mathematics

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Existence Results for Block Matrix Operator of Fractional Orders in Banach Algebras

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Received: 14 May 2019; Accepted: 15 August 2019; Published: 17 September 2019

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Abstract: This paper is concerned with proving the existence of solutions for a coupled system of quadratic integral equations of fractional order in Banach algebras. This result is a direct application of a fixed point theorem of Banach algebras. Some particular cases, examples and remarks are illustrated. Finally, the stability of solutions for that coupled system are studied.

Keywords: fractional order operators; quadratic integral equations; coupled system; 2×2 block operator matrix

MSC: Primary 26A33; Secondary 45D05; 60G22; 33E30

1. Introduction and Preliminaries

Operators which have an operator matrix representation occur in various fields such as system theory, quantum mechanics, hydrodynamics and magnetohydro-dynamics (see [1–3]).

According to their origin, they may have rather different structure, and their study may require quite different approaches.

Let $\mathbb{A}$ be an operator which has the form

$$\mathbb{A} = \begin{pmatrix} T_1 & T_2.T_2' \\ T_3 & T_4 \end{pmatrix} \tag{1}$$

where $T_1,\ T_2,\ T_2',\ T_3,\ T_4$ are nonlinear operators defined on Banach algebras. This kind of operators is studied by many researchers [4–6].

Amar and et al. [7] introduced and studied a coupled system of differential equations under boundary conditions of Rotenberg's model type, the last one arising in growing cell populations. The entries of block operator matrix associated to this system are nonlinear and act on the Banach space.

Kaddachi and et al. [4] concentrated on answering the question: Under which conditions on its entries does the 2×2 operator matrix (Equation (1)) acting on a product of Banach algebras has a fixed point? In [4], some fixed point theorems of a 2×2 block operator matrix defined on nonempty bounded closed convex subsets of Banach algebras are studied, where the entries are nonlinear operators. Furthermore, the obtained results are applied to a coupled system of nonlinear equations.

Let $X = \mathbb{C}(I, \mathbb{R})$, $I = [0, b]$ and $\alpha, \beta > 0$. In this work, the following coupled system of fractional order

$$\begin{aligned} v(t) &= f_1(t, v(t)) + g_1(t, w(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u_1(s, w(s))\, ds, \ t \in I, \\ w(t) &= f_2(t, w(t)) + g_2(t, v(t)) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} u_2(s, v(s))\, ds, \ t \in I. \end{aligned} \tag{2}$$

is studied in Banach algebras; some particular cases are given; and some examples and remarks are illustrated. Finally, the stability of solutions for the coupled system in Equation (2) is studied.

The solution of Equation (2) may be defined by a vector function $\begin{pmatrix} v \\ w \end{pmatrix} \in X \times X$ that satisfies (2).

Now, we introduce the following definitions of fractional operators.

Definition 1 ([8]). *The Riemann–Liouville fractional integral of order $\beta > 0$ of the function $f : [a, b] \to \mathbb{R}$ is given by*

$$I_a^\beta f(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds, \ \ t > a \tag{3}$$

and when $a = 0$, we have $I^\beta f(t) = I_0^\beta f(t)$, $t > 0$.

Definition 2 ([8]). *The Riemann–Liouville fractional derivative of order $\alpha \in (0, 1)$ of a function f is defined as*

$${}_R D_a^\alpha f(t) = \frac{d}{dt} \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s)\, ds, \ \ t \in [a, b]$$

or

$${}_R D_a^\alpha f(t) = \frac{d}{dt} I_a^{1-\alpha} f(t), \ t \in [a, b].$$

2. Existence Theorem

Coupled systems of integral and differential equations are studied in many papers [9–13].

Especially, the investigation for coupled systems of fractional differential equations appears in many studies (e.g., [9,11,14–17]).

Assume that

(i) $u_j : I \times \mathbb{R} \to \mathbb{R}$, $j = 1, 2$ satisfies the Carathéodory condition and

$$|u_j(s, v)| \le m_j(s) \in L^1[I] \quad \forall\, (s, v) \in I \times \mathbb{R},$$

$k_j = \max_{s \in I} I^{\gamma_j} m_j(s)$ for any $\gamma_1 \le \alpha$ and $\gamma_2 \le \beta$.

(ii) f_j, $g_j : I \times \mathbb{R} \to \mathbb{R}$ are continuous and $M_j = \max_{(s,v) \in I \times \mathbb{R}} |f_j(s, v)|$, $j = 1, 2$ $N_j = \max_{(s,v) \in I \times \mathbb{R}} |g_j(s, v)|$, $j = 1, 2$ respectively.

(iii) There exist constants l_j and h_j, which satisfy

$$|f_j(s, v) - f_j(s, w)| \le l_j\, |v - w|, \ \ j = 1, 2$$

and

$$|g_j(s, v) - g_j(s, w)| \le h_j\, |v - w|, \ \ j = 1, 2$$

$\forall\, s \in I$ and $v,\ w \in \mathbb{R}$.

Theorem 1. *Let Assumptions (i)–(iii) be satisfied. Moreover, if* $h_1\, k_1\, h_2\, k_2 < (1-l_1)(1-l_2)\Gamma(\alpha-\gamma_1+1)\Gamma(\beta-\gamma_2+1)$, *then the exists at least one solution for Equation (2) in* $X \times X$.

Proof. Consider the operators $T_1,\ T_2,\ T_3,\ T_4$ and T_2' on X defined by:

$$\begin{cases} (T_1 v)(t) = f_1(t, v(t)) \\ (T_2 w)(t) = g_1(t, w(t)) \\ (T_3 w)(t) = g_2(t, w(t))\ \int_0^t \frac{(t\ -\ s)^{\beta-1}}{\Gamma(\beta)}\ u_2(s, w(s))\, ds \\ (T_4 v)(t) = f_2(t, v(t)) \\ (T_2' w)(t) = \int_0^t \frac{(t\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\ u_1(s, w(s))\, ds. \end{cases} \tag{4}$$

The coupled system in Equation (2) may have the form:

$$\begin{cases} v(t) = T_1 v(t) + T_2 w(t).T_2' w(t) \\ w(t) = T_4 w(t) + T_3 v(t), \end{cases} \tag{5}$$

and

$$\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} T_1 & T_2.T_2' \\ T_3 & T_4 \end{pmatrix} . \begin{pmatrix} v \\ w \end{pmatrix}. \tag{6}$$

Define

$$S = \{v \in X,\ ||v|| \leq M_1 + \frac{N_1\, k_1\, b^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)}\ \}$$

$$S' = \{w \in X,\ ||w|| \leq M_2 + \frac{N_2\, k_2\, b^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)}\ \}.$$

For, let $v_1,\ v_2 \in S$. Thus,

$$||T_1 v_1(t) - T_1 v_2(t)|| \leq l_1 ||v_1 - v_2|| \tag{7}$$

and

$$||T_2 v_1(t) - T_2 v_2(t)|| \leq h_1 ||v_1 - v_2||. \tag{8}$$

In addition, set

$$T_3 v_1(t) = G v_1(t).U v_1(t) = (G.U) v_1(t) \tag{9}$$

where $G v_1(t) = g_2(t, v_1(t))$ and $U v_1(t) = I^{\beta} u_2(t, v_1(t))$

$$\begin{aligned} ||T_3 v_1(t) - T_3 v_2(t)|| &= ||(G.U) v_1(t) - (G.U) v_2(t)|| \\ &\leq ||U v_1(t)||.||G v_1(t) - G v_2(t)|| \\ &\leq \frac{k_2 h_2\, b^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)}.||v_1 - v_2||. \end{aligned} \tag{10}$$

Furthermore,

$$\begin{aligned} |T_3 w(t)| &\leq |g_2(t, w(t))| \int_0^t \frac{(t\ -\ s)^{\beta-1}}{\Gamma(\beta)}\ |u_2(s, w(s))|\, ds \\ &\leq \frac{N_2\, k_2\, b^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)}, \end{aligned} \tag{11}$$

for each $t_1,\ t_2 \in I$ and $t_1 < t_2$, we get

$$\begin{aligned}
|(T_3 w)(t_2) \ - \ (T_3 w)(t_1)| \quad &= \quad |g_2(t_2, w(t_2))\ I^{\beta}\ u_2(t_2, w(t_2)) \ - \ g_2(t_1, w(t_1))\ I^{\beta}\ u_2(t_1, w(t_1)) \\
&+ \quad g_2(t_1, w(t_1))\ I^{\beta}\ u_2(t_2, w(t_2)) \ - \ g_2(t_1, w(t_1))\ I^{\beta}\ u_2(t_2, w((t_2))| \\
&\leq \quad |g_2(t_2, w(t_2)) \ - \ g_2(t_1, w(t_1))|\ I^{\beta}\ |u_2(t_2, w(t_2))| \\
&+ \quad |g(t_1, w(t_1))|\ |\ I^{\beta}\ u_2(t_2, w(t_2)) \ - \ I^{\beta}\ u_2(t_1, w(t_1))\ |,
\end{aligned} \tag{12}$$

but

$$|I^{\beta}\ u_2(t_2, w(t_2)) \ - \ I^{\beta}\ u_2(t_1, w(t_1))| \ \leq \ \int_{t_1}^{t_2} \frac{(t_2 \ - \ s)^{\beta-1}}{\Gamma(\beta)}\ |u_2(s, w(s))|\ ds. \tag{13}$$

Then,

$$|I^{\beta}\ u_2(t_2, w(t_2)) \ - \ I^{\beta}\ u_2(t_1, w(t_1))| \leq \ k_2\ \frac{(t_2 - t_1)^{\beta-\gamma_2}}{\Gamma(\beta - \gamma_2 + 1)}. \tag{14}$$

Then, we get

$$|\ (T_3 w)(t_2) \ - \ (T_3 w)(t_1)\ | \ \leq \ \frac{k_2\ h_2\ b^{\beta-\gamma_2}}{\Gamma(\beta - \gamma_2 + 1)}\ |w(t_2) - w(t_1)| \ + \ \frac{k_2\ N_2}{\Gamma(\beta - \gamma_2 + 1)}\ (t_2 - \ \ t_1)^{\beta-\gamma_2} \tag{15}$$

Then, $\overline{US'}$ is relatively compact.
We prove that $T_3(S) \subseteq (I - T_4)(S')$, for $v_1 \in S$.
Now, we can introduce a function $\phi_{v_1} : X \to X$ by

$$w \to T_3 v_1 + T_4 v_2, \tag{16}$$

then the function ϕ_{v_1} is a contraction with a constant $l_2 + \frac{h_2\ k_2\ b^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)}$. Then, there exists a unique point $w \in X$ where $T_3 v + T_4 w = w$ implies $T_3 v = (I - T_4)w$. Thus,

$$T_3(s) \subseteq (I - T_4)X. \tag{17}$$

For $w \in X,\ ,\ \exists\, s^* \in I$ such that

$$\begin{aligned}
||w||_{\infty} = |w(s^*)| \quad &= \quad |T_3 v(s^*) + T_4 w(s^*)| \\
&\leq \quad |g_2(s^*, v(s^*)) I^{\beta} u_2(s^*, v(s^*)) + f_2(s^*, w(s^*))| \\
&\leq \quad \frac{N_2\ k_2\ b^{\beta-\gamma_2}}{\Gamma(\beta - \gamma_2 + 1)} + M_2.
\end{aligned} \tag{18}$$

Then, $T_3(S) \subseteq (I - T_4)(S')$.
For $v \in S'$, then

$$\begin{aligned}
|T_2' v(t_n) - T_2' v(t)| \quad &= \quad |I^{\alpha} u_1(t_n, v(t_n)) - I^{\alpha} u_1(t, v(t))| \\
&\leq \quad \frac{k_1}{\Gamma(\alpha - \gamma_1 + 1)} |t_n - t|,
\end{aligned} \tag{19}$$

since $t_n \to t$ and u_1 is continuous in the second argument, then by Lebesgue Dominated Convergence Theorem, we have

$$u_1(t_n, v(t_n)) \to u_1(t, v(t)) \ \text{ in } \mathbb{R} \Rightarrow T_2' v(t_n) \to T_2' v(t) \ \text{ in } \mathbb{R},$$

thus $T_2' v \in \mathbb{C}(I,\ \mathbb{R})$.

Defining $T' = T_2'(1 - T_4)^{-1}T_3$, , Assumption (ii) implies that

$$\begin{aligned} M &= ||T'(S)|| = \sup_{w \in S} |T'(w)| \\ &\leq \sup_{t \in I} |\int_0^t \frac{(t\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\ u_1(s, w(s))\ ds| \\ &\leq \frac{k_1\ b^{\alpha-\gamma_1}}{\Gamma(\alpha - \gamma_1 + 1)}, \end{aligned} \tag{20}$$

and therefore $h_1\ k_1\ h_2\ k_2 < (1 - l_1)(1 - l_2)\Gamma(\alpha - \gamma_1 + 1)\Gamma(\beta - \gamma_2 + 1)$.

Let $v_1,\ v_2 \in S$, then $\forall\ s \in I$. We obtain

$$|T_1 v_1(s) + T_2(I - T_4)^{-1}T_3 v_1(s) T_2'(I - T_4)^{-1}T_3 v_2| \quad \leq \quad M_1 + \frac{N_1 k_1\ b^{\alpha-\gamma_1}}{\Gamma(\alpha - \gamma_1 + 1)}. \tag{21}$$

This implies that

$$T_1 v_1 + T_2(I - T_4)^{-1}T_3 v_1 T_2'(I - T_4)^{-1}T_3 v_2 \in S \text{ for any } v_1,\ v_2 \in S. \tag{22}$$

Now, all conditions of Theorem 4.2 in [4] are verified and our results follows. □

Example 1. *Let $I = [0, 1]$. Consider the fractional order coupled system*

$$\begin{aligned} v(t) &= t\,sin(\frac{v(t)}{4}) + \frac{w(t)}{3 + t} I^{3/2}\ \frac{2t\ w(t)}{1 + w(t)}, \quad t\ \in\ I \\ w(t) &= \frac{t + sin(\frac{w(t)}{2})}{2 + t^2} + \frac{v(t)}{4} I^{2/3} \frac{v(t)}{1 + v(t)}, \quad t\ \in\ I. \end{aligned} \tag{23}$$

Set

$$f_1(s, v) = s\,sin(\frac{v(s)}{4}), \quad u_1(s, w) = \frac{2s\ w(s)}{1 + w(s)}, \quad g_1(s, w) \quad = \quad \frac{w(s)}{3 + s},$$

$$f_2(s, w) = \frac{s + sin(\frac{w(s)}{2})}{2 + s^2}, \quad u_2(s, v) = \frac{v(s)}{1 + v(s)}, \quad g_2(s, w) \quad = \quad \frac{w(s)}{4}.$$

Then, we easily get

- $|u_1(s, v)| \ \leq\ 2\,s = m_1(s)$ *and* $|u_2(s, v)| \ \leq\ 1 = m_2(s)$.
  Choose $\gamma_1 = \gamma_2 = 1/2$, *then we can obtain* $k_1 = \frac{8}{3\sqrt{\pi}}$ *and* $k_2 = \frac{2}{\sqrt{\pi}}$

$$\begin{aligned} |g_1(t, w_1) - g_1(t, w_2)| &\leq \frac{1}{3}\ |w_1 - w_2| \\ |g_2(t, v_1) - g_2(t, v_2)| &\leq \frac{1}{4}\ |v_1 - v_2| \end{aligned}$$

and

$$|f_i(t, v_1) - f_i(t, v_2)| \quad \leq \quad \frac{1}{2}\ |v_1 - v_2|, \ \ i = 1, 2.$$

Then, the inequality $h_1\ k_1\ h_2\ k_2 < (1 - l_1)(1 - l_2)\Gamma(\alpha - \gamma + 1)\Gamma(\beta - \gamma + 1)$ *is verified.*

3. Stability of Solutions of the Coupled System

Here, asymptotic stability on $\mathbb{R}_+$ of the solution $z = (v,\ w)$ of the coupled system in Equation (2) is studied.

Definition 3. *A pair $z_1 = (v_1,\ w_1)$ is said to be an asymptotically stable solution of Equation (2) if for any $\varepsilon > 0$ there exists $T' = T'(\varepsilon) > 0$ such that for very $t \geq T'$ and for every other solution $z_2 = (v_2,\ w_2)$ of (2),*

$$|z_1(t) - z_2(t)| \leq \varepsilon.$$

Given two solutions z_1 and z_2 of Equation (2), then we have

$$\begin{aligned}
|v_1(t) - v_2(t)| &\leq |f_1(t, v_1(t)) - f_1(t, v_2(t))| \\
&+ |g_1(t, w_1(t)) \int_0^t \frac{(t\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\, u_1(s, w_1(s))\, ds \\
&- g_1(t, w_2(t)) \int_0^t \frac{(t\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\, u_1(s, w_2(s))\, ds| \\
&\leq l_1|v_1(t) - v_2(t)| \\
&+ g_1(t, w_1(t))| \int_0^t \frac{(t\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\, |u_1(s, w_1(s)) - u_1(s, w_2(s))|\, ds \\
&+ |g_1(t, w_2(t)) - g_1(t, w_1(t))| \int_0^t \frac{(t\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\, |u_1(s, w_2(s))|\, ds \\
&\leq l_1|v_1(t) - v_2(t)| + 2|g_1(t, w_1(t))| \int_0^t \frac{(t\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\, m_1(s)\, ds \\
&+ |g_1(t, w_2(t)) - g_1(t, w_1(t))| \int_0^t \frac{(t\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\, m_1(s)\, ds \\
&\leq l_1|v_1(t) - v_2(t)| + \frac{2\, k_1\, N_1\, b^{\alpha-\gamma_1}}{\Gamma(\alpha - \gamma_1 + 1)} + \frac{h_1\, k_1\, b^{\alpha-\gamma_1}}{\Gamma(\alpha - \gamma_1 + 1)}\, |w_1 - w_2|,
\end{aligned}$$

then

$$(1 - l_1)||v_1(t) - v_2(t)|| \leq \frac{2\, k_1\, N_1\, b^{\alpha-\gamma_1}}{\Gamma(\alpha - \gamma_1 + 1)} + \frac{h_1\, k_1\, b^{\alpha-\gamma_1}}{\Gamma(\alpha - \gamma_1 + 1)}||w_1 - w_2||. \tag{24}$$

In the same fashion, we obtain

$$(1 - l_2)||w_1(t) - w_2(t)|| \leq \frac{2\, k_2\, N_2\, b^{\beta-\gamma_2}}{\Gamma(\beta - \gamma_2 + 1)} + \frac{h_2\, k_2\, b^{\beta-\gamma_2}}{\Gamma(\beta - \gamma_2 + 1)}\, ||v_1 - v_2||, \tag{25}$$

and

$$[1 - l_1 - \tfrac{h_2\, k_2\, b^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)}]\, ||v_1 - v_2|| + [1 - l_2 - \tfrac{h_1\, k_1\, b^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)}]\, ||w_1 - w_2|| \leq \tfrac{2\, k_1\, N_1\, b^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + \tfrac{2\, k_2\, N_2\, b^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)} \tag{26}$$

Let $\Lambda = \min\{1 - l_1 - \frac{h_2\, k_2\, b^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)},\ 1 - l_2 - \frac{h_1\, k_1\, b^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)}\}$, then

$$||v_1 - v_2|| + ||w_1 - w_2|| \leq \Lambda^{-1}[\frac{2\, k_1\, N_1\, b^{\alpha-\gamma_1}}{\Gamma(\alpha - \gamma_1 + 1)} + \frac{2\, k_2\, N_2\, b^{\beta-\gamma_2}}{\Gamma(\beta - \gamma_2 + 1)}]. \tag{27}$$

Since

$$z_1(t) - z_2(t) = (v_1(t) - v_2(t),\ w_1(t) - w_2(t)),$$

then

$$
\begin{aligned}
||z_1(t) - z_2(t)|| &\leq ||v_1(t) - v_2(t)|| + ||w_1(t) - w_2(t)|| \\
&\leq \Lambda^{-1}[\frac{2\, k_1\, N_1\, b^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + \frac{2\, k_2\, N_2\, b^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)}] \\
&\leq \varepsilon.
\end{aligned} \tag{28}
$$

Then, we obtain the following theorem.

Theorem 2. *Let assumptions of Theorem 1 be satisfied,*

$$
l_1\Gamma(\beta-\gamma_2+1) + h_2\, k_2\, b^{\beta-\gamma_2} < \Gamma(\beta-\gamma_2+1),\;\; l_2\Gamma(\alpha-\gamma_1+1) + h_1\, k_1\, b^{\alpha-\gamma_1} < \Gamma(\alpha-\gamma_1+1),
$$

and

$$
\Lambda^{-1}[\frac{2\, k_1\, N_1\, b^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + \frac{2\, k_2\, N_2\, b^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)}] \leq \varepsilon.
$$

Then, the solution of Equation (2) is asymptotically stable on $\mathbb{R}_+$.

4. Further Results

Consequently, we have the following results in $X \times X$.

(i) Letting $\alpha,\ \beta \to 1,$ then we have the coupled system of quadratic integral equations

$$
\begin{aligned}
v(t) &= f_1(t, v(t)) + g_1(t, w(t)) \int_0^t u_1(s, w(s))\, ds, \\
w(t) &= f_2(t, w(t)) + g_2(t, v(t)) \int_0^t u_2(s, v(s))\, ds.
\end{aligned}
$$

(ii) Letting $g_1 = g_2 = 0,$ we get the coupled system of functional equations

$$
\begin{aligned}
v(t) &= f_1(t, v(t)) \\
w(t) &= f_2(t, w(t)).
\end{aligned}
$$

(iii) Putting $f_1(t, w) = a_1(t),\ f_2(t, v) = a_2(t),$ then we have the coupled system of quadratic integral equations of fractional order

$$
\begin{aligned}
v(t) &= a_1(t) + g_1(t, w(t)) \int_0^t \frac{(t\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\, u_1(s, w(s))\, ds, \\
w(t) &= a_2(t) + g_2(t, v(t)) \int_0^t \frac{(t\ -\ s)^{\beta-1}}{\Gamma(\beta)}\, u_2(s, v(s))\, ds.
\end{aligned}
$$

(v) Letting $f_1(t, w) = a_1(t),\quad f_2(t, v) = a_2(t),$ then we get the coupled system of fractional integral equations

$$
\begin{aligned}
v(t) &= a_1(t) + \int_0^t \frac{(t\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\, u_1(s, w(s))\, ds, \\
w(t) &= a_2(t) + \int_0^t \frac{(t\ -\ s)^{\beta-1}}{\Gamma(\beta)}\, u_2(s, v(s))\, ds.
\end{aligned}
$$

System of Fractional Differential Equations

Let

$$\begin{aligned} {}_R D^{\alpha} v(t) &= u_1(t, w(t)), \ t \in J \ \text{ and } \ v(0) = 0, \ \ 0 < \alpha < 1 \\ {}_R D^{\beta} w(t) &= u_2(t, v(t)), \ t \in J \ \text{ and } \ w(0) = 0, \ \ 0 < \beta < 1 \end{aligned} \tag{29}$$

where ${}_R D^{\alpha}$ is a Riemann–Liouville fractional derivative of order $0 < \alpha < 1$.

Theorem 3. *Let assumptions of Theorem 1 be satisfied. Then, there exists at least one solution for Equation (29) in* $X \times X$.

The proof is straight forward as in [11].

By direct calculations, we can prove an existence result for the following coupled systems

$$\begin{aligned} v'(t) &= f_1(t, w(t)), \ \ v(0) = v_0, \\ w'(t) &= f_2(t, v(t)), \ \ w(0) = w_0. \end{aligned} \tag{30}$$

5. Conclusions

The theory of block operator matrices opens up a new line of attack of mathematical problems. During the past years, several papers are devoted to the investigation of linear operator matrices defined by 2×2 block operator matrices (Equation (1)).

In this paper, we prove an existence theorem of solutions for a coupled system of quadratic integral equations of fractional order in Banach algebras, by a direct application of a block operator fixed point theorem [4]. This coupled system includes many key coupled systems of integral and differential equations that arise in nonlinear analysis and their applications. Some examples and remarks are illustrated. Finally, we study the stability of solutions for the coupled system in Equation (2).

Author Contributions: Conceptualization, H.H.G.; methodology, A.M.A. and D.B.; validation, H.H.G., A.M.A. and D.B.; formal analysis, H.H.G.; investigation, H.H.G.; writing–original draft preparation, A.M.A. and D.B.; writing–review and editing, A.M.A. and D.B.; visualization, A.M.A. and D.B.; supervision, H.H.G.; project administration, D.B.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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