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Time-Consistent Investment-Reinsurance Strategies for the Insurer and the Reinsurer under the Generalized Mean-Variance Criteria

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Abstract: Most of the existing literature on optimal investment-reinsurance only studies from the perspective of insurers and also treats the investment-reinsurance decision as a continuous process. However, in practice, the benefits of reinsurers cannot be ignored, nor can decision-makers engage in continuous trading. Under the discrete-time framework, we first propose a multi-period investment-reinsurance optimization problem considering the joint interests of the insurer and the reinsurer, among which their performance is measured by two generalized mean-variance criteria. We derive the time-consistent investment-reinsurance strategies for the proposed model by maximizing the weighted sum of the insurer's and the reinsurer's mean-variance objectives. We discuss the time-consistent investment-reinsurance strategies under two special premium principles. Finally, we provide some numerical simulations to show the impact of the intertemporal restrictions on the time-consistent investment-reinsurance strategies. These results indicate that the intertemporal restrictions will urge the insurer and the reinsurer to shrink the position invested in the risky asset; however, for the time-consistent reinsurance strategy, the impact of the intertemporal restrictions depends on who is the leader in the proposed model.

Keywords: investment and reinsurance; insurer and reinsurer; generalized mean-variance criteria; time-consistent strategy

1. Introduction

Since the insurer and the reinsurer can be allowed to invest their wealth in the securities market, they can obtain profits not only by collecting premiums but also by investing in securities. Different from the other institutional investors, the insurer and the reinsurer will face the double risks that exist in both the insurance market and the securities market. In order to reduce the risk of claims, the insurer can purchase reinsurance contracts from the reinsurer and transfer part of the risk of claims to the reinsurer, because the reinsurer is more risk-seeking than the insurer. Therefore, how to design a suitable reinsurance contract is also the concern of the insurer and the reinsurer. Obviously, the setting of the reinsurance contract depends on the mutual agreement between the insurer and the reinsurer. However, most of the existing literature mainly focuses on the optimal investment-reinsurance problems only from the perspective of the insurer, while the interest of the reinsurer is generally ignored (e.g., Schmidli [1], Zeng and Li [2], Zhu et al. [3], Huang et al. [4], Hu and Wang [5], Deng et al. [6] and so on). Actually, the optimal reinsurance contract for the insurer may not be optimal or even unacceptable for the reinsurer. That is, the reinsurance contract should be designed to take into account the interests of both the insurer and the reinsurer. To address this problem, we will propose an investment-reinsurance optimization problem considering the joint interests of the insurer and the reinsurer, and the corresponding investment-reinsurance strategy will be investigated.

As far as we know, some researchers have paid attention to the joint interests of the insurer and the reinsurer. Li et al. [7] considered the weighted sum of an insurer's and a reinsurer's mean-variance objectives and aimed to find the corresponding time-consistent reinsurance-investment strategy. Li et al. [8] discussed the optimal investment-reinsurance strategy by maximizing the expected exponential utility of the weighted sum of the insurer's and the reinsurer's terminal wealth. Under the mean-variance criterion, Zhao et al. [9] also discussed the time-consistent investment-reinsurance strategy by maximizing the utility of a weighted sum of the insurer's and the reinsurer's surplus processes. Zhou et al. [10] derived the optimal investment-reinsurance strategy with consideration of the joint interests of the insurer and the reinsurer, and also assumed that the decision-maker was an ambiguity-averse manager. Huang et al. [11] investigated a robust optimal investment and reinsurance problem on considering the product of the insurer's and the reinsurer's utilities. Obviously, the above optimization models on the joint interests of the insurer and the reinsurer can be classified into the following two categories. The first kind of model is built by maximizing the utility of a weighted sum of the insurer's and the reinsurer's surplus processes, while the second one is constructed by maximizing the weighted sum/product of the insurer's and the reinsurer's utilities. The former assumes that the insurer and the reinsurer have the same risk aversion coefficients, while the latter considers that they have different risk aversion coefficients. Actually, the latter is more compatible with reality, since the reinsurer is more risk-seeking compared to the insurer. In this paper, we mainly focus on deriving the corresponding time-consistent strategies by maximizing the weighted sum of the insurer's and the reinsurer's mean-variance objectives.

Additionally, the above literature is limited to the study of continuous-time problems, while the discrete-time problems are always ignored by researchers. In fact, the discrete-time setting is more realistic to decision-makers, because they cannot trade continuously since it will generate a lot of transaction costs. Brandt [12] also pointed out that the continuous-time strategies are often inadmissible in discrete time because they may cause negative wealth. Especially for insurers and reinsurers, their surplus processes are more likely to be negative values, because they bear the double risks from the insurance market and the securities market. More importantly, Zhu et al. [13] also presented that the bankruptcies occurring in the earlier stages of investments were greater than those in the later stages. The main reason is that the classical multi-period mean-variance optimizations only consider the performance of the terminal wealth. To deal with this problem, Costa and Nabholz [14] proposed a generalized mean-variance model considering the intertemporal restrictions (i.e., the investors will maximize the weighted sum of the mean-variance objectives over all the periods). Under this framework, the terminal and intermediate performance of the portfolio will be included in the decision-making. There are still many studies on this subject, such as Costa and Araujo [15], Costa and de Oliveira [16], Cui et al. [17], He et al. [18], Zhou et al. [19], Xiao et al. [20] and so on. Inspired by the works mentioned above, in this paper, we will build a generalized multiperiod mean-variance investment-reinsurance optimization model considering the joint interests of the insurer and the reinsurer.

However, the proposed model cannot be directly solved by using the dynamic programming approach because the variance measure does not satisfy the expected iterated property. To the best of our knowledge, there are two methods to solve this problem. The first method is the embedding scheme proposed by Li and Ng [21], and the derived optimal strategy is called the pre-commitment strategy. However, some researchers point out that the precommitment strategy does not satisfy time consistency since the future changes are not taken into account. The second method is provided by Basak and Chabakauri [22] and Björk and Murgoci [23], called the game method, which can provide a time-consistent strategy for decision-makers. Under the game framework, the decision-makers treat this optimization problem as a noncooperative game, and its Nash equilibrium solution is defined as the time-consistent strategy. Since then, many researchers have applied the game method to derive the time-consistent solutions of the various multiperiod mean-variance optimization problems, such as Björk and Murgoci [24], Bensoussan et al. [25], Wu and Zeng [26], Zhou et al. [19], Xiao et al. [20] and so on. Based on the game method shown above, in this paper, we will investigate the time-consistent investment-reinsurance strategies for the generalized mutiperiod mean-variance optimization problem considering the joint interests of the insurer and the reinsurer. In this framework, the intermediate and terminal performance of the insurer and the reinsurer can be both considered.

Motivated by the above studies, we assume the insurer and the reinsurer can invest their wealth in one risky asset and one risk-free asset, as well as the insurer can purchase a proportional reinsurance contract from the reinsurer. We use two generalized mean-variance criteria to measure the performance of the insurer and the reinsurer, and further propose a generalized multi-period investment-reinsurance optimization considering the joint interests of the insurer and the reinsurer by maximizing the weighted sum of the insurer's and the reinsurer's mean-variance objectives. We apply the game method to derive the time-consistent investment-reinsurance strategies for the proposed model and also discuss the time-consistent strategies under the expected and variance value principles. Finally, we provide some numerical simulations to show the impact of the intertemporal restrictions on the time-consistent strategies.

Different from the existing literature, this paper has four contributions. (i) We first propose a generalized multi-period investment-reinsurance optimization problem under the discrete-time framework, and the joint interests of the insurer and the reinsurer are also considered. Actually, it is more realistic to consider the joint interests of the insurer and the reinsurer in the discrete-time framework. On the one hand, it avoids the negative wealth that may be caused by the accumulation of transaction costs in the case of continuous transactions. On the other hand, it ensures that the reinsurance contract is optimal for both the insurer and the reinsurer. (ii) We consider the impact of intertemporal restrictions on the decision-making, that is, the insurer and the reinsurer not only consider the terminal performance but are also concerned with the intermediate performance of their portfolios, which is absent from the existing continuous-time literature (e.g., Li et al. [7], Zhao et al. [9], Huang et al. [11] and so on). In this framework, the insurer and the reinsurer can adjust the intertemporal restrictions dynamically according to their own risk appetites and the market environment. (iii) We first derive the time-consistent investment-reinsurance strategies rather than the traditional precommitment strategies. Compared with the precommitment strategies, the time-consistent strategies might be more suitable for the decision-makers who are more rational and sophisticated, since they take possible future revisions into account. (iv) We first investigate the impact of the intertemporal restrictions on the time-consistent strategies. The interesting finding is that the intertemporal restrictions will urge the insurer and the reinsurer to shrink the position invested in the risky asset. However, for the time-consistent reinsurance strategy, the role of the intertemporal restrictions depends on who is the leader in the proposed model. When the insurer is the leader, the intertemporal restrictions will reduce the retention level of claims, while for the case that the reinsurer is the leader, only when the impact of the reinsurer's leading role is higher than that of the intertemporal restrictions will the intertemporal restrictions shrink the retention level of claims.

The remainder of this paper is organized as follows. In Section 2, we construct a generalized multi-period mean-variance optimization model considering the joint interests of the insurer and the reinsurer. In Section 3, we derive the time-consistent strategies by using the game method. In Section 4, we give some numerical examples to show the differences of the time-consistent strategies under different settings. Finally, we summarize the conclusions of this paper.

2. Generalized Multi-Period Mean-Variance Investment-Reinsurance Optimization Considering Both the Insurer and the Reinsurer

In this paper, we assume that both an insurer and a reinsurer will simultaneously enter the financial market at time 0 to carry out investment-reinsurance activities. Suppose that the insurer and the reinsurer have the initial wealth of w_0^1 and w_0^2, respectively, and they plan to take all the wealth into the capital market within a time horizon T. We suppose that the insurer and the reinsurer are both allowed to invest their wealth in a risk-free asset and a risky asset (note that they invest in the risky asset with different random returns), where the risk-free asset takes a determinate return s_t and the random return on the risky asset invested by the insurer is e_t^1, while the risky asset invested by the reinsurer has the random return e_t^2. Let u_t^1 and u_t^2 represent the amount that the insurer and the reinsurer invest in the risky asset at the beginning of the time period t, respectively. Assume that w_t^1 and w_t^2 denote the insurer's and the reinsurer's wealth at time period t, then the wealth allocated in the risk-free asset can be expressed as $w_t^1 - u_t^1$ and $w_t^2 - u_t^2$, respectively. In addition to the investment, we also assume that a proportional reinsurance contract is applied between the insurer and the reinsurer. The proportion covered by the insurer at the time period t is denoted by q_t, where $q_t \in [0,1]$, $t = 0,1,...,T-1$. Under this reinsurance contract, the insurer only requires to bear the claim amount $q_t z_t$ when facing a claim z_t at the time period t. In this case, q_t can also be treated as the retention level of the claim z_t. Meanwhile, the reinsurer undertakes the rest of the claim amount $(1-q_t)z_t$, as well as obtains a premium $\delta_t(q_t)$ from the insurer. Let c_t denote the premium income of the insurer at the time period t (note that c_t is assumed to be a determinate value), then the remaining premium of the insurer can be expressed as $c_t - \delta_t(q_t)$, $t = 0,1,...,T-1$. Therefore, the wealth process of the insurer and the reinsurer can be shown as follows.

$$\begin{aligned} w_{t+1}^1 &= s_t(w_t^1 - u_t^1) + e_t^1 u_t^1 + c_t - \delta_t(q_t) - q_t z_t \\ &= s_t w_t^1 + c_t + P_t^1 u_t^1 - \delta_t(q_t) - q_t z_t, \end{aligned} \tag{1}$$

and:

$$\begin{aligned} w_{t+1}^2 &= s_t(w_t^2 - u_t^2) + e_t^2 u_t^2 + \delta_t(q_t) - (1-q_t) z_t \\ &= s_t w_t^2 + P_t^2 u_t^2 + \delta_t(q_t) - (1-q_t) z_t, \end{aligned} \tag{2}$$

where, $P_t^1 = e_t^1 - s_t$ and $P_t^2 = e_t^2 - s_t$ denote the excess return of risky assets 1 and 2, respectively, and $t = 0,1,...,T-1$.

Apparently, the insurer and the reinsurer have conflicts of interest because of the reinsurance contract. Therefore, the formulation of the investment-reinsurance strategy should consider both the insurer and the reinsurer. In addition, since the insurer and the reinsurer not only face the risk of claims but also bear the investment risk of the securities market, they are more risk-averse compared to other institutional investors. More importantly, Zhu et al. [13] and Zhou et al. [19] showed that the precommitment and time-consistent strategies derived from the classical mean-variance model will lead to higher bankruptcy probabilities in the earlier periods of an investment. The cause is that the classical mean-variance model only considers the terminal performance of a portfolio and its intermediate performance is ignored. Therefore, the classical multiperiod mean-variance model may not be the best choice for the insurer and the reinsurer. To address this problem, we use two generalized mean-variance criteria to measure the performance of the insurer and the reinsurer. In this setting, the intermediate and terminal performance can be both considered. Simultaneously, we take the weighted sum of the insurer's and the reinsurer's mean-variance criteria into account, so as to measure the joint interests of the insurer and the reinsurer. Under this generalized mean-variance framework, the corresponding multiperiod investment-reinsurance optimization problem can be formulated as follows.

$$\max_{\pi} \alpha \sum_{t=1}^{T} \xi_t^1 [E(w_t^1) - \eta_t^1 Var(w_t^1)] + (1-\alpha) \sum_{t=1}^{T} \xi_t^2 [E(w_t^2) - \eta_t^2 Var(w_t^2)] \tag{3}$$

$$s.t. \begin{cases} w_{t+1}^1 = s_t w_t^1 + c_t + P_t^1 u_t^1 - \delta_t(q_t) - q_t z_t, t = 0,1,...,T-1, \\ w_{t+1}^2 = s_t w_t^2 + P_t^2 u_t^2 + \delta_t(q_t) - (1-q_t) z_t, t = 0,1,...,T-1. \end{cases}$$

Here, we let $\pi = (\pi_0, \pi_1, ..., \pi_{T-1})$, $\pi_k = (u_k^1, u_k^2, q_k)$, $k = 0, 1, ..., T-1$, and the weight α satisfy the condition that $\alpha \in [0, 1]$. Note that α can be regarded as the weighing coefficient between the benefit of the insurer and that of the reinsurer. Intuitively, $\alpha > 0.5$ indicates that the benefit of the insurer is more concerned (i.e., the insurer is the leader); $\alpha < 0.5$ means that the benefit of the reinsurer is paid more attention (i.e., the reinsurer is the leader); while for the case that $\alpha = 0.5$, we consider that the interest of the insurer and that of the reinsurer are equally important. Additionally, we also assume that ξ_t^1 and ξ_t^2 are both the 0-1 variables, among which $\xi_t^1 = 1$ ($\xi_t^2 = 1$) means that the insurer (reinsurer) will consider the intertemporal restriction at the time period t, otherwise, the intertemporal restriction is ignored here, $t = 1, 2, ..., T$. Further, parameters η_t^1 and η_t^2 denote the risk aversion coefficients of the insurer and the reinsurer at time period t, respectively, $t = 1, 2, ..., T$.

Similar to the existing literature, we further assume that $e_t = [e_t^1, e_t^2]'$ and z_t are statistically independent, $t = 0, 1, ..., T-1$. In other words, the covariance $Cov(e_{t_1}, e_{t_2}) = 0$ for $t_1 \neq t_2$, and the covariance $Cov(e_{t_1}, z_{t_2}) = 0$, where $t_1, t_2 = 0, 1, ..., T-1$. Let $\mu_t^i = E(P_t^i)$, $\sigma_t^i = Var(P_t^i)$, $\theta_t = Cov(P_t^1, P_t^2)$, $\tilde{\mu}_t = E(z_t)$ and $\tilde{\sigma}_t = Var(z_t)$, where $i = 1, 2$ and $t = 0, 1, ..., T-1$. In addition, for convenience, we define the notations that $\sum_{t=k}^{l}(\cdot) = 0$ and $\prod_{t=k}^{l}(\cdot) = 1$ for $k > l$. Under this framework, we aim to provide some suitable investment-reinsurance strategies for both the insurer and the reinsurer.

3. Time-Consistent Solution of the Generalized Model

To the best of our knowledge, the expected iterated property is absent from the variance measure, thus Equation (3) is also a time-inconsistent one, that is, it cannot be directly solved by using the dynamic programming approach. Similar to the classical multiperiod mean-variance model, we have two approaches to solve Equation (3). The first approach is the embedding scheme provided by Li and Ng [21], and the derived strategy is called the precommitment strategy. However, some researchers point out the precommitment strategy does not satisfy time consistency since it does not take the future modifications into account. The second one is proposed by Basak and Chabakauri [22] and Björk and Murgoci [23], named as the game approach, and it can provide a time-consistent strategy for decision-makers. In this paper, we mainly focus on providing a time-consistent investment-reinsurance strategy for the rational decision-makers who will consider the decision modifications in the future. Similar to Björk and Murgoci [23], we define the following time-varying mean-variance optimization sub-objective.

$$J_k(w_k^1, w_k^2, \pi) = \alpha \sum_{t=k}^{T} \xi_t^1 [E(w_t^1) - \eta_t^1 Var(w_t^1)] + (1-\alpha) \sum_{t=k}^{T} \xi_t^2 [E(w_t^2) - \eta_t^2 Var(w_t^2)], \tag{4}$$

where $k = 0, 1, ..., T-1$. Then, the time-consistent solution of Equation (3) can be defined similarly as follows.

Definition 1. *Consider a fixed control policy $\hat{\pi} = (\hat{\pi}_0, \hat{\pi}_1, ..., \hat{\pi}_{T-1})$. For $k = 0, 1, ..., T-1$, we define that:*

$$\pi(k) = (\pi_k, \hat{\pi}_{k+1}, ..., \hat{\pi}_{T-1}),$$
$$\hat{\pi}(k) = (\hat{\pi}_k, \hat{\pi}_{k+1}, ..., \hat{\pi}_{T-1}),$$

where $\pi_k = (u_k^1, u_k^2, q_k)$ is arbitrarily control variable. Then π is said to be a time-consistent investment-reinsurance strategy if for all $k = 0, 1, ..., T-1$, it satisfies:

$$\max_{\pi_k} J_k(w_k^1, w_k^2, \pi(k)) = J_k(w_k^1, w_k^2, \hat{\pi}(k)).$$

Additionally, if the time-consistent investment-reinsurance strategy $\hat{\pi}$ exists, the corresponding value function can be defined as:

$$V_k(w_k^1, w_k^2) = J_k(w_k^1, w_k^2, \hat{\pi}(k)).$$

According the definition of time-consistent strategy, we have the following proposition.

Proposition 1. *The value function $V_k(w_k^1, w_k^2)$ satisfies the following recursive relation.*

$$\begin{array}{l} V_k(w_k^1, w_k^2) \\ = \max_{\pi_k} \left\{ \begin{array}{l} E_k[V_{k+1}(w_{k+1}^1, w_{k+1}^2)] - \alpha \sum\limits_{m=k+2}^{T} \xi_m^1 \eta_m^1 Var_k[f_{k+1,m}(w_{k+1}^1)] \\ -(1-\alpha) \sum\limits_{m=k+2}^{T} \xi_m^2 \eta_m^2 Var_k[g_{k+1,m}(w_{k+1}^2)] \\ +\alpha \xi_{k+1}^1 [E_k(w_{k+1}^1) - \eta_{k+1}^1 Var_k(w_{k+1}^1)] \\ +(1-\alpha)\xi_{k+1}^2 [E_k(w_{k+1}^2) - \eta_{k+1}^2 Var_k(w_{k+1}^2)], \\ for\ k = 0, 1, ..., T-2, \end{array} \right\} \end{array} \tag{5}$$

as well as the boundary condition:

$$V_{T-1}(w_{T-1}^1, w_{T-1}^2) = \max_{\pi_{T-1}} \left\{ \begin{array}{l} \alpha[\xi_T^1 E_{T-1}(w_T^1) - \xi_T^1 \eta_T^1 Var_{T-1}(w_T^1)] \\ +(1-\alpha)[\xi_T^2 E_{T-1}(w_T^2) - \xi_T^2 \eta_T^2 Var_{T-1}(w_T^2)] \end{array} \right\}, \tag{6}$$

where:

$$f_{k,\tau}(w_k^1) = \begin{cases} E_k[f_{k+1,\tau}(w_{k+1}^1)],\ for\ \tau > k,\ \tau, k = 0, 1, ..., T-1, \\ w_k^1,\ for\ \tau = k,\ k = 0, 1, ..., T-1, \end{cases} \tag{7}$$

$$g_{k,\tau}(w_k^2) = \begin{cases} E_k[g_{k+1,\tau}(w_{k+1}^2)],\ for\ \tau > k,\ \tau, k = 0, 1, ..., T-1, \\ w_k^2,\ for\ \tau = k,\ k = 0, 1, ..., T-1. \end{cases} \tag{8}$$

Proof. See Appendix A. □

According to Proposition 1, we can derive the following theorem.

Theorem 1. *Suppose that $\alpha \in (0,1)$, for the multi-period mean-variance investment-reinsurance optimization problem Equation (3), the time-consistent investment-reinsurance strategies $\{\hat{\pi}_t = (\hat{u}_t^1, \hat{u}_t^2, \hat{q}_t), t = 0, 1, ..., T-1\}$ can be expressed as follows.*

$$\hat{u}_t^1 = \frac{\left(\sum\limits_{m=t+1}^{T} \xi_m^1 \prod\limits_{i=t+1}^{m-1} s_i \right) \mu_t^1}{2 \left(\sum\limits_{m=t+1}^{T} \xi_m^1 \eta_m^1 \prod\limits_{i=t+1}^{m-1} (s_i)^2 \right) \sigma_t^1}, \tag{9}$$

$$\hat{u}_t^2 = \frac{\left(\sum\limits_{m=t+1}^{T} \xi_m^2 \prod\limits_{i=t+1}^{m-1} s_i \right) \mu_t^2}{2 \left(\sum\limits_{m=t+1}^{T} \xi_m^2 \eta_m^2 \prod\limits_{i=t+1}^{m-1} (s_i)^2 \right) \sigma_t^2}, \tag{10}$$

$$\hat{q}_t = \arg\max_{0 \le q_t \le 1} \left\{ \begin{array}{l} \alpha \left(\sum\limits_{m=t+1}^{T} \xi_m^1 \prod\limits_{i=t+1}^{m-1} s_i \right) [-\delta_t(q_t) - q_t \tilde{\mu}_t] \\ -\alpha \left(\sum\limits_{m=t+1}^{T} \xi_m^1 \eta_m^1 \prod\limits_{i=t+1}^{m-1} (s_i)^2 \right) \tilde{\sigma}_t (q_t)^2 \\ +(1-\alpha) \left(\sum\limits_{m=t+1}^{T} \xi_m^2 \prod\limits_{i=t+1}^{m-1} s_i \right) [\delta_t(q_t) - (1-q_t)\tilde{\mu}_t] \\ -(1-\alpha) \left(\sum\limits_{m=t+1}^{T} \xi_m^2 \eta_m^2 \prod\limits_{i=t+1}^{m-1} (s_i)^2 \right) \tilde{\sigma}_t (1-q_t)^2 \end{array} \right\}. \tag{11}$$

Additionally, the value function $V_t(w_t^1, w_t^2)$, and the functions $f_{t,\tau}(w_t^1)$ and $g_{t,\tau}(w_t^2)$ are given as follows.

$$V_t(w_t^1, w_t^2) = \alpha \left(\sum_{m=t+1}^{T} \xi_m^1 \prod_{i=t}^{m-1} s_i \right) w_t^1 + (1-\alpha) \left(\sum_{m=t+1}^{T} \xi_m^2 \prod_{i=t}^{m-1} s_i \right) w_t^2 + \kappa_t, \tag{12}$$

$$f_{t,\tau}(w_t^1) = \prod_{i=t}^{\tau-1} s_i w_t^1 + \gamma_{t,\tau}, t \geq \tau,\ \tau = 0, 1, ..., T-1, \tag{13}$$

$$g_{t,\tau}(w_t^2) = \prod_{i=t}^{\tau-1} s_i w_t^2 + \rho_{t,\tau}, t \geq \tau,\ \tau = 0, 1, ..., T-1. \tag{14}$$

For convenience, we define that $\gamma_{t,\tau} = 0$ and $\rho_{t,\tau} = 0$ for $t = \tau$, then the parameters κ_t, $\gamma_{t,\tau}$ and $\rho_{t,\tau}$ satisfy the following equations:

$$\begin{cases} \kappa_t = & \sum_{k=t}^{T} \left[\frac{\alpha \left(\sum_{m=k+1}^{T} \xi_m^1 \prod_{i=k+1}^{m-1} s_i \right) (\mu_k^1)^2}{4 \left(\sum_{m=k+1}^{T} \xi_m^1 \eta_m^1 \prod_{i=k+1}^{m-1} (s_i)^2 \right) \sigma_k^1} + \frac{(1-\alpha) \left(\sum_{m=k+1}^{T} \xi_m^2 \prod_{i=k+1}^{m-1} s_i \right) (\mu_k^2)^2}{4 \left(\sum_{m=k+1}^{T} \xi_m^2 \eta_m^2 \prod_{i=k+1}^{m-1} (s_i)^2 \right) \sigma_k^2} \right] \\ & + \alpha \sum_{k=t}^{T} \left[\left(\sum_{m=k+1}^{T} \xi_m^1 \prod_{i=k+1}^{m-1} s_i \right) [c_k - \delta_k(\hat{q}_k) - \hat{q}_k \tilde{\mu}_k] \right] \\ & - \alpha \sum_{k=t}^{T} \left[\left(\sum_{m=k+1}^{T} \xi_m^1 \eta_m^1 \prod_{i=k+1}^{m-1} (s_i)^2 \right) \tilde{\sigma}_k (\hat{q}_k)^2 \right] \\ & + (1-\alpha) \sum_{k=t}^{T} \left[\left(\sum_{m=k+1}^{T} \xi_m^2 \prod_{i=k+1}^{m-1} s_i \right) [\delta_k(\hat{q}_k) - (1-\hat{q}_k) \tilde{\mu}_k] \right] \\ & - (1-\alpha) \sum_{k=t}^{T} \left[\left(\sum_{m=k+1}^{T} \xi_m^2 \eta_m^2 \prod_{i=k+1}^{m-1} (s_i)^2 \right) \tilde{\sigma}_k (1-\hat{q}_k)^2 \right], \\ \gamma_{t,\tau} = & \sum_{k=t}^{\tau} \left[\frac{\left(\sum_{m=k+1}^{T} \xi_m^1 \prod_{i=k+1}^{m-1} s_i \right) (\mu_k^1)^2}{2 \left(\sum_{m=k+1}^{T} \xi_m^1 \eta_m^1 \prod_{i=k+1}^{m-1} (s_i)^2 \right) \sigma_k^1} \right] + \sum_{k=t}^{\tau} \left[\prod_{i=k+1}^{\tau-1} s_i [c_k - \delta_k(\hat{q}_k) - \hat{q}_k \tilde{\mu}_k] \right], \\ \rho_{t,\tau} = & \sum_{k=t}^{\tau} \left[\frac{\left(\sum_{m=k+1}^{T} \xi_m^2 \prod_{i=k+1}^{m-1} s_i \right) (\mu_k^2)^2}{2 \left(\sum_{m=k+1}^{T} \xi_m^2 \eta_m^2 \prod_{i=k+1}^{m-1} (s_i)^2 \right) \sigma_k^2} \right] + \sum_{k=t}^{\tau} \left[\prod_{i=k+1}^{\tau-1} s_i [\delta_k(\hat{q}_k) - (1-\hat{q}_k) \tilde{\mu}_k] \right]. \end{cases} \tag{15}$$

Proof. See Appendix B. □

Theorem 1 only shows the time-consistent strategies when the weight coefficient α satisfies the condition that $\alpha \in (0, 1)$. While for the case that $\alpha = 1$ or $\alpha = 0$, that is, Equation (3) only considers the benefit of the insurer or the reinsurer, then the corresponding time-consistent investment-reinsurance strategies can be derived similarly. The details are shown as follows.

Remark 1. *Suppose that $\alpha = 1$, Equation (3) degenerates into the investment-reinsurance optimization problem only considering the interest of the insurer. In this situation, the time-consistent strategies $\{\hat{\pi}_t = (\hat{u}_t^1, \hat{q}_t), t = 0, ..., T-1\}$ can be obtained similarly as follows.*

$$\hat{u}_t^1 = \frac{\left(\sum_{m=t+1}^{T} \xi_m^1 \prod_{i=t+1}^{m-1} s_i \right) \mu_t^1}{2 \left(\sum_{m=t+1}^{T} \xi_m^1 \eta_m^1 \prod_{i=t+1}^{m-1} (s_i)^2 \right) \sigma_t^1}, \tag{16}$$

$$\hat{q}_t = \arg\max_{0\le q_t\le 1}\left\{ \left(\sum_{m=t+1}^{T}\xi_m^1\prod_{i=t+1}^{m-1}s_i\right)\left[-\tilde{\mu}_t q_t-\delta_t(q_t)\right]-\left(\sum_{m=t+1}^{T}\xi_m^1\eta_m^1\prod_{i=t+1}^{t-1}(s_i)^2\right)(q_t)^2\tilde{\sigma}_t\right\}. \tag{17}$$

In addition, the value function $V_t(w_t^1, w_t^2)$ and the function $f_{t,\tau}(w_t^1)$ can be reduced as:

$$V_t(w_t^1,w_t^2)=\left(\sum_{m=t+1}^{T}\xi_m^1\prod_{i=t+1}^{m-1}s_i\right)w_t^1+\kappa_t, \tag{18}$$

$$f_{t,\tau}(w_t^1)=\prod_{i=t}^{\tau-1}s_i w_t^1+\gamma_{t,\tau},\ t\ge\tau,\ \tau=0,1,...,T-1, \tag{19}$$

where κ_t and $\gamma_{t,\tau}$ (note that $\gamma_{t,\tau}=1$ for $t=\tau$) satisfy the following equations:

$$\begin{cases}\kappa_t = & \sum_{k=t}^{T-1}\left[\left(\sum_{m=k+1}^{T}\xi_m^1\prod_{i=k+1}^{m-1}s_i\right)\{c_k-\delta_k(\hat{q}_k)-\tilde{\mu}_k\hat{q}_k\}\right] \\ & +\sum_{k=t}^{T-1}\left[\frac{\left(\sum_{m=k+1}^{T}\xi_m^1\prod_{i=k+1}^{m-1}s_i\right)(\mu_k^1)^2}{4\left(\sum_{m=k+1}^{T}\xi_m^1\eta_m^1\prod_{i=k+1}^{m-1}(s_i)^2\right)\sigma_k^1}\right]-\sum_{k=t}^{T-1}\left[\left(\sum_{m=k+1}^{T}\xi_m^1\eta_m^1\prod_{i=k+1}^{m-1}(s_i)^2\right)(\hat{q}_k)^2\tilde{\sigma}_k\right], \\ \gamma_{t,\tau} = & \sum_{k=t}^{\tau-1}\left[\left(\prod_{i=k+1}^{\tau-1}s_i\right)\{c_k-\delta_k(\hat{q}_k)-\tilde{\mu}_k\hat{q}_k\}\right]+\sum_{k=t}^{\tau-1}\left[\frac{\left(\sum_{m=k+1}^{T}\xi_m^1\prod_{i=k+1}^{m-1}s_i\right)\left(\prod_{i=k+1}^{\tau-1}s_i\right)(\mu_k^1)^2}{2\left(\sum_{m=k+1}^{T}\xi_m^1\eta_m^1\prod_{i=k+1}^{m-1}(s_i)^2\right)\sigma_k^1}\right].\end{cases} \tag{20}$$

As shown in Remark 1, we can find that the function $V_t(w_t^1, w_t^2)$ and $f_{t,\tau}(w_t^1, w_t^2)$ only depend on the insurer's wealth at the time period t, i.e., w_t^1. In this situation, decision-makers only consider the benefit of the insurer, this is also the traditional approach to deal with the optimal investment-reinsurance problem. However, compared with the existing literature on the study of optimal investment and reinsurance in the continue-time setting (e.g., Schmidli [1], Zeng and Li [2], Zhu et al. [3], and Deng et al. [6] and so on), the proposed strategies in Remark 1 also consider the intermediate performance of the insurer.

Remark 2. *Suppose that $\alpha = 0$, Equation (3) degenerates into the investment-reinsurance optimization problem only considering the interest of the reinsurer. In this case, the time-consistent strategies $\{(\hat{u}_t^2, \hat{q}_t), t = 0, ..., T-1\}$ can be similarly obtained as follows.*

$$\hat{u}_t^2=\frac{\left(\sum_{m=t+1}^{T}\xi_m^2\prod_{i=t+1}^{m-1}s_i\right)\mu_t^2}{2\left(\sum_{m=t+1}^{T}\xi_m^2\eta_m^2\prod_{i=t+1}^{m-1}(s_i)^2\right)\sigma_t^2}, \tag{21}$$

$$\hat{q}_t=\arg\max_{0\le q_t\le 1}\left\{\begin{array}{l}\left(\sum_{m=t+1}^{T}\xi_m^2\prod_{i=t+1}^{m-1}s_i\right)\left[\delta_t(q_t)-\tilde{\mu}_t(1-q_t)\right] \\ -\left(\sum_{m=t+1}^{T}\xi_m^2\eta_m^2\prod_{i=t+1}^{m-1}(s_i)^2\right)\tilde{\sigma}_t(1-q_t)^2\end{array}\right\}. \tag{22}$$

Meanwhile, the value function and the function $g_{t,\tau}(w_t^1, w_t^2)$ can be reduced as follows.

$$V_t(w_t^1,w_t^2)=\prod_{i=t}^{T-1}s_i w_t^2+\kappa_t, \tag{23}$$

$$g_{t,\tau}(w_t^2) = \prod_{i=t}^{\tau-1} s_i w_t^2 + \rho_{t,\tau},\ t \geq \tau,\ \tau = 0, 1, ..., T-1, \tag{24}$$

where κ_t and $\rho_{t,\tau}$ (note that $\rho_{t,\tau} = 0$ for $t = \tau$) satisfy the following equations:

$$\begin{cases} \kappa_t = & \sum\limits_{k=t}^{T-1} \left[\left(\sum\limits_{m=k+1}^{T} \xi_m^2 \prod\limits_{i=k+1}^{m-1} s_i \right) \{\delta_k(\hat{q}_k) - \tilde{\mu}_k(1-\hat{q}_k)\} \right] + \sum\limits_{k=t}^{T-1} \left[\dfrac{\left(\sum\limits_{m=k+1}^{T} \xi_m^2 \prod\limits_{i=k+1}^{m-1} s_i \right) (\mu_k^2)^2}{4\left(\sum\limits_{m=k+1}^{T} \xi_m^2 \eta_m^2 \prod\limits_{i=k+1}^{m-1} (s_i)^2 \right) \sigma_k^2} \right] \\ & - \sum\limits_{k=t}^{T-1} \left[\left(\sum\limits_{m=k+1}^{T} \xi_m^2 \eta_m^2 \prod\limits_{i=k+1}^{m-1} (s_i)^2 \right) \tilde{\sigma}_k (1-\hat{q}_k)^2 \right], \\ \rho_{t,\tau} = & \sum\limits_{k=t}^{\tau-1} \left[\left(\prod\limits_{i=k+1}^{\tau-1} s_i \right) \{\delta_k(\hat{q}_k) - \tilde{\mu}_k(1-\hat{q}_k)\} \right] + \sum\limits_{k=t}^{\tau-1} \left[\dfrac{\left(\sum\limits_{m=k+1}^{T} \xi_m^2 \prod\limits_{i=k+1}^{m-1} s_i \right) \left(\prod\limits_{i=k+1}^{\tau-1} s_i \right) (\mu_k^2)^2}{2\left(\sum\limits_{m=k+1}^{T} \xi_m^2 \eta_m^2 \prod\limits_{i=k+1}^{m-1} (s_i)^2 \right) \sigma_k^2} \right]. \end{cases} \tag{25}$$

Remark 2 shows the time-consistent strategies when Equation (3) only considers the interest of the reinsurer. In this situation, the value function $V_t(w_t^1, w_t^2)$ and the function $f_{t,\tau}(w_t^1)$ only depend on the wealth value w_t^2 (i.e., the reinsurer's wealth at time period t).

Additionally, from Theorem 1 and Remarks 1 and 2, we can find that the reinsurance strategy and the investment strategies are independent of each other, that is, some changes in the reinsurance premium $\delta_t(q_t)$ will not affect the form of the investment strategies shown in Theorem 1. In the following, we will discuss the time-consistent strategies under some classical premium principles (e.g., the expected value principle and the variance value principle). The detailed results are presented in Sections 3.1 and 3.2, respectively.

3.1. Time-Consistent Investment-Reinsurance Strategies under the Expected Value Principle

In this section, we refer to Waters [27] and assume that the reinsurance premium $\delta_t(q_t)$ is calculated according to the expected value principle, i.e., $\delta_t(q_t) = (1+\beta_t)(1-q_t)\tilde{\mu}_t$, where β_t ($\beta_t > 0$) is the safety loading of the reinsurer. For convenience, we define the following notation:

$$b_t = \frac{\left[\alpha \left(\sum\limits_{m=t+1}^{T} \xi_m^1 \prod\limits_{i=t+1}^{m-1} s_i \right) - (1-\alpha) \left(\sum\limits_{m=t+1}^{T} \xi_m^2 \prod\limits_{i=t+1}^{m-1} s_i \right) \right] \beta_t \tilde{\mu}_t + 2(1-\alpha) \left(\sum\limits_{m=t+1}^{T} \xi_m^2 \eta_m^2 \prod\limits_{i=t+1}^{m-1} (s_i)^2 \right) \tilde{\sigma}_t}{2\left[\alpha \left(\sum\limits_{m=t+1}^{T} \xi_m^1 \eta_m^1 \prod\limits_{i=t+1}^{m-1} (s_i)^2 \right) + (1-\alpha) \left(\sum\limits_{m=t+1}^{T} \xi_m^2 \eta_m^2 \prod\limits_{i=t+1}^{m-1} (s_i)^2 \right) \right] \tilde{\sigma}_t}.$$

Based on the conclusions shown in Theorem 1 and Remarks 1 and 2, we can derive the time-consistent strategies under some special settings. For details see Corollary 1 and Remarks 3–5.

Corollary 1. *Suppose that $\alpha \in (0,1)$ and the reinsurance premium $\delta_t(q_t)$ is calculated according to the above expected value principle. In this situation, the time-consistent investment strategies for Equation (3) are coincident with these in Theorem 1, while the corresponding time-consistent reinsurance strategy (i.e., $\hat{q}_t$, $t = 0, 1, ..., T-1$) can be reduced as follows.*

$$\hat{q}_t = \begin{cases} 0, & if \quad b_t < 0, \\ b_t, & if \quad 0 \leq b_t \leq 1, \\ 1, & if \quad b_t > 1. \end{cases} \tag{26}$$

As shown in Corollary 1, we can find that the time-consistent reinsurance strategy $\hat{q}_t$ is dependent with the notation b_t, since the value of b_t will determine whether the time-consistent strategy q_t is to take the interior point or the boundary point. When b_t is a negative value, the time-consistent reinsurance strategy q_t is equal to 0, that is, the reinsurer will bear all risk of claims; when $b_t > 1$, the time-consistent reinsurance strategy can be expressed as $q_t = 1$, in this situation, the insurer will undertake all the risk of claims; otherwise, q_t will be to take interior point b_t. Apparently, the intertemporal restrictions will affect the form of b_t, that is, the time-consistent reinsurance q_t is also influenced by the intertemporal restrictions.

Remark 3. *Suppose that $\alpha \in (0,1)$, $\xi_t^1 = \xi_t^2 = 0$ for $t = 1,2,...,T-1$, $\xi_T^1 = \xi_T^2 = 1$ and the reinsurance premium $\delta_t(q_t)$ is calculated according to the above expected value principle. Therefore, the time-consistent investment-reinsurance strategies for Equation (3), i.e., $\hat{\pi}_t = \{(\hat{u}_t^1, \hat{u}_t^2, \hat{q}_t), t = 0,1,...,T-1\}$, can be reduced as:*

$$\begin{cases} \hat{u}_t^1 = \dfrac{\mu_t^1}{2\eta_T^1\left(\prod\limits_{i=t+1}^{T-1} s_i\right)\sigma_t^1}, \\ \hat{u}_t^2 = \dfrac{\mu_t^2}{2\eta_T^2\left(\prod\limits_{i=t+1}^{T-1} s_i\right)\sigma_t^2}, \end{cases} \tag{27}$$

$$\hat{q}_t = \begin{cases} 0, & if \quad b_t < 0, \\ b_t, & if \quad 0 \le b_t \le 1, \\ 1, & if \quad b_t > 1. \end{cases} \tag{28}$$

In this case, b_t can be reduced as $b_t = \frac{(1-\alpha)\eta_T^2}{\alpha\eta_T^1 + (1-\alpha)\eta_T^2} + \frac{(2\alpha-1)\beta_t\tilde{\mu}_t}{2\left[\alpha\eta_T^1 + (1-\alpha)\eta_T^2\right]\prod_{i=t+1}^{T-1} s_i\tilde{\sigma}_t}$.

Remark 3 shows that the insurer and the reinsurer only consider the performance of terminal wealth and the reinsurance premium is calculated according to the expected value principle. Compared with Corollary 1 and Remark 3, we can find that the latter only considers the terminal risk aversion coefficients η_T^1 and η_T^2, while the former is dependent on all the risk aversion coefficients at the different time periods (i.e., η_t^1 and η_t^2 for $t = 1,2,...,T$).

Remark 4. *Suppose that $\alpha = 1$ and the reinsurance premium $\delta_t(q_t)$ is calculated according to the above expected value principle. In this case, the time-consistent investment strategy for Equation (3) is same as that in Remark 1. However, the time-consistent reinsurance strategy (i.e., $\hat{q}_t, t = 0,...,T-1$) can be reduced as follows.*

$$\hat{q}_t = \frac{\left(\sum\limits_{m=t+1}^{T} \xi_m^1 \prod\limits_{i=t+1}^{m-1} s_i\right)\beta_t\tilde{\mu}_t}{2\left(\sum\limits_{m=t+1}^{T} \xi_m^1\eta_m^1 \prod\limits_{i=t+1}^{m-1} (s_i)^2\right)\tilde{\sigma}_t} \wedge 1. \tag{29}$$

Remark 4 shows the time-consistent investment-reinsurance strategies for the insurer under the expected value principle. In this case, the insurer's decision only depends on its own performance, while the performance of the reinsurer is ignored here. However, the intertemporal restrictions still restrict the formulation of the time-consistent strategy.

Remark 5. *Suppose that $\alpha = 0$ and the reinsurance premium $\delta_t(q_t)$ is calculated according to the above expected value principle. In this situation, the time-consistent investment strategy for Equation (3) is consistent with that in Remark 2. In addition, the time-consistent reinsurance strategy (i.e., $\hat{q}_t, t = 0,...,T-1$) can be expressed as follows.*

$$\hat{q}_t = 0 \vee \left(1 - \frac{\left(\sum\limits_{m=t+1}^{T} \xi_m^2 \prod\limits_{i=t+1}^{m-1} s_i\right)\beta_t\tilde{\mu}_t}{2\left(\sum\limits_{m=t+1}^{T} \xi_m^2\eta_m^2 \prod\limits_{i=t+1}^{m-1} (s_i)^2\right)\tilde{\sigma}_t}\right) \wedge 1. \tag{30}$$

Remark 5 shows the time-consistent investment-reinsurance strategies for the reinsurer under the expected value principle. Compared with the results shown in Remark 4, the proposed strategies in Remark 5 are derived from the other extreme, that is, the decision-maker only considers the performance of the reinsurer.

3.2. Time-Consistent Investment and Reinsurance Strategies under the Variance Value Principle

In this section, we assume that the reinsurance premium $\delta_t(q_t)$ is calculated according to the variance value principle (the readers can refer to Waters [27]), that is, $\delta_t(q_t) = (1-q_t)\tilde{\mu}_t + \beta_t(1-q_t)^2\tilde{\sigma}_t, t = 0,1,...,T-1$. Similarly, we can derive the time-consistent strategies under some special settings. For details see Corollary 2 and Remarks 6–8.

Corollary 2. *Suppose that $\alpha \in (0,1)$ and the reinsurance premium $\delta_t(q_t)$ is calculated according to the above variance value principle. For Equation (3), its time-consistent investment strategies are also coincident with these in Theorem 1. In addition, the corresponding time-consistent reinsurance strategy ($\hat{q}_t,\ t = 0,1,...,T-1$) can be expressed as:*

$$\hat{q}_t = \begin{cases} 0, & if \quad \hat{m}_t \neq 0 \text{ and } \hat{b}_t < 0, \\ \hat{b}_t, & if \quad \hat{m}_t \neq 0 \text{ and } 0 \leq \hat{b}_t \leq 1, \\ 1, & if \quad \hat{m}_t \neq 0 \text{ and } \hat{b}_t > 1, \\ \forall q_t \in [0,1], & if \quad \hat{m}_t = 0, \end{cases} \tag{31}$$

where:

$$\begin{cases} \hat{b}_t = \dfrac{\left[\alpha\left(\sum\limits_{m=t+1}^{T}\xi_m^1\prod\limits_{i=t+1}^{m-1}s_i\right)-(1-\alpha)\left(\sum\limits_{m=t+1}^{T}\xi_m^2\prod\limits_{i=t+1}^{m-1}s_i\right)\right]\beta_t+(1-\alpha)\left(\sum\limits_{m=t+1}^{T}\xi_m^2\eta_m^2\prod\limits_{i=t+1}^{m-1}(s_i)^2\right)}{\left[\alpha\left(\sum\limits_{m=t+1}^{T}\xi_m^1\prod\limits_{i=t+1}^{m-1}s_i\right)-(1-\alpha)\left(\sum\limits_{m=t+1}^{T}\xi_m^2\prod\limits_{i=t+1}^{m-1}s_i\right)\right]\beta_t+\left[\alpha\left(\sum\limits_{m=t+1}^{T}\xi_m^1\eta_m^1\prod\limits_{i=t+1}^{m-1}(s_i)^2\right)+(1-\alpha)\left(\sum\limits_{m=t+1}^{T}\xi_m^2\eta_m^2\prod\limits_{i=t+1}^{m-1}(s_i)^2\right)\right]}, \\ \hat{m}_t = \left[\alpha\left(\sum\limits_{m=t+1}^{T}\xi_m^1\prod\limits_{i=t+1}^{m-1}s_i\right)-(1-\alpha)\left(\sum\limits_{m=t+1}^{T}\xi_m^2\prod\limits_{i=t+1}^{m-1}s_i\right)\right]\beta_t \\ \qquad + \left[\alpha\left(\sum\limits_{m=t+1}^{T}\xi_m^1\eta_m^1\prod\limits_{i=t+1}^{m-1}(s_i)^2\right)+(1-\alpha)\left(\sum\limits_{m=t+1}^{T}\xi_m^2\eta_m^2\prod\limits_{i=t+1}^{m-1}(s_i)^2\right)\right]. \end{cases}$$

Obviously, the intertemporal restrictions also restrict the formulation of the proposed strategies in Corollary 2. Compared with Corollary 1 and Corollary 2, we can find that the proposed reinsurance strategy shown in Corollary 1 will be affected by the expectation and variance of the claim z_t (i.e., $\tilde{\mu}_t$ and $\tilde{\sigma}_t$), while that in Corollary 2 is independent with $\tilde{\mu}_t$ and $\tilde{\sigma}_t$. That is, under the variance value principle, the reserve level $\hat{q}_t$ does not change because of the size of $\tilde{\mu}_t$ and $\tilde{\sigma}_t$. However, with the increase of parameters $\tilde{\mu}_t$ and $\tilde{\sigma}_t$, the insurer will pay more premiums to the reinsurer since the reinsurance premium $\delta_t(q_t)$ is an increasing function of both $\tilde{\mu}_t$ and $\tilde{\sigma}_t$.

Remark 6. *Suppose that $\alpha \in (0,1)$, $\xi_t^1 = \xi_t^2 = 0$ for $t = 1,2,...,T-1$, $\xi_T^1 = \xi_T^2 = 1$, and the reinsurance premium $\delta_t(q_t)$ is calculated according to the above variance value principle. The time-consistent investment-reinsurance strategies for Equation (3) (i.e., $\hat{\pi}_t = \{(\hat{u}_t^1, \hat{u}_t^2, \hat{q}_t),\ t = 0,1,...,T-1\}$) can be simplified as:*

$$\begin{cases} \hat{u}_t^1 = \dfrac{\mu_t^1}{2\eta_T^1\left(\prod\limits_{i=t+1}^{T-1}s_i\right)\sigma_t^1}, \\ \hat{u}_t^2 = \dfrac{\mu_t^2}{2\eta_T^2\left(\prod\limits_{i=t+1}^{T-1}s_i\right)\sigma_t^2}, \end{cases} \tag{32}$$

$$\hat{q}_t = \begin{cases} 0, & if \quad m_t \neq 0,\ \hat{b}_t < 0, \\ \hat{b}_t, & if \quad m_t \neq 0,\ 0 \leq \hat{b}_t \leq 1, \\ 1, & if \quad m_t \neq 0,\ \hat{b}_t > 1, \\ \forall q_t \in [0,1], & if \quad m_t = 0, \end{cases} \tag{33}$$

where:

$$\begin{cases} \hat{b}_t = \dfrac{(2\alpha-1)\beta_t + (1-\alpha)\eta_T^2 \left(\prod\limits_{i=t+1}^{T-1} s_i \right)}{(2\alpha-1)\beta_t + \left[\alpha\eta_T^1 + (1-\alpha)\eta_T^2\right] \left(\prod\limits_{i=t+1}^{T-1} s_i \right)}, \\ \hat{m}_t = (2\alpha-1)\beta_t + \left[\alpha\eta_T^1 + (1-\alpha)\eta_T^2\right] \left(\prod\limits_{i=t+1}^{T-1} s_i \right). \end{cases}$$

Remark 6 shows the time-consistent strategies for the insurer and the reinsurer who only consider the performance of their terminal wealth, among which the reinsurance premium is calculated according to the variance value principle. Compared with Corollary 2 and Remark 6, we can find that the proposed reinsurance strategy not only relies on the terminal risk aversion coefficients η_T^1 and η_T^2, but also has nothing to do with the expectation and variance of a claim.

Remark 7. *Suppose that $\alpha = 1$ and the reinsurance premium $\delta_t(q_t)$ is calculated according to the variance value principle. In this situation, the time-consistent investment strategies for Equation (3) are also coincident with those in Remark 1, while the time-consistent reinsurance strategy (i.e., $\hat{q}_t,\ t = 0, ..., T-1$) can be simplified as follows.*

$$\hat{q}_t = \frac{\left(\sum\limits_{m=t+1}^{T} \xi_m^1 \prod\limits_{i=t+1}^{m-1} s_i \right) \beta_t}{\left(\sum\limits_{m=t+1}^{T} \xi_m^1 \eta_m^1 \prod\limits_{i=t+1}^{m-1} (s_i)^2 \right) + \left(\sum\limits_{m=t+1}^{T} \xi_m^1 \prod\limits_{i=t+1}^{m-1} s_i \right) \beta_t}. \tag{34}$$

Remark 7 shows the time-consistent strategies when the decision-makers only consider the performance of the insurer rather than that of the reinsurer. Compared with the time-consistent reinsurance strategy shown in Corollary 2 and Remark 7, we can find that the former might be a boundary point, while the latter has to be an interior value.

Remark 8. *Suppose that $\alpha = 0$ and the reinsurance premium $\delta_t(q_t)$ is calculated according to the above variance value principle. The time-consistent investment strategy are consistent with these in Remark 2. However, the corresponding time-consistent reinsurance strategy (i.e., $\hat{q}_t,\ t = 0, ..., T-1$) is reduced as follows.*

$$\hat{q}_t = \begin{cases} 0,\ if\ \beta_t > \dfrac{\left(\sum\limits_{m=t+1}^{T} \xi_m^2 \eta_m^2 \prod\limits_{i=t+1}^{m-1} (s_i)^2 \right)}{\left(\sum\limits_{m=t+1}^{T} \xi_m^2 \prod\limits_{i=t+1}^{m-1} s_i \right)}, \\ 1,\ if\ \beta_t < \dfrac{\left(\sum\limits_{m=t+1}^{T} \xi_m^2 \eta_m^2 \prod\limits_{i=t+1}^{m-1} (s_i)^2 \right)}{\left(\sum\limits_{m=t+1}^{T} \xi_m^2 \prod\limits_{i=t+1}^{m-1} s_i \right)}, \\ \forall q_t \in [0,1],\ if\ \beta_t = \dfrac{\left(\sum\limits_{m=t+1}^{T} \xi_m^2 \eta_m^2 \prod\limits_{i=t+1}^{m-1} (s_i)^2 \right)}{\left(\sum\limits_{m=t+1}^{T} \xi_m^2 \prod\limits_{i=t+1}^{m-1} s_i \right)}. \end{cases} \tag{35}$$

Remark 8 shows the time-consistent strategies when the decision-makers only consider the performance of the reinsurer, among which the reinsurance premium is according to the variance value principle. In this case, the safety loading coefficient β_t can be treated as the risk compensation coefficient of the reinsurer. That is, the larger the β_t, the more the reinsurer can obtain risk compensations from the insurer. Remark 8 indicates that when β_t is large enough, the reinsurer is willing to assume all the risk of claims; otherwise, the reinsurer is reluctant to assume any claim risks, since the insurer cannot offer a suitable reinsurance premium as the risk compensation.

4. Numerical Analysis

In this section, we will provide some numerical simulations to show the results presented in Section 3. We assume that the initial wealth of the insurer and the reinsurer are $w_0^1 = 1$ and $w_0^2 = 1$, respectively. Using the data provided by Li and Ng [21], we let $\mu_t^1 = 1.162$, $\mu_t^2 = 1.228$, $\sigma_t^1 = 0.0146$, $\sigma_t^2 = 0.0289$, $\theta_t = 0.0145$ and $s_t = 1.04$, $t = 0, 1, ..., T-1$. Additionally, we suppose that the safety loading coefficient of the reinsurer satisfies $\beta_t = 0.8$, and the claim z_t follows an exponential distribution with the rate parameter $\lambda_t = 10$, $t = 0, 1, ..., T-1$. In the following, we will show the evolutions process of the time-consistent strategies under different settings and investigate how the intertemporal restrictions affect the time-consistent investment and reinsurance strategies. In addition, we assume that ξ_t^1 and ξ_t^2 have the following situations.

- Case I. Suppose that $\xi_t^1 = \xi_t^2 = 1$ in Equation (3), $t = 1, 2, ..., T$. In this setting, the insurer's and the reinsurer's decisions will take all the intertemporal restrictions into account.
- Case II. Suppose that $\xi_t = \xi_t^2 = 0$ for $t = 1, 2, ..., T-1$ and $\xi_T^1 = \xi_T^2 = 1$ in Equation (3). In this case, the decision-makers only consider the performance of their terminal wealth.

Based on the above parameter settings, we will discuss the impact of the intertemporal restrictions on the time-consistent investment and reinsurance strategies. Since the reinsurance premium $\delta_t(q_t)$ and the weight coefficient α have nothing to do with the investment strategies of the insurer and the reinsurer, we do not have to repeat the evolution process of time-consistent investment strategies when the $\delta_t(q_t)$ and α take different settings. However, the impacts of the $\delta_t(q_t)$ and the weight α on the formulation of the time-consistent reinsurance strategy cannot be ignored. Under this conception, we can obtain the following simulations.

4.1. Simulations of the Insurer's and the Reinsurer's Time-Consistent Investment Strategies

In this section, we will discuss how the intertemporal restrictions affect the time-consistent investment strategies. Motivated by Xiao et al. [20], we assume that the risk aversion coefficients of the insurer and the reinsurer satisfy the following two exponential functions: $\eta_t^1 = \eta_T^1 \times \phi_1^{T-t}$ and $\eta_t^2 = \eta_T^2 \times \phi_2^{T-t}$, among which ϕ_1 and ϕ_2 are both greater than the return of the risk-free asset, $t = 1, 2, ..., T$. Additionally, since the insurer is more risk-averse compared to the reinsurer, we also assume that $\eta_T^1 > \eta_T^2$ and $\phi_1 > \phi_2$. Under this assumption, we let $\eta_T^1 = 2$, $\eta_T^1 = 1$, $\phi_1 = 1.08$ and $\phi_1 = 1.06$. Therefore, we can derive the corresponding simulation paths of the time-consistent investment strategies shown in Sections 3.1 and 3.2. For details see Figure 1.

Figure 1 shows the evolution paths of the insurer's and the reinsurer's time-consistent investment strategies when the investment horizon is $T = 100$. As shown in Figure 1, we find that compared with the time-consistent investment strategies without considering intertemporal restrictions, the insurer and the reinsurer will both reduce investment position invested in the risky asset when the intertemporal restrictions are all considered in the decision-making. This indicates that the insurer and the reinsurer will increase the amount invested in risk-free assets (i.e., $w_t^1 - \hat{u}_t^1$ or $w_t^2 - \hat{u}_t^2$), so as to reduce the risk in the earlier periods. In addition, the above time-consistent strategies are all increasing functions of the time period t, no matter that the intertemporal restrictions are considered or not. That is, the closer the insurer and the reinsurer get to the end of their investment, the more wealth they invest in the risky asset. This cause is that with the accumulation of the insurer's and the reinsurer's wealth, they have enough ability to bear the investment risk.

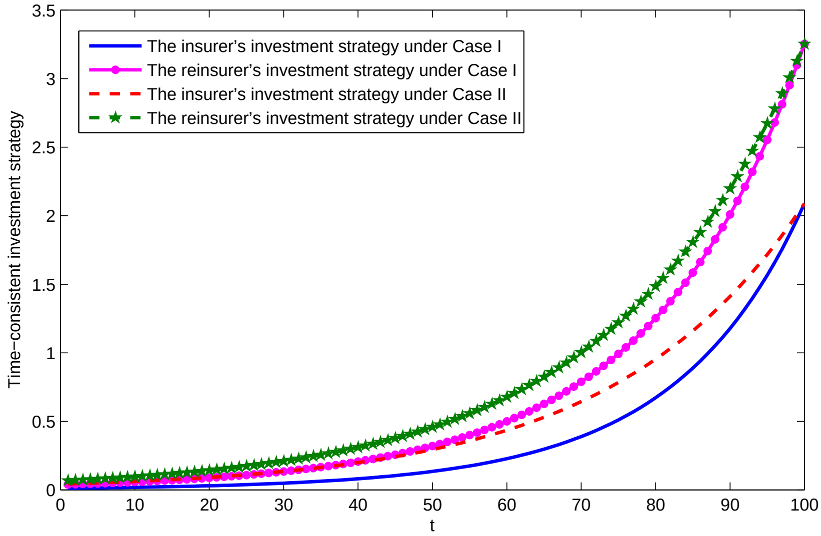

Figure 1. Time-consistent investment strategies for the insurer and the reinsurer ($T = 100$).

In Figure 1, we assume that the risk aversion coefficients of the insurer and the reinsurer follow the exponential distributions mentioned above. In the following, we want to check the above conclusions whether is true when the risk aversion coefficient appears in other forms. Inspired by Wang and Chen [28], we assume the risk aversion coefficients are described by the following linear functions: $\eta_t^1 = \eta_T^1 + k_1 \times (T - t)$ and $\eta_t^2 = \eta_T^2 + k_2 \times (T - t)$, $t = 1, 2, ..., T$, where parameters k_1 and k_1 are both greater than 0. Similar to Figure 1, we also assume that $\eta_T^1 > \eta_T^2$ and $k_1 > k_2$, since the insurer is more risk-averse than the reinsurer.

Let $T = 100$, $\eta_T^1 = 2$, $\eta_T^2 = 1$, $k_1 = 0.5$ and $k_2 = 0.2$, and then we can derive the corresponding simulations presented in Figure 2. As shown in Figure 2, we find that the intertemporal restrictions also have restriction effects on the time-consistent investment strategies; meanwhile, these investment strategies are all increasing functions of the time period t. Apparently, the above conclusions are coincident with these in Figure 1.

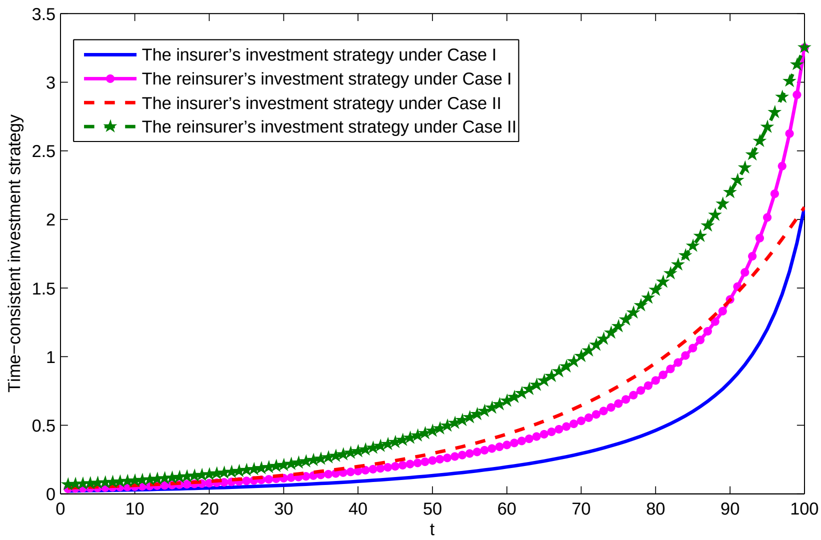

Figure 2. Time-consistent investment strategy for the insurer ($\alpha = 0.6$ and $T = 100$).

4.2. Simulations of the Time-Consistent Reinsurance Strategy

In the following, we will discuss the evolution of the time-consistent reinsurance strategy under different settings. More importantly, we will investigate the impacts of the reinsurance premium $\delta_t(q_t)$ and the weight coefficient α on the time-consistent reinsurance strategy. In this section, we assume that the insurer and the reinsurer will adopt the classical expected and variance value principles to make the time-consistent reinsurance strategy. Using the parameters shown in Figure 1, we can derive the corresponding simulation path of the time-consistent reinsurance strategy under the expected value principle and the variance value principle, respectively. For detailed simulation results see Figures 3–6.

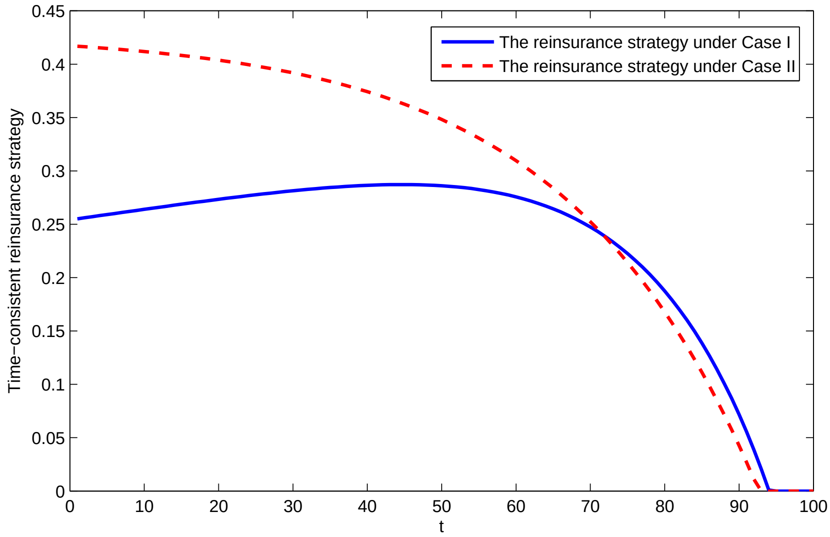

Figure 3. Time-consistent reinsurance strategy under the expected value principle ($\alpha = 0.4$ and $T = 100$).

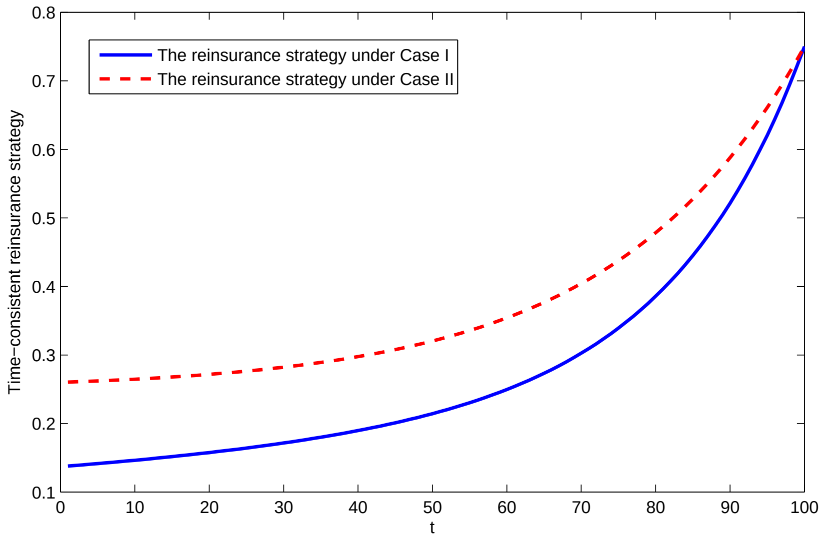

Figure 4. Time-consistent reinsurance strategy under the expected value principle ($\alpha = 0.6$ and $T = 100$).

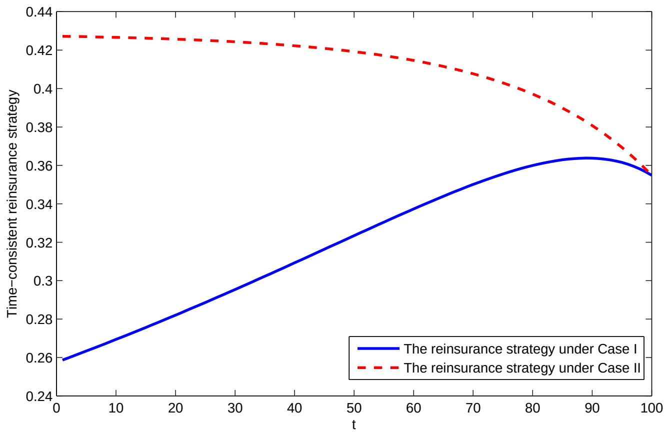

Figure 5. Time-consistent reinsurance strategy under the variance value principle ($\alpha = 0.4$ and $T = 100$).

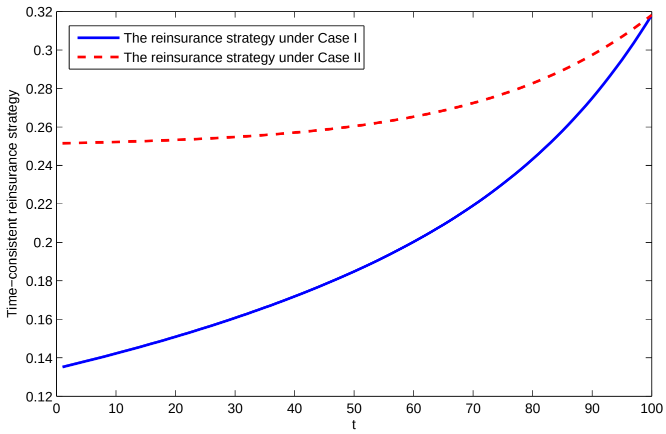

Figure 6. Time-consistent reinsurance strategy under the variance value principle ($\alpha = 0.6$ and $T = 100$).

Under the expected value principle, Figures 3 and 4 show the evolution of the time-consistent reinsurance strategy with different weight coefficients α (i.e., $\alpha = 0.4$ and $\alpha = 0.6$). Under Case II, we find that the time-consistent reinsurance strategy is a decreasing function of the time period t when the weight coefficient takes $\alpha = 0.4$, while for the situation that $\alpha = 0.6$, the time-consistent reinsurance strategy increases with the time period t. Actually, the condition $\alpha = 0.4$ indicates that the reinsurer is paid more attention (i.e., the reinsurer is the leader), in this case, the reinsurer wants to get more reinsurance business from the insurer. Therefore, the retention level of the claim $\hat{q}_t$ will decrease with the time period t. On the other hand, the condition $\alpha = 0.6$ means that the insurer is getting more attention (i.e., the insurer is the leader), in this situation, the insurer will pass most of claim risks to the reinsurer, because the bankruptcy probability is higher in the earlier periods. However, with the accumulation of the insurer's wealth, the insurer is willing to increase the retention level of the claim $\hat{q}_t$, that is, $\hat{q}_t$ increases with the time period t. Note that this monotonicity is not necessarily true for the time-consistent reinsurance strategy with consideration of the intertemporal restrictions.

We further consider the impact of the intertemporal restrictions on the time-consistent reinsurance strategy (i.e., the time-consistent reinsurance strategy under Case I in Figures 3 and 4). As shown in Figures 3 and 4, we find that the intertemporal restrictions have a significant impact on reinsurance strategy. However, there are large differences in the impact of intertemporal restrictions on the time-consistent investment strategies and the time-consistent reinsurance strategy. As mentioned in Section 4.1, we have concluded that the intertemporal restrictions will make the insurer and the reinsurer reduce the positions invested in the risky asset. While for the reinsurance strategy, the intertemporal restrictions do not necessarily cause the decision-makers to reduce the retention level $\hat{q}_t$ (e.g., the time-consistent reinsurance strategy under Case I in Figure 3). The cause is that when $\alpha = 0.4$ (i.e., the reinsurer has the leading role) and the intertemporal restrictions are taken into consideration, the leading role urges the reinsurer to increase reinsurance proportion $(1 - \hat{q}_t)$, but the intertemporal restrictions cause the reinsurer to reduce reinsurance business as much as possible (i.e., the reinsurer wants the insurer to increase the retention level $\hat{q}_t$). Therefore, only when the impact of the intertemporal restrictions is higher than that of the reinsurer's leading role, the intertemporal restrictions can induce the reinsurer to shrink the reinsurance proportion $(1 - \hat{q}_t)$. However, when $\alpha = 0.4$ (i.e., the insurer has the leading role) and the intertemporal restrictions are taken into consideration, the target of the intertemporal restrictions and that of the insurer's leading role is consistent, that is, reducing the retention level $\hat{q}_t$. Generally speaking, the role of the intertemporal restrictions on the time-consistent reinsurance strategy depends on who is the leader in Equation (3).

Similar to Figures 3 and 4, we can derive the evolution path of the time-consistent reinsurance strategy under the variance value principle. For details see Figures 5 and 6.

Figures 5 and 6 show the corresponding evolution path of the time-consistent reinsurance strategy under the variance value principle. When the insurer and the reinsurer do not consider the impact of the intertemporal restrictions (i.e., Case II), we find that the time-consistent reinsurance strategy $\hat{q}_t$ decreases with the time period t when the weight coefficient takes $\alpha = 0.4$, while for the case that $\alpha = 0.6$, the time-consistent reinsurance strategy $\hat{q}_t$ is an increasing function of the time period t. Obviously, the above conclusion is coincident with that in Figures 3 and 4. In addition, under the different weight coefficients α, it is not difficult to find that the role of the intertemporal restrictions is basically consistent with that in Figures 3 and 4.

In Figures 3–6, we assume that the risk aversion coefficients η_t^1 and η_t^2 follow the two given exponential distributions, respectively. Using the parameters shown in Figure 2, that is, the risk aversion coefficients η_t^1 and η_t^2 both described by the linear functions, we will further discuss the evolution path of the time-consistent reinsurance strategy under the expected value principle and the variance value principle, respectively. For details see Figures 7–10.

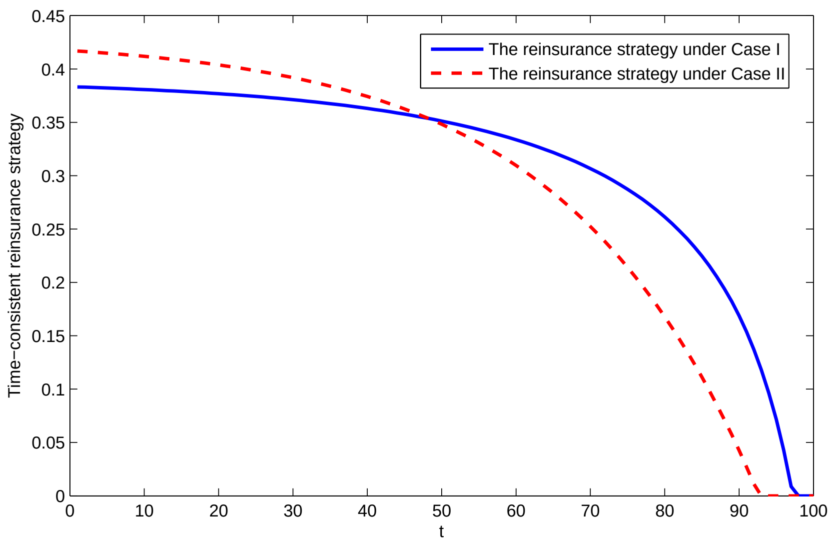

Figure 7. Time-consistent reinsurance strategy under the expected value principle ($\alpha = 0.4$ and $T = 100$).

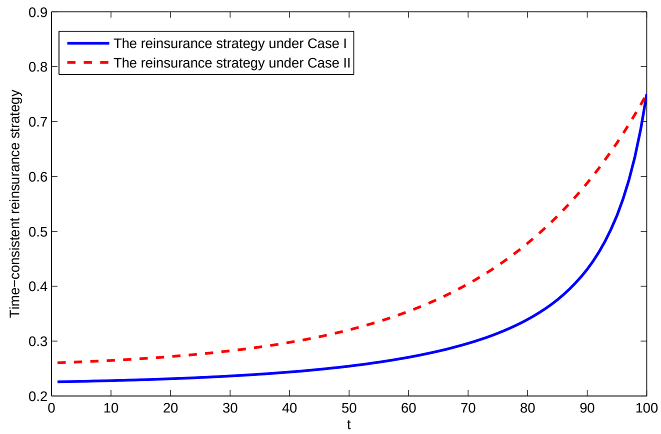

Figure 8. Time-consistent reinsurance strategy under the expected value principle ($\alpha = 0.6$ and $T = 100$).

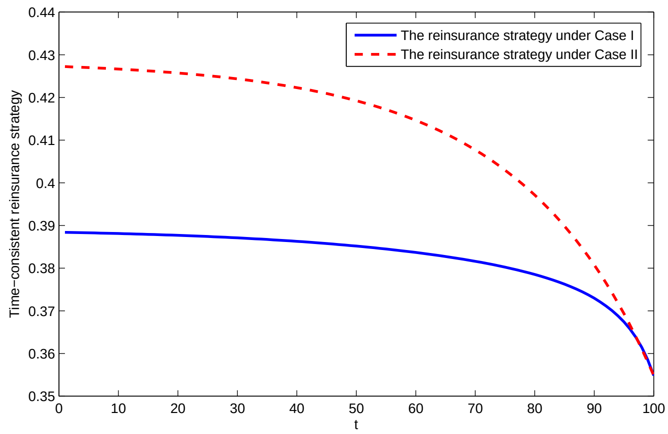

Figure 9. Time-consistent reinsurance strategy under the variance value principle ($\alpha = 0.4$ and $T = 100$).

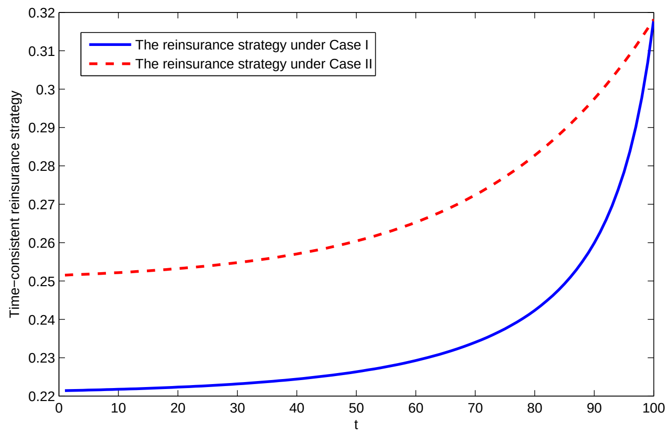

Figure 10. Time-consistent reinsurance strategy under the variance value principle ($\alpha = 0.6$ and $T = 100$).

As shown in Figures 7–10, we find that the evolution trends of the above reinsurance strategy are basically consistent with those in Figures 3–6. In other words, under the two risk aversion coefficient functions mentioned above, our conclusions are robust to some extent.

5. Conclusions

In this paper, we first propose a multi-period investment-reinsurance optimization problem with consideration of the joint interests of the insurer and the reinsurer under the generalized mean-variance framework. The proposed model is constructed by maximizing the weighted sum of the insurer's and the reinsurer's mean-variance objectives. We use a game method to derive the time-consistent investment-reinsurance strategies, and also obtain the exact expression of the time-consistent reinsurance strategy under two special premium principles. Finally, we provide some numerical simulations to present the evolution process of the above time-consistent strategies, so as to show the impact of the intertemporal restrictions on the time-consistent strategies. Some interesting findings are concluded as follows: (a) The intertemporal restrictions will urge the insurer and the reinsurer to shrink the positions invested in the risky asset. (b) The role of the intertemporal restrictions on the time-consistent reinsurance strategy depends on who is the leader in the proposed model. When the insurer is the leader, the intertemporal restrictions will reduce the retention level of a claim, while for the case that the reinsurer is the leader, only when the impact of the reinsurer's leading role is higher than that of the intertemporal restrictions will the intertemporal restrictions reduce the retention level of a claim.

These interesting findings also provide some useful advice for insurers and reinsurers on the actual investment-reinsurance issue. In the framework of the proposed model, insurers and reinsurers can adjust the intertemporal restriction conditions when the securities market is in different states. For example, when the securities market is in a bull market, insurers and reinsurers can appropriately reduce the intertemporal restrictions to obtain a higher terminal return, while for the case that the securities market is in a bear market, they can increase the number of the intertemporal restrictions, so as to prevent the bankruptcies that occur in the investment-reinsurance process. In addition, the proposed reinsurance strategy takes into account the common interests of the insurer and the reinsurer as well as the impact of intertemporal restrictions on the reinsurance strategy. Similarly, the insurer and the reinsurer can adjust the reinsurance strategy dynamically according to their own risk appetites and the market environment, which also provides a new idea for the actual formulation of the reinsurance contract.

While the proposed model can cover many classical ones, there are also some limits in our study. First, we assume that the risk aversion coefficient does not depend on decision-makers' current wealth level; however, some researchers point out that the greater the wealth of decision-makers, the less risk-averse they are likely to be. Obviously, the optimal investment-reinsurance problem with the above state-dependent risk aversion can be regarded as one of research directions. Second, this paper assumes that the returns of risky assets are statistically independent among different time periods. However, some empirical studies show that the returns of risky assets always exhibit a certain degree of dependency among different time periods. Therefore, our work can be further investigated under the weak assumption that the returns of risky assets have the serially correlated structure.

Author Contributions: Supervision, Z.Z.; Writing—original draft, H.X.; Writing—review and editing, T.R. and Y.B.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. The Proof of Proposition 1

When $k = T-1$, according the definition of time-consistent strategy, we have:

$$V_{T-1}(w_{T-1}^1, w_{T-1}^2) = \max_{\pi_{T-1}} \left\{ \begin{array}{l} \alpha[\xi_T^1 E_{T-1}(w_T^1) - \xi_T^1 \eta_T^1 Var_{T-1}(w_T^1)] \\ +(1-\alpha)[\xi_T^2 E_{T-1}(w_T^2) - \xi_T^2 \eta_T^2 Var_{T-1}(w_T^2)] \end{array} \right\}. \tag{A1}$$

This indicates that Proposition 1 holds for $k = T-1$. Then for $k = 0, 1, ..., T-2$, the function $J_k(w_k^1, w_k^2, \pi)$ can be expressed as:

$$\begin{array}{rcl} J_k(w_k^1, w_k^2, \pi) & = & \alpha \sum_{t=k+1}^{T} \xi_t^1 [E_k(w_t^1) - \eta_t^1 Var_k(w_t^1)] + (1-\alpha) \sum_{t=k+1}^{T} \xi_t^2 [E_k(w_t^2) - \eta_t^2 Var_k(w_t^2)] \\ & = & \alpha \sum_{t=k+2}^{T} \xi_t^1 [E_k(w_t^1) - \eta_t^1 Var_k(w_t^1)] + (1-\alpha) \sum_{t=k+2}^{T} \xi_t^2 [E_k(w_t^2) - \eta_t^2 Var_k(w_t^2)] \\ & & +\alpha \xi_{k+1}^1 [E_k(w_{k+1}^1) - \eta_{k+1}^1 Var_k(w_{k+1}^1)] \\ & & +(1-\alpha) \xi_{k+1}^2 [E_k(w_{k+1}^2) - \eta_{k+1}^2 Var_k(w_{k+1}^2)]. \end{array} \tag{A2}$$

By using the law of iterated expectation and the law of total variance, we have:

$$E_k(w_t^i) = E_k[E_{k+1}(w_t^i)], i = 1, 2, \tag{A3}$$

$$Var_k(w_t^i) = E_k[Var_{k+1}(w_t^i)] + Var_k[E_{k+1}(w_t^i)], i = 1, 2. \tag{A4}$$

Then, $J_k(w_k^1, w_k^2, \pi)$ can be rewritten as:

$$\begin{array}{rl} & J_k(w_k^1, w_k^2, \pi) \\ = & E_k \left\{ \alpha \sum_{t=k+2}^{T} \xi_t^1 [E_{k+1}(w_t^1) - \eta_t^1 Var_{k+1}(w_t^1)] + (1-\alpha) \sum_{t=k+2}^{T} \xi_t^2 [E_{k+1}(w_t^2) - \eta_t^2 Var_{k+1}(w_t^2)] \right\} \\ & -\alpha \sum_{t=k+2}^{T} \xi_t^1 \eta_t^1 Var_k[E_{k+1}(w_t^1)] - (1-\alpha) \sum_{t=k+2}^{T} \xi_t^2 \eta_t^2 Var_k[E_{k+1}(w_t^2)] \\ & +\alpha \xi_{k+1}^1 [E_k(w_{k+1}^1) - \eta_{k+1}^1 Var_k(w_{k+1}^1)] + (1-\alpha) \xi_{k+1}^2 [E_k(w_{k+1}^2) - \eta_{k+1}^2 Var_k(w_{k+1}^2)], \\ = & E_k[J_{k+1}(w_{k+1}^1, w_{k+1}^2, \pi)] - \alpha \sum_{t=k+2}^{T} \xi_t^1 \eta_t^1 Var_k[E_{k+1}(w_t^1)] - (1-\alpha) \sum_{t=k+2}^{T} \xi_t^2 \eta_t^2 Var_k[E_{k+1}(w_t^2)] \\ & +\alpha \xi_{k+1}^1 [E_k(w_{k+1}^1) - \eta_{k+1}^1 Var_k(w_{k+1}^1)] + (1-\alpha) \xi_{k+1}^2 [E_k(w_{k+1}^2) - \eta_{k+1}^2 Var_k(w_{k+1}^2)]. \end{array} \tag{A5}$$

Let $f_{k,t}(w_k^1) = E_k(w_t^1)|_{\hat{\pi}(k)} = E_k[f_{k+1,t}(w_{k+1}^1)]$ and $g_{k,t}(w_k^2) = E_k(w_t^2)|_{\hat{\pi}(k)} = E_k[g_{k+1,t}(w_{k+1}^2)]$. Additionally, due to the fact that $V_k(w_k^1, w_k^2) = \max_{\pi_k} J_k(w_k^1, w_k^2, \pi(k)) = J_k(w_k^1, w_k^2, \hat{\pi}(k))$, we can derive the following iterative formula:

$$\begin{array}{rl} & V_k(w_k^1, w_k^2) \\ = & \max_{\pi_k} \left\{ \begin{array}{l} E_k[V_{k+1}(w_{k+1}^1, w_{k+1}^2)] - \alpha \sum\limits_{t=k+2}^{T} \xi_t^1 \eta_t^1 Var_k[f_{k+1,t}(w_{k+1}^1)] \\ -(1-\alpha) \sum\limits_{t=k+2}^{T} \xi_t^2 \eta_t^2 Var_k[g_{k+1,t}(w_{k+1}^2)] \\ +\alpha \xi_{k+1}^1 [E_k(w_{k+1}^1) - \eta_{k+1}^1 Var_k(w_{k+1}^1)] \\ +(1-\alpha) \xi_{k+1}^2 [E_k(w_{k+1}^2) - \eta_{k+1}^2 Var_k(w_{k+1}^2)] \end{array} \right\}, \\ & k = 0, 1, ..., T-2. \end{array} \tag{A6}$$

Therefore, we complete the proof of Proposition 1.

Appendix B. The proof of Theorem 1

When $t = T-1$, we have:

$$\begin{array}{rl} & V_{T-1}(w_{T-1}^1, w_{T-1}^2) \\ = & \max_{\pi_{T-1}} \left\{ \begin{array}{l} \alpha[\xi_T^1 E_{T-1}(w_T^1) - \xi_T^1 \eta_T^1 Var_{T-1}(w_T^1)] \\ +(1-\alpha)[\xi_T^2 E_{T-1}(w_T^2) - \xi_T^2 \eta_T^2 Var_{T-1}(w_T^2)] \end{array} \right\} \\ = & \max_{\pi_{T-1}} \left\{ \begin{array}{l} \alpha \xi_T^1 [s_{T-1} w_{T-1}^1 + \mu_{T-1}^1 u_{T-1}^1 + c_{T-1} - \delta_{T-1}(q_{T-1}) - q_{T-1} \tilde{\mu}_{T-1}] \\ -\alpha \xi_T^1 \eta_T^1 [\sigma_{T-1}^1 (u_{T-1}^1)^2 + \tilde{\sigma}_{T-1}(q_{T-1})^2] \\ +(1-\alpha) \xi_T^2 [s_{T-1} w_{T-1}^2 + \mu_{T-1}^2 u_{T-1}^2 + \delta_{T-1}(q_{T-1}) - (1-q_{T-1}) \tilde{\mu}_{T-1}] \\ -(1-\alpha) \xi_T^2 \eta_T^2 [\sigma_{T-1}^2 (u_{T-1}^2)^2 + \tilde{\sigma}_{T-1}(1-q_{T-1})^2] \end{array} \right\}. \end{array}$$

Then, we have:

$$\hat{u}_{T-1}^{1}=\frac{\mu_{T-1}^{1}}{2\eta_{T}^{1}\sigma_{T-1}^{1}}, \tag{A7}$$

$$\hat{u}_{T-1}^{2}=\frac{\mu_{T-1}^{2}}{2\eta_{T}^{2}\sigma_{T-1}^{2}}, \tag{A8}$$

$$\hat{q}_{T-1}=\arg\max_{0\le q_{T-1}\le 1}\left\{\begin{array}{l}(\xi_T^2-\alpha\xi_T^1-\alpha\xi_T^2)\delta_{T-1}(q_{T-1})\\ -\tilde{\mu}_{T-1}[\alpha\xi_T^1 q_{T-1}+(1-\alpha)\xi_T^2(1-q_{T-1})]\\ -[\alpha\xi_T^1\eta_T^1(q_{T-1})^2+(1-\alpha)\xi_T^2\eta_T^2\tilde{\sigma}_{T-1}(1-q_{T-1})^2]\end{array}\right\}. \tag{A9}$$

Then, we have:

$$\begin{aligned}&V_{T-1}(w_{T-1}^1,w_{T-1}^2)\\ =&\left\{\begin{array}{l}\alpha\xi_T^1 s_{T-1}w_{T-1}^1+(1-\alpha)\xi_T^2 s_{T-1}w_{T-1}^2+\frac{\alpha(\mu_{T-1}^1)^2}{4\eta_T^1\sigma_{T-1}^1}+\frac{(1-\alpha)(\mu_{T-1}^2)^2}{4\eta_T^2\sigma_{T-1}^2}+\alpha\xi_T^1 c_{T-1}\\ +(\xi_T^2-\alpha\xi_T^1-\alpha\xi_T^2)\delta_{T-1}(\hat{q}_{T-1})-\tilde{\mu}_{T-1}[\alpha\xi_T^1\hat{q}_{T-1}+(1-\alpha)\xi_T^2(1-\hat{q}_{T-1})]\\ -[\alpha\xi_T^1\eta_T^1(\hat{q}_{T-1})^2\tilde{\sigma}_{T-1}+(1-\alpha)\xi_T^2\eta_T^2(1-\hat{q}_{T-1})^2]\tilde{\sigma}_{T-1}\end{array}\right\}\\ =&\alpha\xi_T^1 s_{T-1}w_{T-1}^1+(1-\alpha)\xi_T^2 s_{T-1}w_{T-1}^2+\kappa_{T-1},\end{aligned} \tag{A10}$$

where:

$$\kappa_{T-1}=\left\{\begin{array}{l}\frac{\alpha(\mu_{T-1}^1)^2}{4\eta_T^1\sigma_{T-1}^1}+\frac{(1-\alpha)(\mu_{T-1}^2)^2}{4\eta_T^2\sigma_{T-1}^2}+\alpha\xi_T^1 c_{T-1}+(\xi_T^2-\alpha\xi_T^1-\alpha\xi_T^2)\delta_{T-1}(\hat{q}_{T-1})\\ -\tilde{\mu}_{T-1}[\alpha\xi_T^1\hat{q}_{T-1}+(1-\alpha)\xi_T^2(1-\hat{q}_{T-1})]\\ -[\alpha\xi_T^1\eta_T^1(\hat{q}_{T-1})^2+(1-\alpha)\xi_T^2\eta_T^2(1-\hat{q}_{T-1})^2]\tilde{\sigma}_{T-1}\end{array}\right\}. \tag{A11}$$

Assume that Theorem 1 holds for $t=j+1,j+2,...,T-1$, then when $t=j$ we have:

$$\begin{aligned}&V_j(w_j^1,w_j^2)\\ =&\max_{\pi_j}\left\{\begin{array}{l}\alpha\left(\sum\limits_{m=j+2}^{T}\xi_m^1\prod\limits_{i=j+1}^{m-1}s_i\right)E_j(w_{j+1}^1)+(1-\alpha)\left(\sum\limits_{m=j+2}^{T}\xi_m^2\prod\limits_{i=j+1}^{m-1}s_i\right)E_j(w_{j+1}^2)+\kappa_{j+1}\\ -\alpha\sum\limits_{m=j+2}^{T}\xi_m^1\eta_m^1 Var_j[f_{j+1,m}(w_{j+1}^1)]-(1-\alpha)\sum\limits_{m=j+2}^{T}\xi_m^2\eta_m^2 Var_j[g_{j+1,m}(w_{j+1}^2)]\\ +\alpha\xi_{j+1}^1[E_j(w_{j+1}^1)-\eta_{j+1}^1 Var_j(w_{j+1}^1)]+(1-\alpha)\xi_{j+1}^2[E_j(w_{j+1}^2)-\eta_{j+1}^2 Var_j(w_{j+1}^2)]\end{array}\right\}\\ =&\max_{\pi_j}\left\{\begin{array}{l}\alpha\left(\sum\limits_{m=j+1}^{T}\xi_m^1\prod\limits_{i=j+1}^{m-1}s_i\right)[s_jw_j^1+\mu_j^1u_j^1+c_j-\delta_j(q_j)-q_j\tilde{\mu}_j]\\ -\alpha\left(\sum\limits_{m=j+1}^{T}\xi_m^1\eta_m^1\prod\limits_{i=j+1}^{m-1}(s_i)^2\right)[\sigma_j^1(u_j^1)^2+\tilde{\sigma}_j(q_j)^2]+\kappa_{j+1}\\ +(1-\alpha)\left(\sum\limits_{m=j+1}^{T}\xi_m^2\prod\limits_{i=j+1}^{m-1}s_i\right)[s_jw_j^2+\mu_j^2u_j^2+\delta_j(q_j)-(1-q_j)\tilde{\mu}_j]\\ -(1-\alpha)\left(\sum\limits_{m=j+1}^{T}\xi_m^2\eta_m^2\prod\limits_{i=j+1}^{m-1}(s_i)^2\right)[\sigma_j^2(u_j^2)^2+\tilde{\sigma}_j(1-q_j)^2]\end{array}\right\}.\end{aligned} \tag{A12}$$

Then, we have:

$$\hat{u}_j^1=\frac{\left(\sum\limits_{m=j+1}^{T}\xi_m^1\prod\limits_{i=j+1}^{m-1}s_i\right)\mu_j^1}{2\left(\sum\limits_{m=j+1}^{T}\xi_m^1\eta_m^1\prod\limits_{i=j+1}^{m-1}(s_i)^2\right)\sigma_j^1}, \tag{A13}$$

$$\hat{u}_j^2 = \frac{\left(\sum_{m=j+1}^{T} \xi_m^2 \prod_{i=j+1}^{m-1} s_i\right) \mu_j^2}{2\left(\sum_{m=j+1}^{T} \xi_m^2 \eta_m^2 \prod_{i=j+1}^{m-1} (s_i)^2\right) \sigma_j^2}, \tag{A14}$$

$$\hat{q}_j = \arg\max_{0\le q_j\le 1} \left\{ \begin{array}{l} \alpha\left(\sum_{m=j+1}^{T} \xi_m^1 \prod_{i=j+1}^{m-1} s_i\right)[-\delta_j(q_j) - q_j\tilde{\mu}_j] \\ -\alpha\left(\sum_{m=j+1}^{T} \xi_m^1 \eta_m^1 \prod_{i=j+1}^{m-1} (s_i)^2\right)\tilde{\sigma}_j(q_j)^2 \\ +(1-\alpha)\left(\sum_{m=j+1}^{T} \xi_m^2 \prod_{i=j+1}^{m-1} s_i\right)[\delta_j(q_j) - (1-q_j)\tilde{\mu}_j] \\ -(1-\alpha)\left(\sum_{m=j+1}^{T} \xi_m^2 \eta_m^2 \prod_{i=j+1}^{m-1} (s_i)^2\right)\tilde{\sigma}_j(1-q_j)^2 \end{array} \right\}. \tag{A15}$$

Then, we have:

$$\begin{aligned} &V_j(w_j^1, w_j^2) \\ = &\left\{ \begin{array}{l} \alpha\left(\sum_{m=j+1}^{T} \xi_m^1 \prod_{i=j}^{m-1} s_i\right) w_j^1 + (1-\alpha)\left(\sum_{m=j+1}^{T} \xi_m^2 \prod_{i=j}^{m-1} s_i\right) w_j^2 \\ + \frac{\alpha\left(\sum_{m=j+1}^{T} \xi_m^1 \prod_{i=j+1}^{m-1} s_i\right)(\mu_j^1)^2}{4\left(\sum_{m=j+1}^{T} \xi_m^1 \eta_m^1 \prod_{i=j+1}^{m-1} (s_i)^2\right)\sigma_j^1} + \frac{(1-\alpha)\left(\sum_{m=j+1}^{T} \xi_m^2 \prod_{i=j+1}^{m-1} s_i\right)\mu_j^2}{4\left(\sum_{m=j+1}^{T} \xi_m^2 \eta_m^2 \prod_{i=j+1}^{m-1} (s_i)^2\right)\sigma_j^2} \\ +\alpha\left(\sum_{m=j+1}^{T} \xi_m^1 \prod_{i=j+1}^{m-1} s_i\right)[c_j - \delta_j(\hat{q}_j) - \hat{q}_j\tilde{\mu}_j] \\ -\alpha\left(\sum_{m=j+1}^{T} \xi_m^1 \eta_m^1 \prod_{i=j+1}^{m-1} (s_i)^2\right)\tilde{\sigma}_j(\hat{q}_j)^2 + \kappa_{j+1} \\ +(1-\alpha)\left(\sum_{m=j+1}^{T} \xi_m^2 \prod_{i=j+1}^{m-1} s_i\right)[\delta_j(\hat{q}_j) - (1-\hat{q}_j)\tilde{\mu}_j] \\ -(1-\alpha)\left(\sum_{m=j+1}^{T} \xi_m^2 \eta_m^2 \prod_{i=j+1}^{m-1} (s_i)^2\right)\tilde{\sigma}_j(1-\hat{q}_j)^2 \end{array} \right\} \\ = &\ \alpha\left(\sum_{m=j+1}^{T} \xi_m^1 \prod_{i=j}^{m-1} s_i\right) E_j(w_j^1) + (1-\alpha)\left(\sum_{m=j+1}^{T} \xi_m^2 \prod_{i=j}^{m-1} s_i\right) E(w_j^2) + \kappa_j, \end{aligned} \tag{A16}$$

$$\begin{aligned} f_{j,\tau}(w_j^1) &= \prod_{i=j}^{\tau-1} s_i w_j^1 + \frac{\left(\sum_{m=j+1}^{T} \xi_m^1 \prod_{i=j+1}^{m-1} s_i\right)(\mu_j^1)^2}{2\left(\sum_{m=j+1}^{T} \xi_m^1 \eta_m^1 \prod_{i=j+1}^{m-1} (s_i)^2\right)\sigma_j^1} + \prod_{i=j+1}^{\tau-1} s_i[c_j - \delta_j(\hat{q}_j) - \hat{q}_j\tilde{\mu}_j] + \gamma_{j+1,\tau} \\ &= \prod_{i=j}^{\tau-1} s_i w_j^1 + \gamma_{j,\tau}, \end{aligned} \tag{A17}$$

$$\begin{aligned} g_{j,\tau}(w_j^2) &= \prod_{i=j}^{\tau-1} s_i w_j^2 + \frac{\left(\sum_{m=j+1}^{T} \xi_m^2 \prod_{i=j+1}^{m-1} s_i\right)(\mu_j^2)^2}{2\left(\sum_{m=j+1}^{T} \xi_m^2 \eta_m^2 \prod_{i=j+1}^{m-1} (s_i)^2\right)\sigma_j^2} + \prod_{i=j+1}^{\tau-1} s_i[\delta_j(\hat{q}_j) - (1-\hat{q}_j)\tilde{\mu}_j] + \rho_{j+1,\tau} \\ &= \prod_{i=j}^{\tau-1} s_i w_j^2 + \rho_{j,\tau}. \end{aligned} \tag{A18}$$

Then, we have:

$$\begin{cases} \kappa_j = & \kappa_{j+1} + \frac{\alpha\left(\sum\limits_{m=j+1}^{T}\xi_m^1\prod\limits_{i=j+1}^{m-1}s_i\right)(\mu_j^1)^2}{4\left(\sum\limits_{m=j+1}^{T}\xi_m^1\eta_m^1\prod\limits_{i=j+1}^{m-1}(s_i)^2\right)\sigma_j^1} + \frac{(1-\alpha)\left(\sum\limits_{m=j+1}^{T}\xi_m^2\prod\limits_{i=j+1}^{m-1}s_i\right)(\mu_j^2)^2}{4\left(\sum\limits_{m=j+1}^{T}\xi_m^2\eta_m^2\prod\limits_{i=j+1}^{m-1}(s_i)^2\right)\sigma_j^2} \\ & +\alpha\left(\sum\limits_{m=j+1}^{T}\xi_m^1\prod\limits_{i=j+1}^{m-1}s_i\right)[c_j-\delta_j(\hat{q}_j)-\hat{q}_j\tilde{\mu}_j] - \alpha\left(\sum\limits_{m=j+1}^{T}\xi_m^1\eta_m^1\prod\limits_{i=j+1}^{m-1}(s_i)^2\right)\tilde{\sigma}_j(\hat{q}_j)^2 \\ & +(1-\alpha)\left(\sum\limits_{m=j+1}^{T}\xi_m^2\prod\limits_{i=j+1}^{m-1}s_i\right)[\delta_j(\hat{q}_j)-(1-\hat{q}_j)\tilde{\mu}_j] \\ & -(1-\alpha)\left(\sum\limits_{m=j+1}^{T}\xi_m^2\eta_m^2\prod\limits_{i=j+1}^{m-1}(s_i)^2\right)\tilde{\sigma}_j(1-\hat{q}_j)^2, \\ \gamma_{j,\tau} = & \gamma_{j+1,\tau} + \frac{\left(\sum\limits_{m=j+1}^{T}\xi_m^1\prod\limits_{i=j+1}^{m-1}s_i\right)(\mu_j^1)^2}{2\left(\sum\limits_{m=j+1}^{T}\xi_m^1\eta_m^1\prod\limits_{i=j+1}^{m-1}(s_i)^2\right)\sigma_j^1} + \prod\limits_{i=j+1}^{\tau-1}s_i[c_j-\delta_j(\hat{q}_j)-\hat{q}_j\tilde{\mu}_j], \\ \rho_{j,\tau} = & \rho_{j+1,\tau} + \frac{\left(\sum\limits_{m=j+1}^{T}\xi_m^2\prod\limits_{i=j+1}^{m-1}s_i\right)(\mu_j^2)^2}{2\left(\sum\limits_{m=j+1}^{T}\xi_m^2\eta_m^2\prod\limits_{i=j+1}^{m-1}(s_i)^2\right)\sigma_j^2} + \prod\limits_{i=j+1}^{\tau-1}s_i[\delta_j(\hat{q}_j)-(1-\hat{q}_j)\tilde{\mu}_j]. \end{cases} \tag{A19}$$

This indicates that:

$$\begin{cases} \kappa_j = & \sum\limits_{k=j}^{T}\left[\frac{\alpha\left(\sum\limits_{m=k+1}^{T}\xi_m^1\prod\limits_{i=k+1}^{m-1}s_i\right)(\mu_k^1)^2}{4\left(\sum\limits_{m=k+1}^{T}\xi_m^1\eta_m^1\prod\limits_{i=k+1}^{m-1}(s_i)^2\right)\sigma_k^1} + \frac{(1-\alpha)\left(\sum\limits_{m=k+1}^{T}\xi_m^2\prod\limits_{i=k+1}^{m-1}s_i\right)(\mu_k^2)^2}{4\left(\sum\limits_{m=k+1}^{T}\xi_m^2\eta_m^2\prod\limits_{i=k+1}^{m-1}(s_i)^2\right)\sigma_k^2}\right] \\ & +\alpha\sum\limits_{k=j}^{T}\left[\left(\sum\limits_{m=k+1}^{T}\xi_m^1\prod\limits_{i=k+1}^{m-1}s_i\right)[c_k-\delta_k(\hat{q}_k)-\hat{q}_k\tilde{\mu}_k]\right] \\ & -\alpha\sum\limits_{k=j}^{T}\left[\left(\sum\limits_{m=k+1}^{T}\xi_m^1\eta_m^1\prod\limits_{i=k+1}^{m-1}(s_i)^2\right)\tilde{\sigma}_k(\hat{q}_k)^2\right] \\ & +(1-\alpha)\sum\limits_{k=j}^{T}\left[\left(\sum\limits_{m=k+1}^{T}\xi_m^2\prod\limits_{i=k+1}^{m-1}s_i\right)[\delta_k(\hat{q}_k)-(1-\hat{q}_k)\tilde{\mu}_k]\right] \\ & -(1-\alpha)\sum\limits_{k=j}^{T}\left[\left(\sum\limits_{m=k+1}^{T}\xi_m^2\eta_m^2\prod\limits_{i=k+1}^{m-1}(s_i)^2\right)\tilde{\sigma}_k(1-\hat{q}_k)^2\right], \\ \gamma_{j,\tau} = & \sum\limits_{k=j}^{\tau}\left[\frac{\left(\sum\limits_{m=k+1}^{T}\xi_m^1\prod\limits_{i=k+1}^{m-1}s_i\right)(\mu_k^1)^2}{2\left(\sum\limits_{m=k+1}^{T}\xi_m^1\eta_m^1\prod\limits_{i=k+1}^{m-1}(s_i)^2\right)\sigma_k^1}\right] + \sum\limits_{k=j}^{\tau}\left[\prod\limits_{i=k+1}^{\tau-1}s_i[c_k-\delta_k(\hat{q}_k)-\hat{q}_k\tilde{\sigma}_k]\right], \\ \rho_{j,\tau} = & +\sum\limits_{k=j}^{\tau}\left[\frac{\left(\sum\limits_{m=j+1}^{T}\xi_m^2\prod\limits_{i=k+1}^{m-1}s_i\right)(\mu_k^2)^2}{2\left(\sum\limits_{m=k+1}^{T}\xi_m^2\eta_m^2\prod\limits_{i=k+1}^{m-1}(s_i)^2\right)\sigma_k^2}\right] + \sum\limits_{k=j}^{\tau}\left[\prod\limits_{i=k+1}^{\tau-1}s_i[\delta_k(\hat{q}_k)-(1-\hat{q}_k)\tilde{\mu}_k]\right]. \end{cases} \tag{A20}$$

From the above proof, we can conclude that Theorem 1 holds for all $t = 0, 1, .., T-1$.

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