advantages of the HA operator include: the correlation between inputs and itself is also taken into account, and no redundancy exists in the aggregated values. To date, the HA operator has been widely extended with various fuzzy sets, such as IFSs [29] and linguistic neutrosohpic numbers [30]. The limitations of this operator are related to two aspects. (5) Until now, no HA operator has been modified to aggregate picture fuzzy information; and (6) there is a hypothesis in the HA operator that each input is relevant with the remaining inputs, but this is not true at all times. More commonly, parts of arguments have relationships with each other, while no connections exist among several parts. Apparently, the conventional HA operator is helpless to handle this situation.

For coping with defect (6), the concepts of partitioned HA (PHA) and partitioned geometric HA (PGHA) operators are put forward by Liu et al. [31]. The function of these operators is to address such a general circumstance where the inputs are classified into several partitions, and the interrelationships are just found among the inputs in the same partition but not found among those in distinct partitions. Motivated by this idea, this study proposes some PHA operators within a picture fuzzy condition to circumvent weaknesses (5) and (6).

The main contributions of this study are outlined in the following.

First, new operational laws of PFNs are defined to capture the interactions among four membership degrees, so that limitation (3) is overcome.

Second, some PHA operators are extended with PFNs based on the new interaction operation rules. They can settle the situation where criteria need to be partitioned, as correlations can be seen only in the same part rather that in different parts. Special cases and important properties are discussed. Hence, drawbacks (4), (5), and (6) are all defeated.

Third, novel methods with the proposed aggregation operations are proposed to cope with complex decision-making issues in picture fuzzy circumstances. In the case study, PFNs are suggested to describe hotel evaluation information, and our methods are adopted to select the optimal hotel. This surmounts limitation (1) and justifies the feasibility of the proposed methods.

Fourth, full discussions with sensitivity analyses and comparison analyses are taken to confirm the strengths of our methods.

The remainder of this study is arranged as follows. In Section 2, the preliminaries of PFNs and PHA operators are introduced in brief. Section 3 defines the new interaction operation rules of PFNs, and several picture fuzzy PHA operators are based on these operations. Then, new decision-making methods with these aggregation operators are recommended in Section 4. In Section 5, a case of hotel selection is studied to show the advantages of PFNs and the practicability of our methods. Section 6 makes some discussions by analyzing the influence of parameters and comparing with other existing approaches for demonstrating the superiority of the proposed methods. Last, some necessary conclusions are provided.

2. Preliminaries

2.1. Picture Fuzzy Numbers

Definition 1 ([8])**.** The PFS (picture fuzzy set) is an object on a universe Ω *with* $a(\chi) = \{< \chi, p_a(\chi), m_a(\chi), n_a(\chi) > | \chi \in \Omega\}$, *where* $p_a(\chi) \in [0,1]$ *is the positive membership degree,* $m_a(\chi) \in [0,1]$ *is the neutral membership degree,* $n_a(\chi) \in [0,1]$ *is the negative membership degree, and* $v_a(\chi) = 1 - p_a(\chi) - m_a(\chi) - n_a(\chi) \in [0,1]$ *is the refusal membership degree of* χ *in* a.

In particular, the PFS is degenerated to a PFN (picture fuzzy number) $a = (p_a, m_a, n_a)$ if Ω has only one element.

Definition 2 ([22])**.** *Let* $a = (p_a, m_a, n_a)$ *and* $b = (p_b, m_b, n_b)$ *be two arbitrary PFNs (picture fuzzy numbers), then the operational laws between them are*

(1) $a \oplus_W b = (p_a, m_a, n_a) \oplus_W (p_b, m_b, n_b) = (1 - (1 - p_a)(1 - p_b), m_a m_b, (n_a + m_a)(n_b + m_b) - m_a m_b)$;
(2) $a \otimes_W b = (p_a, m_a, n_a) \otimes_W (p_b, m_b, n_b) = ((p_a + m_a)(p_b + m_b) - m_a m_b, m_a m_b, 1 - (1 - n_a)(1 - n_b))$;
(3) $\delta \cdot a = (1 - (1 - p_a)^\delta, (m_a)^\delta, (n_a + m_a)^\delta - (m_a)^\delta), \delta \in (0, +\infty)$;
(4) $a^\delta = ((p_a + m_a)^\delta - (m_a)^\delta, (m_a)^\delta, 1 - (1 - n_a)^\delta), \delta \in (0, +\infty)$.

Example 1. *Let* $a_1 = (p_1, m_1, n_1) = (0.5, 0.3, 0.1)$ *and* $a_2 = (p_2, m_2, n_2) = (0.6, 0, 0.3)$ *be two PFNs. According to Definition 2, the aggregation values of* a_1 *and* a_2 *are* $a_1 \oplus a_2 = (0.8, 0, 0.12)$ *and* $a_1 \otimes a_2 = (0.48, 0, 0.37)$. *That is, because* $m_2 = 0$ *in* a_2*, the value of* m_1 *in* a_1 *has no influence on the aggregation values of* a_1 *and* a_2 *Clearly, this is an imperfection.*

Definition 3 ([22])**.** *Assume* $a = (p_a, m_a, n_a)$ *is a PFN, the score function and the accuracy function of* a are

$$E(a) = p_a - n_a, \tag{1}$$

$$F(a) = p_a + m_a + n_a. \tag{2}$$

Definition 4 ([22])**.** *Given two PFNs* $a = (p_a, m_a, n_a)$ *and* $b = (p_b, m_b, n_b)$*, the comparison method is*

(1) *if* $E(a) < E(b)$*, then* $a \prec b$;
(2) *if* $E(a) = E(b)$ *and* $F(a) > F(b)$*, then* $a \succ b$;
(3) *when* $E(a) = E(b)$ *and* $F(a) = F(b)$*, then* $a \sim b$.

2.2. Partitioned Heronian Averaging Operators

HA operator is a powerful tool to explore the relationship among inputs. The definitions of HA and geometric HA (GHA) operators are given in the following.

Definition 5 ([29]). *Suppose α_i ($i = 1, 2, \ldots, y$) is a group of non-negative real numbers and $\beta, \eta \geq 0$, then the HA aggregation operator is*

$$HA^{\beta,\eta}(\alpha_1, \alpha_2, \cdots, \alpha_y) = \left(\frac{2}{y(y+1)} \sum_{i=1}^{y} \sum_{j=1}^{y} (\alpha_i)^{\beta} (\alpha_j)^{\eta}\right)^{\frac{1}{\beta+\eta}}, \tag{3}$$

and the GHA operator is

$$GHA^{\beta,\eta}(\alpha_1, \alpha_2, \cdots, \alpha_y) = \frac{1}{\beta+\eta}\left(\prod_{i=1}^{y} \prod_{j=1}^{y} \beta\alpha_i + \eta\alpha_j\right)^{\frac{2}{y(y+1)}}. \tag{4}$$

To reflect a situation where argument values in the same group correlate with each other and the inputs in different groups are dissociated, Liu et al. [31] put forward the concepts of the PHA aggregation operator and PGHA operator as follows.

Definition 6 ([31]). *Assume α_i ($i = 1, 2, \ldots, y$) is a set of non-negative real numbers, and they are divided into x independent groups $S_1, S_2, \ldots, S_x$, where $S_k = \left\{\alpha_{k1}, \alpha_{k2}, \ldots \alpha_{k|S_k|}\right\}$ ($k = 1, 2, \ldots, x$), the cardinality of S_k is $|S_k|$ and $\sum_{k=1}^{x} |S_k| = y$. Let $\beta, \eta \geq 0$, then the PHA operator is*

$$PHA^{\beta,\eta}(\alpha_1, \alpha_2, \cdots, \alpha_y) = \frac{1}{x}\left(\sum_{k=1}^{x} \left(\frac{2}{|S_k|(|S_k|+1)} \sum_{i=1}^{|S_k|} \sum_{j=1}^{|S_k|} (\alpha_{ki})^{\beta} (\alpha_{kj})^{\eta}\right)^{\frac{1}{\beta+\eta}}\right), \tag{5}$$

and the PGHA operator is

$$PGHA^{\beta,\eta}(\alpha_1, \alpha_2, \cdots, \alpha_y) = \left(\prod_{k=1}^{x} \left(\frac{1}{\beta+\eta}\left(\prod_{i=1}^{|S_k|} \prod_{j=1}^{|S_k|} \beta\alpha_{ki} + \eta\alpha_{kj}\right)^{\frac{2}{|S_k|(|S_k|+1)}}\right)\right)^{\frac{1}{x}}. \tag{6}$$

3. Some Picture Fuzzy Interaction Partitioned Heronian Averaging Operators

In this section, the interaction operational rules of PNFs are first defined. Thereafter, some picture fuzzy interaction PHA operators are presented based on the new rules.

3.1. Interaction Operational Laws of Picture Fuzzy Numbers

The In this subsection, new operational laws—the interaction operational laws of PFNs—are proposed to overcome the limitations of the existing operations mentioned in Section 2.2.

The interaction operations take the interaction among true, hesitant and false membership degrees into account, which are shown as follows.

Definition 7. *Given two PFNs $a_1 = (p_1, m_1, n_1)$ and $a_2 = (p_2, m_2, n_2)$, then the interaction operational laws are defined as*

(1) $a_1 \oplus a_2 = \left(1 - \prod_{i=1}^{2}(1-p_i), \prod_{i=1}^{2}(1-p_i) - \prod_{i=1}^{2}(1-p_i-m_i), \prod_{i=1}^{2}(1-p_i-m_i) - \prod_{i=1}^{2}(1-p_i-m_i-n_i)\right)$;

(2) $a_1 \otimes a_2 = \left(\prod_{i=1}^{2}(1-n_i-m_i) - \prod_{i=1}^{2}(1-n_i-m_i-p_i), \prod_{i=1}^{2}(1-n_i) - \prod_{i=1}^{2}(1-n_i-m_i), 1 - \prod_{i=1}^{2}(1-n_i)\right)$;

(3) $\delta \cdot a_1 = \left(1 - (1-p_1)^{\delta}, (1-p_1)^{\delta} - (1-p_1-m_1)^{\delta}, (1-p_1-m_1)^{\delta} - (1-p_1-m_1-n_1)^{\delta}\right), \delta \in (0, +\infty)$;

(4) $(a_1)^{\delta} = \left((1-n_1-m_1)^{\delta} - (1-n_1-m_1-p_1)^{\delta}, (1-n_1)^{\delta} - (1-n_1-m_1)^{\delta}, 1 - (1-n_1)^{\delta}\right), \delta \in (0, +\infty)$.

Example 2. *Let $a_1 = (0.5, 0.3, 0.1)$ and $a_2 = (0.6, 0, 0.3)$, which are the same as Example 1. Based on Definition 7, the interaction aggregation values of a_1 and a_2 are $a_1 \oplus a_2 = (0.8, 0.12, 0.07)$ and $a_1 \otimes a_2 = (0.41, 0.21, 0.37)$, which are more reasonable than the results in Example 1.*

3.2. Picture Fuzzy Interaction Partitioned Heronian Averaging Operator

In this subsection, the picture fuzzy interaction PHA (*PFIPHA*) and picture fuzzy weighted interaction PHA (*PFWIPHA*) operators are defined, and some important properties are proved.

Definition 8. *Assume $a_i = (p_i, m_i, n_i)$ $(i = 1, 2, \ldots, y)$ is a set of PFNs, and they can be partitioned into x distinct sorts $S_1, S_2, \ldots, S_x$, where $S_k = \left\{a_{k1}, a_{k2}, \ldots, a_{k|s_K|}\right\}$ $(k = 1, 2, \ldots, x)$. Then the PFIPHA operator is defined as*

$$PFIPHA^{\beta,\eta}(a_1, a_2, \ldots, a_y) = \frac{1}{x}\left(\sum_{k=1}^{x}\left(\frac{2}{\left|S_k\right|\left(\left|S_k\right|+1\right)}\sum_{i=1}^{|S_k|}\sum_{j=i}^{|S_k|}(a_{ki})^{\beta}\otimes(a_{kj})^{\eta}\right)^{\frac{1}{\beta+\eta}}\right) \tag{7}$$

where $\beta, \eta \geq 0$, $|S_k|$ is the cardinality of S_k, and $\sum\limits_{k=1}^{x}|S_k| = y$.

Theorem 1. *If $a_i = (p_i, m_i, n_i)$ $(i = 1, 2, \ldots, y)$ is a group of PFNs, and $\beta, \eta \geq 0$, then their aggregated result using Equation (7) is still a PFN, and the following is true:*

$$\begin{gathered} PFIPHA^{\beta,\eta}(a_1, a_2, \ldots, a_y) = \\ \left(1-\left(\prod_{k=1}^{x}\left(1-(1-N_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}, \left(\prod_{k=1}^{x}\left(1-(1-N_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}-\right. \\ \left.\left(\prod_{k=1}^{x}\left(1+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-(1-O_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}, \left(\prod_{k=1}^{x}\left(1+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-(1-O_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}\left((M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}\right) \end{gathered} \tag{8}$$

where $M_k^{\beta,\eta} = \left(\prod\limits_{i=1,j=i}^{|S_k|}A_k^{\beta,\eta}\right)^{t_k}$, $N_k^{\beta,\eta} = \left(\prod\limits_{i=1,j=i}^{|S_k|}B_k^{\beta,\eta}\right)^{t_k}$, $O_k^{\beta,\eta} = \left(\prod\limits_{i=1,j=i}^{|S_k|}C_k^{\beta,\eta}\right)^{t_k}$, $t_k = \frac{2}{|S_k|(|S_k|+1)}$, $A_k^{\beta,\eta} = (v_{ki})^{\beta}(v_{kj})^{\eta}$, $B_k^{\beta,\eta} = 1-(u_{ki})^{\beta}(u_{kj})^{\eta}+(v_{ki})^{\beta}(v_{kj})^{\eta}$, $C_k^{\beta,\eta} = 1+(v_{ki})^{\beta}(v_{kj})^{\eta}-(r_{ki})^{\beta}(r_{kj})^{\eta}$, $r_{ki} = 1-n_{ki}$, $r_{kj} = 1-n_{kj}$, $u_{ki} = 1-n_{ki}-m_{ki}$, $u_{kj} = 1-n_{kj}-m_{kj}$, $v_{ki} = 1-n_{ki}-m_{ki}-p_{ki}$ and $v_{kj} = 1-n_{kj}-m_{kj}-p_{kj}$.

Note that the Proof of Theorem 1 can be seen in the Appendix A.

For the *PFIPHA* operator, the following properties should be satisfied.

Property 1. *(Idempotency) Suppose $a_i = (p_i, m_i, n_i)$ $(i = 1, 2, \ldots, y)$ is a collection of PFNs, and $a_i = a = (p, m, n)$ for all $i = 1, 2, \ldots, y$, then $PFIPHA^{\beta,\eta}(a_1, a_2, \ldots, a_y) = a$.*

Proof. Because $a_1 = a_2 = \cdots = a_y = (p, m, n)$, then for all $i = 1, 2, \ldots, y$, $r_i = 1-n_i = r$, $u_i = 1-n_i-m_i = u$ and $v_i = 1-n_i-m_i-p_i = v \Rightarrow A_k^{\beta,\eta} = (v_{ki})^{\beta}(v_{kj})^{\eta} = v^{\beta+\eta}$, $B_k^{\beta,\eta} = 1-(u_{ki})^{\beta}(u_{kj})^{\eta}+(v_{ki})^{\beta}(v_{kj})^{\eta} = 1-u^{\beta+\eta}+v^{\beta+\eta}$ and $C_k^{\beta,\eta} = 1+(v_{ki})^{\beta}(v_{kj})^{\eta}-(r_{ki})^{\beta}(r_{kj})^{\eta} = 1+v^{\beta+\eta}-r^{\beta+\eta} \Rightarrow$ $M_k^{\beta,\eta} = \left(\prod\limits_{i=1,j=i}^{|S_k|}A_k^{\beta,\eta}\right)^{t_k} = \left(\prod\limits_{i=1,j=i}^{|S_k|}v^{\beta+\eta}\right)^{t_k} = v^{\beta+\eta}$, $N_k^{\beta,\eta} = \left(\prod\limits_{i=1,j=i}^{|S_k|}B_k^{\beta,\eta}\right)^{t_k} = \left(\prod\limits_{i=1,j=i}^{|S_k|}(1-u^{\beta+\eta}+v^{\beta+\eta})\right)^{t_k} = 1-u^{\beta+\eta}+v^{\beta+\eta}$ and $O_k^{\beta,\eta} = \left(\prod\limits_{i=1,j=i}^{|S_k|}C_k^{\beta,\eta}\right)^{t_k} = \left(\prod\limits_{i=1,j=i}^{|S_k|}(1+v^{\beta+\eta}-r^{\beta+\eta})\right)^{t_k} = 1+v^{\beta+\eta}-r^{\beta+\eta}$, then

$$\begin{gathered} PFIPHA^{\beta,\eta}(a_1, a_2, \ldots, a_y) = \\ \left(1-\left(\prod_{k=1}^{x}\left(1-(1-N_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}, \left(\prod_{k=1}^{x}\left(1-(1-N_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}-\right. \\ \left.\left(\prod_{k=1}^{x}\left(1+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-(1-O_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}, \left(\prod_{k=1}^{x}\left(1+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-(1-O_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}\left((M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}\right) \\ = \end{gathered}$$

$$\left(1-\left(\prod_{k=1}^{x}\left(1-\left(1-(1-u^{\beta+\eta}+v^{\beta+\eta})+v^{\beta+\eta}\right)^{\frac{1}{\beta+\eta}}+(v^{\beta+\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}},\right.$$
$$\left(\prod_{k=1}^{x}\left(1-\left(1-(1-u^{\beta+\eta}+v^{\beta+\eta})+v^{\beta+\eta}\right)^{\frac{1}{\beta+\eta}}+(v^{\beta+\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}\left(1+(v^{\beta+\eta})^{\frac{1}{\beta+\eta}}-\left(1-(1+v^{\beta+\eta}-r^{\beta+\eta})+v^{\beta+\eta}\right)^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}},$$
$$\left.\left(\prod_{k=1}^{x}\left(1+(v^{\beta+\eta})^{\frac{1}{\beta+\eta}}-\left(1-(1+v^{\beta+\eta}-r^{\beta+\eta})+v^{\beta+\eta}\right)^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}\left((v^{\beta+\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}\right)$$
$$=$$
$$\left(1-\left(\prod_{k=1}^{x}\left(1-(u^{\beta+\eta})^{\frac{1}{\beta+\eta}}+(v^{\beta+\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}},\left(\prod_{k=1}^{x}\left(1-(u^{\beta+\eta})^{\frac{1}{\beta+\eta}}+(v^{\beta+\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}\left(1+(v^{\beta+\eta})^{\frac{1}{\beta+\eta}}-(r^{\beta+\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}},\right.$$
$$\left.\left(\prod_{k=1}^{x}\left(1+(v^{\beta+\eta})^{\frac{1}{\beta+\eta}}-(r^{\beta+\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}\left((v^{\beta+\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}\right)$$
$$=\left(1-\left(\prod_{k=1}^{x}(1-u+v)\right)^{\frac{1}{x}},\left(\prod_{k=1}^{x}(1-u+v)\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}(1+v-r)\right)^{\frac{1}{x}},\left(\prod_{k=1}^{x}(1+v-r)\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}(v)\right)^{\frac{1}{x}}\right)$$
$$=(1-(1-u+v),(1-u+v)-(1+v-r),\,(1+v-r)-v)=(u-v,r-u,1-r)$$
$$=((1-n-m)-(1-n-m-p),(1-n)-(1-n-m),1-(1-n))=(p,m,n)=a.\ \square$$

Now, the Proof is completed.

Property 2. *(Commutativity) Let $a_i=(p_i,m_i,n_i)$ and $a_i^*=(p_i^*,m_i^*,n_i^*)$ $(i=1,2,\ldots,y)$ be two sets of PFNs. If $(a_1^*,a_2^*,\cdots,a_y^*)$ is an arbitrary permutation of $(a_1,a_2,\cdots,a_y)$, then $PFIPHA^{\beta,\eta}(a_1,a_2,\ldots,a_y)=PFIPHA^{\beta,\eta}(a_1^*,a_2^*,\cdots,a_y^*)$.*

Proof. According to Equation (8), $PFIPHA^{\beta,\eta}(a_1,a_2,\ldots,a_y)=$
$$\left(1-\left(\prod_{k=1}^{x}\left(1-(1-N_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}},\left(\prod_{k=1}^{x}\left(1-(1-N_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}-\right.$$
$$\left.\left(\prod_{k=1}^{x}\left(1+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-(1-O_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}},\left(\prod_{k=1}^{x}\left(1+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-(1-O_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}\left((M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}\right),$$
and $PFIPHA^{\beta,\eta}(a_1^*,a_2^*,\cdots,a_y^*)=$
$$\left(1-\left(\prod_{k=1}^{x}\left(1-(1-N_k^{*\beta,\eta}+M_k^{*\beta,\eta})^{\frac{1}{\beta+\eta}}+(M_k^{*\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}},\left(\prod_{k=1}^{x}\left(1-(1-N_k^{*\beta,\eta}+M_k^{*\beta,\eta})^{\frac{1}{\beta+\eta}}+(M_k^{*\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}-\right.$$
$$\left.\left(\prod_{k=1}^{x}\left(1+(M_k^{*\beta,\eta})^{\frac{1}{\beta+\eta}}-(1-O_k^{*\beta,\eta}+M_k^{*\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}},\left(\prod_{k=1}^{x}\left(1+(M_k^{*\beta,\eta})^{\frac{1}{\beta+\eta}}-(1-O_k^{*\beta,\eta}+M_k^{*\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}\left((M_k^{*\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}\right).$$
As $(a_1^*,a_2^*,\cdots,a_y^*)$ is an arbitrary permutation of $(a_1,a_2,\cdots,a_y)$, it is clearly that $PFIPHA^{\beta,\eta}(a_1,a_2,\ldots,a_y)=PFIPHA^{\beta,\eta}(a_1^*,a_2^*,\cdots,a_y^*)$. $\square$

Several special cases of the $PFIPHA^{\beta,\eta}$ operator are discussed as follows.

Special case 1:

When $\eta\to 0$, $PFIPHA^{\beta,0}(a_1,a_2,\ldots,a_y)=$
$$\left(1-\left(\prod_{k=1}^{x}\left(1-\left(1-\left(\prod_{i=1}^{|S_k|}\left(1-(u_{ki})^{\beta}+(v_{ki})^{\beta}\right)\right)^{t_k}+\left(\prod_{i=1}^{|S_k|}(v_{ki})^{\beta}\right)^{t_k}\right)^{\frac{1}{\beta}}+\left(\left(\prod_{i=1}^{|S_k|}(v_{ki})^{\beta}\right)^{t_k}\right)^{\frac{1}{\beta}}\right)\right)^{\frac{1}{x}},\right.$$
$$\left(\prod_{k=1}^{x}\left(1-\left(1-\left(\prod_{i=1}^{|S_k|}\left(1-(u_{ki})^{\beta}+(v_{ki})^{\beta}\right)\right)^{t_k}+\left(\prod_{i=1}^{|S_k|}(v_{ki})^{\beta}\right)^{t_k}\right)^{\frac{1}{\beta}}+\left(\left(\prod_{i=1}^{|S_k|}(v_{ki})^{\beta}\right)^{t_k}\right)^{\frac{1}{\beta}}\right)\right)^{\frac{1}{x}}-$$
$$\left(\prod_{k=1}^{x}\left(1+\left(\left(\prod_{i=1}^{|S_k|}(v_{ki})^{\beta}\right)^{t_k}\right)^{\frac{1}{\beta}}-\left(1-\left(\prod_{i=1}^{|S_k|}\left(1+(v_{ki})^{\beta}-(r_{ki})^{\beta}\right)\right)^{t_k}+\left(\prod_{i=1}^{|S_k|}(v_{ki})^{\beta}\right)^{t_k}\right)^{\frac{1}{\beta}}\right)\right)^{\frac{1}{x}},$$
$$\left.\left(\prod_{k=1}^{x}\left(1+\left(\left(\prod_{i=1}^{|S_k|}(v_{ki})^{\beta}\right)^{t_k}\right)^{\frac{1}{\beta}}-\left(1-\left(\prod_{i=1}^{|S_k|}\left(1+(v_{ki})^{\beta}-(r_{ki})^{\beta}\right)\right)^{t_k}+\left(\prod_{i=1}^{|S_k|}(v_{ki})^{\beta}\right)^{t_k}\right)^{\frac{1}{\beta}}\right)\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}\left(\left(\left(\prod_{i=1}^{|S_k|}(v_{ki})^{\beta}\right)^{t_k}\right)^{\frac{1}{\beta}}\right)\right)^{\frac{1}{x}}\right).$$

Special case 2:

When $\beta \to 0$, then $PFIPHA^{0,\eta}(a_1, a_2, \ldots, a_y) =$

$$\left(1-\left(\prod_{k=1}^{x}\left(1-\left(1-\left(\prod_{i=1,j=i}^{|S_k|}\left(1-(u_{kj})^{\eta}+(v_{ki})^{\eta}\right)\right)^{t_k}+\left(\prod_{i=1,j=i}^{|S_k|}(v_{kj})^{\eta}\right)^{t_k}\right)^{\frac{1}{\eta}}+\left(\left(\prod_{i=1,j=i}^{|S_k|}(v_{kj})^{\eta}\right)^{t_k}\right)^{\frac{1}{\eta}}\right)\right)^{\frac{1}{x}},\right.$$

$$\left(\prod_{k=1}^{x}\left(1-\left(1-\left(\prod_{i=1,j=i}^{|S_k|}\left(1-(u_{kj})^{\eta}+(v_{kj})^{\eta}\right)\right)^{t_k}+\left(\prod_{i=1,j=i}^{|S_k|}(v_{kj})^{\eta}\right)^{t_k}\right)^{\frac{1}{\eta}}+\left(\left(\prod_{i=1,j=i}^{|S_k|}(v_{kj})^{\eta}\right)^{t_k}\right)^{\frac{1}{\eta}}\right)\right)^{\frac{1}{x}}-$$

$$\left(\prod_{k=1}^{x}\left(1+\left(\left(\prod_{i=1,j=i}^{|S_k|}(v_{kj})^{\eta}\right)^{t_k}\right)^{\frac{1}{\eta}}-\left(1-\left(\prod_{i=1,j=i}^{|S_k|}\left(1+(v_{kj})^{\eta}-(r_{kj})^{\eta}\right)\right)^{t_k}+\left(\prod_{i=1,j=i}^{|S_k|}(v_{kj})^{\eta}\right)^{t_k}\right)^{\frac{1}{\eta}}\right)\right)^{\frac{1}{x}},$$

$$\left.\left(\prod_{k=1}^{x}\left(1+\left(\left(\prod_{i=1,j=i}^{|S_k|}(v_{kj})^{\eta}\right)^{t_k}\right)^{\frac{1}{\eta}}-\left(1-\left(\prod_{i=1,j=i}^{|S_k|}\left(1+(v_{kj})^{\eta}-(r_{kj})^{\eta}\right)\right)^{t_k}+\left(\prod_{i=1,j=i}^{|S_k|}(v_{kj})^{\eta}\right)^{t_k}\right)^{\frac{1}{\eta}}\right)\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}\left(\left(\left(\prod_{i=1,j=i}^{|S_k|}(v_{kj})^{\eta}\right)^{t_k}\right)^{\frac{1}{\eta}}\right)\right)^{\frac{1}{x}}\right)$$

Special case 3:

When $\beta = 1$ and $\eta \to 0$, then $PFIPHA^{1,0}(a_1, a_2, \ldots, a_y) =$

$$\left(1-\left(\prod_{k=1}^{x}\left(\prod_{i=1}^{|S_k|}(1-u_{ki}+v_{ki})\right)^{t_k}\right)^{\frac{1}{x}},\left(\prod_{k=1}^{x}\left(\prod_{i=1}^{|S_k|}(1-u_{ki}+v_{ki})\right)^{t_k}\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}\left(\prod_{i=1}^{|S_k|}(1+v_{ki}-r_{ki})\right)^{t_k}\right)^{\frac{1}{x}},\left(\prod_{k=1}^{x}\left(\prod_{i=1}^{|S_k|}(1+v_{ki}-r_{ki})\right)^{t_k}\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}\left(\prod_{i=1}^{|S_k|}v_{ki}\right)^{t_k}\right)^{\frac{1}{x}}\right).$$

Special case 4:

When $\beta = \eta = 1$, then $PFIPHA^{1,0}(a_1, a_2, \ldots, a_y) =$

$$\left(1-\left(\prod_{k=1}^{x}\left(1-\left(1-\left(\prod_{i=1,j=i}^{|S_k|}(1-u_{ki}u_{kj}+v_{ki}v_{kj})\right)^{t_k}+\left(\prod_{i=1,j=i}^{|S_k|}v_{ki}v_{kj}\right)^{t_k}\right)^{\frac{1}{2}}+\left(\left(\prod_{i=1,j=i}^{|S_k|}v_{ki}v_{kj}\right)^{t_k}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{x}},\right.$$

$$\left(\prod_{k=1}^{x}\left(1-\left(1-\left(\prod_{i=1,j=i}^{|S_k|}(1-u_{ki}u_{kj}+v_{ki}v_{kj})\right)^{t_k}+\left(\prod_{i=1,j=i}^{|S_k|}v_{ki}v_{kj}\right)^{t_k}\right)^{\frac{1}{2}}+\left(\left(\prod_{i=1,j=i}^{|S_k|}v_{ki}v_{kj}\right)^{t_k}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{x}}-$$

$$\left(\prod_{k=1}^{x}\left(1+\left(\left(\prod_{i=1,j=i}^{|S_k|}v_{ki}v_{kj}\right)^{t_k}\right)^{\frac{1}{2}}-\left(1-\left(\prod_{i=1,j=i}^{|S_k|}(1+v_{ki}v_{kj}-r_{ki}r_{kj})\right)^{t_k}+\left(\prod_{i=1,j=i}^{|S_k|}v_{ki}v_{kj}\right)^{t_k}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{x}},$$

$$\left.\left(\prod_{k=1}^{x}\left(1+\left(\left(\prod_{i=1,j=i}^{|S_k|}v_{ki}v_{kj}\right)^{t_k}\right)^{\frac{1}{2}}-\left(1-\left(\prod_{i=1,j=i}^{|S_k|}(1+v_{ki}v_{kj}-r_{ki}r_{kj})\right)^{t_k}+\left(\prod_{i=1,j=i}^{|S_k|}v_{ki}v_{kj}\right)^{t_k}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}\left(\left(\left(\prod_{i=1,j=i}^{|S_k|}v_{ki}v_{kj}\right)^{t_k}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{x}}\right).$$

Definition 9. *If $a_i = (p_i, m_i, n_i)$ $(i = 1, 2, \ldots, y)$ is a set of PFNs, and they can be partitioned into x distinct sorts $S_1, S_2, \ldots, S_x$, where $S_k = \left\{a_{k1}, a_{k2}, \ldots, a_{k|s_K|}\right\}$ $(k = 1, 2, \ldots, x)$; $(w_1, w_2, \cdots, w_y)$ is the weight vector of $(a_1, a_2, \cdots, a_y)$, where $w_i \in [0, 1]$ and $\sum_{i=1}^{y} w_i = 1$, then the PFWIPHA operator is defined as*

$$PFWIPHA^{\beta,\eta}(a_1, a_2, \ldots, a_y) = \frac{1}{x}\left(\sum_{k=1}^{x}\left(\frac{2}{|S_k|(|S_k|+1)}\sum_{i=1}^{|S_k|}\sum_{j=i}^{|S_k|}(w_i a_{ki})^{\beta} \otimes (w_j a_{kj})^{\eta}\right)^{\frac{1}{\beta+\eta}}\right) \tag{9}$$

where $\beta, \eta \geq 0$, $|S_k|$ is the cardinality of S_k, and $\sum_{k=1}^{x} |S_k| = y$.

Theorem 2. *Suppose $a_i = (p_i, m_i, n_i)$ $(i = 1, 2, \ldots, y)$ is a group of PFNs, $(w_1, w_2, \cdots, w_y)$ is the weight vector of $(a_1, a_2, \cdots, a_y)$, $w_i \in [0, 1]$, $\sum_{i=1}^{y} w_i = 1$ and $\beta, \eta \geq 0$, then their aggregated result using Equation (9) is still a PFN, and the following is true:*

$$\begin{aligned} & PFWIPHA^{\beta,\eta}(a_1, a_2, \ldots, a_y) = \\ & \Bigg(1 - \left(\prod_{k=1}^{x}\left(1 - (1 + J_k^{\beta,\eta} - L_k^{\beta,\eta})^{\frac{1}{\beta+\eta}} + (J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}, \left(\prod_{k=1}^{x}\left(1 - (1 + J_k^{\beta,\eta} - L_k^{\beta,\eta})^{\frac{1}{\beta+\eta}} + (J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}} - \\ & \left(\prod_{k=1}^{x}\left(1 + (J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}} - (1 - Q_k^{\beta,\eta} + J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}, \left(\prod_{k=1}^{x}\left(1 + (J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}} - (1 - Q_k^{\beta,\eta} + J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}} - \left(\prod_{k=1}^{x}\left((J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}\Bigg) \end{aligned} \tag{10}$$

where $J_k^{\beta,\eta} = \left(\prod_{i=1,j=i}^{|S_k|} G_k^{\beta,\eta}\right)^{t_k}$, $L_k^{\beta,\eta} = \left(\prod_{i=1,j=i}^{|S_k|}\left(1 - H_k^{\beta,\eta} + G_k^{\beta,\eta}\right)\right)^{t_k}$, $Q_k^{\beta,\eta} = \left(\prod_{i=1,j=i}^{|S_k|}\left(1 + G_k^{\beta,\eta} - I_k^{\beta,\eta}\right)\right)^{t_k}$, $t_k = \frac{2}{|S_k|(|S_k|+1)}$, $G_k^{\beta,\eta} = \left((v_{ki})^{w_i}\right)^{\beta}\left((v_{kj})^{w_j}\right)^{\eta}$, $H_k^{\beta,\eta} = \left(1 + (v_{ki})^{w_i} - (e_{ki})^{w_i}\right)^{\beta}\left(1 + (v_{kj})^{w_j} - (e_{kj})^{w_j}\right)^{\eta}$, $I_k^{\beta,\eta} = \left(1 - (f_{ki})^{w_i} + (v_{ki})^{w_i}\right)^{\beta}\left(1 - (f_{kj})^{w_j} + (v_{kj})^{w_j}\right)^{\eta}$, $e_{ki} = 1 - p_{ki}$, $e_{kj} = 1 - p_{kj}$, $f_{ki} = 1 - p_{ki} - m_{ki}$, $f_{kj} = 1 - p_{kj} - m_{kj}$, $v_{ki} = 1 - p_{ki} - m_{ki} - n_{ki}$ *and* $v_{kj} = 1 - p_{kj} - m_{kj} - n_{kj}$.

Note that the Proof of Theorem 2 can be seen in the Appendix B.

3.3. Picture Fuzzy Interaction Partitioned Geometric Heronian Averaging Operator

In this subsection, the picture fuzzy interaction partitioned geometric HA (*PFIPGHA*) and the picture fuzzy weighted interaction partitioned geometric HA (*PFWIPGHA*) operators are discussed. Different from arithmetic aggregation operators, the geometric aggregation operators emphasize the equilibrium of all inputs and the harmonization (rather than the complementarity) among their individual values [32]. Therefore, the following discussion of *PFIPGHA* and *PFWIPGHA* operators are also necessary.

Definition 10. *Assume $a_i = (p_i, m_i, n_i)$ $(i = 1, 2, \ldots, y)$ is a set of PFNs, and they can be partitioned into x distinct sorts $S_1, S_2, \ldots, S_x$, where $S_k = \left\{a_{k1}, a_{k2}, \ldots, a_{k|s_K|}\right\}$ $(k = 1, 2, \ldots, x)$. Then the PFIPGHA operator is defined as*

$$PFIPGHA^{\beta,\eta}(a_1, a_2, \ldots, a_y) = \left(\prod_{k=1}^{x}\left(\frac{1}{\beta+\eta}\left(\prod_{i=1,j=i}^{|S_k|}(\beta a_{ki} \oplus \eta a_{kj})\right)^{\frac{2}{|S_k|(|S_k|+1)}}\right)\right)^{\frac{1}{x}} \tag{11}$$

where $\beta, \eta \geq 0$, $|S_k|$ is the cardinality of S_k and $\sum_{k=1}^{x} |S_k| = y$.

Theorem 3. *If $a_i = (p_i, m_i, n_i)$ $(i = 1, 2, \ldots, y)$ is a group of PFNs, and $\beta, \eta \geq 0$, then their aggregated result using Equation (11) is still a PFN, and the following is true:*

$$\begin{aligned} & PFIPGHA^{\beta,\eta}(a_1, a_2, \ldots, a_y) = \\ & \Bigg(\left(\prod_{k=1}^{x}\left(1 + \left((R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}} - \left(1 - (U_k^{\beta,\eta})^{t_k} + (R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}} - \left(\prod_{k=1}^{x}\left(\left((R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}, \left(\prod_{k=1}^{x}\left(1 - \left(1 + (R_k^{\beta,\eta})^{t_k} - (V_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}} + \left((R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}} - \\ & \left(\prod_{k=1}^{x}\left(1 + \left((R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}} - \left(1 - (U_k^{\beta,\eta})^{t_k} + (R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}, 1 - \left(\prod_{k=1}^{x}\left(1 - \left(1 + (R_k^{\beta,\eta})^{t_k} - (V_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}} + \left((R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}\Bigg), \end{aligned} \tag{12}$$

here $R_k^{\beta,\eta} = \prod\limits_{i=1,j=i}^{|S_k|}\left((v_{ki})^{\beta}(v_{kj})^{\eta}\right)$, $U_k^{\beta,\eta} = \prod\limits_{i=1,j=i}^{|S_k|}\left(1+(v_{ki})^{\beta}(v_{kj})^{\eta}-(e_{ki})^{\beta}(e_{kj})^{\eta}\right)$, $V_k^{\beta,\eta} = \prod\limits_{i=1,j=i}^{|S_k}\left(1-(f_{ki})^{\beta}(f_{kj})^{\eta}+(v_{ki})^{\beta}(v_{kj})^{\eta}\right)$, $t_k = \frac{2}{|S_k|(|S_k|+1)}$, $e_{ki} = 1-p_{ki}$, $e_{kj} = 1-p_{kj}$, $f_{ki} = 1-p_{ki}-m_{ki}$, $f_{kj} = 1-p_{kj}-m_{kj}$, $v_{ki} = 1-p_{ki}-m_{ki}-n_{ki}$ *and* $v_{kj} = 1-p_{kj}-m_{kj}-n_{kj}$.

Note that the Proof of Theorem 1 can be seen in the Appendix C.

For the *PFIPGHA* operator, the following properties should be satisfied.

Property 3. (*Idempotency*) *Suppose* $a_i = (p_i, m_i, n_i)$ $(i = 1, 2, \ldots, y)$ *is a collection of PFNs, and* $a_i = a = (p, m, n)$ *for all* $i = 1, 2, \ldots, y$, *then* $PFIPGHA^{\beta,\eta}(a_1, a_2, \ldots, a_y) = a$.

Property 4. (*Commutativity*) *Let* $a_i = (p_i, m_i, n_i)$ *and* $a_i^* = (p_i^*, m_i^*, n_i^*)$ $(i = 1, 2, \ldots, y)$ *be two sets of PFNs. If* $(a_1^*, a_2^*, \cdots, a_y^*)$ *is an arbitrary permutation of* $(a_1, a_2, \cdots, a_y)$, *then* $PFIPGHA^{\beta,\eta}(a_1, a_2, \ldots, a_y) = PFIPGHA^{\beta,\eta}(a_1^*, a_2^*, \cdots, a_y^*)$.

Note that the proofs of Property 3 and 4 are similar to those of Property 1 and 2, respectively. Thus, they are omitted to save space in this study.

Several special cases of the $PFIPGHA^{\beta,\eta}$ operator are discussed as follows.

Special case 1:

When $\eta \to 0$, $PFIPGHA^{\beta,0}(a_1, a_2, \ldots, a_y) =$

$$\begin{aligned}
&\Bigg(\left(\left(\prod_{k=1}^{x}\left(1+\left(\left(\prod_{i=1}^{|S_k|}\left((g_{ki})^{\beta}\right)\right)^{t_k}\right)^{\frac{1}{\beta}}-\left(1-\left(\prod_{i=1}^{|S_k|}\left(1+(g_{ki})^{\beta}-(e_{ki})^{\beta}\right)\right)^{t_k}+\left(\prod_{i=1}^{|S_k|}\left((g_{ki})^{\beta}\right)\right)^{t_k}\right)^{\frac{1}{\beta}}\right)\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}\left(\left(\left(\prod_{i=1}^{|S_k|}\left((g_{ki})^{\beta}\right)\right)^{t_k}\right)^{\frac{1}{\beta}}\right)\right)\right)^{\frac{1}{x}},\\
&\left(\prod_{k=1}^{x}\left(1-\left(1+\left(\prod_{i=1}^{|S_k|}\left((g_{ki})^{\beta}\right)\right)^{t_k}-\left(\prod_{i=1}^{|S_k}\left(1-(f_{ki})^{\beta}+(g_{ki})^{\beta}\right)\right)^{t_k}\right)^{\frac{1}{\beta}}+\left(\left(\prod_{i=1}^{|S_k|}\left((g_{ki})^{\beta}\right)\right)^{t_k}\right)^{\frac{1}{\beta}}\right)\right)^{\frac{1}{x}}-\\
&\left(\prod_{k=1}^{x}\left(1+\left(\left(\prod_{i=1}^{|S_k|}\left((g_{ki})^{\beta}\right)\right)^{t_k}\right)^{\frac{1}{\beta}}-\left(1-\left(\prod_{i=1}^{|S_k|}\left(1+(g_{ki})^{\beta}-(e_{ki})^{\beta}\right)\right)^{t_k}+\left(\prod_{i=1}^{|S_k|}\left((g_{ki})^{\beta}\right)\right)^{t_k}\right)^{\frac{1}{\beta}}\right)\right)^{\frac{1}{x}},\\
&1-\left(\prod_{k=1}^{x}\left(1-\left(1+\left(\prod_{i=1}^{|S_k|}\left((g_{ki})^{\beta}\right)\right)^{t_k}-\left(\prod_{i=1}^{|S_k}\left(1-(f_{ki})^{\beta}+(g_{ki})^{\beta}\right)\right)^{t_k}\right)^{\frac{1}{\beta}}+\left(\left(\prod_{i=1}^{|S_k|}\left((g_{ki})^{\beta}\right)\right)^{t_k}\right)^{\frac{1}{\beta}}\right)\right)^{\frac{1}{x}}\Bigg).
\end{aligned}$$

Special case 2:

When $\beta \to 0$, then $PFIPGHA^{0,\eta}(a_1, a_2, \ldots, a_y) =$

$$\begin{aligned}
&\Bigg(\left(\left(\prod_{k=1}^{x}\left(1+\left(\left(\prod_{i=1,j=i}^{|S_k|}\left((g_{kj})^{\eta}\right)\right)^{t_k}\right)^{\frac{1}{\eta}}-\left(1-\left(\prod_{i=1,j=i}^{|S_k|}\left(1+(g_{kj})^{\eta}-(e_{kj})^{\eta}\right)\right)^{t_k}+\left(\prod_{i=1,j=i}^{|S_k|}\left((g_{kj})^{\beta}\right)\right)^{t_k}\right)^{\frac{1}{\eta}}\right)\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}\left(\left(\left(\prod_{i=1,j=i}^{|S_k|}\left((g_{kj})^{\eta}\right)\right)^{t_k}\right)^{\frac{1}{\eta}}\right)\right)\right)^{\frac{1}{x}},\\
&\left(\prod_{k=1}^{x}\left(1-\left(1+\left(\prod_{i=1,j=i}^{|S_k|}\left((g_{kj})^{\eta}\right)\right)^{t_k}-\left(\prod_{i=1,j=i}^{|S_k}\left(1-(f_{kj})^{\eta}+(g_{kj})^{\eta}\right)\right)^{t_k}\right)^{\frac{1}{\eta}}+\left(\left(\prod_{i=1,j=i}^{|S_k|}\left((g_{kj})^{\eta}\right)\right)^{t_k}\right)^{\frac{1}{\eta}}\right)\right)^{\frac{1}{x}}-\\
&\left(\prod_{k=1}^{x}\left(1+\left(\left(\prod_{i=1,j=i}^{|S_k|}\left((g_{kj})^{\eta}\right)\right)^{t_k}\right)^{\frac{1}{\eta}}-\left(1-\left(\prod_{i=1,j=i}^{|S_k|}\left(1+(g_{kj})^{\eta}-(e_{kj})^{\eta}\right)\right)^{t_k}+\left(\prod_{i=1,j=i}^{|S_k|}\left((g_{kj})^{\eta}\right)\right)^{t_k}\right)^{\frac{1}{\eta}}\right)\right)^{\frac{1}{x}},\\
&1-\left(\prod_{k=1}^{x}\left(1-\left(1+\left(\prod_{i=1,j=i}^{|S_k|}\left((g_{kj})^{\eta}\right)\right)^{t_k}-\left(\prod_{i=1,j=i}^{|S_k}\left(1-(f_{kj})^{\eta}+(g_{kj})^{\eta}\right)\right)^{t_k}\right)^{\frac{1}{\eta}}+\left(\left(\prod_{i=1,j=i}^{|S_k|}\left((g_{kj})^{\eta}\right)\right)^{t_k}\right)^{\frac{1}{\eta}}\right)\right)^{\frac{1}{x}}\Bigg).
\end{aligned}$$

Special case 3:

When $\beta = 1$ and $\eta \to 0$, then $PFIPGHA^{1,0}(a_1, a_2, \ldots, a_y) =$

$$\left(\left(\prod_{k=1}^{x}\left(\prod_{i=1}^{|S_k|}(1+g_{ki}-e_{ki})\right)^{t_k}\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}\left(\prod_{i=1}^{|S_k|}g_{ki}\right)^{t_k}\right)^{\frac{1}{x}}, \left(\prod_{k=1}^{x}\left(\prod_{i=1}^{|S_k}(1-f_{ki}+g_{ki})\right)^{t_k}\right)^{\frac{1}{x}} - \left(\prod_{k=1}^{x}\left(\prod_{i=1}^{|S_k|}(1+g_{ki}-e_{ki})\right)^{t_k}\right)^{\frac{1}{x}}, 1-\left(\prod_{k=1}^{x}\left(\prod_{i=1}^{|S_k}(1-f_{ki}+g_{ki})\right)^{t_k}\right)^{\frac{1}{x}}\right).$$

Special case 4:

When $\beta = \eta = 1$, then $PFIPHA^{1,0}(a_1, a_2, \ldots, a_y) =$

$$\Bigg(\Bigg(\prod_{k=1}^{x}\Bigg(1+\Bigg(\Bigg(\prod_{i=1,j=i}^{|S_k|}\big((g_{ki})(g_{kj})\big)\Bigg)^{t_k}\Bigg)^{\frac{1}{2}}-\Bigg(1-\Bigg(\prod_{i=1,j=i}^{|S_k|}\big(1+(g_{ki})(g_{kj})-(e_{ki})(e_{kj})\big)\Bigg)^{t_k}+\Bigg(\prod_{i=1,j=i}^{|S_k|}\big((g_{ki})(g_{kj})\big)\Bigg)^{t_k}\Bigg)^{\frac{1}{2}}\Bigg)\Bigg)^{\frac{1}{x}}-\Bigg(\prod_{k=1}^{x}\Bigg(\Bigg(\Bigg(\prod_{i=1,j=i}^{|S_k|}\big((g_{ki})(g_{kj})\big)\Bigg)^{t_k}\Bigg)^{\frac{1}{2}}\Bigg)\Bigg)^{\frac{1}{x}},$$

$$\Bigg(\prod_{k=1}^{x}\Bigg(1-\Bigg(1+\Bigg(\prod_{i=1,j=i}^{|S_k|}\big((g_{ki})(g_{kj})\big)\Bigg)^{t_k}-\Bigg(\prod_{i=1,j=i}^{|S_k}\big(1-(f_{ki})(f_{kj})+(g_{ki})(g_{kj})\big)\Bigg)^{t_k}\Bigg)^{\frac{1}{2}}+\Bigg(\Bigg(\prod_{i=1,j=i}^{|S_k|}\big((g_{ki})(g_{kj})\big)\Bigg)^{t_k}\Bigg)^{\frac{1}{2}}\Bigg)\Bigg)^{\frac{1}{x}}-$$

$$\Bigg(\prod_{k=1}^{x}\Bigg(1+\Bigg(\Bigg(\prod_{i=1,j=i}^{|S_k|}\big((g_{ki})(g_{kj})\big)\Bigg)^{t_k}\Bigg)^{\frac{1}{2}}-\Bigg(1-\Bigg(\prod_{i=1,j=i}^{|S_k|}\big(1+(g_{ki})(g_{kj})-(e_{ki})(e_{kj})\big)\Bigg)^{t_k}+\Bigg(\prod_{i=1,j=i}^{|S_k|}\big((g_{ki})(g_{kj})\big)\Bigg)^{t_k}\Bigg)^{\frac{1}{2}}\Bigg)\Bigg)^{\frac{1}{x}},$$

$$1-\Bigg(\prod_{k=1}^{x}\Bigg(1-\Bigg(1+\Bigg(\prod_{i=1,j=i}^{|S_k|}\big((g_{ki})(g_{kj})\big)\Bigg)^{t_k}-\Bigg(\prod_{i=1,j=i}^{|S_k}\big(1-(f_{ki})(f_{kj})+(g_{ki})(g_{kj})\big)\Bigg)^{t_k}\Bigg)^{\frac{1}{2}}+\Bigg(\Bigg(\prod_{i=1,j=i}^{|S_k|}\big((g_{ki})(g_{kj})\big)\Bigg)^{t_k}\Bigg)^{\frac{1}{2}}\Bigg)\Bigg)^{\frac{1}{x}}\Bigg).$$

Definition 11. *If $a_i = (p_i, m_i, n_i)$ $(i = 1, 2, \ldots, y)$ is a set of PFNs, and they can be partitioned into x distinct sorts $S_1, S_2, \ldots, S_x$, where $S_k = \left\{a_{k1}, a_{k2}, \ldots, a_{k|s_K|}\right\}$ $(k = 1, 2, \ldots, x)$; $(w_1, w_2, \cdots, w_y)$ is the weight vector of $(a_1, a_2, \cdots, a_y)$, where $w_i \in [0, 1]$ and $\sum_{i=1}^{y} w_i = 1$, then the PFWIPGHA operator is defined as*

$$PFWIPGHA^{\beta,\eta}(a_1, a_2, \ldots, a_y) = \left(\prod_{k=1}^{x}\left(\frac{1}{\beta+\eta}\left(\prod_{i=1,j=i}^{|S_k|}\beta(a_{ki})^{w_i} \oplus \eta(a_{kj})^{w_j}\right)^{\frac{2}{|S_k|(|S_k|+1)}}\right)\right)^{\frac{1}{x}}, \tag{13}$$

here $\beta, \eta \geq 0$, $|S_k|$ is the cardinality of S_k and $\sum_{k=1}^{x} |S_k| = y$.

Theorem 4. *Suppose $a_i = (p_i, m_i, n_i)$ $(i = 1, 2, \ldots, y)$ is a group of PFNs, $(w_1, w_2, \cdots, w_y)$ is the weight vector of $(a_1, a_2, \cdots, a_y)$, $w_i \in [0, 1]$, $\sum_{i=1}^{y} w_i = 1$ and $\beta, \eta \geq 0$, then their aggregated result using Equation (13) is still a PFN, and the following is true:*

$$\begin{gathered} PFWIPGHA^{\beta,\eta}(a_1, a_2, \ldots, a_y) = \\ \Bigg(\Bigg(\prod_{k=1}^{x}\Big(1-(O_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-\big(1-P_k^{\beta,\eta}+O_k^{\beta,\eta}\big)^{\frac{1}{\beta+\eta}}\Big)\Bigg)^{\frac{1}{x}}-\Bigg(\prod_{k=1}^{x}\Big((O_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\Big)\Bigg)^{\frac{1}{x}}, \Bigg(\prod_{k=1}^{x}\Big(1-\big(1+O_k^{\beta,\eta}-Y_k^{\beta,\eta}\big)^{\frac{1}{\beta+\eta}}+(O_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\Big)\Bigg)^{\frac{1}{x}} \\ -\Bigg(\prod_{k=1}^{x}\Big(1-(O_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-\big(1-P_k^{\beta,\eta}+O_k^{\beta,\eta}\big)^{\frac{1}{\beta+\eta}}\Big)\Bigg)^{\frac{1}{x}}, 1-\Bigg(\prod_{k=1}^{x}\Big(1-\big(1+O_k^{\beta,\eta}-Y_k^{\beta,\eta}\big)^{\frac{1}{\beta+\eta}}+(O_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\Big)\Bigg)^{\frac{1}{x}}\Bigg) \end{gathered} \tag{14}$$

where $O_k^{\beta,\eta} = \left(\prod_{i=1,j=i}^{|S_k|} G_k^{\beta,\eta}\right)^{t_k}$, $P_k^{\beta,\eta} = \left(\prod_{i=1,j=i}^{|S_k|}\left(1+G_k^{\beta,\eta}-T_k^{\beta,\eta}\right)\right)^{t_k}$, $Y_k^{\beta,\eta} = \left(\prod_{i=1,j=i}^{|S_k|}\left(1-Z_k^{\beta,\eta}+G_k^{\beta,\eta}\right)\right)^{t_k}$, $t_k = \frac{2}{|S_k|(|S_k|+1)}$, $G_k^{\beta,\eta} = \left((v_{ki})^{w_i}\right)^{\beta}\left((v_{kj})^{w_j}\right)^{\eta}$, $T_k^{\beta,\eta} = \left(1-(u_{ki})^{w_i}+(v_{ki})^{w_i}\right)^{\beta}\left(1-(u_{kj})^{w_j}+(v_{kj})^{w_j}\right)^{\eta}$, $Z_k^{\beta,\eta} = \left(1+(v_{ki})^{w_i}-(r_{ki})^{w_i}\right)^{\beta}\left(1+(v_{kj})^{w_j}-(r_{kj})^{w_j}\right)^{\eta}$, $r_{ki} = 1-n_{ki}$, $r_{kj} = 1-n_{kj}$, $u_{ki} = 1-n_{ki}-m_{ki}$, $u_{kj} = 1-n_{kj}-m_{kj}$, $v_{ki} = 1-n_{ki}-m_{ki}-p_{ki}$ and $v_{kj} = 1-n_{kj}-m_{kj}-p_{kj}$.

Note that the Proof of Theorem 4 can be seen in the Appendix D.

4. Decision Making Methods with the Proposed Aggregation Operators

In this section, decision making methods with the proposed aggregation operators are explored to address complex decision making issues in a picture fuzzy environment.

4.1. Problem Description

With respect to a group decision making problem with picture fuzzy information, assume that there are h alternatives, denoted as $\{\pi_1, \pi_2, \cdots, \pi_h\}$, and l criteria, denoted as $\left\{\phi_1, \phi_2, \cdots, \phi_l\right\}$. The weight vector of these criteria is $\{w_1, w_2, \cdots, w_l\}$, where $w_1 + w_2 + \cdots + w_l = 1$ and $0 \le w_1, w_2, \cdots, w_l \le 1$. Then, the decision makers are asked to rank all the alternatives or select the best alternative. Suppose the evaluation matrix provided by decision makers is $A = (a_{ij})_{h\times l}$, where $a_{ij} = (p_{ij}, m_{ij}, n_{ij})$, which is a PFN, represents the evaluation value of alternative π_i $(i = 1, 2, \cdots, h)$ under criterion ϕ_j $(j = 1, 2, \cdots, l)$.

4.2. Decision Making Procedures

In this subsection, decision making methods based on the proposed aggregation operators are suggested to solve the decision making problem described in Section 4.1. The specific decision-making procedures are stated as follows.

Step 1: Normalize the initial decision making matrix.

Because benefit and cost criteria may be contained in an initial evaluation matrix at the same time, they are usually transferred into the same type in the first step. The normalization equation of PFNs is

$$b_{ij} = \begin{cases} (p_{ij}, m_{ij}, n_{ij}) & for\ benefit\ criteria\ \phi_j \\ (n_{ij}, m_{ij}, p_{ij}) & for\ \cos t\ criteria\ \phi_j \end{cases}. \tag{15}$$

Thus, the normalized evaluation matrix can be denoted as $B = (b_{ij})_{h\times l}$.

Step 2: Calculate the overall preference degree of each alternative.

Based on the *PFWIPHA* or *PFWIPGHA* operator defined in Section 3, the evaluation values in each row of evaluation matrix B are aggregated, and then the overall preference degree of each alternative is calculated as

$$a_i = PFWIPHA^{\beta,\eta}(a_{i1}, a_{i2}, \ldots, a_{il})\ (i = 1, 2, \cdots, h), \tag{16}$$

or

$$a_i = PFWIPGHA^{\beta,\eta}(a_{i1}, a_{i2}, \ldots, a_{il})\ (i = 1, 2, \cdots, h). \tag{17}$$

Step 3: Compute the score function or accuracy function.

Based on Equation (1), the score function $E(a_i)$ can be calculated. If two score function values are equal, then the accuracy function $F(a_i)$ should be computed using Equation (2).

Step 4: Obtain the ranking order.

According to the comparison method defined in Definition 4, the ranking order of all the alternatives is obtained, and the best alternative is selected as π^*.

5. Case Study

In this section, a hotel selection case is studied to justify the practicability of the proposed method.

Recently, five college students plan to book a hotel in advance for their trip next week. They browse hotel evaluation information in TripAdvisor.com. As a very popular tourism website, TripAdvisor.com has many true comments about hotels, restaurants and tourist attractions. Accordingly, they choose four satisfactory hotels based on their price, comfortability, service, location, and convenience. In the following, the presented methods are suggested to select the optimal hotel. The program codes of the presented method run under the MATLAB R2016b software. The operation platform is a laptop with a Windows 10 operating system, an Intel(R) Core(TM) i5-8250U CPU with 1.80 GHz, and 12G random-access memory (RAM). The advantage of MATLAB lies in numerical calculation. It can efficiently solve complex problems, dynamically simulate the system, and display the numerical results with powerful graphics functions.

First, they need to fill out a questionnaire of selected hotels (See Appendix E) for giving their respective opinions of these four alternatives (denoted as $\pi_1, \pi_2, \pi_3, \pi_4$) under six evaluation criteria. They are ϕ_1 (price), ϕ_2 (comfortability), ϕ_3 (service), ϕ_4 (location) and ϕ_5 (convenience). Then, their answers can be counted and represented by PFNs. For example, if two students think the price of hotel π_1 is high, one student holds that the price is medium, and two students believe that the price is low, then their evaluations can be described by a PFN $a_{11} = (0.4, 0.2, 0.4)$. Thus, when they give their all evaluations, an original evaluation matrix A with PFNs can be obtained, as shown in Table 1.

Table 1. Initial decision making matrix A.

A	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
π_1	$(0.4, 0.2, 0.4)$	$(0.6, 0.2, 0.2)$	$(0.4, 0.2, 0.2)$	$(0.6, 0, 0.2)$	$(0.6, 0, 0.4)$
π_2	$(0.2, 0.4, 0.4)$	$(0.8, 0.2, 0)$	$(0.4, 0.4, 0.2)$	$(0.6, 0.4, 0)$	$(0.4, 0.2, 0.2)$
π_3	$(0.4, 0.4, 0.2)$	$(0.6, 0, 0.4)$	$(0.8, 0, 0.2)$	$(0.6, 0.2, 0.2)$	$(0.4, 0.2, 0.2)$
π_4	$(0.2, 0.4, 0.2)$	$(0.4, 0.4, 0.2)$	$(0.6, 0.4, 0)$	$(0.4, 0.2, 0.4)$	$(0.8, 0.2, 0)$

Case 1: Select the best hotel with the method based on the *PFWIPHA* operator.

Step 1: Normalize the initial decision making matrix.

Since ϕ_2, ϕ_3, ϕ_4 and ϕ_5 are benefit criteria, while ϕ_1 belongs to cost criterion, they should be normalized based on Equation (15). The normalized evaluation matrix B is shown in Table 2.

Table 2. Normalized evaluation matrix B.

B	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
π_1	$(0.4, 0.2, 0.4)$	$(0.6, 0.2, 0.2)$	$(0.4, 0.2, 0.2)$	$(0.6, 0, 0.2)$	$(0.6, 0, 0.4)$
π_2	$(0.4, 0.4, 0.2)$	$(0.8, 0.2, 0)$	$(0.4, 0.4, 0.2)$	$(0.6, 0.4, 0)$	$(0.4, 0.2, 0.2)$
π_3	$(0.2, 0.4, 0.4)$	$(0.6, 0, 0.4)$	$(0.8, 0, 0.2)$	$(0.6, 0.2, 0.2)$	$(0.4, 0.2, 0.2)$
π_4	$(0.2, 0.4, 0.2)$	$(0.4, 0.4, 0.2)$	$(0.6, 0.4, 0)$	$(0.4, 0.2, 0.4)$	$(0.8, 0.2, 0)$

Step 2: Calculate the overall preference degree of each alternative.

Suppose $\beta = \eta = 1$, and all criteria have the same importance, namely, $w_1 = w_2 = w_3 = w_4 = w_5 = 0.2$. In general, a higher price means a large probability of good comfortability and service. Likewise, the convenience of transport may have great relations with the location of a hotel. Thus, according to this correlation pattern, these criteria are partitioned into two parts: $S_1 = \{\phi_1, \phi_2, \phi_3\}$ and $S_2 = \{\phi_4, \phi_5\}$. Then, based on the *PFWIPHA* operator in Equation (16), the overall preference degree of each hotel is computed as: $a_2 = (0.2492, 0.7506, 0)$, $a_3 = (0.2526, 0.0859, 0.6615)$ and $a_4 = (0.2142, 0.7858, 0)$.

Step 3: Compute the score function.

Using Equation (1), the score function values are calculated as: $E(a_1) = -0.2809$, $E(a_2) = 0.2494$, $E(a_3) = -0.4088$ and $E(a_4) = 0.2142$.

Step 4: Obtain the ranking order.

since $E(a_2) > E(a_4) > E(a_1) > E(a_3)$, the final ranking order is $\pi_2 \succ \pi_4 \succ \pi_1 \succ \pi_3$, and the optimal hotel is π_2.

Case 2: Select the best hotel with the method based on the *PFWIPGHA* operator.

Step 1: Normalize the initial decision making matrix.

The normalized evaluation matrix is the same as matrix B, which is shown in Table 2.

Step 2: Calculate the overall preference degree of each alternative.

Suppose $\beta = \eta = 1$, and all criteria have the same importance, namely, $w_1 = w_2 = w_3 = w_4 = w_5 = 0.2$. In accordance with the correlation pattern, these criteria are partitioned into two parts: $S_1 = \{\phi_1, \phi_2, \phi_3\}$ and $S_2 = \{\phi_4, \phi_5\}$. Then using the *PFWIPGHA* operator in Equation (17), the overall preference degree of each hotel is computed as: $a_1 = (0.7502, 0.0511, 0.1987)$, $a_2 = (0.7884, 0.1197, 0.0919)$, $a_3 = (0.7675, 0.1094, 0.1231)$ and $a_4 = (0.7805, 0.1251, 0.0943)$.

Step 3: Compute the score function and accuracy function.

Using Equation (1), the score function values are calculated as: $E(a_1) = 0.5514$, $E(a_2) = 0.6965$, $E(a_3) = 0.6443$ and $E(a_4) = 0.6862$.

Step 4: Obtain the ranking order.

Since $E(a_2) > E(a_4) > E(a_3) > E(a_1)$, the final ranking order is $\pi_2 \succ \pi_4 \succ \pi_3 \succ \pi_1$, and the optimal hotel is π_2.

It can be seen that the best alternative is always π_2 no matter which aggregation operator (*PFWIPHA* or *PFWIPGHA*) is used.

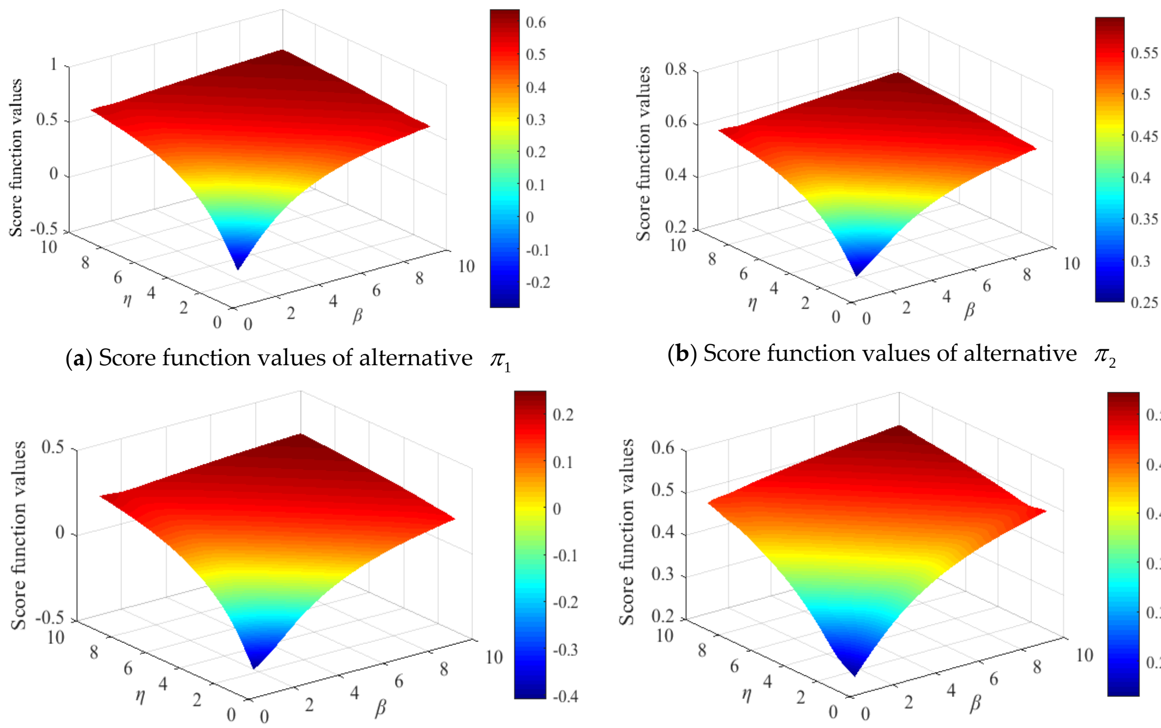
6. Discussions

In this section, the influence of parameters in the proposed method is investigated by sensitivity analyses and the advantages of our method are demonstrated through comparison analyses

6.1. Sensitivity Analyses

In this subsection, the impacts of parameters β and η on the final ranking orders under *PFWIPHA* and *PFWIPGHA* operators are discussed, respectively.

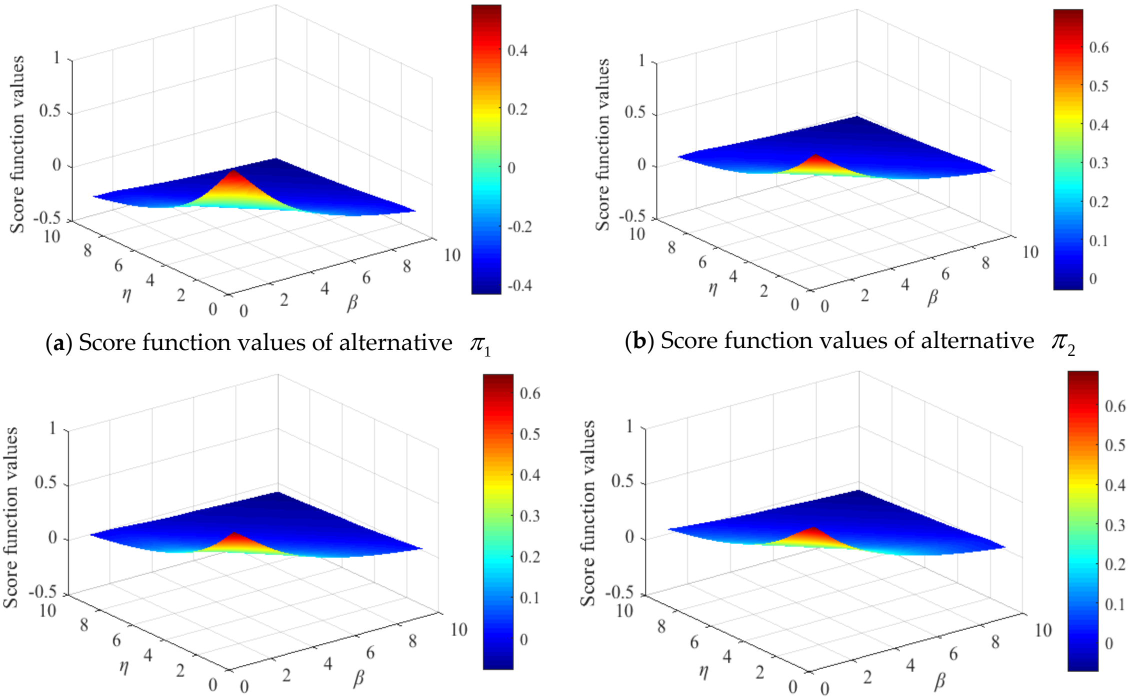
First, the score function values of each alternative under *PFWIPHA* and *PFWIPGHA* operators are calculated by assigning different β and η values, as shown in Figures 1 and 2.



(**a**) Score function values of alternative π_1 (**b**) Score function values of alternative π_2

(**c**) Score function values of alternative π_3 (**d**) Score function values of alternative π_4

Figure 1. Score function values of four alternatives with $\beta,\eta \in (1,10)$ under *PFWIPHA* operators.



(**a**) Score function values of alternative π_1

(**b**) Score function values of alternative π_2

(**c**) Score function values of alternative π_3

(**d**) Score function values of alternative π_4

Figure 2. Score function values of alternative π_1 with $\beta,\eta \in (1,10)$ under *PFWIPGHA* operators.

From Figure 1, it can be seen that the ranking orders may change when dissimilar β or η values are allocated under the *PFWIPHA* operator. Even though, the best alternative is π_2 in many cases, and the worst alternative is always π_3. And the preference relations between alternative π_1 and π_3 (that is, $\pi_1 \succ \pi_3$), and that among π_2, π_3 and π_4 (that is, $\pi_2 \succ \pi_4 \succ \pi_3$) are always true.

From Figure 2, similar conclusions can be reached under the *PFWIPGHA* operator. For example, dissimilar rankings are derived with different values of β and η, but the optimal alternative is still π_2 in most circumstances whereas the worst alternative is always π_1. Likewise, the preference relations between alternative π_1 and π_4 (that is, $\pi_4 \succ \pi_1$), and that among π_1, π_2 and π_3 (that is, $\pi_2 \succ \pi_3 \succ \pi_1$) always exist.

All of these indicate that both *PFWIPHA* and *PFWIPGHA* operators have a certain degree of stability, and do not lose flexibility at the same time. Moreover, an interesting phenomenon is that with the growth of β or η value, the score function of the aggregated values increase under the *PFWIPHA* operator, while they decrease under the *PFWIPGHA* operator. Thus, for an optimistic DM, he/she can choose a larger β or η value under the *PFWIPHA* operator, or a smaller β or η value under the *PFWIPGHA* operator. In contrast, for a pessimistic DM, he/she can choose a smaller β or η value under the *PFWIPHA* operator, or a larger β or η value under the *PFWIPGHA* operator.

6.2. Comparison Analyses

In this subsection, different approaches based on the existing aggregation operators of PFNs are compared in terms of ranking results and other characteristics to demonstrate the advantages of our method.

(1) Comparisons in terms of ranking results

First, the ranking results using the existing approaches with dissimilar aggregation operators (the picture fuzzy weighted averaging (*PFWA*), picture fuzzy weighted geometric averaging (*PFWGA*), *PFWIPHA* and *PFWIPGHA* operators) are listed in Table 3.

Table 3. Ranking orders with dissimilar aggregation operators.

Aggregation Operators	Score Function Values	Ranking Orders	Best Alternatives
PFWA operator [33]	$E(a_1) = 0.1519, E(a_2) = 0.3488,$ $E(a_3) = 0.1067, E(a_4) = 0.3886.$	$\pi_4 \succ \pi_2 \succ \pi_1 \succ \pi_3$	ϕ_2
PFWGA operator [22]	$E(a_1) = 0.3486, E(a_2) = 0.3226,$ $E(a_3) = 0.4169, E(a_4) = 0.3023.$	$\pi_3 \succ \pi_1 \succ \pi_2 \succ \pi_4$	π_3
PFWIPHA operator	$E(a_1) = -0.2809, E(a_2) = 0.2494,$ $E(a_3) = -0.4088, E(a_4) = 0.2142.$	$\pi_2 \succ \pi_4 \succ \pi_1 \succ \pi_3$	π_2
PFWIPGHA operator	$E(a_1) = 0.5514, E(a_2) = 0.6965,$ $E(a_3) = 0.6443, E(a_4) = 0.6862.$	$\pi_2 \succ \pi_4 \succ \pi_3 \succ \pi_1$	π_2

It is clear that the four ranking orders in Table 3 are dissimilar with each other. For seeking out the optimal ranking order among them, the technique proposed by Jahan et al. [34] is employed. As exhibited in Table 4, the numbers of times a host is assigned to diverse ranks is counted. For example, the hotel π_1 has once a ranking of 2, twice a ranking of 3 and once a ranking of 4.

Table 4. Numbers of times a hotel assigned to diverse ranks.

Hotels	Ranks			
	1	2	3	4
π_1		1	2	1
π_2	2	1	1	
π_3	1		1	2
π_4	1	2		1

Then, based on Table 4, the smoothing of hotels assignment over ranks is computed (See Table 5).

Table 5. Smoothing of hotels assignment over ranks Λ_{iZ}.

Hotels	Ranks			
	1	2	3	4
π_1	0	1	3	4
π_2	2	3	4	4
π_3	1	1	2	4
π_4	1	3	3	4

According to Table 5, the following maximizing objective function is performed:

$$\begin{aligned} &Max\ \Theta = \sum_{i=1}^{4}\sum_{z=1}^{4}(\Lambda_{iz}\cdot\tfrac{4^2}{z}\cdot\nabla_{iz}) \\ &s.t.\begin{cases} \sum\limits_{i=1}^{4}\nabla_{iz} = 1, z = 1,2,3,4 \\ \sum\limits_{z=1}^{4}\nabla_{iz} = 1, i = 1,2,3,4 \\ \nabla_{iz} = 0\ or\ 1,\ \forall i, z \end{cases} \end{aligned} \tag{18}$$

Through solving the programming model (18), the optimal ranking order $\pi_2 \succ \pi_4 \succ \pi_1 \succ \pi_3$ is derived.

For clarity, the optimal ranking order and four ranking orders in Table 3 are simultaneously depicted in the same figure (See Figure 3). It is clear that the optimal ranking is the same with the ranking order through adopting *PFWIPHA* operator. The difference between the optimal ranking and the ranking result by utilizing *PFWIPGHA* operator is small. Conversely, a larger discrepancy can be seen among the optimal ranking order, the ranking order with *PFWA* operator, and the ranking order with *PFWGA* operator. All of these indicate that the presented method is more suitable than the existing methods in disposing decision making problems with interactions and partitions of inputs.

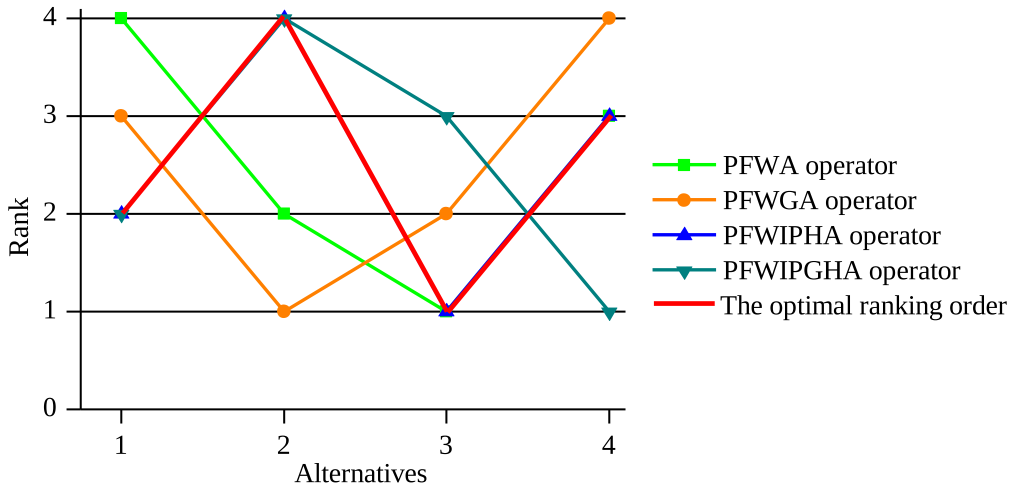


Figure 3. Optimal ranking order among different orders.

(2) Further comparisons in terms of other features

In view of other characteristics (such as whether consider the relationships among membership functions, the correlations among criteria, and the partition the inputs), further comparisons are made in Table 6.

Table 6. Further comparisons in terms of other features.

Aggregation Operators	Operational Rules	Interactions among Membership Functions	Interrelationships among Aggregating Arguments	Partition of the Inputs
PFWA operator [33]	Tradition	Unconsidered	Unconsidered	Unconsidered
PFWGA operator [22]	Tradition	Unconsidered	Unconsidered	Unconsidered
PFWIPHA operator	Interaction	Considered	Considered	Considered
PFWIPGHA operator	Interaction	Considered	Considered	Considered

From Table 6, it can be seen that both the *PFWA* [21] and *PFWGA* [8] operators are based on the traditional operational laws of PFNs, where the interactions among membership functions are not considered. They cannot guarantee the accuracy of aggregated values when 'zero' occurs in the neutral membership degrees. In contrast, the present method (the *PFWIPHA* or *PFWIPGHA* operator) is based on new interaction operational rules of PFNs, which considers the interactions among membership functions. Furthermore, interrelationships among criteria are considered in our method. It is true that traditional HA and GHA operators take the interrelationships among aggregating arguments into account as well, but they hold the supposition that each argument is in relation to the rest of the arguments. Compared with them, a great advantage of our method is that the partition of the input arguments is allowed, so that the interactions of inputs are only considered in the same part rather than different parts.

(3) Strengths analyses and summary

Based on the above comparison analyses, the strengths of our method are summarized as follows.

First, the *PFWIPHA* and *PFWIPGHA* operators suggested in this study are based on new interaction operational rules of PFNs. That is, they consider the interactions among dissimilar membership functions, and can avoid irrational outcomes even when the neutral membership degree is allocated as zero.

Second, the proposed method borrows from the ideas of PHA and *PGHA* operators, so that it can do well with the situation where arguments are divided into several portions. Thus, when the interaction operations are combined, our method can weigh the interrelationships of relevant inputs in the same panel, and prevent any consideration of the interrelationships of uncorrelated inputs in different panels.

Third, as our method under the *PFWIPHA* or *PFWIPGHA* operator is modified by two parameters, it is more flexible than the *PFWA* and *PFWGA* operators. On the other hand, when only one part exists in the inputs, the *PFWIPGA* or *PFWIPGHA* operator is regarded as the *IHA* or *GHA* operator with PFNs, respectively. In this sense, our method is more universal and powerful.

7. Conclusions

HA operators are effective aggregation operators to dispose interrelationships among criteria, while PHA operators aim to do with such correlations only in the same partition. As a result, several aggregation operators, such as *PFIPHA*, *PFWIPHA*, *PFIPGHA* and *PFWIPGHA* operators were presented by combining the PHA operators with picture fuzzy information. Many significant properties, such as the properties of idempotency and commutativity, were proved. Four special cases of *PFIPHA* and *PFIPGHA* with different parameter values were discussed, respectively. Moreover, the interaction operational rules of PFNs were first proposed to overcome the shortcoming of the existing operations. Then, the new decision-making methods based on these operators were used to solve the hotel selection problem by a case study. Sensitivity analyses and comparison analyses revealed that our methods are more flexible and general when processing complicated evaluation issues within a picture fuzzy environment.

However, one weakness of our methods is that the criteria weights are assigned previously and directly. This may be not very reasonable in some cases. Thus, it is worth exploring the weights determination methods in the future. On the other hand, the extended PHA operators can be well used to solve other real problems under picture fuzzy conditions. It is something that we are heading toward.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. The Proof of Theorem 1

Proof. Based on the interaction operational laws defined in Definition 7, $(a_{ki})^{\beta}$ $= \left((1-n_{ki}-m_{ki})^{\beta}-(1-n_{ki}-m_{ki}-p_{ki})^{\beta},(1-n_{ki})^{\beta}-(1-n_{ki}-m_{ki})^{\beta},1-(1-n_{ki})^{\beta}\right)$ and $(a_{kj})^{\eta}$ $=$ $\left((1-n_{kj}-m_{kj})^{\eta}-(1-n_{kj}-m_{kj}-p_{kj})^{\eta},(1-n_{kj})^{\eta}-(1-n_{kj}-m_{kj})^{\eta},1-(1-n_{kj})^{\eta}\right)$.

Let $r_{ki} = 1-n_{ki}$, $r_{kj} = 1-n_{kj}$, $u_{ki} = 1-n_{ki}-m_{ki}$, $u_{kj} = 1-n_{kj}-m_{kj}$, $v_{ki} = 1-n_{ki}-m_{ki}-p_{ki}$ and $v_{kj} = 1-n_{kj}-m_{kj}-p_{kj}$, then $(a_{ki})^{\beta} = \left((u_{ki})^{\beta}-(v_{ki})^{\beta},(r_{ki})^{\beta}-(u_{ki})^{\beta},1-(r_{ki})^{\beta}\right)$, $(a_{kj})^{\eta}$ $= \left((u_{kj})^{\eta}-(v_{kj})^{\eta},(r_{kj})^{\eta}-(u_{kj})^{\eta},1-(r_{kj})^{\eta}\right)$ and

$(a_{ki})^{\beta}\otimes(a_{kj})^{\eta} = \left((u_{ki})^{\beta}(u_{kj})^{\eta}-(v_{ki})^{\beta}(v_{kj})^{\eta},(r_{ki})^{\beta}(r_{kj})^{\eta}-(u_{ki})^{\beta}(u_{kj})^{\eta},1-(r_{ki})^{\beta}(r_{kj})^{\eta}\right)$,

$$\Rightarrow \sum_{i=1}^{|S_k|}\sum_{j=i}^{|S_k|}(a_{ki})^{\beta}\otimes(a_{kj})^{\eta} =$$

$$\Bigg(1-\prod_{i=1,j=i}^{|S_k|}\left(1-(u_{ki})^{\beta}(u_{kj})^{\eta}+(v_{ki})^{\beta}(v_{kj})^{\eta}\right),\prod_{i=1,j=i}^{|S_k|}\left(1-(u_{ki})^{\beta}(u_{kj})^{\eta}+(v_{ki})^{\beta}(v_{kj})^{\eta}\right)-\prod_{i=1,j=i}^{|S_k|}\left(1+(v_{ki})^{\beta}(v_{kj})^{\eta}-(r_{ki})^{\beta}(r_{kj})^{\eta}\right),$$
$$\prod_{i=1,j=i}^{|S_k|}\left(1+(v_{ki})^{\beta}(v_{kj})^{\eta}-(r_{ki})^{\beta}(r_{kj})^{\eta}\right)-\prod_{i=1,j=i}^{|S_k|}\left((v_{ki})^{\beta}(v_{kj})^{\eta}\right)\Bigg).$$

Let $t_k = \frac{2}{|S_k|(|S_k|+1)}$, $A_k^{\beta,\eta} = (v_{ki})^{\beta}(v_{kj})^{\eta}$, $B_k^{\beta,\eta} = 1-(u_{ki})^{\beta}(u_{kj})^{\eta}+(v_{ki})^{\beta}(v_{kj})^{\eta}$ and $C_k^{\beta,\eta} = 1+(v_{ki})^{\beta}(v_{kj})^{\eta}-(r_{ki})^{\beta}(r_{kj})^{\eta}$, then

$$\frac{2}{|S_k|(|S_k|+1)}\sum_{i=1}^{|S_k|}\sum_{j=i}^{|S_k|}(a_{ki})^{\beta}\otimes(a_{kj})^{\eta} = \left(1-\left(\prod_{i=1,j=i}^{|S_k|}B_k^{\beta,\eta}\right)^{t_k},\left(\prod_{i=1,j=i}^{|S_k|}B_k^{\beta,\eta}\right)^{t_k} - \left(\prod_{i=1,j=i}^{|S_k|}C_k^{\beta,\eta}\right)^{t_k},\left(\prod_{i=1,j=i}^{|S_k|}C_k^{\beta,\eta}\right)^{t_k}-\left(\prod_{i=1,j=i}^{|S_k|}A_k^{\beta,\eta}\right)^{t_k}\right) \Rightarrow$$

$$\left(\frac{2}{|S_k|(|S_k|+1)}\sum_{i=1}^{|S_k|}\sum_{j=i}^{|S_k|}(a_{ki})^{\beta}\otimes(a_{kj})^{\eta}\right)^{\frac{1}{\beta+\eta}} = \left(\begin{array}{l}\left(\left(1-\left(\prod_{i=1,j=i}^{|S_k|}B_k^{\beta,\eta}\right)^{t_k}+\left(\prod_{i=1,j=i}^{|S_k|}A_k^{\beta,\eta}\right)^{t_k}\right)^{\frac{1}{\beta+\eta}}-\left(\left(\prod_{i=1,j=i}^{|S_k|}A_k^{\beta,\eta}\right)^{t_k}\right)^{\frac{1}{\beta+\eta}},\left(1-\left(\prod_{i=1,j=i}^{|S_k|}C_k^{\beta,\eta}\right)^{t_k}+\left(\prod_{i=1,j=i}^{|S_k|}A_k^{\beta,\eta}\right)^{t_k}\right)^{\frac{1}{\beta+\eta}}-\right.\\ \left.\left(1+\left(\prod_{i=1,j=i}^{|S_k|}A_k^{\beta,\eta}\right)^{t_k}-\left(\prod_{i=1,j=i}^{|S_k|}B_k^{\beta,\eta}\right)^{t_k}\right)^{\frac{1}{\beta+\eta}},1-\left(1-\left(\prod_{i=1,j=i}^{|S_k|}C_k^{\beta,\eta}\right)^{t_k}+\left(\prod_{i=1,j=i}^{|S_k|}A_k^{\beta,\eta}\right)^{t_k}\right)^{\frac{1}{\beta+\eta}}\right)\end{array}\right).$$

Let $M_k^{\beta,\eta} = \left(\prod_{i=1,j=i}^{|S_k|}A_k^{\beta,\eta}\right)^{t_k}$, $N_k^{\beta,\eta} = \left(\prod_{i=1,j=i}^{|S_k|}B_k^{\beta,\eta}\right)^{t_k}$ and $O_k^{\beta,\eta} = \left(\prod_{i=1,j=i}^{|S_k|}C_k^{\beta,\eta}\right)^{t_k}$, then

$$\sum_{k=1}^{x}\left(\frac{2}{|S_k|(|S_k|+1)}\sum_{i=1}^{|S_k|}\sum_{j=i}^{|S_k|}(a_{ki})^{\beta}\otimes(a_{kj})^{\eta}\right)^{\frac{1}{\beta+\eta}} = \left(\begin{array}{l}1-\prod_{k=1}^{x}\left(1-(1-N_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right),\prod_{k=1}^{x}\left(1-(1-N_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)-\\ \prod_{k=1}^{x}\left(1+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-(1-O_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right),\prod_{k=1}^{x}\left(1+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-(1-O_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)-\prod_{k=1}^{x}\left((M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\end{array}\right),$$

$$\Rightarrow \frac{1}{x}\left(\sum_{k=1}^{x}\left(\frac{2}{|S_k|(|S_k|+1)}\sum_{i=1}^{|S_k|}\sum_{j=i}^{|S_k|}(a_{ki})^{\beta}\otimes(a_{kj})^{\eta}\right)^{\frac{1}{\beta+\eta}}\right) = \left(\begin{array}{l}1-\left(\prod_{k=1}^{x}\left(1-(1-N_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}},\left(\prod_{k=1}^{x}\left(1-(1-N_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}-\\ \left(\prod_{k=1}^{x}\left(1+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-(1-O_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}},\left(\prod_{k=1}^{x}\left(1+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-(1-O_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}\left((M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}\end{array}\right).$$

Thus, $PFIPHA^{\beta,\eta}(a_1,a_2,\ldots,a_y) =$

$$\left(\begin{array}{l}1-\left(\prod_{k=1}^{x}\left(1-(1-N_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}},\left(\prod_{k=1}^{x}\left(1-(1-N_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}-\\ \left(\prod_{k=1}^{x}\left(1+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-(1-O_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}},\left(\prod_{k=1}^{x}\left(1+(M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-(1-O_k^{\beta,\eta}+M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}\left((M_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}\end{array}\right).\ \square$$

Appendix B. The Proof of Theorem 2

Proof. According to the interaction operational laws defined in Definition 7, $w_i a_{ki} = \left(1-(1-p_{ki})^{w_i},(1-p_{ki})^{w_i}-(1-p_{ki}-m_{ki})^{w_i},(1-p_{ki}-m_{ki})^{w_i}-(1-p_{ki}-m_{ki}-n_{ki})^{w_i}\right)$, $w_j a_{kj} = \left(1-(1-p_{kj})^{w_j},(1-p_{kj})^{w_j}-(1-p_{kj}-m_{kj})^{w_j},(1-p_{kj}-m_{kj})^{w_j}-(1-p_{kj}-m_{kj}-n_{kj})^{w_j}\right)$, $(w_{ki}a_{ki})^{\beta} =$

$$\left(\begin{array}{l}\left(1+(1-p_{ki}-m_{ki}-n_{ki})^{w_i}-(1-p_{ki})^{w_i}\right)^{\beta}-\left((1-p_{ki}-m_{ki}-n_{ki})^{w_i}\right)^{\beta},\left(1-(1-p_{ki}-m_{ki})^{w_i}+(1-p_{ki}-m_{ki}-n_{ki})^{w_i}\right)^{\beta}-\\ \left(1+(1-p_{ki}-m_{ki}-n_{ki})^{w_i}-(1-p_{ki})^{w_i}\right)^{\beta},1-\left(1-(1-p_{ki}-m_{ki})^{w_i}+(1-p_{ki}-m_{ki}-n_{ki})^{w_i}\right)^{\beta}\end{array}\right)$$

and $(w_j a_{kj})^{\eta} =$

$$\left(\begin{array}{l}\left(1+(1-p_{kj}-m_{kj}-n_{kj})^{w_j}-(1-p_{kj})^{w_j}\right)^{\eta}-\left((1-p_{kj}-m_{kj}-n_{kj})^{w_j}\right)^{\eta},\left(1-(1-p_{kj}-m_{kj})^{w_j}+(1-p_{kj}-m_{kj}-n_{kj})^{w_j}\right)^{\eta}-\\ \left(1+(1-p_{kj}-m_{kj}-n_{kj})^{w_j}-(1-p_{kj})^{w_j}\right)^{\eta},1-\left(1-(1-p_{kj}-m_{kj})^{w_j}+(1-p_{kj}-m_{kj}-n_{kj})^{w_j}\right)^{\eta}\end{array}\right).$$

Let $e_{ki} = 1-p_{ki}$, $e_{kj} = 1-p_{kj}$, $f_{ki} = 1-p_{ki}-m_{ki}$, $f_{kj} = 1-p_{kj}-m_{kj}$, $v_{ki} = 1-p_{ki}-m_{ki}-n_{ki}$ and $v_{kj} = 1-p_{kj}-m_{kj}-n_{kj}$, then

$(w_{ki}a_{ki})^{\beta} = \left(\left(1+(v_{ki})^{w_i}-(e_{ki})^{w_i}\right)^{\beta}-\left((v_{ki})^{w_i}\right)^{\beta},\left(1-(f_{ki})^{w_i}+(v_{ki})^{w_i}\right)^{\beta}-\left(1+(v_{ki})^{w_i}-(e_{ki})^{w_i}\right)^{\beta},1-\left(1-(f_{ki})^{w_i}+(v_{ki})^{w_i}\right)^{\beta}\right)$,

$(w_ja_{kj})^{\eta} = \left(\left(1+(v_{kj})^{w_j}-(e_{kj})^{w_j}\right)^{\eta}-\left((v_{kj})^{w_j}\right)^{\eta},\left(1-(f_{kj})^{w_j}+(v_{kj})^{w_j}\right)^{\eta}-\left(1+(v_{kj})^{w_j}-(e_{kj})^{w_j}\right)^{\eta},1-\left(1-(f_{kj})^{w_j}+(v_{kj})^{w_j}\right)^{\eta}\right)$,

and $(w_ia_{ki})^{\beta}\otimes(w_ja_{kj})^{\eta} =$

$$\left(\begin{array}{l}\left(1+(v_{ki})^{w_i}-(e_{ki})^{w_i}\right)^{\beta}\left(1+(v_{kj})^{w_j}-(e_{kj})^{w_j}\right)^{\eta}-\left((v_{ki})^{w_i}\right)^{\beta}\left((v_{kj})^{w_j}\right)^{\eta},\left(1-(f_{ki})^{w_i}+(v_{ki})^{w_i}\right)^{\beta}\left(1-(f_{kj})^{w_j}+(v_{kj})^{w_j}\right)^{\eta}-\\ \left(1+(v_{ki})^{w_i}-(e_{ki})^{w_i}\right)^{\beta}\left(1+(v_{kj})^{w_j}-(e_{kj})^{w_j}\right)^{\eta},1-\left(1-(f_{ki})^{w_i}+(v_{ki})^{w_i}\right)^{\beta}\left(1-(f_{kj})^{w_j}+(v_{kj})^{w_j}\right)^{\eta}\end{array}\right).$$

Let $G_k^{\beta,\eta} = \left((v_{ki})^{w_i}\right)^{\beta}\left((v_{kj})^{w_j}\right)^{\eta}$, $H_k^{\beta,\eta} = \left(1+(v_{ki})^{w_i}-(e_{ki})^{w_i}\right)^{\beta}\left(1+(v_{kj})^{w_j}-(e_{kj})^{w_j}\right)^{\eta}$ and $I_k^{\beta,\eta} = \left(1-(f_{ki})^{w_i}+(v_{ki})^{w_i}\right)^{\beta}\left(1-(f_{kj})^{w_j}+(v_{kj})^{w_j}\right)^{\eta}$, then $(w_ia_{ki})^{\beta}\otimes(w_ja_{kj})^{\eta} = \left(H_k^{\beta,\eta}-G_k^{\beta,\eta},I_k^{\beta,\eta}-H_k^{\beta,\eta},1-I_k^{\beta,\eta}\right),\Rightarrow \sum_{i=1}^{|S_k|}\sum_{j=i}^{|S_k|}(w_ia_{ki})^{\beta}\otimes(w_ja_{kj})^{\eta} = \left(1-\prod_{i=1,j=i}^{|S_k|}\left(1-H_k^{\beta,\eta}+G_k^{\beta,\eta}\right),\prod_{i=1,j=i}^{|S_k|}\left(1-H_k^{\beta,\eta}+G_k^{\beta,\eta}\right)-\prod_{i=1,j=i}^{|S_k|}\left(1+G_k^{\beta,\eta}-I_k^{\beta,\eta}\right),\prod_{i=1,j=i}^{|S_k|}\left(1+G_k^{\beta,\eta}-I_k^{\beta,\eta}\right)-\prod_{i=1,j=i}^{|S_k|}\left(G_k^{\beta,\eta}\right)\right)$.

Let $t_k = \frac{2}{|S_k|(|S_k|+1)}$, then $\frac{2}{|S_k|(|S_k|+1)}\sum_{i=1}^{|S_k|}\sum_{j=i}^{|S_k|}(w_ia_{ki})^{\beta}\otimes(w_ja_{kj})^{\eta} =$

$$\left(1-\left(\prod_{i=1,j=i}^{|S_k|}(1-H_k^{\beta,\eta}+G_k^{\beta,\eta})\right)^{t_k},\left(\prod_{i=1,j=i}^{|S_k|}(1-H_k^{\beta,\eta}+G_k^{\beta,\eta})\right)^{t_k}-\left(\prod_{i=1,j=i}^{|S_k|}(1+G_k^{\beta,\eta}-I_k^{\beta,\eta})\right)^{t_k},\left(\prod_{i=1,j=i}^{|S_k|}(1+G_k^{\beta,\eta}-I_k^{\beta,\eta})\right)^{t_k}-\left(\prod_{i=1,j=i}^{|S_k|}G_k^{\beta,\eta}\right)^{t_k}\right)\Rightarrow$$

$$\left(\frac{2}{|S_k|(|S_k|+1)}\sum_{i=1}^{|S_k|}\sum_{j=i}^{|S_k|}(w_ia_{ki})^{\beta}\otimes(w_ja_{kj})^{\eta}\right)^{\frac{1}{\beta+\eta}} =$$

$$\left(\begin{array}{l}\left(1+\left(\prod_{i=1,j=i}^{|S_k|}G_k^{\beta,\eta}\right)^{t_k}-\left(\prod_{i=1,j=i}^{|S_k|}\left(1-H_k^{\beta,\eta}+G_k^{\beta,\eta}\right)\right)^{t_k}\right)^{\frac{1}{\beta+\eta}}-\left(\left(\prod_{i=1,j=i}^{|S_k|}G_k^{\beta,\eta}\right)^{t_k}\right)^{\frac{1}{\beta+\eta}},\left(1-\left(\prod_{i=1,j=i}^{|S_k|}\left(1+G_k^{\beta,\eta}-I_k^{\beta,\eta}\right)\right)^{t_k}+\left(\prod_{i=1,j=i}^{|S_k|}G_k^{\beta,\eta}\right)^{t_k}\right)^{\frac{1}{\beta+\eta}}-\\ \left(1+\left(\prod_{i=1,j=i}^{|S_k|}G_k^{\beta,\eta}\right)^{t_k}-\left(\prod_{i=1,j=i}^{|S_k|}\left(1-H_k^{\beta,\eta}+G_k^{\beta,\eta}\right)\right)^{t_k}\right)^{\frac{1}{\beta+\eta}},1-\left(1-\left(\prod_{i=1,j=i}^{|S_k|}\left(1+G_k^{\beta,\eta}-I_k^{\beta,\eta}\right)\right)^{t_k}+\left(\prod_{i=1,j=i}^{|S_k|}G_k^{\beta,\eta}\right)^{t_k}\right)^{\frac{1}{\beta+\eta}}\end{array}\right).$$

Let $J_k^{\beta,\eta} = \left(\prod_{i=1,j=i}^{|S_k|}G_k^{\beta,\eta}\right)^{t_k}$, $L_k^{\beta,\eta} = \left(\prod_{i=1,j=i}^{|S_k|}\left(1-H_k^{\beta,\eta}+G_k^{\beta,\eta}\right)\right)^{t_k}$ and $Q_k^{\beta,\eta} = \left(\prod_{i=1,j=i}^{|S_k|}\left(1+G_k^{\beta,\eta}-I_k^{\beta,\eta}\right)\right)^{t_k}$, then $\left(\frac{2}{|S_k|(|S_k|+1)}\sum_{i=1}^{|S_k|}\sum_{j=i}^{|S_k|}(w_ia_{ki})^{\beta}\otimes(w_ja_{kj})^{\eta}\right)^{\frac{1}{\beta+\eta}} = \left((1+J_k^{\beta,\eta}-L_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-(J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}},(1-Q_k^{\beta,\eta}+J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-(1+J_k^{\beta,\eta}-L_k^{\beta,\eta})^{\frac{1}{\beta+\eta}},1-(1-Q_k^{\beta,\eta}+J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right),\Rightarrow\sum_{k=1}^{x}\left(\frac{2}{|S_k|(|S_k|+1)}\sum_{i=1}^{|S_k|}\sum_{j=i}^{|S_k|}(w_ia_{ki})^{\beta}\otimes(w_ja_{kj})^{\eta}\right)^{\frac{1}{\beta+\eta}} =$

$$\left(\begin{array}{l}1-\prod_{k=1}^{x}\left(1-(1+J_k^{\beta,\eta}-L_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}+(J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right),\prod_{k=1}^{x}\left(1-(1+J_k^{\beta,\eta}-L_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}+(J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)-\\ \prod_{k=1}^{x}\left(1+(J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-(1-Q_k^{\beta,\eta}+J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right),\prod_{k=1}^{x}\left(1+(J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-(1-Q_k^{\beta,\eta}+J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)-\prod_{k=1}^{x}\left((J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\end{array}\right)\Rightarrow$$

$$\frac{1}{x}\left(\sum_{k=1}^{x}\left(\frac{2}{|S_k|(|S_k|+1)}\sum_{i=1}^{|S_k|}\sum_{j=i}^{|S_k|}(w_ia_{ki})^{\beta}\otimes(w_ja_{kj})^{\eta}\right)^{\frac{1}{\beta+\eta}}\right) =$$

$$\left(\begin{array}{l}1-\left(\prod_{k=1}^{x}\left(1-(1+J_k^{\beta,\eta}-L_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}+(J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}},\left(\prod_{k=1}^{x}\left(1-(1+J_k^{\beta,\eta}-L_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}+(J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}-\\ \left(\prod_{k=1}^{x}\left(1+(J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-(1-Q_k^{\beta,\eta}+J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}},\left(\prod_{k=1}^{x}\left(1+(J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-(1-Q_k^{\beta,\eta}+J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}\left((J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}\end{array}\right).$$

Thus, $PFWIPHA^{\beta,\eta}(a_1,a_2,\ldots,a_y) =$

$$\left(\begin{array}{l}1-\left(\prod_{k=1}^{x}\left(1-(1+J_k^{\beta,\eta}-L_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}+(J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}},\left(\prod_{k=1}^{x}\left(1-(1+J_k^{\beta,\eta}-L_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}+(J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}-\\ \left(\prod_{k=1}^{x}\left(1+(J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-(1-Q_k^{\beta,\eta}+J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}},\left(\prod_{k=1}^{x}\left(1+(J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-(1-Q_k^{\beta,\eta}+J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}\left((J_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}\end{array}\right).\ \square$$

Appendix C. The proof of Theorem 3

Proof. According to the interaction operational laws defined in Definition 7, $\beta a_{ki} \oplus \eta a_{kj} = \big(1-(1-p_{ki})^{\beta}(1-p_{kj})^{\eta}, (1-p_{ki})^{\beta}(1-p_{kj})^{\eta} - (1-p_{ki}-m_{ki})^{\beta}(1-p_{kj}-m_{kj})^{\eta}, (1-p_{ki}-m_{ki})^{\beta}(1-p_{kj}-m_{kj})^{\eta} - (1-p_{ki}-m_{ki}-n_{ki})^{\beta}(1-p_{kj}-m_{kj}-n_{kj})^{\eta}\big)$

Let $e_{ki} = 1-p_{ki}$, $e_{kj} = 1-p_{kj}$, $f_{ki} = 1-p_{ki}-m_{ki}$, $f_{kj} = 1-p_{kj}-m_{kj}$, $v_{ki} = 1-p_{ki}-m_{ki}-n_{ki}$ and $v_{kj} = 1-p_{kj}-m_{kj}-n_{kj}$, then $\beta a_{ki} \oplus \eta a_{kj} = \big(1-(e_{ki})^{\beta}(e_{kj})^{\eta}, (e_{ki})^{\beta}(e_{kj})^{\eta} - (f_{ki})^{\beta}(f_{kj})^{\eta}, (f_{ki})^{\beta}(f_{kj})^{\eta} - (v_{ki})^{\beta}(v_{kj})^{\eta}\big)$,

$$\Rightarrow \prod_{i=1,j=i}^{|S_k|}(\beta a_{ki} \oplus \eta a_{kj}) =$$

$$\left(\begin{array}{l}\prod\limits_{i=1,j=i}^{|S_k|}\big(1+(v_{ki})^{\beta}(v_{kj})^{\eta}-(e_{ki})^{\beta}(e_{kj})^{\eta}\big) - \prod\limits_{i=1,j=i}^{|S_k|}\big((v_{ki})^{\beta}(v_{kj})^{\eta}\big), \prod\limits_{i=1,j=i}^{|S_k|}\big(1-(f_{ki})^{\beta}(f_{kj})^{\eta}+(v_{ki})^{\beta}(v_{kj})^{\eta}\big) - \\ \prod\limits_{i=1,j=i}^{|S_k|}\big(1+(v_{ki})^{\beta}(v_{kj})^{\eta}-(e_{ki})^{\beta}(e_{kj})^{\eta}\big), 1-\prod\limits_{i=1,j=i}^{|S_k}\big(1-(f_{ki})^{\beta}(f_{kj})^{\eta}+(v_{ki})^{\beta}(v_{kj})^{\eta}\big)\end{array}\right).$$

Let $t_k = \frac{2}{|S_k|(|S_k|+1)}$, then $\left(\prod\limits_{i=1,j=i}^{|S_k|}(\beta a_{ki} \oplus \eta a_{kj})\right)^{\frac{2}{|S_k|(|S_k|+1)}} =$

$$\left(\begin{array}{l}\left(\prod\limits_{i=1,j=i}^{|S_k|}\big(1+(v_{ki})^{\beta}(v_{kj})^{\eta}-(v_{ki})^{\beta}(v_{kj})^{\eta}\big)\right)^{t_k} - \left(\prod\limits_{i=1,j=i}^{|S_k|}\big((v_{ki})^{\beta}(v_{kj})^{\eta}\big)\right)^{t_k}, \left(\prod\limits_{i=1,j=i}^{|S_k}\big(1-(f_{ki})^{\beta}(f_{kj})^{\eta}+(v_{ki})^{\beta}(v_{kj})^{\eta}\big)\right)^{t_k} - \\ \left(\prod\limits_{i=1,j=i}^{|S_k|}\big(1+(v_{ki})^{\beta}(v_{kj})^{\eta}-(e_{ki})^{\beta}(e_{kj})^{\eta}\big)\right)^{t_k}, 1-\left(\prod\limits_{i=1,j=i}^{|S_k}\big(1-(f_{ki})^{\beta}(f_{kj})^{\eta}+(v_{ki})^{\beta}(v_{kj})^{\eta}\big)\right)^{t_k}\end{array}\right).$$

Let $R_k^{\beta,\eta} = \prod\limits_{i=1,j=i}^{|S_k|}\big((v_{ki})^{\beta}(v_{kj})^{\eta}\big)$, $U_k^{\beta,\eta} = \prod\limits_{i=1,j=i}^{|S_k|}\big(1+(v_{ki})^{\beta}(v_{kj})^{\eta}-(e_{ki})^{\beta}(e_{kj})^{\eta}\big)$ and $V_k^{\beta,\eta} = \prod\limits_{i=1,j=i}^{|S_k|}\big(1-(f_{ki})^{\beta}(f_{kj})^{\eta}+(v_{ki})^{\beta}(v_{kj})^{\eta}\big)$, then $\frac{1}{\beta+\eta}\left(\prod\limits_{i=1,j=i}^{|S_k|}(\beta a_{ki} \oplus \eta a_{kj})\right)^{\frac{2}{|S_k|(|S_k|+1)}} =$

$$\left(1-\left(1-(U_k^{\beta,\eta})^{t_k}+(R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}}, \left(1-(U_k^{\beta,\eta})^{t_k}+(R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}} - \left(1+(R_k^{\beta,\eta})^{t_k}-(V_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}}, \left(1+(R_k^{\beta,\eta})^{t_k}-(V_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}} - \left((R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}}\right),$$

$$\Rightarrow \prod_{k=1}^{x}\left(\frac{1}{\beta+\eta}\left(\prod_{i=1,j=i}^{|S_k|}(\beta a_{ki} \oplus \eta a_{kj})\right)^{\frac{2}{|S_k|(|S_k|+1)}}\right) =$$

$$\left(\begin{array}{l}\prod\limits_{k=1}^{x}\left(1+\left((R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}} - \left(1-(U_k^{\beta,\eta})^{t_k}+(R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}}\right) - \prod\limits_{k=1}^{x}\left(\left((R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}}\right), \prod\limits_{k=1}^{x}\left(1-\left(1+(R_k^{\beta,\eta})^{t_k}-(V_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}} + \left((R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}}\right) - \\ \prod\limits_{k=1}^{x}\left(1+\left((R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}} - \left(1-(U_k^{\beta,\eta})^{t_k}+(R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}}\right), 1-\prod\limits_{k=1}^{x}\left(1-\left(1+(R_k^{\beta,\eta})^{t_k}-(V_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}} + \left((R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}}\right)\end{array}\right) \Rightarrow$$

$$\left(\prod_{k=1}^{x}\left(\frac{1}{\beta+\eta}\left(\prod_{i=1,j=i}^{|S_k|}(\beta a_{ki} \oplus \eta a_{kj})\right)^{\frac{2}{|S_k|(|S_k|+1)}}\right)\right)^{\frac{1}{x}} =$$

$$\left(\begin{array}{l}\left(\prod\limits_{k=1}^{x}\left(1+\left((R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}} - \left(1-(U_k^{\beta,\eta})^{t_k}+(R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}} - \left(\prod\limits_{k=1}^{x}\left(\left((R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}, \left(\prod\limits_{k=1}^{x}\left(1-\left(1+(R_k^{\beta,\eta})^{t_k}-(V_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}} + \left((R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}} - \\ \left(\prod\limits_{k=1}^{x}\left(1+\left((R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}} - \left(1-(U_k^{\beta,\eta})^{t_k}+(R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}, 1-\left(\prod\limits_{k=1}^{x}\left(1-\left(1+(R_k^{\beta,\eta})^{t_k}-(V_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}} + \left((R_k^{\beta,\eta})^{t_k}\right)^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}\end{array}\right).$$

□

Appendix D. The Proof of Theorem 4

Proof. Based on the interaction operational laws defined in Definition 7, $(a_{ki})^{w_i} = \big((1-n_{ki}-m_{ki})^{w_i} - (1-n_{ki}-m_{ki}-p_{ki})^{w_i}, (1-n_{ki})^{w_i} - (1-n_{ki}-m_{ki})^{w_i}, 1-(1-n_{ki})^{w_i}\big)$ and $(a_{kj})^{w_j} = \big((1-n_{kj}-m_{kj})^{w_j} - (1-n_{kj}-m_{kj}-p_{kj})^{w_j}, (1-n_{kj})^{w_j} - (1-n_{kj}-m_{kj})^{w_j}, 1-(1-n_{kj})^{w_j}\big)$.

Let $r_{ki} = 1-n_{ki}$, $r_{kj} = 1-n_{kj}$, $u_{ki} = 1-n_{ki}-m_{ki}$, $u_{kj} = 1-n_{kj}-m_{kj}$, $v_{ki} = 1-n_{ki}-m_{ki}-p_{ki}$ and $v_{kj} = 1-n_{kj}-m_{kj}-p_{kj}$, then
$\beta(a_{ki})^{w_i} =$

$\left(1-\left(1-(u_{ki})^{w_i}+(v_{ki})^{w_i}\right)^{\beta},\left(1-(u_{ki})^{w_i}+(v_{ki})^{w_i}\right)^{\beta}-\left(1+(v_{ki})^{w_i}-(r_{ki})^{w_i}\right)^{\beta},\left(1+(v_{ki})^{w_i}-(r_{ki})^{w_i}\right)^{\beta}-\left((v_{ki})^{w_i}\right)^{\beta}\right)$ and $\eta(a_{kj})^{w_j}$ $=\left(1-\left(1-(u_{kj})^{w_j}+(v_{kj})^{w_j}\right)^{\eta},\left(1-(u_{kj})^{w_j}+(v_{kj})^{w_j}\right)^{\eta}-\left(1+(v_{kj})^{w_j}-(r_{kj})^{w_j}\right)^{\eta},\left(1+(v_{kj})^{w_j}-(r_{kj})^{w_j}\right)^{\eta}-\left((v_{kj})^{w_j}\right)^{\eta}\right)$, $\Rightarrow$ $\beta(a_{ki})^{w_i}\oplus\eta(a_{kj})^{w_j}=$

$$\begin{pmatrix}1-\left(1-(u_{ki})^{w_i}+(v_{ki})^{w_i}\right)^{\beta}\left(1-(u_{kj})^{w_j}+(v_{kj})^{w_j}\right)^{\eta},\left(1-(u_{ki})^{w_i}+(v_{ki})^{w_i}\right)^{\beta}\left(1-(u_{kj})^{w_j}+(v_{kj})^{w_j}\right)^{\eta}-\\ \left(1+(v_{ki})^{w_i}-(r_{ki})^{w_i}\right)^{\beta}\left(1+(v_{kj})^{w_j}-(r_{kj})^{w_j}\right)^{\eta},\left(1+(v_{ki})^{w_i}-(r_{ki})^{w_i}\right)^{\beta}\left(1+(v_{kj})^{w_j}-(r_{kj})^{w_j}\right)^{\eta}-\left((v_{ki})^{w_i}\right)^{\beta}\left((v_{kj})^{w_j}\right)^{\eta}\end{pmatrix}.$$

Let $G_k^{\beta,\eta}=\left((v_{ki})^{w_i}\right)^{\beta}\left((v_{kj})^{w_j}\right)^{\eta}$, $T_k^{\beta,\eta}=\left(1-(u_{ki})^{w_i}+(v_{ki})^{w_i}\right)^{\beta}\left(1-(u_{kj})^{w_j}+(v_{kj})^{w_j}\right)^{\eta}$ and $Z_k^{\beta,\eta}=\left(1+(v_{ki})^{w_i}-(r_{ki})^{w_i}\right)^{\beta}\left(1+(v_{kj})^{w_j}-(r_{kj})^{w_j}\right)^{\eta}$, then $\prod_{i=1,j=i}^{|S_k|}\beta(a_{ki})^{w_i}\oplus\eta(a_{kj})^{w_j}=\left(\prod_{i=1,j=i}^{|S_k|}\left(1+G_k^{\beta,\eta}-T_k^{\beta,\eta}\right)-\prod_{i=1,j=i}^{|S_k|}G_k^{\beta,\eta},\prod_{i=1,j=i}^{|S_k|}\left(1-Z_k^{\beta,\eta}+G_k^{\beta,\eta}\right)-\prod_{i=1,j=i}^{|S_k|}\left(1+G_k^{\beta,\eta}-T_k^{\beta,\eta}\right),1-\prod_{i=1,j=i}^{|S_k|}\left(1-Z_k^{\beta,\eta}+G_k^{\beta,\eta}\right)\right)$.

Let $t_k=\frac{2}{|S_k|(|S_k|+1)}$, then $\left(\prod_{i=1,j=i}^{|S_k|}\beta(a_{ki})^{w_i}\oplus\eta(a_{kj})^{w_j}\right)^{\frac{2}{|S_k|(|S_k|+1)}}=\left(\left(\prod_{i=1,j=i}^{|S_k|}\left(1+G_k^{\beta,\eta}-T_k^{\beta,\eta}\right)\right)^{t_k}-\left(\prod_{i=1,j=i}^{|S_k|}G_k^{\beta,\eta}\right)^{t_k},\left(\prod_{i=1,j=i}^{|S_k|}\left(1-Z_k^{\beta,\eta}+G_k^{\beta,\eta}\right)\right)^{t_k}-\left(\prod_{i=1,j=i}^{|S_k|}\left(1+G_k^{\beta,\eta}-T_k^{\beta,\eta}\right)\right)^{t_k},1-\left(\prod_{i=1,j=i}^{|S_k|}\left(1-Z_k^{\beta,\eta}+G_k^{\beta,\eta}\right)\right)^{t_k}\right)$, $\Rightarrow$ $\frac{1}{\beta+\eta}\left(\prod_{i=1,j=i}^{|S_k|}\beta(a_{ki})^{w_i}\oplus\eta(a_{kj})^{w_j}\right)^{\frac{2}{|S_k|(|S_k|+1)}}=$

$$\begin{pmatrix}1-\left(1-\left(\prod_{i=1,j=i}^{|S_k|}\left(1+G_k^{\beta,\eta}-T_k^{\beta,\eta}\right)\right)^{t_k}+\left(\prod_{i=1,j=i}^{|S_k|}G_k^{\beta,\eta}\right)^{t_k}\right)^{\frac{1}{\beta+\eta}},\left(1-\left(\prod_{i=1,j=i}^{|S_k|}\left(1+G_k^{\beta,\eta}-T_k^{\beta,\eta}\right)\right)^{t_k}+\left(\prod_{i=1,j=i}^{|S_k|}G_k^{\beta,\eta}\right)^{t_k}\right)^{\frac{1}{\beta+\eta}}-\\ \left(1+\left(\prod_{i=1,j=i}^{|S_k|}G_k^{\beta,\eta}\right)^{t_k}-\left(\prod_{i=1,j=i}^{|S_k|}\left(1-Z_k^{\beta,\eta}+G_k^{\beta,\eta}\right)\right)^{t_k}\right)^{\frac{1}{\beta+\eta}},\left(1+\left(\prod_{i=1,j=i}^{|S_k|}G_k^{\beta,\eta}\right)^{t_k}-\left(\prod_{i=1,j=i}^{|S_k|}\left(1-Z_k^{\beta,\eta}+G_k^{\beta,\eta}\right)\right)^{t_k}\right)^{\frac{1}{\beta+\eta}}-\left(\left(\prod_{i=1,j=i}^{|S_k|}G_k^{\beta,\eta}\right)^{t_k}\right)^{\frac{1}{\beta+\eta}}\end{pmatrix}.$$

Let $O_k^{\beta,\eta}=\left(\prod_{i=1,j=i}^{|S_k|}G_k^{\beta,\eta}\right)^{t_k}$, $P_k^{\beta,\eta}=\left(\prod_{i=1,j=i}^{|S_k|}\left(1+G_k^{\beta,\eta}-T_k^{\beta,\eta}\right)\right)^{t_k}$ and $Y_k^{\beta,\eta}=\left(\prod_{i=1,j=i}^{|S_k|}\left(1-Z_k^{\beta,\eta}+G_k^{\beta,\eta}\right)\right)^{t_k}$, then $\prod_{k=1}^{x}\left(\frac{1}{\beta+\eta}\left(\prod_{i=1,j=i}^{|S_k|}\beta(a_{ki})^{w_i}\oplus\eta(a_{kj})^{w_j}\right)^{\frac{2}{|S_k|(|S_k|+1)}}\right)=$

$$\begin{pmatrix}\prod_{k=1}^{x}\left(1-(O_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-\left(1-P_k^{\beta,\eta}+O_k^{\beta,\eta}\right)^{\frac{1}{\beta+\eta}}\right)-\prod_{k=1}^{x}\left((O_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right),\prod_{k=1}^{x}\left(1-\left(1+O_k^{\beta,\eta}-Y_k^{\beta,\eta}\right)^{\frac{1}{\beta+\eta}}+(O_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)-\\ \prod_{k=1}^{x}\left(1-(O_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-\left(1-P_k^{\beta,\eta}+O_k^{\beta,\eta}\right)^{\frac{1}{\beta+\eta}}\right),1-\prod_{k=1}^{x}\left(1-\left(1+O_k^{\beta,\eta}-Y_k^{\beta,\eta}\right)^{\frac{1}{\beta+\eta}}+(O_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\end{pmatrix}, \Rightarrow$$

$\left(\prod_{k=1}^{x}\left(\frac{1}{\beta+\eta}\left(\prod_{i=1,j=i}^{|S_k|}\beta(a_{ki})^{w_i}\oplus\eta(a_{kj})^{w_j}\right)^{\frac{2}{|S_k|(|S_k|+1)}}\right)\right)^{\frac{1}{x}}=$

$$\begin{pmatrix}\left(\prod_{k=1}^{x}\left(1-(O_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-\left(1-P_k^{\beta,\eta}+O_k^{\beta,\eta}\right)^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}-\left(\prod_{k=1}^{x}\left((O_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}},\left(\prod_{k=1}^{x}\left(1-\left(1+O_k^{\beta,\eta}-Y_k^{\beta,\eta}\right)^{\frac{1}{\beta+\eta}}+(O_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}\\ -\left(\prod_{k=1}^{x}\left(1-(O_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}-\left(1-P_k^{\beta,\eta}+O_k^{\beta,\eta}\right)^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}},1-\left(\prod_{k=1}^{x}\left(1-\left(1+O_k^{\beta,\eta}-Y_k^{\beta,\eta}\right)^{\frac{1}{\beta+\eta}}+(O_k^{\beta,\eta})^{\frac{1}{\beta+\eta}}\right)\right)^{\frac{1}{x}}\end{pmatrix}. \square$$

Appendix E. Questionnaire of Evaluating Hotels

Please make a choice in each line.

Table A1. Questionnaire of evaluation hotels.

Hotels	Criteria	Evaluation		
π_1	ϕ_1 (Price)	□ High	□ Medium	□ Low
	ϕ_2 (Comfortability)	□ Good	□ Fair	□ Poor
	ϕ_3 (Service)	□ Good	□ Fair	□ Poor
	ϕ_4 (Location)	□ Good	□ Fair	□ Poor
	ϕ_5 (Convenience)	□ Good	□ Fair	□ Poor
π_2	ϕ_1 (Price)	□ High	□ Medium	□ Low
	ϕ_2 (Comfortability)	□ Good	□ Fair	□ Poor
	ϕ_3 (Service)	□ Good	□ Fair	□ Poor
	ϕ_4 (Location)	□ Good	□ Fair	□ Poor
	ϕ_5 (Convenience)	□ Good	□ Fair	□ Poor
π_3	ϕ_1 (Price)	□ High	□ Medium	□ Low
	ϕ_2 (Comfortability)	□ Good	□ Fair	□ Poor
	ϕ_3 (Service)	□ Good	□ Fair	□ Poor
	ϕ_4 (Location)	□ Good	□ Fair	□ Poor
	ϕ_5 (Convenience)	□ Good	□ Fair	□ Poor
π_4	ϕ_1 (Price)	□ High	□ Medium	□ Low
	ϕ_2 (Comfortability)	□ Good	□ Fair	□ Poor
	ϕ_3 (Service)	□ Good	□ Fair	□ Poor
	ϕ_4 (Location)	□ Good	□ Fair	□ Poor
	ϕ_5 (Convenience)	□ Good	□ Fair	□ Poor

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