mathematics

MDPI

Article

Fourier Truncation Regularization Method for a Time-Fractional Backward Diffusion Problem with a Nonlinear Source

Fan Yang *, Ping Fan, Xiao-Xiao Li and Xin-Yi Ma

School of Science, Lanzhou University of Technology, Lanzhou 730050, China; nwldg@lut.edu.cn (P.F.); xiaoxiaoli@lut.edu.cn (X.-X.L.); xinyima@lut.edu.cn (X.-Y.M.)

* Correspondence: fanyang@lut.edu.cn; Tel.: +86-151-1708-4315

Received: 4 August 2019 ; Accepted: 4 September 2019 ; Published: 19 September 2019

check for updates

Abstract: In present paper, we deal with a backward diffusion problem for a time-fractional diffusion problem with a nonlinear source in a strip domain. We all know this nonlinear problem is severely ill-posed, i.e., the solution does not depend continuously on the measurable data. Therefore, we use the Fourier truncation regularization method to solve this problem. Under an a priori hypothesis and an a priori regularization parameter selection rule, we obtain the convergence error estimates between the regular solution and the exact solution at $0 \leq x < 1$.

Keywords: time-fractional diffusion problem; ill-posed problem; Fourier truncation method; error estimate

MSC: 35R25; 47A52; 35R30

---

1. Introduction

Over the past decade, fractional diffusion problems have become important in engineering and science [1]. This term is used to describe a large number of problems in biological, chemical, control theory, mechanical engineering, finance, and fractional dynamics [2–4]. Fractional order derivatives and integrals can be used to describe the memory and hereditary properties of different substances [5], leading to the fractional-order model, which is very useful. Information regarding the development of fractional differential operators can be found in [6,7].

The fractional diffusion equation, which can be used to describe the superdiffusion and superdiffusion phenomena [8], is obtained by converting the derivative of the heat equation regarding time into the derivative of the fractional order. A large number of scholars have studied the time-fractional diffusion equation. In [9], an iteration regularization method is used to consider the inverse heat conduction problem for a time-fractional diffusion equation. In [10], the mollification regularization method is used to consider the Cauchy problem of the time-fractional diffusion equation. In [11], the spectral regularization method is applied to solve the inverse problem of the time-fractional inverse diffusion problem. In [12], two different regularization methods are used to solve a Riesz–Feller space-fractional backward diffusion problem. In [13], the spectral regularization method is used to solve the Cauchy problem of the time-fractional advection-dispersion equation. In [14], a new regularization method is applied to solve a time-fractional inverse diffusion problem. In [15], the optimal regularization method is used to solve an inverse heat conduction problem for the fractional diffusion equation. In [16], the Landweber iterative regularization method is applied to identify the initial value problem of the time–space fractional diffusion-wave equation. In [17,18], The quasi-boundary value method is used to identify the initial value of the heat equation and the time-fractional diffusion equation on a spherically symmetric domain. In [19,20], the authors considered the nonlinear equation with variable exponents and proved a finite-time blow-up result for the solutions with negative initial energy and for certain solutions with positive energy.

However, in the above papers on inverse problem of the fractional diffusion equation, the equation is homogeneous and the problem is linear. In [21], a new modified regularization method can be used to solve the backward problem for the nonlinear space-fractional diffusion equation. In [22], two new modified regularization methods are applied to solve the backward problem for a nonlinear Riesz–Feller space fractional diffusion equation. However, in [21,22], the fractional diffusion equation is a space-fractional diffusion equation. In this paper, we investigate the following nonlinear time-fractional diffusion equation:

$$\begin{cases} D_t^\alpha u(x,t) - u_{xx}(x,t) = f(x,t,u(x,t)), & t>0, x\in\mathbb{R}, \\ u(x,0)=0, & x\in\mathbb{R}, \\ u(1,t)=g(t), & t>0, \\ u(x,t)|_{x\to\infty} bounded, & t>0, \end{cases} \tag{1}$$

where the time-fractional derivative $D_t^\alpha u(x,t)$ is the Caputo fractional derivative of order $\alpha(0<\alpha\le 1)$ defined by [16]:

$$\begin{cases} D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{du(x,s)}{ds}\frac{ds}{(t-s)^\alpha}, 0<\alpha<1, \\ \\ D_t^\alpha u(x,t) = \frac{du(x,t)}{dt}, \alpha=1, \end{cases} \tag{2}$$

where $f(x,t,u(x,t))$ is a known nonlinear heat source. $u(1,t)=g(t)$ is an additional condition and can be used to identify $u(x,t)$ as $0\le x<1$. The measurable data $g^\delta(t)\in L^2(\mathbb{R})$ meets

$$\|g^\delta - g\|_{L^2(\mathbb{R})} \le \delta, \tag{3}$$

where $\|\cdot\|$ represents $L^2(\mathbb{R})$ norm and $\delta>0$ is a noise level.

The structure of this article is as follows. Some auxiliary results are given in Section 2. The Fourier truncation regularization method is presented in Section 3. The error estimates are given in Section 4, and a simple conclusion is given in Section 5.

2. Some Auxiliary Results

Suppose $\widehat{f}(\xi)$ denotes the Fourier transform of $f(t)$ defined by

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi t} f(t)dt. \tag{4}$$

In order to use the Fourier transform for variable t, we let $u(x,t)=0$, as $t<0$. Using the Fourier transform for variable t, we obtain

$$\begin{cases} (i\xi)^{\frac{\alpha}{2}}\widehat{u}(x,\xi) - \widehat{u}_{xx}(x,t) = \widehat{f(s,t,u)}(x,\xi), & \xi\in\mathbb{R}, x\in\mathbb{R}, \\ \widehat{u}(x,0)=0, & x\in\mathbb{R}, \\ \widehat{u}(1,\xi)=\widehat{g}(\xi), & \xi\in\mathbb{R}, \\ \widehat{u}(x,\xi)|_{x\to\infty} bounded, & \xi\in\mathbb{R}. \end{cases} \tag{5}$$

Through solving this ordinary differential equation, we obtain the solution of Problem (1) in the following frequency domain:

$$\widehat{u}(x,\xi) = e^{(i\xi)^{\frac{\alpha}{2}}(1-x)}\widehat{g}(\xi) + \int_x^1 e^{(i\xi)^{\frac{\alpha}{2}}(s-x)}\frac{\widehat{f(s,t,u)}(s,\xi)}{2(i\xi)^{\frac{\alpha}{2}}}ds. \tag{6}$$

Then,

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi t} (e^{(i\xi)^{\frac{\alpha}{2}}(1-x)} \widehat{g}(\xi) + \int_x^1 e^{(i\xi)^{\frac{\alpha}{2}}(s-x)} \frac{\widehat{f(s,t,u)}(s,\xi)}{2(i\xi)^{\frac{\alpha}{2}}} ds) d\xi. \tag{7}$$

Using the properties of the Fourier transform for the time-fractional derivative, we obtain

$$\mid (i\xi)^{\frac{\alpha}{2}} \mid = \begin{cases} \mid \xi \mid^{\frac{\alpha}{2}} (cos(\frac{\alpha\pi}{4}) - isin(\frac{\alpha\pi}{4})), & \xi < 0, \\ \mid \xi \mid^{\frac{\alpha}{2}} (cos(\frac{\alpha\pi}{4}) + isin(\frac{\alpha\pi}{4})), & \xi \geq 0. \end{cases} \tag{8}$$

Using (7) and (8), we find $\mid (i\xi)^{\frac{\alpha}{2}} \mid$ has the positive real part $\mid \xi \mid^{\frac{\alpha}{2}} cos(\frac{\alpha\pi}{4})$. As a result of $0 \leq x < s < 1$, as $|\xi|$ becomes large, $|e^{(i\xi)^{\frac{\alpha}{2}}(1-x)}|$ and $|e^{(i\xi)^{\frac{\alpha}{2}}(s-x)}|$ increase quickly. The solution for $0 \leq x < 1$ can be destroyed by the small errors in the high-frequency components of the measurable data $g^{\delta}(t)$. Thus, Problem (1) is severely ill-posed. Therefore, the Fourier truncation regularization method is used to solve Problem (1). First, we introduce an important Lemma.

Lemma 1. *If $w(x) \in C(0,1)$ and satisfies*

$$w(x) \leq C_1 + 2C_2(1-x) \int_x^1 w(y) dy, \tag{9}$$

then,

$$w(x) \leq C_1 e^{2C_2(1-x)}, \tag{10}$$

where $C_1 \geq 0, C_2 \geq 0$ are constants.

Proof. Assume $W(x) = w(1-x), W(1-x) = w(x), \forall x \in (0,1)$.

Then,

$$W(1-x) \leq C_1 + 2C_2(1-x) \int_x^1 w(y) dy. \tag{11}$$

Assuming $z = 1 - y$, we have

$$W(1-x) \leq C_1 + 2C_2(1-x) \int_0^{1-x} w(1-z) dz \leq C_1 + 2C_2 \int_0^{1-x} w(1-z) dz. \tag{12}$$

Assuming $\tau = 1 - x, \tau \in (0, x)$, then we have

$$W(\tau) \leq C_1 + 2C_2 \int_0^{\tau} W(z) dz. \tag{13}$$

Using the Gronwall inequality for (13), we have

$$W(\tau) \leq e^{\int_0^{\tau} 2C_2 ds} \int_0^{\tau} C_1 ds. \tag{14}$$

Then,

$$W(\tau) \leq C_1 \tau e^{2C_2\tau} \leq C_1 e^{2C_2\tau}, \tag{15}$$

and so,

$$W(1-x) \leq C_1 e^{2C_2(1-x)}. \tag{16}$$

Hence,

$$w(x) \leq C_1 e^{2C_2(1-x)}. \tag{17}$$

□

3. Fourier Regularization Method and Results

Define the Fourier regularization solution of Problem (1) as follows:

$$u^{\delta}_{\xi_{max}}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi t} (e^{(i\xi)^{\frac{\alpha}{2}}(1-x)} \widehat{g}^{\delta}(\xi) \mathcal{X}_{max}(\xi) + \int_x^1 e^{(i\xi)^{\frac{\alpha}{2}}(s-x)} \frac{\widehat{f(s,t,u^{\delta}_{\xi_{max}(\xi)})}(s,\xi)}{2(i\xi)^{\frac{\alpha}{2}}} ds \mathcal{X}_{max}(\xi)) d\xi, \quad (18)$$

where $\mathcal{X}_{max}$ is the characteristic function of the interval $[-\xi_{\max}, \xi_{\max}]$, i.e.,

$$\mathcal{X}_{max}(\xi) = \begin{cases} 1, \mid \xi \mid \leq \xi_{max}, \\ 0, \mid \xi \mid \geq \xi_{max}. \end{cases} \quad (19)$$

The Fourier truncation regularization method is a very effective method for dealing with ill-posed problems. Many authors have used it to deal with different ill-posed problems, such as in [23–32]. In [33], the authors extended the Fourier method to the general filtering method and solved the semi-linear ill-posed problem in the general framework. Moreover, they obtained excellent results. Hereafter, the existence, uniqueness, and stability of the solution for Problem (18) is considered.

Theorem 1. *Suppose $g \in L^2(\mathbb{R})$, assume $f \in L^{\infty}([0,1] \times \mathbb{R} \times \mathbb{R})$ satisfy $f(x,t,0) = 0$ and*

$$\mid f(x,t,\omega) - f(x,t,\nu) \mid \leq k \mid \omega - \nu \mid, \quad (20)$$

for constant $k > 0$ independent of x, t, ω, ν. Therefore, a unique solution $u^{\delta}_{\xi_{max}}(x,t)$ to problem (18) exists.

Proof. For $\omega(x,t) \in C([0,1]; L^2(\mathbb{R}))$, define the operator as follows:

$$G(\omega)(x,t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(i\xi)^{\frac{\alpha}{2}}(1-x)} \widehat{g}(\xi) \mathcal{X}_{max}(\xi) e^{i\xi t} d\xi + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi t} \int_x^1 e^{(i\xi)^{\frac{\alpha}{2}}(s-x)} \frac{\widehat{f(s,t,\omega)}(s,\xi)}{2(i\xi)^{\frac{\alpha}{2}}} ds \mathcal{X}_{max}(\xi) d\xi.$$

Therefore, for $\omega, \nu \in C([0,1]; L^2(\mathbb{R}))$, we first prove the following inequality:

$$\|G^p(\omega_{\xi_{max}})(x,\cdot) - G^p(\nu_{\xi_{max}})(x,\cdot)\| \leq (\frac{k}{2})^p (e^{\xi_{max}^{\frac{\alpha}{2}}} \xi_{max}^{-\frac{\alpha}{2}})^p \frac{1}{\sqrt{p!}} (1-x)^{\frac{p}{2}} |||\omega_{\xi_{max}} - \nu_{\xi_{max}}|||, \quad (21)$$

where $\|\cdot\|$ is the norm in $L^2(\mathbb{R})$, and $|||\cdot|||$ is the sup norm of $C([0,1]; L^2(\mathbb{R}))$. We use the method of induction to prove Inequality (21).

When $p = 1$, we have

$$\begin{aligned}
&\|G(\omega_{\xi_{max}})(x,\cdot) - G(\nu_{\xi_{max}})(x,\cdot)\|^2 = \|\widehat{G}(\omega_{\xi_{max}})(x,\cdot) - \widehat{G}(\nu_{\xi_{max}})(x,\cdot)\|^2 \\
&= \int_{-\infty}^{\infty} | \int_x^1 e^{(i\xi)^{\frac{\alpha}{2}}(s-x)} \frac{\widehat{f(s,t,\omega_{\xi_{max}})}(s,\xi) - \widehat{f(s,t,\nu_{\xi_{max}})}(s,\xi)}{2(i\xi)^{\frac{\alpha}{2}}} ds \mathcal{X}_{max}(\xi)|^2 d\xi \\
&= \int_{-\xi_{max}}^{\xi_{max}} [\int_x^1 \frac{e^{(i\xi)^{\frac{\alpha}{2}}(s-x)}}{2(i\xi)^{\frac{\alpha}{2}}} (\widehat{f(s,t,\omega_{\xi_{max}})}(s,\xi) - \widehat{f(s,t,\nu_{\xi_{max}})}(s,\xi)) ds]^2 d\xi \\
&\leq \int_{-\xi_{max}}^{\xi_{max}} [\int_x^1 \frac{e^{2(i\xi)^{\frac{\alpha}{2}}(s-x)}}{4(i\xi)^{\alpha}} ds \int_x^1 |\widehat{f(s,t,\omega_{\xi_{max}})}(s,\xi) - \widehat{f(s,t,\nu_{\xi_{max}})}(s,\xi)|^2 ds] d\xi
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{e^{2(i\xi_{max})^{\frac{\alpha}{2}}}}{4(i\xi_{max})^{\alpha}}(1-x)\int_{-\xi_{max}}^{\xi_{max}}[\int_{x}^{1}|f(s,\widehat{t,\omega_{\xi_{max}}})(s,\xi)-f(s,\widehat{t,\nu_{\xi_{max}}})(s,\xi)|^{2}ds]d\xi\\
&\leq \frac{e^{2\xi_{max}^{\frac{\alpha}{2}}}}{4\xi_{max}^{\alpha}}(1-x)\int_{x}^{1}[\int_{-\infty}^{\infty}|f(s,\widehat{t,\omega_{\xi_{max}}})(s,\xi)-f(s,\widehat{t,\nu_{\xi_{max}}})(s,\xi)|^{2}d\xi]ds\\
&= \frac{e^{2\xi_{max}^{\frac{\alpha}{2}}}}{4\xi_{max}^{\alpha}}(1-x)\int_{x}^{1}[\int_{-\infty}^{\infty}|f(s,t,\omega_{\xi_{max}})(s,t)-f(s,t,\nu_{\xi_{max}}(s,t)|^{2}dt]ds\\
&\leq \frac{e^{2\xi_{max}^{\frac{\alpha}{2}}}}{4\xi_{max}^{\alpha}}(1-x)k^{2}\int_{x}^{1}[\int_{-\infty}^{\infty}|\omega_{\xi_{max}}(s,t)-\nu_{\xi_{max}}(s,t)|^{2}dt]ds\\
&= \frac{e^{2\xi_{max}^{\frac{\alpha}{2}}}}{4\xi_{max}^{\alpha}}(1-x)k^{2}\int_{x}^{1}\|\omega_{\xi_{max}}(s,\cdot)-\nu_{\xi_{max}}(s,\cdot)\|^{2}ds\\
&\leq \frac{e^{2\xi_{max}^{\frac{\alpha}{2}}}}{4\xi_{max}^{\alpha}}(1-x)k^{2}(1-x)|||\omega_{\xi_{max}}-\nu_{\xi_{max}}|||^{2}\\
&\leq \frac{e^{2\xi_{max}^{\frac{\alpha}{2}}}}{4\xi_{max}^{\alpha}}(1-x)k^{2}|||\omega_{\xi_{max}}-\nu_{\xi_{max}}|||^{2}.
\end{aligned}
$$

When $p=j$, the inequality satisfies

$$
\|G^{j}(\omega_{\xi_{max}})(x,\cdot)-G^{j}(\nu_{\xi_{max}})(x,\cdot)\|^{2}\leq(\frac{k^{2}}{4})^{j}(\frac{e^{2\xi_{max}^{\frac{\alpha}{2}}}}{\xi_{max}^{\alpha}})^{j}\frac{(1-x)^{j}}{j!}|||\omega_{\xi_{max}}-\nu_{\xi_{max}}|||^{2}. \tag{22}
$$

Therefore, when $p=j+1$, we have

$$
\begin{aligned}
&\|G^{j+1}(\omega_{\xi_{max}})(x,\cdot)-G^{j+1}(\nu_{\xi_{max}})(x,\cdot)\|^{2}=\|\widehat{G}(G^{j}(\omega_{\xi_{max}}))(x,\cdot)-\widehat{G}(G^{j}(\nu_{\xi_{max}}))(x,\cdot)\|^{2}\\
&=\int_{-\infty}^{\infty}|\int_{x}^{1}e^{(i\xi)^{\frac{\alpha}{2}}(s-x)}\frac{f(s,t,G^{j}\widehat{(\omega_{\xi_{max}}(s,\xi)))}-f(s,t,G^{j}\widehat{(\nu_{\xi_{max}}(s,\xi)))}}{2(i\xi)^{\frac{\alpha}{2}}}ds\mathcal{X}_{max}(\xi)|^{2}d\xi\\
&\leq\int_{-\xi_{max}}^{\xi_{max}}[\int_{x}^{1}\frac{e^{2(i\xi)^{\frac{\alpha}{2}}(s-x)}}{4(i\xi)^{\alpha}}ds\int_{x}^{1}|f(s,t,G^{j}\widehat{(\omega_{\xi_{max}}(s,\xi)))}-f(s,t,G^{j}\widehat{(\nu_{\xi_{max}}(s,\xi)))}|^{2}ds]d\xi\\
&\leq\frac{e^{2(i\xi_{max})^{\frac{\alpha}{2}}}}{4(i\xi_{max})^{\alpha}}(1-x)\int_{-\xi_{max}}^{\xi_{max}}[\int_{x}^{1}|f(s,t,G^{j}\widehat{(\omega_{\xi_{max}}(s,\xi)))}-f(s,t,G^{j}\widehat{(\nu_{\xi_{max}}(s,\xi)))}|^{2}ds]d\xi\\
&\leq\frac{e^{2\xi_{max}^{\frac{\alpha}{2}}}}{4\xi_{max}^{\alpha}}(1-x)\int_{x}^{1}[\int_{-\infty}^{\infty}|f(s,t,G^{j}\widehat{(\omega_{\xi_{max}}(s,\xi)))}-f(s,t,G^{j}\widehat{(\nu_{\xi_{max}}(s,\xi)))}|^{2}d\xi]ds\\
&=\frac{e^{2\xi_{max}^{\frac{\alpha}{2}}}}{4\xi_{max}^{\alpha}}(1-x)\int_{x}^{1}[\int_{-\infty}^{\infty}|f(s,t,G^{j}(\omega_{\xi_{max}}(s,\xi)))-f(s,t,G^{j}(\nu_{\xi_{max}}(s,\xi)))|^{2}dt]ds\\
&\leq\frac{e^{2\xi_{max}^{\frac{\alpha}{2}}}}{4\xi_{max}^{\alpha}}(1-x)k^{2}\int_{x}^{1}[\int_{-\infty}^{\infty}|G^{j}(\omega_{\xi_{max}}(s,t))-G^{j}(\nu_{\xi_{max}}(s,t))|^{2}dt]ds
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{e^{2\xi_{max}^{\frac{\alpha}{2}}}}{4\xi_{max}^{\alpha}}(1-x)k^2\int_x^1 \|G^j(\omega_{\xi_{max}}(s,\cdot)) - G^j(\nu_{\xi_{max}}(s,\cdot))\|^2 ds \\
&\leq \frac{e^{2\xi_{max}^{\frac{\alpha}{2}}}}{4\xi_{max}^{\alpha}}(1-x)k^2\int_x^1 (\frac{k^2}{4})^j(\frac{e^{2\xi_{max}^{\frac{\alpha}{2}}}}{\xi_{max}^{\alpha}})^j\frac{(1-s)^j}{j!}|||\omega_{\xi_{max}} - \nu_{\xi_{max}}|||^2 ds \\
&\leq (\frac{k^2}{4})^{j+1}(\frac{e^{2\xi_{max}^{\frac{\alpha}{2}}}}{\xi_{max}^{\alpha}})^{j+1}\frac{|||\omega_{\xi_{max}} - \nu_{\xi_{max}}|||^2}{j!}\int_x^1 (1-s)^j ds \\
&= (\frac{e^{2\xi_{max}^{\frac{\alpha}{2}}}k^2}{4\xi_{max}^{\alpha}})^{j+1}\frac{(1-x)^{j+1}}{(j+1)!}|||\omega_{\xi_{max}} - \nu_{\xi_{max}}|||^2.
\end{aligned}
$$

Then,

$$
\|G^{j+1}(\omega_{\xi_{max}})(x,\cdot) - G^{j+1}(\nu_{\xi_{max}})(x,\cdot)\| \leq (\frac{e^{\xi_{max}^{\frac{\alpha}{2}}}k}{2\xi_{max}^{\frac{\alpha}{2}}})^{j+1}\frac{(1-x)^{\frac{j+1}{2}}}{\sqrt{(j+1)!}}|||\omega_{\xi_{max}} - \nu_{\xi_{max}}|||. \tag{23}
$$

Applying the method of induction, for all $\omega, \nu \in C([0,1]; L^2(\mathbb{R}))$, we have

$$
\|G^p(\omega_{\xi_{max}})(x,\cdot) - G^p(\nu_{\xi_{max}})(x,\cdot)\| \leq (\frac{k}{2})^p(e^{\xi_{max}^{\frac{\alpha}{2}}}\xi_{max}^{-\frac{\alpha}{2}})^p\frac{1}{\sqrt{p!}}(1-x)^{\frac{p}{2}}|||\omega_{\xi_{max}} - \nu_{\xi_{max}}|||.
$$

Considering the operator $G : C(x; L^2(\mathbb{R})) \to C(x; L^2(\mathbb{R}))$, we can obtain

$$
\lim_{p\to\infty}(\frac{k}{2})^p(e^{\xi_{max}^{\frac{\alpha}{2}}}\xi_{max}^{-\frac{\alpha}{2}})^p\frac{1}{\sqrt{p!}}(1-x)^{\frac{p}{2}} = 0. \tag{24}
$$

Therefore, a positive number p_0 exists such that $0 < (\frac{k}{2})^{p_0}(e^{\xi_{max}^{\frac{\alpha}{2}}}\xi_{max}^{-\frac{\alpha}{2}})^{p_0}\frac{1}{\sqrt{p_0!}}(1-x)^{\frac{p_0}{2}} < 1$. Thus, G^{p_0} is a contractive mapping, stating that equation $G^{p_0}(\omega) = \omega$ has the unique solution $u_{\xi_{max}}(x,t) \in C(x; L^2(\mathbb{R}))$. Note that $G(G^{p_0}(u_{\xi_{max}})) = G(u_{\xi_{max}})$, so $G^{p_0}(G(u_{\xi_{max}})) = u_{\xi_{max}}$. According to the uniqueness of the fixed point, for the equation $G(\omega) = \omega$, there exists the unique solution $u_{\xi_{max}}$. □

Theorem 2. *Suppose f satisfies Inequality (20), $u_{\xi_{max}}$ and $u^{\delta}_{\xi_{max}}$ are the solutions of Equation (18), respectively. For $0 < x < 1$, we have*

$$
\|u_{\xi_{max}} - u^{\delta}_{\xi_{max}}\| \leq \sqrt{2}\delta e^{\xi_{max}^{\frac{\alpha}{2}}(1-x)}e^{\frac{1}{4}\xi_{max}^{-\alpha}k^2(1-x)}. \tag{25}
$$

Proof. Using Parseval's equality, we get

$$
\begin{aligned}
&\|u_{\xi_{max}}(x,\cdot) - u^{\delta}_{\xi_{max}}(x,\cdot)\|^2 = \|\widehat{u}_{\xi_{max}}(x,\cdot) - \widehat{u}^{\delta}_{\xi_{max}}(x,\cdot)\|^2 \\
&= \int_{-\infty}^{\infty} |[e^{(i\xi)^{\frac{\alpha}{2}}(1-x)}(\widehat{g}(\xi) - \widehat{g}^{\delta}(\xi)) \\
&+ \int_x^1 e^{(i\xi)^{\frac{\alpha}{2}}(s-x)}\frac{f(s,\widehat{t,u_{\xi_{max}}})(s,\xi) - f(s,t,\widehat{u^{\delta}_{\xi_{max}}})(s,\xi)}{2(i\xi)^{\frac{\alpha}{2}}}ds]\mathcal{X}_{max}(\xi)|^2 d\xi
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-\xi_{max}}^{\xi_{max}} |[e^{(i\xi)^{\frac{\alpha}{2}}(1-x)}(\widehat{g}(\xi) - \widehat{g}^{\delta}(\xi)) \\
&+ \int_x^1 e^{(i\xi)^{\frac{\alpha}{2}}(s-x)} \frac{f(s,\widehat{t,u_{\xi_{max}}})(s,\xi) - f(s,\widehat{t,u^{\delta}_{\xi_{max}}})(s,\xi)}{2(i\xi)^{\frac{\alpha}{2}}} ds]|^2 d\xi \\
&\leq 2\int_{-\xi_{max}}^{\xi_{max}} [e^{(i\xi)^{\frac{\alpha}{2}}(1-x)}(\widehat{g}(\xi) - \widehat{g}^{\delta}(\xi))]^2 d\xi \\
&+ \frac{1}{2}\int_{-\xi_{max}}^{\xi_{max}} [\int_x^1 \frac{e^{(i\xi)^{\frac{\alpha}{2}}(s-x)}}{(i\xi)^{\frac{\alpha}{2}}}(f(s,\widehat{t,u_{\xi_{max}}})(s,\xi) - f(s,\widehat{t,u^{\delta}_{\xi_{max}}})(s,\xi) ds]^2 d\xi \\
&\leq 2e^{2\xi_{max}^{\frac{\alpha}{2}}(1-x)} \int_{-\infty}^{\infty} |\widehat{g}(\xi) - \widehat{g}^{\delta}(\xi)|^2 d\xi \\
&\quad + \frac{1}{2}\int_{-\xi_{max}}^{\xi_{max}} [\int_x^1 \frac{e^{\xi_{max}^{\frac{\alpha}{2}}(s-x)}}{\xi_{max}^{\frac{\alpha}{2}}}(f(s,\widehat{t,u_{\xi_{max}}})(s,\xi) - f(s,\widehat{t,u^{\delta}_{\xi_{max}}})(s,\xi)) ds]^2 d\xi \\
&\leq 2e^{2\xi_{max}^{\frac{\alpha}{2}}(1-x)} \|\widehat{g}(\xi) - \widehat{g}^{\delta}(\xi)\|^2 \\
&\quad + \frac{1}{2}\int_{-\xi_{max}}^{\xi_{max}} [\int_x^1 e^{-2\xi_{max}^{\frac{\alpha}{2}}x} ds \int_x^1 \frac{e^{2\xi_{max}^{\frac{\alpha}{2}}s}}{\xi_{max}^{\alpha}}(f(s,\widehat{t,u_{\xi_{max}}})(s,\xi) - f(s,\widehat{t,u^{\delta}_{\xi_{max}}})(s,\xi))^2 ds] d\xi \\
&\leq 2e^{2\xi_{max}^{\frac{\alpha}{2}}(1-x)} \|\widehat{g}(\xi) - \widehat{g}^{\delta}(\xi)\|^2 \\
&\quad + \frac{1}{2}(1-x)e^{-2\xi_{max}^{\frac{\alpha}{2}}x} \int_{-\infty}^{\infty}\int_x^1 \frac{e^{2\xi_{max}^{\frac{\alpha}{2}}s}}{\xi_{max}^{\alpha}}(f(s,\widehat{t,u_{\xi_{max}}})(s,\xi) - f(s,\widehat{t,u^{\delta}_{\xi_{max}}})(s,\xi))^2 ds d\xi \\
&= 2e^{2\xi_{max}^{\frac{\alpha}{2}}(1-x)} \|\widehat{g}(\xi) - \widehat{g}^{\delta}(\xi)\|^2 + \frac{1}{2}(1-x)e^{-2\xi_{max}^{\frac{\alpha}{2}}x} \int_x^1\int_{-\infty}^{\infty} \frac{e^{2\xi_{max}^{\frac{\alpha}{2}}s}}{\xi_{max}^{\alpha}}(f(s,t,u_{\xi_{max}}) - f(s,t,u^{\delta}_{\xi_{max}}))^2 dt ds \\
&\leq 2e^{2\xi_{max}^{\frac{\alpha}{2}}(1-x)} \|\widehat{g}(\xi) - \widehat{g}^{\delta}(\xi)\|^2 + \frac{1}{2}(1-x)k^2e^{-2\xi_{max}^{\frac{\alpha}{2}}x} \int_x^1\int_{-\infty}^{\infty} \frac{e^{2\xi_{max}^{\frac{\alpha}{2}}s}}{\xi_{max}^{\alpha}}|u_{\xi_{max}}(s,t) - u^{\delta}_{\xi_{max}}(s,t)|^2 dt ds \\
&= 2e^{2\xi_{max}^{\frac{\alpha}{2}}(1-x)}\delta^2 + \frac{1}{2}(1-x)k^2e^{-2\xi_{max}^{\frac{\alpha}{2}}x}\xi_{max}^{-\alpha}\int_x^1 e^{2\xi_{max}^{\frac{\alpha}{2}}s}\|u_{\xi_{max}}(s,\cdot) - u^{\delta}_{\xi_{max}}(s,\cdot)\|^2 ds.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\|u_{\xi_{max}}(x,\cdot) - u^{\delta}_{\xi_{max}}(x,\cdot)\|^2 \\
&\leq 2e^{2\xi_{max}^{\frac{\alpha}{2}}(1-x)}\delta^2 + \frac{1}{2}(1-x)k^2e^{-2\xi_{max}^{\frac{\alpha}{2}}x}\xi_{max}^{-\alpha}\int_x^1 e^{2\xi_{max}^{\frac{\alpha}{2}}s}\|u_{\xi_{max}}(s,\cdot) - u^{\delta}_{\xi_{max}}(s,\cdot)\|^2 ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&e^{2\xi_{max}^{\frac{\alpha}{2}}x}\|u_{\xi_{max}}(x,\cdot) - u^{\delta}_{\xi_{max}}(x,\cdot)\|^2 \\
&\leq 2e^{2\xi_{max}^{\frac{\alpha}{2}}}\delta^2 + \frac{1}{2}(1-x)k^2\xi_{max}^{-\alpha}\int_x^1 e^{2\xi_{max}^{\frac{\alpha}{2}}s}\|u_{\xi_{max}}(s,\cdot) - u^{\delta}_{\xi_{max}}(s,\cdot)\|^2 ds.
\end{aligned}
$$

Using Lemma 1, we obtain

$$
e^{2\xi_{max}^{\frac{\alpha}{2}}x}\|u_{\xi_{max}}(x,\cdot) - u^{\delta}_{\xi_{max}}(x,\cdot)\|^2 \leq 2e^{2\xi_{max}^{\frac{\alpha}{2}}}\delta^2 e^{\frac{1}{2}\xi_{max}^{-\alpha}k^2(1-x)}, \tag{26}
$$

then,

$$
\|u_{\xi_{max}}(x,\cdot) - u^{\delta}_{\xi_{max}}(x,\cdot)\|^2 \leq 2e^{2\xi_{max}^{\frac{\alpha}{2}}(1-x)}\delta^2 e^{\frac{1}{2}\xi_{max}^{-\alpha}k^2(1-x)}. \tag{27}
$$

Hence,

$$
\|u_{\xi_{max}}(x,\cdot) - u^{\delta}_{\xi_{max}}(x,\cdot)\| \leq \sqrt{2}e^{\xi_{max}^{\frac{\alpha}{2}}(1-x)}\delta e^{\frac{1}{4}\xi_{max}^{-\alpha}k^2(1-x)}. \tag{28}
$$

□

Remark 1. *From Theorem 2, we can see* $\delta \to 0$, $\|u_{\xi_{max}}(x,\cdot) - u^{\delta}_{\xi_{max}}(x,\cdot)\| \to 0$. *Therefore, the solution of Problem (18) depends continuously on* $g \in L^2(\mathbb{R})$.

4. Error Estimate

The error estimates at $0 < x < 1$ *and* $x = 0$ are given in this section.

- **The error estimate at $0 < x < 1$.**

In order to obtain the error estimate at $0 < x < 1$, we impose an a priori bound on $u(x,t)$, i.e.,

$$\int_{-\infty}^{\infty} e^{2x\xi^{\frac{\alpha}{2}}} |\widehat{u}(x,\xi)|^2 d\xi \le E^2, \ \forall x \in (0,1), \tag{29}$$

where E is a constant.

Theorem 3. *Assume that* f *satisfies Equation (20).* $u(x,t)$ *is the exact solution of Problem (1) and* $u^{\delta}_{\xi_{max}}(x,t)$ *is the solution of (18). Suppose conditions (3) and (29) hold. Choosing the regularization parameter*

$$\xi_{max} = (ln(\frac{E}{\delta}))^{\frac{2}{\alpha}}, \tag{30}$$

we get the following estimate at $0 < x < 1$*:*

$$\|u(x,\cdot) - u^{\delta}_{\xi_{max}}(x,\cdot)\| \le 2\sqrt{2}E^{1-x}\delta^{x} e^{\frac{1}{4}(ln\frac{E}{\delta})^{-2}k^2(1-x)}. \tag{31}$$

Proof. Applying Parseval's equality, we obtain

$$\begin{aligned}
&\|u(x,\cdot) - u_{\xi_{max}}(x,\cdot)\|^2 = \|\widehat{u}(x,\cdot) - \widehat{u}_{\xi_{max}}(x,\cdot)\|^2 \\
&= \int_{-\infty}^{\infty} |e^{(i\xi)^{\frac{\alpha}{2}}(1-x)}(1-\mathcal{X}_{max}(\xi))\widehat{g}(\xi) + \int_x^1 e^{(i\xi)^{\frac{\alpha}{2}}(s-x)} \frac{\widehat{f(s,t,u)}(s,\xi)}{2(i\xi)^{\frac{\alpha}{2}}} ds \\
&\quad - \int_x^1 e^{(i\xi)^{\frac{\alpha}{2}}(s-x)} \frac{\widehat{f(s,t,u_{\xi_{max}})}(s,\xi)}{2(i\xi)^{\frac{\alpha}{2}}} ds \mathcal{X}_{max}(\xi)|^2 d\xi \\
&= \int_{-\infty}^{\infty} |e^{(i\xi)^{\frac{\alpha}{2}}(1-x)}(1-\mathcal{X}_{max}(\xi))\widehat{g}(\xi) + \int_x^1 e^{(i\xi)^{\frac{\alpha}{2}}(s-x)} \frac{\widehat{f(s,t,u)}(s,\xi)}{2(i\xi)^{\frac{\alpha}{2}}} ds \\
&- \int_x^1 e^{(i\xi)^{\frac{\alpha}{2}}(s-x)} \frac{\widehat{f(s,t,u)}(s,\xi)}{2(i\xi)^{\frac{\alpha}{2}}} ds \mathcal{X}_{max}(\xi) d\xi + \int_x^1 e^{(i\xi)^{\frac{\alpha}{2}}(s-x)} \frac{\widehat{f(s,t,u)}(s,\xi)}{2(i\xi)^{\frac{\alpha}{2}}} ds \mathcal{X}_{max} d\xi \\
&\quad - \int_x^1 e^{(i\xi)^{\frac{\alpha}{2}}(s-x)} \frac{\widehat{f(s,t,u_{\xi_{max}})}(s,\xi)}{2(i\xi)^{\frac{\alpha}{2}}} ds \mathcal{X}_{max}(\xi)|^2 d\xi \\
&= \int_{-\infty}^{\infty} |[e^{(i\xi)^{\frac{\alpha}{2}}(1-x)}\widehat{g}(\xi) + \int_x^1 e^{(i\xi)^{\frac{\alpha}{2}}(s-x)} \frac{\widehat{f(s,t,u)}(s,\xi)}{2(i\xi)^{\frac{\alpha}{2}}} ds](1-\mathcal{X}_{max}(\xi)) \\
&+ \int_x^1 \frac{e^{(i\xi)^{\frac{\alpha}{2}}(s-x)}}{2(i\xi)^{\frac{\alpha}{2}}} (\widehat{f(s,t,u)}(s,\xi) - \widehat{f(s,t,u_{\xi_{max}})}(s,\xi)) ds \mathcal{X}_{max}(\xi)|^2 d\xi \\
&= \int_{-\infty}^{\infty} |(1-\mathcal{X}_{max}(\xi))\widehat{u}(x,\xi) + \int_x^1 \frac{e^{(i\xi)^{\frac{\alpha}{2}}(s-x)}}{2(i\xi)^{\frac{\alpha}{2}}} (\widehat{f(s,t,u)}(s,\xi) - \widehat{f(s,t,u_{\xi_{max}})}(s,\xi)) ds \mathcal{X}_{max}(\xi)|^2 d\xi
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\int_{-\infty}^{\infty}|(1-\mathcal{X}_{max}(\xi))\widehat{u}(x,\xi)|^2 d\xi \\
&+2\int_{-\infty}^{\infty}\int_x^1 \frac{e^{(i\xi)^{\frac{\alpha}{2}}(s-x)}}{2(i\xi)^{\frac{\alpha}{2}}}(\widehat{f(s,t,u)}(s,\xi)-\widehat{f(s,t,u_{\xi_{max}})}(s,\xi))ds\mathcal{X}_{max}(\xi)|^2 d\xi \\
&= 2\int_{|\xi|>\xi_{max}}|\widehat{u}(x,\xi)|^2 d\xi \\
&+2\int_{-\xi_{max}}^{\xi_{max}}|\int_x^1 \frac{e^{(i\xi)^{\frac{\alpha}{2}}(s-x)}}{2(i\xi)^{\frac{\alpha}{2}}}(\widehat{f(s,t,u)}(s,\xi)-\widehat{f(s,t,u_{\xi_{max}})}(s,\xi))ds|^2 d\xi \\
&\leq 2\int_{|\xi|>\xi_{max}} e^{-2x\xi^{\frac{\alpha}{2}}}e^{2x\xi^{\frac{\alpha}{2}}}|\widehat{u}(x,\xi)|^2 d\xi \\
&\quad+\frac{1}{2}\int_{-\xi_{max}}^{\xi_{max}}|\int_x^1 \frac{e^{\xi_{max}^{\frac{\alpha}{2}}(s-x)}}{\xi_{max}^{\frac{\alpha}{2}}}(\widehat{f(s,t,u)}(s,\xi)-\widehat{f(s,t,u_{\xi_{max}})}(s,\xi))ds|^2 d\xi \\
&\leq 2e^{-2x\xi^{\frac{\alpha}{2}}}E^2+\frac{1}{2}\int_{-\xi_{max}}^{\xi_{max}}|\int_x^1 e^{-2\xi_{max}^{\frac{\alpha}{2}}x}ds\int_x^1\frac{e^{2\xi_{max}^{\frac{\alpha}{2}}s}}{\xi_{max}^{\alpha}}|\widehat{f(s,t,u)}(s,\xi)-\widehat{f(s,t,u_{\xi_{max}})}(s,\xi)|^2 ds|d\xi \\
&\leq 2e^{-2x\xi^{\frac{\alpha}{2}}}E^2+\frac{1}{2}(1-x)e^{-2\xi_{max}^{\frac{\alpha}{2}}x}\int_{-\xi_{max}}^{\xi_{max}}\int_x^1\frac{e^{2\xi_{max}^{\frac{\alpha}{2}}s}}{\xi_{max}^{\alpha}}|\widehat{f(s,t,u)}(s,\xi)-\widehat{f(s,t,u_{\xi_{max}})}(s,\xi)|^2 dsd\xi \\
&= 2e^{-2x\xi^{\frac{\alpha}{2}}}E^2+\frac{1}{2}(1-x)e^{-2\xi_{max}^{\frac{\alpha}{2}}x}\int_x^1\frac{e^{2\xi_{max}^{\frac{\alpha}{2}}s}}{\xi_{max}^{\alpha}}\int_{-\xi_{max}}^{\xi_{max}}|\widehat{f(s,t,u)}(s,\xi)-\widehat{f(s,t,u_{\xi_{max}})}(s,\xi)|^2 d\xi ds \\
&\leq 2e^{-2x\xi^{\frac{\alpha}{2}}}E^2+\frac{1}{2}(1-x)e^{-2\xi_{max}^{\frac{\alpha}{2}}x}\int_x^1\frac{e^{2\xi_{max}^{\frac{\alpha}{2}}s}}{\xi_{max}^{\alpha}}\int_{-\infty}^{\infty}|f(s,t,u)-f(s,t,u_{\xi_{max}})|^2 dtds \\
&\leq 2e^{-2x\xi^{\frac{\alpha}{2}}}E^2+\frac{1}{2}(1-x)e^{-2\xi_{max}^{\frac{\alpha}{2}}x}k^2\int_x^1\frac{e^{2\xi_{max}^{\frac{\alpha}{2}}s}}{\xi_{max}^{\alpha}}\int_{-\infty}^{\infty}|u(s,t)-u_{\xi_{max}}(s,t)|^2 dtds \\
&= 2e^{-2x\xi^{\frac{\alpha}{2}}}E^2+\frac{1}{2}(1-x)k^2 e^{-2\xi_{max}^{\frac{\alpha}{2}}x}\xi_{max}^{-\alpha}\int_x^1 e^{2\xi_{max}^{\frac{\alpha}{2}}s}\|u(s,\cdot)-u_{\xi_{max}}(s,\cdot)\|^2 ds.
\end{aligned}
$$

Then,

$$\|u(x,\cdot)-u_{\xi_{max}}(x,\cdot)\|^2\leq 2e^{-2x\xi^{\frac{\alpha}{2}}}E^2+\frac{1}{2}(1-x)k^2 e^{-2\xi_{max}^{\frac{\alpha}{2}}x}\xi_{max}^{-\alpha}\int_x^1 e^{2\xi_{max}^{\frac{\alpha}{2}}s}\|u(s,\cdot)-u_{\xi_{max}}(s,\cdot)\|^2 ds.$$

Therefore,

$$e^{2x\xi_{max}^{\frac{\alpha}{2}}}\|u(x,\cdot)-u_{\xi_{max}}(x,\cdot)\|^2\leq 2E^2+\frac{1}{2}(1-x)k^2\xi_{max}^{-\alpha}\int_x^1 e^{2\xi_{max}^{\frac{\alpha}{2}}s}\|u(s,\cdot)-u_{\xi_{max}}(s,\cdot)\|^2 ds. \tag{32}$$

Using Lemma 1, we obtain

$$e^{2x\xi_{max}^{\frac{\alpha}{2}}}\|u(x,\cdot)-u_{\xi_{max}}(x,\cdot)\|^2\leq 2E^2 e^{\frac{1}{2}(1-x)k^2\xi_{max}^{-\alpha}}. \tag{33}$$

Then,

$$\|u(x,\cdot)-u_{\xi_{max}}(x,\cdot)\|\leq\sqrt{2}Ee^{\frac{1}{4}(1-x)k^2\xi_{max}^{-\alpha}}e^{-x\xi_{max}^{\frac{\alpha}{2}}}. \tag{34}$$

Applying the triangle inequality, combining (25) and (34) with (30), we obtain

$$
\begin{aligned}
\|u(x,\cdot)-u_{\xi_{max}}^{\delta}(x,\cdot)\| &\leq \|u(x,\cdot)-u_{\xi_{max}}(x,\cdot)\|+\|u_{\xi_{max}}(x,\cdot)-u_{\xi_{max}}^{\delta}(x,\cdot)\| \\
&\leq \sqrt{2}Ee^{\frac{1}{4}(1-x)k^2\xi_{max}^{-\alpha}}e^{-x\xi_{max}^{\frac{\alpha}{2}}}+\sqrt{2}e^{\xi_{max}^{\frac{\alpha}{2}}(1-x)}\delta e^{\frac{1}{4}\xi_{max}^{-\alpha}k^2(1-x)} \\
&= 2\sqrt{2}e^{\frac{1}{4}(1-x)k^2(ln\frac{E}{\delta})^{-2}}E^{1-x}\delta^x.
\end{aligned}
$$

□

Remark 2. *From Theorem 3, we only get that the error estimate is bound but not the convergence, as* $\delta \to 0$ *when* $x = 0$*. In order to give the convergence error estimate at* $x = 0$*, we must give an a priori error bound of* $u(0, t)$ *as follows:*

$$\int_{-\infty}^{\infty} |\widehat{u}(0,\xi)|^2 (1+\xi^2)^p d\xi \le E^2, \tag{35}$$

where E is a constant.

- **The error estimate at** $x = 0$**.**

Theorem 4. *By choosing the regularization parameter*

$$\xi_{max} = (ln(\frac{E}{\sqrt{\delta}}))^{\frac{2}{\alpha}}, \tag{36}$$

we get the error estimate as follows:

$$\|u(0,\cdot) - u^{\delta}_{\xi_{max}}(0,\cdot)\| \le 2\sqrt{2}E\delta^{\frac{1}{2}} e^{\frac{1}{4}(ln\frac{E}{\sqrt{\delta}})^{-2}k^2}. \tag{37}$$

Proof. Applying Parseval's equality, we obtain

$$\begin{aligned}
&\|u(0,\cdot) - u_{\xi_{max}}(0,\cdot)\|^2 = \|\widehat{u}(0,\cdot) - \widehat{u}_{\xi_{max}}(0,\cdot)\|^2 \\
&= \int_{-\infty}^{\infty} |e^{(i\xi)^{\frac{\alpha}{2}}}(1 - \mathcal{X}_{max}(\xi))\widehat{g}(\xi) + \int_0^1 e^{(i\xi)^{\frac{\alpha}{2}}s} \frac{\widehat{f(s,t,u)}(s,\xi)}{2(i\xi)^{\frac{\alpha}{2}}} ds \\
&\quad - \int_0^1 e^{(i\xi)^{\frac{\alpha}{2}}s} \frac{\widehat{f(s,t,u_{\xi_{max}})}(s,\xi)}{2(i\xi)^{\frac{\alpha}{2}}} ds \mathcal{X}_{max}(\xi)|^2 d\xi \\
&= \int_{-\infty}^{\infty} |e^{(i\xi)^{\frac{\alpha}{2}}}(1 - \mathcal{X}_{max}(\xi))\widehat{g}(\xi) + \int_0^1 e^{(i\xi)^{\frac{\alpha}{2}}s} \frac{\widehat{f(s,t,u)}(s,\xi)}{2(i\xi)^{\frac{\alpha}{2}}} ds \\
&\quad - \int_0^1 e^{(i\xi)^{\frac{\alpha}{2}}s} \frac{\widehat{f(s,t,u)}(s,\xi)}{2(i\xi)^{\frac{\alpha}{2}}} ds \mathcal{X}_{max}(\xi) d\xi + \int_0^1 e^{(i\xi)^{\frac{\alpha}{2}}s} \frac{\widehat{f(s,t,u)}(s,\xi)}{2(i\xi)^{\frac{\alpha}{2}}} ds \mathcal{X}_{max}(\xi) d\xi \\
&\quad - \int_0^1 e^{(i\xi)^{\frac{\alpha}{2}}s)} \frac{\widehat{f(s,t,u_{\xi_{max}})}(s,\xi)}{2(i\xi)^{\frac{\alpha}{2}}} ds \mathcal{X}_{max}(\xi)|^2 d\xi \\
&= \int_{-\infty}^{\infty} |[e^{(i\xi)^{\frac{\alpha}{2}}}\widehat{g}(\xi) + \int_0^1 e^{(i\xi)^{\frac{\alpha}{2}}s)} \frac{\widehat{f(s,t,u)}(s,\xi)}{2(i\xi)^{\frac{\alpha}{2}}} ds](1 - \mathcal{X}_{max}(\xi)) \\
&\quad + \int_0^1 \frac{e^{(i\xi)^{\frac{\alpha}{2}}s}}{2(i\xi)^{\frac{\alpha}{2}}} (\widehat{f(s,t,u)}(s,\xi) - \widehat{f(s,t,u^{\delta}_{\xi_{max}})}(s,\xi)) ds \mathcal{X}_{max}(\xi)|^2 d\xi \\
&= \int_{-\infty}^{\infty} |(1 - \mathcal{X}_{max}(\xi))\widehat{u}(0,\xi) + \int_0^1 \frac{e^{(i\xi)^{\frac{\alpha}{2}}s}}{2(i\xi)^{\frac{\alpha}{2}}} (\widehat{f(s,t,u)}(s,\xi) - \widehat{f(s,t,u^{\delta}_{\xi_{max}})}(s,\xi)) ds \mathcal{X}_{max}(\xi)|^2 d\xi \\
&\le 2\int_{-\infty}^{\infty} |(1 - \mathcal{X}_{max}(\xi))\widehat{u}(0,\xi)|^2 d\xi + 2\int_{-\infty}^{\infty}\int_0^1 \frac{e^{(i\xi)^{\frac{\alpha}{2}}s}}{2(i\xi)^{\frac{\alpha}{2}}} (\widehat{f(s,t,u)}(s,\xi) - \widehat{f(s,t,u^{\delta}_{\xi_{max}})}(s,\xi)) ds \mathcal{X}_{max}(\xi)|^2 d\xi \\
&= 2\int_{|\xi|>\xi_{max}} |\widehat{u}(0,\xi)|^2 d\xi + 2\int_{-\xi_{max}}^{\xi_{max}} |\int_0^1 \frac{e^{(i\xi)^{\frac{\alpha}{2}}s}}{2(i\xi)^{\frac{\alpha}{2}}} (\widehat{f(s,t,u)}(s,\xi) - \widehat{f(s,t,u^{\delta}_{\xi_{max}})}(s,\xi)) ds|^2 d\xi
\end{aligned}$$

$$\leq 2\int_{|\xi|>\xi_{max}}(1+\xi^2)^{-p}(1+\xi^2)^p|\widehat{u}(0,\xi)|^2d\xi$$
$$+\frac{1}{2}\int_{-\xi_{max}}^{\xi_{max}}|\int_0^1\frac{e^{\xi_{max}^{\frac{\alpha}{2}}s}}{\xi_{max}^{\frac{\alpha}{2}}}(f(\widehat{s,t,u})(s,\xi)-f(\widehat{s,t,u_{\xi_{max}}^{\delta}})(s,\xi))ds|^2d\xi$$
$$\leq 2\sup_{|\xi|>\xi_{max}}\{(1+\xi^2)^{-p}\}E^2+\frac{1}{2}\int_{-\xi_{max}}^{\xi_{max}}|\int_0^1\frac{e^{2\xi_{max}^{\frac{\alpha}{2}}s}}{\xi_{max}^{\alpha}}|f(\widehat{s,t,u})(s,\xi)-f(\widehat{s,t,u_{\xi_{max}}^{\delta}})(s,\xi)|^2ds|d\xi$$
$$\leq 2(1+\xi_{max}^2)^{-p}E^2+\frac{1}{2}\int_{-\xi_{max}}^{\xi_{max}}\int_0^1\frac{e^{2\xi_{max}^{\frac{\alpha}{2}}s}}{\xi_{max}^{\alpha}}|f(\widehat{s,t,u})(s,\xi)-f(\widehat{s,t,u_{\xi_{max}}^{\delta}})(s,\xi)|^2dsd\xi$$
$$\leq 2\xi_{max}^{-2p}E^2+\frac{1}{2}\int_0^1\frac{e^{2\xi_{max}^{\frac{\alpha}{2}}s}}{\xi_{max}^{\alpha}}\int_{-\xi_{max}}^{\xi_{max}}|f(\widehat{s,t,u})(s,\xi)-f(\widehat{s,t,u_{\xi_{max}}^{\delta}})(s,\xi)|^2d\xi ds$$
$$\leq 2\xi_{max}^{-2p}E^2+\frac{1}{2}\int_0^1\frac{e^{2\xi_{max}^{\frac{\alpha}{2}}s}}{\xi_{max}^{\alpha}}\int_{-\infty}^{\infty}|f(s,t,u)-f(s,t,u_{\xi_{max}})|^2dtds$$
$$\leq 2\xi_{max}^{-2p}E^2+\frac{1}{2}k^2\int_0^1\frac{e^{2\xi_{max}^{\frac{\alpha}{2}}s}}{\xi_{max}^{\alpha}}\int_{-\infty}^{\infty}|u(s,t)-u_{\xi_{max}}(s,t)|^2dtds$$
$$=2\xi_{max}^{-2p}E^2+\frac{1}{2}k^2\xi_{max}^{-\alpha}\int_0^1 e^{2\xi_{max}^{\frac{\alpha}{2}}s}\|u(s,\cdot)-u_{\xi_{max}}(s,\cdot)\|^2ds.$$

Then,

$$\|u(0,\cdot)-u_{\xi_{max}}(0,\cdot)\|^2\leq 2\xi_{max}^{-2p}E^2+\frac{1}{2}k^2\xi_{max}^{-\alpha}\int_0^1 e^{2\xi_{max}^{\frac{\alpha}{2}}s}\|u(s,\cdot)-u_{\xi_{max}}(s,\cdot)\|^2ds.$$

Therefore,

$$e^{2\cdot 0\xi_{max}^{\frac{\alpha}{2}}}\|u(0,\cdot)-u_{\xi_{max}}(0,\cdot)\|^2\leq 2\xi_{max}^{-2p}E^2+\frac{1}{2}k^2\xi_{max}^{-\alpha}\int_0^1 e^{2\xi_{max}^{\frac{\alpha}{2}}s}\|u(s,\cdot)-u_{\xi_{max}}(s,\cdot)\|^2ds. \tag{38}$$

Using Lemma 1, we obtain

$$e^{2\cdot 0\cdot\xi_{max}^{\frac{\alpha}{2}}}\|u(0,\cdot)-u_{\xi_{max}}(0,\cdot)\|^2\leq 2\xi_{max}^{-2p}E^2e^{\frac{1}{2}k^2\xi_{max}^{-\alpha}}. \tag{39}$$

Then,

$$\|u(0,\cdot)-u_{\xi_{max}}(0,\cdot)\|\leq\sqrt{2}\xi_{max}^{-p}Ee^{\frac{1}{4}k^2\xi_{max}^{-\alpha}}. \tag{40}$$

Applying the triangle inequality, combining (25) and (40) with (36), we have

$$\begin{aligned}\|u(0,\cdot)-u_{\xi_{max}}^{\delta}(0,\cdot)\| &\leq \|u(0,\cdot)-u_{\xi_{max}}(0,\cdot)\|+\|u_{\xi_{max}}(0,\cdot)-u_{\xi_{max}}^{\delta}(0,\cdot)\|\\ &\leq \sqrt{2}\xi_{max}^{-p}Ee^{\frac{1}{4}k^2\xi_{max}^{-\alpha}}+\sqrt{2}e^{\xi_{max}^{\frac{\alpha}{2}}}\delta e^{\frac{1}{4}\xi_{max}^{-\alpha}k^2}\\ &= 2\sqrt{2}e^{\frac{1}{4}k^2(In\frac{E}{\sqrt{\delta}})^{-2}}E\delta^{\frac{1}{2}}.\end{aligned}$$

□

5. Conclusions

In this paper, the Fourier truncation regularization method is used to deal with the backward problem for a time-fractional with a nonlinear source. At the same time, the error estimates between the exact solution and the regularization solution are obtained at $0 \leq x < 1$.

Author Contributions: Conceptualization, X.-X.L.; Formal analysis, X.-Y.M.; Methodology, F.Y. and P.F.

Funding: The project is supported by the National Natural Science Foundation of China (No. 11561045, No. 11961044), the Doctor Fund of Lan Zhou University of Technology.

Acknowledgments: The authors would like to thanks the editor and the referees for their valuable comments and suggestions that improve the quality of our paper.

Conflicts of Interest: The authors declare that they have no conflict of interest.

References

1. Diethelm, K. *The Analysis of Fractional Differential Equations*; Springer: Belin, Germany, 2010.
2. Metzler, R.; Klafter, J. The random walk's guide to anomalous diffusion: A fractional dynamics approach. *Phys. Rep.* **2000**, *339*, 1–77. [CrossRef]
3. Podlubny, I. *Fractional Differential Equations: An introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*; Academic: San Diego, CA, USA, 1999; Volume 198.
4. Scalas, E.; Gorenflo, R.; Mainardi, F. Fractional calculus and continuous-time finance. *Physica A* **2000**, *284*, 376–384. [CrossRef]
5. Gorenflo, R.; Mainardi, F. Some recent advances in theory and simulation of fractional diffusion processes. *J. Comput. Appl. Math.* **2009**, *229*, 400–415. [CrossRef]
6. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; Wiley: New York, NY, USA, 1993.
7. Oldham, K.B;. Spanier, J. *The Fractional Calculus*; Academic Press: New York, NY, USA, 1974.
8. Sokolov, I.M.; Klafter, J. From diffusion to anomalous diffusion: A century after Einsteins Brownian motion. *Chaos* **2005**, *15*, 1–77. [CrossRef] [PubMed]
9. Cheng, H.; Fu, C.L. An iteration regularization for a time-fractional inverse diffusion problem. *Appl. Math. Model.* **2012**, *36*, 5642–5649. [CrossRef]
10. Zheng, G.H.; Wei, T. A new regularization method for a Cauchy problem of the time fractional diffusion equation. *Adv. Comput. Math.* **2012**, *36*, 377–398. [CrossRef]
11. Zheng, G.H.; Wei, T. Spectral regularization method for solving a time-fractional inverse diffusion problem. *Appl. Math. Comput.* **2011**, *218*, 396–405. [CrossRef]
12. Zheng, G.H.; Wei, T. Two regularization methods for solving a Riesz-Feller space-fractional backward diffusion problem. *Inverse Probl.* **2010**, *26*, 115017. [CrossRef]
13. Zheng, G.H.; Wei, T. Spectral regularization method for a Cauchy problem of the time fractional advection-dispersion equation. *J. Comput. Appl. Math.* **2010**, *233*, 2631–2640. [CrossRef]
14. Zheng, G.H.; Wei, T. A new regularization method for solving a time-fractional inverse diffusion problem. *J. Math. Anal. Appl.* **2011**, *378*, 418–431. [CrossRef]
15. Xiong, X.T.; Guo, H.B.; Liu, X.H. An inverse problem for a fractional diffusion equation. *J. Comput. Appl. Math.* **2012**, *236*, 4474–4484. [CrossRef]
16. Yang, F.; Zhang, Y.; Li, X.X. Landweber iterative method for identifying the initial value problem of the time-space fractional diffusion-wave equation. *Numer. Algorithms* **2019**, 1–22. [CrossRef]
17. Yang, F.; Sun, Y.R.; Li, X.X.; Huang, C.Y. The quasi-boundary value method for identifying the initial value of heat equation on a columnar symmetric domain. *Numer. Algorithms* **2019**, *81*, 623–639. [CrossRef]
18. Yang, F.; Wang, N.; Li, X.X. A quasi-boundary regularization method for identifying the initial value of time-fractional diffusion equation on spherically symmetric domain. *J. Inverse Ill Pose Probl.* **2019**. [CrossRef]
19. Messaoudi, S.A. Blow up inn solutions of a quasilinear wave equation with variable exponent nonlineaities. *Math. Methods Appl. Sci.* **2017**, *40*, 6976–6986. [CrossRef]
20. Messaoudi, S.A.; Bonfoh, A.; Mukiawa, B.S.; Enyi, C.D. The global attractor for a suspension bridge with memory and partially hinged boundary conditions. *Z. Angew. Math. Mech.* **2016**, *97*, 159–172. [CrossRef]
21. Duy, H.D.N.; Huy, T.N.; Dinh, L.L.; Le, G.Q.T. Inverse problem for nonlinear backward space-fractional diffusion equation. *J. Inverse Ill Pose Probl.* **2016**, *25*, 423–443. [CrossRef]
22. Tuan, N.H.; Duy, D.H.; Huy, T.N.; Dinh, H.L.; Nguyen, L.V.T.; Kirane, M. On a Riesz-Feller space fractional backward diffusion problem with a nonlinear source. *J. Comput. Appl. Math.* **2017**, *312*, 103–126. [CrossRef]
23. Xiong, X.T.; Zhou, Q.; Hon, Y.C. An inverse problem for fractional diffusion equationin 2-dimensional case: Stability analysis and regularization. *J. Math. Anal. Appl.* **2012**, *393*, 185–199. [CrossRef]

24. Qian, Z.; Fu, C.L.; Xiong, X.T.; Wei, T. Fourier truncation method for high order numerical derivatives. *Appl. Math. Comput.* **2006**, *181*, 940–948. [CrossRef]
25. Yang, F.; Fu, C.L. Two regularization methods to identify time-dependent heat source through an internal measurement of temperature. *Math. Comput. Model.* **2011**, *53*, 793–804. [CrossRef]
26. Yang, F.; Fu, C.L.; Li, X.X. The inverse source problem for time fractional diffusion equation: stability analysis and regularization. *Inverse Probl. Sci. Eng.* **2015**, *23*, 969–996. [CrossRef]
27. Fu, C.L.; Xiong, X.T.; Qian, Z. Fourier regularization for a backward heat equation. *J. Math. Anal. Appl.* **2007**, *331*, 472–480. [CrossRef]
28. Fu, C.L.; Zhang, Y.X.; Cheng, H.; Ma, Y.J. The a posteriori Fourier method for solving ill-posed problems. *Inverse Probl.* **2012**, *28*, 095002. [CrossRef]
29. Fu, C.L.; Feng, X.L.; Qian, Z. The Fourier regularization for solving the Cauchy problem for the Helmholtz equation. *Appl. Numer. Math.* **2009**, *59*, 2625–2640. [CrossRef]
30. Fu, C.L.; Xiong, X.T.; Fu, P. Fourier regularization method for solving thesurface heat flux from interior observations. *Math. Comput. Model.* **2005**, *42*, 489–498. [CrossRef]
31. Yang, F.; Zhang, P.; Li, X.X. The truncation method for the Cauchy problem of the inhomogeneous Helmholtz equation. *Appl. Anal.* **2019**, *98*, 991–1004. [CrossRef]
32. Yang, F.; Fan, P.; Li, X.X. Fourier Truncation Regularization Method for a Three-Dimensional Cauchy Problem of the Modified Helmholtz Equation with Perturbed Wave Number. *Mathematics* **2019**, *7*, 705. [CrossRef]
33. Doan, V.N.; Nguyen, H.T.; Khoa, V.A.; Vo, V.A. A note on the derivation of filter regularization operators for nonlinear evolution equations. *Appl. Anal.* **2018**, *97*, 3–12. [CrossRef]