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Fuzzy Counterparts of Fischer Diagonal Condition in $\top$-Convergence Spaces

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Abstract: Fischer diagonal condition plays an important role in convergence space since it precisely ensures a convergence space to be a topological space. Generally, Fischer diagonal condition can be represented equivalently both by Kowalsky compression operator and Gähler compression operator. $\top$-convergence spaces are fundamental fuzzy extensions of convergence spaces. Quite recently, by extending Gähler compression operator to fuzzy case, Fang and Yue proposed a fuzzy counterpart of Fischer diagonal condition, and proved that $\top$-convergence space with their Fischer diagonal condition just characterizes strong L-topology—a type of fuzzy topology. In this paper, by extending the Kowalsky compression operator, we present a fuzzy counterpart of Fischer diagonal condition, and verify that a $\top$-convergence space with our Fischer diagonal condition precisely characterizes topological generated L-topology—a type of fuzzy topology. Hence, although the crisp Fischer diagonal conditions based on the Kowalsky compression operator and the on Gähler compression operator are equivalent, their fuzzy counterparts are not equivalent since they describe different types of fuzzy topologies. This indicates that the fuzzy topology (convergence) is more complex and varied than the crisp topology (convergence).

Keywords: fuzzy topology; fuzzy convergence; $\top$-convergence; diagonal condition

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1. Introduction

The notion of convergence space is investigated by generalizing the convergence in topological space [1]. For a set X, let $P(X)$ (respectively, $\mathrm{F}(X)$) denote the power set (respectively, filters set) on X. *A convergence space* is defined by a pair (X, c), where $\mathrm{c} \subseteq \mathrm{F}(X) \times X$ fulfills:

(C1) For every $x \in X, (\dot{x}, x) \in \mathrm{c}$, $\dot{x} = \{A \in P(X) | x \in A\}$.
(C2) For all $\mathrm{F}, \mathrm{G} \in \mathrm{F}(X)$, $\mathrm{F} \subseteq \mathrm{G}$, $(\mathrm{F}, x) \in \mathrm{c} \Longrightarrow (\mathrm{G}, x) \in \mathrm{c}$.

If $(\mathrm{F}, x) \in \mathrm{c}$, we also denote $\mathrm{F} \stackrel{\mathrm{c}}{\longrightarrow} x$, and say that F converges to x.

A convergence space (X, c) is called pretopological if it satisfies either of the following three equivalent conditions:

(**P1**) For $\{\mathrm{F}_t\}_{t\in T} \subseteq \mathrm{F}(X)$ and $x \in X$, $\forall t \in T, \mathrm{F}_t \stackrel{\mathrm{c}}{\longrightarrow} x \Longrightarrow \bigcap\limits_{t\in T} \mathrm{F}_t \stackrel{\mathrm{c}}{\longrightarrow} x$.

(**P2**) For $x \in X$, $\mathrm{U}_{\mathrm{c}}(x) = \bigcap\{\mathrm{F} \in \mathrm{F}(X) | \mathrm{F} \stackrel{\mathrm{c}}{\longrightarrow} x\}$ converges to x. Generally, $\mathrm{U}_{\mathrm{c}} = \{\mathrm{U}_{\mathrm{c}}(x)\}_{x\in X}$ is called the neighborhood system associated with c.

(**P3**) For $x \in X$ and $\mathrm{F} \in \mathrm{F}(X)$, $\mathrm{F} \stackrel{\mathrm{c}}{\longrightarrow} x \Leftrightarrow \mathrm{F} \supseteq \mathrm{U}_{\mathrm{c}}(x)$.

A pretopological convergence space is called topological if it fulfills the next condition:

(**U**) For any $x \in X$, if $A \in \mathrm{U_c}(x)$ then there exists a $B \in \mathrm{U_c}(x)$ such that $A \in \mathrm{U_c}(y)$ for any $y \in B$.

Topological convergence spaces can also be characterized by Kowalsky diagonal condition and Fischer diagonal condition [1,2].

Let T be any set and $\Phi : T \longrightarrow \mathrm{F}(X)$ be a mapping, called a choice mapping of filters. For $\mathrm{F} \in \mathrm{F}(T)$, let $\Phi^{\Rightarrow}(\mathrm{F})$ denote the image of F under the mapping Φ, i.e., the filter on $\mathrm{F}(X)$ generated by $\{\Phi(A)|A \in \mathrm{F}\}$ as a filter base. Then, the *Kowalsky compression operator* on $\Phi^{\Rightarrow}(\mathrm{F}) \in \mathrm{F}(\mathrm{F}(X))$ is defined as

$$K\Phi\mathrm{F} \in \mathrm{F}(X) := \bigcup_{A \in \mathrm{F}} \bigcap_{y \in A} \Phi(y).$$

There is an equivalent statement for $K\Phi\mathrm{F}$ proposed by Gähler [3] (please also see the Remark in Jäger [4]):

$$A \in K\Phi\mathrm{F} \Longleftrightarrow \{t \in T | A \in \Phi(t)\} \in \mathrm{F},$$

which is here called Gähler compression operator to distinguish the two descriptions.

For a convergence space (X, c), the *Fischer diagonal condition* is given as below:

(**FD**) Let T be any set, $\psi : T \longrightarrow X$ and $\Phi : T \longrightarrow \mathrm{F}(X)$ with $\Phi(t) \xrightarrow{\mathrm{c}} \psi(t)$ for every $t \in T$. For any $\mathrm{F} \in \mathrm{F}(X)$, $\psi^{\Rightarrow}(\mathrm{F}) \xrightarrow{\mathrm{c}} x \Longrightarrow K\Phi\mathrm{F} \xrightarrow{\mathrm{c}} x$.

Taking $T = X$ and $\psi = id_X$ in (**FD**), we give the *Kowalsky diagonal condition* (**KD**).

By using the Kowalsky compression operator, the condition (**U**) can be restated as:

(**U**) For any $x \in X$, $\mathrm{U_c}(x) \subseteq K\mathrm{U_c}\mathrm{U_c}(x)$, where $\mathrm{U_c} : X \longrightarrow \mathrm{F}(X)$ is the selection mapping of filters determined by neighborhood system.

It is verified that a pretopological convergence space is topological iff it fulfills (**KD**), and a convergence space is topological iff it fulfills (**FD**).

It is known that a topological space (X, δ) corresponds uniquely to a topological convergence space (X, c) by taking that for each $x \in X$,

$$\mathrm{U_c}(x) = \mathrm{U}_\delta(x) := \{A \subseteq X | \exists B \in \delta \text{ s.t. } x \in B \subseteq A\}.$$

Then, we obtain a bijection between topological convergence spaces and topological spaces. In this case, we say that topological convergence spaces characterize topological spaces, or, in other words, that it establishes the convergence theory associated with topological spaces.

In (**FD**), change the statement

$$\psi^{\Rightarrow}(\mathrm{F}) \xrightarrow{\mathrm{c}} x \Longrightarrow K\Phi\mathrm{F} \xrightarrow{\mathrm{c}} x \text{ as } K\Phi\mathrm{F} \xrightarrow{\mathrm{c}} x \Longrightarrow \psi^{\Rightarrow}(\mathrm{F}) \xrightarrow{\mathrm{c}} x,$$

then the resulted condition is denoted as (**DFD**), called the *dual Fischer diagonal condition*. Interestingly, the condition (**DFD**) precisely characterizes the regularity of convergence space [1,2].

To sum up, both Fischer diagonal condition (**FD**) and its dual condition (**DFD**) are based on Kowalsky (or equivalent, Gähler) compression operator, and (**FD**) describes topologicalness while (**DFD**) characterizes regularity of convergence spaces.

Fuzzy set theory, proposed by Zadeh [5], is a fundamental mathematical tool to deal with uncertain information. Fuzzy set theory has been widely used in many regards such as medical diagnosis, data mining, decision-making, machine learning and so on [6–9]. In addition, that fuzzy set combines traditional mathematics produces many new mathematical branches such as fuzzy algebra, fuzzy topology, fuzzy order, fuzzy logic, etc. In this paper, we focus on the convergence theory associated with lattice-valued topology (i.e., fuzzy topology with the membership values in a complete lattice L).

Lattice-valued convergence spaces are fuzzy extensions of convergence spaces. Compared with the classical convergence spaces, the lattice-value convergence spaces are more complex and varied. L-filters and $\top$-filter are basic tools to study lattice-valued convergence spaces. Based on L-filters, Jäger [10], Flores [11], Fang [12] and Li [13] defined types of L-convergence spaces. Based on $\top$-filters, Fang and Yue [14] introduced $\top$-convergence spaces. Nowadays, these spaces have been widely discussed and developed [4,15–42].

Many interesting lattice-valued versions of Fischer diagonal condition and dual Fischer diagonal condition were proposed to study the topologicalnesses and regularities of many kinds of lattice-valued convergence spaces. Precisely,

Under the background of L-filter:

(1) The Gähler compression operator was presented by Jäger [4]. Then, some types of lattice-valued Fischer diagonal conditions were proposed to discuss the topologicalnesses of different L-convergence spaces [13,18,28]. Particularly, it was proved in [28] that stratified L-convergence space with the considered Fischer diagonal condition precisely characterizes stratified L-topological space. Meanwhile, some lattice-valued dual Fischer diagonal conditions were introduced to study the regularities of different L-convergence spaces [19,27,31].

(2) The Kowalsky compression operator was presented by Flores [15]. Then, he introduced a lattice-valued Fischer diagonal condition and proved that stratified L-convergence space with his diagonal condition can characterize a tower of stratified L-topological spaces. Later, Richardson and his co-author used the dual condition of Flores's diagonal condition to describe the regularity of stratified L-convergence spaces [15,16].

Under the background of $\top$-filter:

(1) The Gähler compression operator was presented by Fang and Yue in $\top$-convergence space [14]. Then, they introduced a lattice-valued Fischer diagonal condition and proved that condition can characterize strong L-topological spaces. They also proposed a lattice-valued dual Fischer diagonal to describe a regularity of $\top$-convergence spaces. Quite recently, the author extended Fang and Yue's diagonal conditions and used them to study the relative topologicalness and the relative regularity in $\top$-convergence spaces [22,26].

(2) The Kowalsky compression operator was presented by the author in $\top$-convergence space [30]. Then, we proposed a lattice-valued version of dual Fischer diagonal condition to discuss the regularity of $\top$-convergence spaces. In preparing the paper [30], we also tried to consider a lattice-valued version of Fischer diagonal condition such that $\top$-convergence spaces with the diagonal condition can characterize a kind of lattice-valued topological spaces. However, we failed to do that.

In this paper, we resolve the above mentioned question. Precisely, we present a lattice-valued Fischer diagonal condition based on Kowalsky compression operator and prove that there is a bijection between $\top$-convergence spaces with our diagonal condition and topological generated L-topological spaces. Thus, our Fischer diagonal condition characterizes precisely topological generated L-topological spaces. Therefore, we establish the convergence theory associated with topological generated L-topological spaces. This is the main contribution of the paper.

This paper is organized as follows. In Section 2, we review some notions and notations as preliminary. In Section 3, we give a lattice-valued Fischer diagonal condition and use it to describe a subclass of $\top$-convergence spaces, called topological generated $\top$-convergence spaces. In Section 4, we establish the relationships between our diagonal condition and Fang and Yue's diagonal condition. Then, we further verify that there is a one-to-one correspondence between topological generated $\top$-convergence spaces and topological generated L-topological spaces. Hence, we establish the convergence theory associated with the topological generated L-topological spaces.

2. Preliminaries

In this section, we recall some notions and conclusions about L-fuzzy sets, $\top$-convergence spaces and L-topological spaces for later use.

By an integral, commutative quantale we mean a pair $L = (L, *)$ such that:

(i) $(L, \leq)$ is a complete lattice, and its top (respectively, bottom) element is denoted as $\top$ (respectively, $\bot$).

(ii) $*$ is a commutative semigroup operation on L satisfying

$$\forall \alpha \in L, \forall \{\beta_i\}_{i \in I} \subseteq L, \alpha * \bigvee_{i \in I} \beta_i = \bigvee_{i \in I} (\alpha * \beta_i).$$

(iii) For every $\alpha \in L$, $\top * \alpha = \alpha$.

For any $\alpha, \beta \in L$, we define

$$\alpha \to \beta = \bigvee \{\gamma \in L : \alpha * \gamma \leq \beta\}.$$

The properties of the binary operations $*$ and $\to$ can be referred to [43–45]. We list some of them used in the sequel.

(1) $\alpha * \beta \leq \gamma \Leftrightarrow \beta \leq \alpha \to \gamma$.

(2) $\alpha * (\alpha \to \beta) \leq \beta$.

(3) $\alpha \to (\beta \to \gamma) = (\alpha * \beta) \to \gamma$.

(4) $(\bigvee\limits_{i \in I} \alpha_i) \to \beta = \bigwedge\limits_{i \in I} (\alpha_i \to \beta)$.

(5) $\alpha \to (\bigwedge\limits_{i \in I} \beta_i) = \bigwedge\limits_{i \in I} (\alpha \to \beta_i)$.

For $\alpha, \beta \in L$, we define $\alpha \ll \beta$ iff for every directed subsets $D \subseteq L$, $\beta \leq \vee D$ always implies the existence of $\gamma \in D$ such that $\alpha \leq \gamma$. Note that $\alpha \ll \beta$ implies $\alpha \leq \beta$. When $L = [0, 1]$, $\alpha \ll \beta \Longleftrightarrow \alpha < \beta$.

A complete lattice $(L, \leq)$ is called meet continuous if, for $\alpha \in L$ and $\{\beta_i\}_{i \in I}$ being directed in L, it holds that $\alpha \wedge \bigvee\limits_{i \in I} \beta_i = \bigvee\limits_{i \in I} (\alpha \wedge \beta_i)$. A complete lattice $(L, \leq)$ is called continuous if $\alpha = \vee\{\beta \in L | \beta \ll \alpha\}$ for each $\alpha \in L$. A continuous lattice is a natural meet continuous lattice [44].

In this paper, if not otherwise stated, we always let $L = (L, *)$ be an integral, commutative quantale with the underlying lattice $(L, \leq)$ being meet continuous.

A mapping $A : X \longrightarrow L$ is called an L-fuzzy set on X, and the family of L-fuzzy sets on X is denoted as L^X. For $A \in P(X)$, let $\top_A$ denote its characteristic mapping. The operations $\bigvee$, $\bigwedge$, $*$ and $\to$ on L can translate onto L^X pointwisely. Precisely, for $A, B, A_i (i \in I) \in L^X$,

$$(\bigvee_{i \in I} A_i)(x) = \bigvee_{i \in I} A_i(x), \quad (\bigwedge_{i \in I} A_i)(x) = \bigwedge_{i \in I} A_i(x),$$

$$(A * B)(x) = A(x) * B(x), (A \to B)(x) = A(x) \to B(x).$$

Let $f : X \longrightarrow Y$ be a mapping. Then, define $f^{\to} : L^X \longrightarrow L^Y$ and $f^{\leftarrow} : L^Y \longrightarrow L^X$ as follows [45]:

$$\forall A \in L^X, \forall y \in Y, f^{\to}(A)(y) = \bigvee_{f(x)=y} A(x),\ \forall B \in L^Y, \forall x \in X, f^{\leftarrow}(B)(x) = B(f(x)).$$

For $A, B \in L^X$, the degree of A in B is defined as follows [46–48]:

$$S_X(A, B) = \bigwedge_{x \in X} (A(x) \to B(x)).$$

Definition 1 ([45,49])**.** *A nonempty subset* $\mathbf{F} \subseteq L^X$ *is called a* $\top$*-filter on* X *if it holds that:*

(1) $A \in \mathbf{F} \Longrightarrow \bigvee_{x \in X} A(x) = \top$.
(2) $A, B \in \mathbf{F} \Longrightarrow A \wedge B \in \mathbf{F}$.
(3) $\bigvee_{B \in \mathbf{F}} S_X(B, A) = \top \Longrightarrow A \in \mathbf{F}$.

The family of $\top$*-filters on* X *is denoted as* $\mathbf{F}_L^{\top}(X)$.

Definition 2 ([45])**.** *A nonempty subset* $\mathbf{B} \subseteq L^X$ *is called a* $\top$*-filter base on* X *if it satisfies:*

(1) $A \in \mathbf{B} \Longrightarrow \bigvee_{x \in X} A(x) = \top$.
(2) $A, B \in \mathbf{B} \Longrightarrow \bigvee_{C \in \mathbf{B}} S_X(C, A \wedge B) = \top$.

Each $\top$*-filter base* $\mathbf{B}$ *generates a* $\top$*-filter* $\mathbf{F_B} := \{A \in L^X | \bigvee_{B \in \mathbf{B}} S_X(B, A) = \top\}$. *In [38], it is proved that for any* $A \in L^X$, $\bigvee_{B \in \mathbf{B}} S_X(B, A) = \bigvee_{B \in \mathbf{F_B}} S_X(B, A)$.

We collect some fundamental facts about $\top$-filters in the following proposition.

Proposition 1 ([14,45])**.**

(1) *For every* $x \in X$, $[x]_{\top} =: \{A \in L^X | A(x) = \top\}$ *is a* $\top$*-filter.*
(2) $\{\mathbf{F}_i\}_{i \in I} \subseteq \mathbf{F}_L^{\top}(X) \Longrightarrow \bigcap_{i \in I} \mathbf{F}_i \in \mathbf{F}_L^{\top}(X)$.
(3) *Let* $f : X \longrightarrow Y$ *be a mapping and* $\mathbf{F} \in \mathbf{F}_L^{\top}(X)$. *Then, define* $f^{\Rightarrow}(\mathbf{F})$ *as the* $\top$*-filter on* Y *generated by the* $\top$*-filter base* $\{f^{\to}(A) | A \in \mathbf{F}\}$. *Furthermore,* $B \in f^{\Rightarrow}(\mathbf{F})$ *iff* $f^{\leftarrow}(B) \in \mathbf{F}$.

Given $\mathbf{F} \in \mathbf{F}_L^{\top}(X)$, then the set $\eta(\mathbf{F}) = \{A \subseteq X | \top_A \in \mathbf{F}\}$ is a filter on X. Conversely, given $\mathrm{F} \in \mathrm{F}(X)$, then the set $\{\top_A | A \in \mathrm{F}\}$ forms a $\top$-filter base and the associated $\top$-filter is denoted as $\varpi(\mathrm{F})$.

Lemma 1 (Lemma 2.6 in [30])**.** *Let* $f : X \longrightarrow Y$, $\mathrm{F} \in \mathrm{F}(X)$ *and* $\mathbf{F} \in \mathbf{F}_L^{\top}(X)$. *Then,*

(1) $\eta\varpi(\mathrm{F}) = \mathrm{F}$.
(2) $\varpi\eta(\mathbf{F}) \subseteq \mathbf{F}$.
(3) *For every* $x \in X, \varpi(\dot{x}) = [x]_{\top}$.
(4) *For every* $x \in X, \eta([x]_{\top}) = \dot{x}$.

(5) $\eta(f^{\Rightarrow}(\mathbf{F})) = f^{\Rightarrow}(\eta(\mathbf{F}))$.

Definition 3 ([14])**.** *A $\top$-convergence space is a pair $(X, \mathbf{q})$, where $\mathbf{q} \subseteq \mathbf{F}_L^{\top}(X) \times X$ satisfies:*

(TC1) *For every $x \in X$, $([x]_{\top}, x) \in \mathbf{q}$.*
(TC2) $(\mathbf{F}, x) \in \mathbf{q}$, $\mathbf{F} \subseteq \mathbf{G} \Longrightarrow (\mathbf{G}, x) \in \mathbf{q}$.

If $(\mathbf{F}, x) \in \mathbf{q}$, we also denote $\mathbf{F} \xrightarrow{\mathbf{q}} x$ and say that $\mathbf{F}$ converges to x.
Obviously, a $\top$-convergence space reduces to a convergence space whenever $L = \{\perp, \top\}$.

Let T be any set, $\phi : T \longrightarrow \mathbf{F}_L^{\top}(X)$ and $\mathbf{F} \in \mathbf{F}_L^{\top}(T)$. In [26], the author defined an extending *Kowalsky compression operator* on $\phi^{\Rightarrow}(\mathbf{F}) \in \mathbf{F}_L^{\top}(\mathbf{F}_L^{\top}(X))$ by

$$k\phi\mathbf{F} := \bigcup_{A \in \eta(\mathbf{F})} \bigcap_{y \in A} \phi(y) \in \mathbf{F}_L^{\top}(X).$$

Lemma 2 (Lemma 3.2 in [30])**.** *Let $f : X \longrightarrow Y$, $\phi : T \longrightarrow \mathbf{F}_L^{\top}(X)$ and $\Phi : T \longrightarrow \mathrm{F}(X)$. Then, for every $\mathbf{F} \in \mathbf{F}_L^{\top}(T)$ and every $\mathrm{F} \in \mathrm{F}(T)$,*

(1) $f^{\Rightarrow}(k\phi\mathbf{F}) = k(f^{\Rightarrow} \circ \phi)\mathbf{F}$.
(2) *put $\Phi_1 = \eta \circ \phi$, then $\eta(k\phi\mathbf{F}) = K\Phi_1\eta(\mathbf{F})$.*
(3) *put $\phi_1 = \varpi \circ \Phi$, then $\eta(k\phi_1\varpi(\mathrm{F})) = K\Phi\mathrm{F}$.*
(4) *taking $\sigma : T \longrightarrow \mathbf{F}_L^{\top}(X)$ with $\sigma(t) \subseteq \phi(t)$ for every $t \in T$, then $k\sigma\mathbf{F} \subseteq k\phi\mathbf{F}$.*

Definition 4 ([45,47])**.** *A subset $\tau \subseteq L^X$ is called an L-topology (L-Top) on X if it contains $\top_X, \top_{\emptyset}$ and is closed with respect to finite meets and arbitrary joins. The pair (X, τ) is called an L-topological space. Furthermore, τ is called stratified (SL-Top) if $\alpha * A \in \tau$ for each $\alpha \in L$ and each $A \in \tau$. A stratified L-topology τ is called strong (STrL-Top) if $\alpha \to A \in \tau$ for each $\alpha \in L$ and each $A \in \tau$.*

Definition 5 ([25,45])**.** *Assume that L to be a continuous lattice. For a topological space (X, δ), all L-fuzzy sets $A \in L^X$ with*

$$A_\alpha = \{x \in X | \alpha \ll A(x)\} \in \delta \text{ for each } \alpha \in L$$

form an L-topology on X. Such space is called a topological generated L-topological space (TGL-Top). The L-topological space generated by (X, δ) is denoted as $(X, \varrho(\delta))$. When $* = \wedge$, $\varrho(\delta)$ is precisely the L-topology generated by $\{\top_A | A \in \delta\}$ and all constant value L-fuzzy sets on X as a subbase.

In [25], Lai and Zhang introduced a kind of L-topological space, called conical neighborhood space (CNS). We do not give the definition of CNS here; please see Definition 5.1 in [25]. Lai and Zhang proved that the mentioned L-topological spaces have the following inclusive relation:

STrL-Top$\subseteq$ CNS $\subseteq$ SL-Top$\subseteq$ L-Top and TGL-Top$\subseteq$ CNS.

By suitable lattice-valued Fischer diagonal condition:

(1) The convergence theory associated with SL-Top is presented in [28].
(2) The convergence theory associated with STrL-Top is given in [14].
(3) The convergence theory associated with CNS is developed in [29].

In the following, we establish the convergence theory associated with TGL-Top by appropriate lattice-valued Fischer diagonal condition.

3. Topological Generated $\top$-Convergence Spaces vs. Fischer Diagonal Condition Based on Extending Kowalsky Compression Operator

In this section, we present a lattice-valued Fischer diagonal condition based on extending Kowalsky compression operator. We also prove that $\top$-convergence spaces with our diagonal condition precisely characterize those spaces generated by topological convergence spaces.

First, we consider the pretopological conditions for a $\top$-convergence space $(X, \mathbf{q})$. It is not difficult to verify that the following three conditions are equivalent.

(**TP1**) For every $i \in I, \mathbf{F}_i \xrightarrow{\mathbf{q}} x \Longrightarrow \bigcap\limits_{i\in I} \mathbf{F}_i \xrightarrow{\mathbf{q}} x$.

(**TP2**) For every $x \in X$, $\mathbf{U_q}(x) = \bigcap\{\mathbf{F} \in \mathbf{F}_L^\top(X) | \mathbf{F} \xrightarrow{\mathbf{q}} x\}$ converges to x. Usually, $\mathbf{U_q} = \{\mathbf{U_q}(x)\}_{x\in X}$ is called the $\top$-neighborhood system associated with $\mathbf{q}$.

(**TP3**) For every $x \in X$, $\mathbf{F} \xrightarrow{\mathbf{q}} x \Leftrightarrow \mathbf{F} \supseteq \mathbf{U_q}(x)$.

Definition 6. *A $\top$-convergence space $(X, \mathbf{q})$ is called pretopological if it fulfills any of (***TP1***)–(***TP3***).*

Second, we consider the topological condition for $\top$-convergence space both by extending Fischer diagonal condition and extending Kowalsky diagonal condition.

We define a lattice-valued extension of Fischer diagonal condition as below:

(**TFD**) Let T be any set, $\psi : T \longrightarrow X$ and $\phi : T \longrightarrow \mathbf{F}_L^\top(X)$ satisfying $\phi(t) \xrightarrow{\mathbf{q}} \psi(t)$ for every $t \in T$. Then, for any $\mathbf{F} \in \mathbf{F}_L^\top(X)$, $\psi^\Rightarrow(\mathbf{F}) \xrightarrow{\mathbf{q}} x \Longrightarrow k\phi\mathbf{F} \xrightarrow{\mathbf{q}} x$.

Taking $T = X$ and $\psi = id_X$ in (**TFD**), then we get the *Lattice-valued Kowalsky diagonal condition* (**TKD**).

Moreover, a lattice-valued version of neighborhood condition (**U**) is given as follows:

(**TU**) For any $x \in X$, $k\mathbf{U_q}\mathbf{U_q}(x) \supseteq \mathbf{U_q}(x)$.

The theorem below presents the relationships between lattice-valued versions of Fischer diagonal condition, Kowalsky diagonal condition and pretopological condition.

Theorem 1. *A $\top$-convergence space $(X, \mathbf{q})$ satisfies* (**TFD**) *iff it is pretopological and satisfies* (**TKD**) *iff it is pretopological and satisfies* (**TU**).

Proof. (**TFD**)$\Rightarrow$(**TP1**). Let $\{\mathbf{F}_t | t \in T\}$ be all $\top$-filters on X converge to x. Then, define

$$\psi : T \longrightarrow X, \phi : T \longrightarrow \mathbf{F}_L^\top(X) \text{ as } \psi(t) \equiv x, \phi(t) = \mathbf{F}_t \text{ for each } t \in T.$$

Take $\mathbf{F}_\perp := \{\{\top_T\}\}$ as the least member of $\mathbf{F}_L^\top(T)$. Then, we observe easily that $\psi^\Rightarrow(\mathbf{F}_\perp) = [x]_\top$ and $k\phi\mathbf{F}_\perp = \bigcap\limits_{t\in T} \phi(t) = \bigcap\limits_{t\in T} \mathbf{F}_t$. By $[x]_\top \xrightarrow{\mathbf{q}} x$ and (**TFD**), we have that $k\phi\mathbf{F}_\perp = \bigcap\limits_{t\in T} \mathbf{F}_t \xrightarrow{\mathbf{q}} x$, i.e., (**TP1**) holds.

(**TFD**)$\Rightarrow$(**TKD**). It is obvious.

(**TP3**)+(**TKD**)$\Rightarrow$(**TU**). By (**TP3**), we have that $\mathbf{U_q}(y) \xrightarrow{\mathbf{q}} y$ for any $y \in X$. Then, by $\mathbf{U_q}(x) \xrightarrow{\mathbf{q}} x$ and (**TKD**), we get that $k\mathbf{U_q}\mathbf{U_q}(x) \xrightarrow{\mathbf{q}} x$, i.e., $k\mathbf{U_q}\mathbf{U_q}(x) \supseteq \mathbf{U_q}(x)$.

(**TP3**)+(**TU**)$\Rightarrow$(**TFD**). Let T be any set, $\psi : T \longrightarrow X$ and $\phi : T \longrightarrow \mathbf{F}_L^\top(X)$ with $\phi(t) \xrightarrow{\mathbf{q}} \psi(t)$ for every $t \in T$. It follows by (**TP3**) that $\phi(t) \supseteq \mathbf{U_q}(\psi(t))$ for any $t \in T$.

Let $\mathbf{F} \in \mathbf{F}_L^\top(X)$ and $\psi^\Rightarrow(\mathbf{F}) \xrightarrow{\mathbf{q}} x$, then it holds by (**TP3**) that $\psi^\Rightarrow(\mathbf{F}) \supseteq \mathbf{U_q}(x)$.

Next, we prove that $k\mathbf{U_q}\psi^{\Rightarrow}(\mathbf{F}) \subseteq k(\mathbf{U_q} \circ \psi)\mathbf{F}$. Indeed, let $A \in k\mathbf{U_q}\psi^{\Rightarrow}(\mathbf{F})$; then, there exists an $A \in \eta(\psi^{\Rightarrow}(\mathbf{F}))$ such that $A \in \mathbf{U_q}(y)$ for any $y \in A$. By Lemma 1(5), it holds that $A \in \psi^{\Rightarrow}(\eta(\mathbf{F}))$, i.e., $\psi^{\leftarrow}(A) \in \eta(\mathbf{F})$. Then, we get that, for every $z \in \psi^{\leftarrow}(A)$, $A \in \mathbf{U_q}(\psi(z)) = (\mathbf{U_q} \circ \psi)(z)$. Thus, $A \in k(\mathbf{U_q} \circ \psi)\mathbf{F}$.

Combinaing the above statements, it follows by Lemma 2 (4) and (**TU**) that

$$k\phi\mathbf{F} \supseteq k(\mathbf{U_q} \circ \psi)\mathbf{F} \supseteq k\mathbf{U_q}\psi^{\Rightarrow}(\mathbf{F}) \supseteq k\mathbf{U_q}\mathbf{U_q}(x) \supseteq \mathbf{U_q}(x).$$

That means $k\phi\mathbf{F} \xrightarrow{\mathbf{q}} x$, as desired. □

For a convergence space (X, c), it is easy to verify that the pair $\pi(X, \mathrm{c}) = (X, \pi(\mathrm{c}))$ defined by

$$\forall \mathbf{F} \in \mathbf{F}_L^{\top}(X), \forall x \in X, \mathbf{F} \xrightarrow{\pi(\mathrm{c})} x \Leftrightarrow \eta(\mathbf{F}) \xrightarrow{\mathrm{c}} x,$$

is a $\top$-convergence space.

Definition 7. *A $\top$-convergence space $(X, \mathbf{q})$ is called topological generated if $(X, \mathbf{q}) = \pi(X, \mathrm{c})$ for some topological convergence space (X, c).*

The next theorem shows that $\top$-convergence spaces generated by convergence spaces connect diagonal condition (**FD**) and lattice-valued diagonal condition (**TFD**) well.

Theorem 2. *Convergence space (X, c) fulfills* (**FD**) *iff $(X, \pi(\mathrm{c}))$ fulfills* (**TFD**).

Proof. $\Longrightarrow$. Let $\psi : T \longrightarrow X$ and $\phi : T \longrightarrow \mathbf{F}_L^{\top}(X)$ with $\forall t \in T, \phi(t) \xrightarrow{\pi(\mathrm{c})} \psi(t)$. Putting $\Phi = \eta \circ \phi$, then, from $\phi(t) \xrightarrow{\pi(\mathrm{c})} \psi(t)$ we get

$$\Phi(t) = \eta(\phi(t)) \xrightarrow{\mathrm{c}} \psi(t), \forall t \in T.$$

Taking any $\psi^{\Rightarrow}(\mathbf{F}) \xrightarrow{\pi(\mathrm{c})} x$, then from Lemma 1(5) we obtain

$$\psi^{\Rightarrow}(\eta(\mathbf{F})) = \eta(\psi^{\Rightarrow}(\mathbf{F})) \xrightarrow{\mathrm{c}} x.$$

It follows from (**FD**) and Lemma 2 (2) that $\eta(k\phi\mathbf{F}) = K\Phi\eta(\mathbf{F}) \xrightarrow{\mathrm{c}} x$, that is, $k\phi\mathbf{F} \xrightarrow{\pi(\mathrm{c})} x$. We verify that (**TFD**) is fulfilled.

$\Longleftarrow$. Let $\psi : T \longrightarrow X$ and $\Phi : T \longrightarrow \mathrm{F}(X)$ with $\forall t \in T, \Phi(t) \xrightarrow{\mathrm{c}} \psi(t)$. Putting $\phi = \varpi \circ \Phi$, then, from Lemma 1(1), we conclude

$$\forall t \in T, \eta \circ \phi(t) = \eta \circ \varpi \circ \Phi(t) = \Phi(t) \xrightarrow{\mathrm{c}} \psi(t), \text{ i.e., } \phi(t) \xrightarrow{\pi(\mathrm{c})} \psi(t).$$

Taking any $\psi^{\Rightarrow}(\mathrm{F}) \xrightarrow{\mathrm{c}} x$, then, from Lemma 1(5), we get

$$\eta\psi^{\Rightarrow}(\varpi(\mathbf{F})) = \psi^{\Rightarrow}((\eta \circ \varpi)(\mathbf{F})) = \psi^{\Rightarrow}(\mathrm{F}) \xrightarrow{\mathrm{c}} x, \text{ that is, } \psi^{\Rightarrow}(\varpi(\mathbf{F})) \xrightarrow{\pi(\mathrm{c})} x.$$

It holds from (**TFD**) that $k\phi\varpi(\mathrm{F}) \xrightarrow{\pi(\mathrm{c})} x$. Further, from Lemma 2 (3), we obtain $\eta(k\phi\varpi(\mathrm{F})) = K\Phi\mathrm{F} \xrightarrow{\mathrm{c}} x$. This shows that the condition (**FD**) is fulfilled. □

The following theorem shows that the lattice-valued Fischer diagonal condition **(TFD)** precisely characterizes topological generated $\top$-convergence spaces.

Theorem 3. *A $\top$-convergence space $(X, \mathbf{q})$ is topological generated iff it satisfies* **(TFD)***.*

Proof. The necessity can be concluded from Theorem 2. We prove the sufficiency below. Assume that $(X, \mathbf{q})$ satisfies **(TFD)**. Then, define $\theta(X, \mathbf{q}) = (X, \theta(\mathbf{q}))$ as

$$\forall x \in X, \forall \mathrm{F} \in \mathrm{F}(X), \mathrm{F} \xrightarrow{\theta(\mathbf{q})} x \Leftrightarrow \mathrm{F} \supseteq \eta(\mathbf{U_q}(x)).$$

Obviously, $(X, \theta(\mathbf{q}))$ is a convergence space. Next, we verify $(X, \mathbf{q}) = \pi\theta(X, \mathbf{q})$.

(1) $\mathbf{q} \subseteq \pi\theta(\mathbf{q})$. It follows by

$$\begin{aligned} \mathbf{F} \xrightarrow{\mathbf{q}} x \quad &\overset{(\mathbf{TP3})}{\Rightarrow} \quad \mathbf{F} \supseteq \mathbf{U_q}(x) \Rightarrow \eta(\mathbf{F}) \supseteq \eta(\mathbf{U_q}(x)) \\ &\Rightarrow \quad \eta(\mathbf{F}) \xrightarrow{\theta(\mathbf{q})} x \Rightarrow \mathbf{F} \xrightarrow{\pi\theta(\mathbf{q})} x. \end{aligned}$$

(2) $\pi\theta(\mathbf{q}) \subseteq \mathbf{q}$. Letting $\mathbf{F} \xrightarrow{\pi\theta(\mathbf{q})} x$, we check below that:

(i) $\mathbf{F} \supseteq k\mathbf{U_q}\mathbf{F}$. Take $A \in k\mathbf{U_q}\mathbf{F}$, then $\exists B \in \eta(\mathbf{F})$ such that $A \in \mathbf{U_q}(y) \subseteq [y]_\top$ for any $y \in B$. It follows that $A(y) = \top$ for any $y \in B$, and then $\top_B \leq A$, which means that $A \in \mathbf{F}$.

(ii) $k\mathbf{U_q}\mathbf{F} \supseteq k\mathbf{U_q}\mathbf{U_q}(x)$. By $\mathbf{F} \xrightarrow{\pi\theta(\mathbf{q})} x$, we have $\eta(\mathbf{F}) \xrightarrow{\theta(\mathbf{q})} x$, i.e., $\eta(\mathbf{F}) \supseteq \eta(\mathbf{U_q}(x))$. Then,

$$\begin{aligned} k\mathbf{U_q}\mathbf{U_q}(x) &= \bigcup_{B \in \eta(\mathbf{U_q}(x))} \bigcap_{y \in B} \mathbf{U_q}(y) \\ &\subseteq \bigcup_{B \in \eta(\mathbf{F})} \bigcap_{y \in B} \mathbf{U_q}(y) = k\mathbf{U_q}\mathbf{F}. \end{aligned}$$

(iii) $k\mathbf{U_q}\mathbf{U_q}(x) \supseteq \mathbf{U_q}(x)$. It follow by **(TU)**.

Combining (i)–(iii), we get that $\mathbf{F} \supseteq \mathbf{U_q}(x)$, thus it follows by **(TP3)** that $\mathbf{F} \xrightarrow{\mathbf{q}} x$.

From Theorem 2 and that $(X, \mathbf{q}) = \pi\theta(X, \mathbf{q})$ satisfies **(TFD)**, we get that $(X, \theta(\mathbf{q}))$ is topological. □

In the following, we give two examples of topological generated $\top$-convergence spaces.

Example 1. *For a set X, the discrete $\top$-convergence structure $\mathbf{q}_d$ is defined as $\mathbf{F} \xrightarrow{\mathbf{q}_d} x \Longleftrightarrow \mathbf{F} \supseteq [x]_\top$ for any $\mathbf{F} \in \mathbf{F}_L^\top(X)$ and any $x \in X$; and the indiscrete $\top$-convergence structure $\mathbf{q}_{ind}$ is defined by $\mathbf{F} \xrightarrow{\mathbf{q}_{ind}} x$ for every $\mathbf{F} \in \mathbf{F}_L^\top(X)$ and every $x \in X$, see [14].*

(1) *$(X, \mathbf{q}_d)$ is topological generated. Indeed, for any $x \in X$, note that $\mathbf{U}_{\mathbf{q}_d}(x) = [x]_\top \xrightarrow{\mathbf{q}_d} x$ and*

$$\begin{aligned} k\mathbf{U}_{\mathbf{q_d}}\mathbf{U}_{\mathbf{q_d}}(x) &= \bigcup_{B\in\eta(\mathbf{U}_{\mathbf{q_d}}(x))}\bigcap_{y\in B}\mathbf{U}_{\mathbf{q_d}}(y), \text{ by Lemma 1(4)} \\ &= \bigcup_{B\in\dot{x}}\bigcap_{y\in B}[y]_\top \\ &= [x]_\top \supseteq \mathbf{U}_{\mathbf{q_d}}(x). \end{aligned}$$

Hence, $(X, \mathbf{q}_d)$ satisfies **(TP2)** *and* **(TU)**. *It follows by Theorems 1 and 3 that $(X, \mathbf{q}_d)$ is topological generated.*

(2) *$(X, \mathbf{q}_{ind})$ is topological generated. Since any $\top$-filter converge to any point, then it follows immediately that $(X, \mathbf{q}_{ind})$ satisfies* **(TFD)**. *Hence, $(X, \mathbf{q}_d)$ is topological generated.*

Finally, we consider the categoric properties of topological generated $\top$-convergence spaces. A mapping $f : (X, \mathbf{q}) \longrightarrow (X', \mathbf{q}')$ between $\top$-convergence spaces is called continuous if $f^{\Rightarrow}(\mathbf{F}) \xrightarrow{\mathbf{q}'} f(x)$ for any $\mathbf{F} \xrightarrow{\mathbf{q}} x$.

We denote the category consisting of $\top$-convergence spaces and continuous mappings as $\top$-**CON**. $\top$-**CON** is a topological category in the sense that each source $(X \xrightarrow{f_i} (X_i, \mathbf{q}_i))_{i\in I}$ has initial structure $\mathbf{q}$ on X defined as follows [14,37]:

$$\mathbf{F} \xrightarrow{\mathbf{q}} x \Longleftrightarrow \forall i \in I, f_i^{\Rightarrow}(\mathbf{F}) \xrightarrow{\mathbf{q}_i} f_i(x).$$

Let $\top(X)$ denote all $\top$-convergence structures on X. For $\mathbf{p}, \mathbf{q} \in \top(X)$, we say $\mathbf{q}$ is finer than $\mathbf{p}$, denoted as $\mathbf{p} \leq \mathbf{q}$, if $id_X : (X, \mathbf{q}) \longrightarrow (X, \mathbf{p})$ is continuous. It is known that $(\top(X), \leq)$ forms a completed lattice [30].

We denote **TTG-CON** as the full subcategory of **T-CON** consisting of topological generated $\top$-convergence spaces.

Theorem 4. **TTG-CON** *is a topological category.*

Proof. We verify that **TTG-CON** has initial structure. Given a source $(X \xrightarrow{f_i} (X_i, \mathbf{q}_i))_{i\in I}$ in **TTG-CON**, take $\mathbf{q}$ as the initial structure in **T-CON**, i.e.,

$$\mathbf{F} \xrightarrow{\mathbf{q}} x \Leftrightarrow \forall i \in I, f_i^{\Rightarrow}(\mathbf{F}) \xrightarrow{\mathbf{q}_i} f_i(x).$$

Next, we show $(X, \mathbf{q}) \in$**TTG-CON**. Let $\psi : T \longrightarrow X$, $\phi : T \longrightarrow \mathbf{F}_L^\top(X)$ such that for every $t \in T, \phi(t) \xrightarrow{\mathbf{q}} \psi(t)$. Then, for any $i \in I, f_i^{\Rightarrow}(\phi(t)) \xrightarrow{\mathbf{q}_i} f_i(\psi(t))$. Putting $\phi_i = f_i^{\Rightarrow} \circ \phi$ and $\psi_i = f_i \circ \psi$, we obtain $\phi_i(t) \xrightarrow{\mathbf{q}_i} \psi_i(t)$.

Take any $\psi^{\Rightarrow}(\mathbf{F}) \xrightarrow{\mathbf{q}} x$. We have

$$\psi_i^{\Rightarrow}(\mathbf{F}) = (f_i \circ \psi)^{\Rightarrow}(\mathbf{F}) = f_i^{\Rightarrow}\psi^{\Rightarrow}(\mathbf{F}) \xrightarrow{\mathbf{q}_i} f_i(x).$$

Since $(X_i, \mathbf{q}_i)$ satisfies **(TFD)**, we obtain that $k\phi_i\mathbf{F} \xrightarrow{\mathbf{q}_i} x$. From Lemma 2(1), we further get for every $i \in I$,

$$f_i^{\Rightarrow}(k\phi\mathbf{F}) = k(f_i^{\Rightarrow} \circ \phi)\mathbf{F} = k\phi_i\mathbf{F} \xrightarrow{\mathbf{q}_i} f_i(x).$$

It follows that $k\phi\mathbf{F} \xrightarrow{\mathbf{q}} x$. Hence, $(X, \mathbf{q})$ satisfies **(TFD)**, as desired. □

Remark 1. *Let $\top_{TG}(X)$ denote all topological generated $\top$-convergence structures on X. Then, from Theorem 4, we conclude that $(\top_{TG}(X), \leq)$ forms a complete lattice.*

Theorem 5. **TTG-CON** *is a reflective subcategory of* **T-CON***.*

Proof. Given $(X, \mathbf{q}) \in$**T-CON**, put

$$\mathbf{rq} = \bigvee\{\mathbf{p}' \leq \mathbf{q} | \mathbf{p}' \in \top_{TG}(X)\}.$$

Then, $\mathbf{rq} \leq \mathbf{q}$ and so $id_X : (X, \mathbf{q}) \longrightarrow (X, \mathbf{rq})$ is continuous. Moreover, from Remark 1, we get $\mathbf{rq} \in \top_{TG}(X)$.

Let $(X, \mathbf{p}) \in$ **TTG-CON** and $f : (X, \mathbf{q}) \longrightarrow (Y, \mathbf{p})$ be a continuous mapping. Take $\mathbf{s}$ as the initial structure of $f : X \longrightarrow (Y, \mathbf{p})$ in **TTG-CON**. Then, $(X, \mathbf{s}) \in$ **TTG-CON** and

$$\mathbf{s} \leq \mathbf{q} \Longrightarrow \mathbf{s} \leq \mathbf{rq} \Longrightarrow id_X : (X, \mathbf{rq}) \longrightarrow (X, \mathbf{s}) \text{ is continuous,}$$

$$f : (X, \mathbf{s}) \longrightarrow (Y, \mathbf{p}) \text{ is continuous.}$$

It follows that $f = f \circ id_X : (X, \mathbf{rq}) \longrightarrow (Y, \mathbf{p})$ is continuous. This shows that the following diagram commutes in Figure 1.

$$\begin{tikzcd}
(X, \mathbf{q}) \arrow[r, "\mathrm{id}_X"] \arrow[d, "f"'] & (X, \mathbf{rq}) \arrow[dl, "f"] \\
(Y, \mathbf{p}) &
\end{tikzcd}$$

Figure 1. Reflective diagram.

Hence, **TTG-CON** is reflective in **T-CON**. □

4. Topological Generated $\top$-Convergence Spaces vs. Topological Generated L-Topological Spaces

In [14], Fang and Yue proposed a lattice-valued Fischer diagonal condition based on Gähler compression operator and proved that there is a bijection between $\top$-convergence spaces with their diagonal condition and strong L-topological spaces. In this section, we establish the relationships between our diagonal condition and Fang and Yue's diagonal condition. Then, analogizing Fang and Yue's result, we further verify that there is a one-to-one correspondence between topological generated $\top$-convergence spaces and topological generated L-topological spaces. Hence, we establish the convergence theory associated with the topological generated L-topological spaces.

Let T be any set and $\phi : T \longrightarrow \mathbf{F}_L^\top(X)$. Take $\mathbf{F} \in \mathbf{F}_L^\top(T)$ and $A \in L^X$, and define $\hat{\phi}(A) \in L^T$ as

$$\forall t \in T, \hat{\phi}(A)(t) = \bigvee_{B \in \phi(t)} S_X(B, A).$$

The *Gähler compression operator* on $\phi^\Rightarrow(\mathbf{F}) \in \mathbf{F}_L^\top(\mathbf{F}_L^\top(X))$ is defined by

$$g\phi\mathbf{F} := \{A \in L^X | \hat{\phi}(A) \in \mathbf{F}\} \in \mathbf{F}_L^\top(X).$$

For a $\top$-convergence space $(X, \mathbf{q})$, an extension of diagonal condition **(FD)** is presented in [14]:

(**TFDW**) Let T be any set, $\psi : T \longrightarrow X$ and $\phi : T \longrightarrow \mathbf{F}_L^{\top}(X)$ with $\phi(t) \xrightarrow{\mathbf{q}} \psi(t)$ for every $t \in T$. For any $\mathbf{F} \in \mathbf{F}_L^{\top}(X)$, $\psi^{\Rightarrow}(\mathbf{F}) \xrightarrow{\mathbf{q}} x \Longrightarrow g\phi\mathbf{F} \xrightarrow{\mathbf{q}} x$.

The next theorem shows that (**TFDW**) is weaker than (**TFD**) generally, but they are equivalent when $L = \{\perp, \top\}$.

Theorem 6. *Let T be any set, $\phi : T \longrightarrow \mathbf{F}_L^{\top}(X)$ and $\mathbf{F} \in \mathbf{F}_L^{\top}(T)$. Then,*

(1) $k\phi\mathbf{F} \subseteq g\phi\mathbf{F}$.
(2) $k\phi\mathbf{F} = g\phi\mathbf{F}$ *when* $L = \{\perp, \top\}$.

Proof. (1) Taking $A \in k\phi\mathbf{F}$, then $\exists B \in \eta(\mathbf{F})$ such that $A \in \phi(y)$ for any $y \in B$. It follows that

$$\forall y \in B, \hat{\phi}(A)(y) = \bigvee_{C \in \phi(y)} S_X(C, A) \geq S_X(A, A) = \top,$$

i.e., $\top_B \leq \hat{\phi}(A)$. By $\top_B \in \mathbf{F}$, we have $\hat{\phi}(A) \in \mathbf{F}$ and so $A \in g\phi\mathbf{F}$.

(2) We verify that $A \in g\phi\mathbf{F} \Longrightarrow A \in k\phi\mathbf{F}$. In fact, $A \in g\phi\mathbf{F}$ means $\hat{\phi}(A) \in \mathbf{F}$. Taking

$$B = \{y \in T | \hat{\phi}(A)(y) = \bigvee_{C \in \phi(y)} S_X(C, A) = \top\},$$

it follows that $A \in \phi(y)$ for any $y \in B$. Note that, when $L = \{\perp, \top\}$, $\top_B = \hat{\phi}(A) \in \mathbf{F}$, i.e., $B \in \eta(\mathbf{F})$. We get that $A \in k\phi\mathbf{F}$. □

In the following, we further verify that (**TFD**)$\Longleftrightarrow$(**TFDW**) for $\top$-convergence spaces generated by convergence spaces.

Lemma 3. *Let $\phi : T \longrightarrow \mathbf{F}_L^{\top}(X)$ and $\Phi : T \longrightarrow \mathrm{F}(X)$.*

(1) *Taking $\Phi_1 = \eta \circ \phi$, then for every $\mathbf{F} \in \mathbf{F}_L^{\top}(T)$, $\eta(g\phi\mathbf{F}) \supseteq K\Phi_1\eta(\mathbf{F})$.*
(2) *Taking $\phi_1 = \varpi \circ \Phi$, then for every $\mathrm{F} \in \mathrm{F}(T)$, $\eta(g\phi_1\varpi(\mathrm{F})) = K\Phi\mathrm{F}$.*

Proof. (1) It follows by

$$\begin{aligned}
A \in K\Phi_1\eta(\mathbf{F}) &\Longrightarrow \exists B \in \eta(\mathbf{F}), \text{ s.t. } \forall y \in B, A \in \Phi_1(y) = (\eta \circ \phi)(y) \\
&\Longrightarrow \exists \top_B \in \mathbf{F} \text{ s.t. } \forall y \in B, \top_A \in \phi(y) \\
&\Longrightarrow \exists \top_B \in \mathbf{F} \text{ s.t. } \forall y \in B, \hat{\phi}(\top_A)(y) = \top \\
&\Longrightarrow \exists \top_B \in \mathbf{F} \text{ s.t. } \top_B \leq \hat{\phi}(\top_A) \\
&\Longrightarrow \hat{\phi}(\top_A) \in \mathbf{F} \\
&\Longrightarrow \top_A \in g\phi\mathbf{F} \\
&\Longrightarrow A \in \eta(g\phi\mathbf{F}).
\end{aligned}$$

(2) Let $A \in \eta(g\phi_1\varpi(\mathrm{F}))$. Then, $\top_A \in g\phi_1\varpi(\mathrm{F})$ and so $\hat{\phi_1}(\top_A) \in \varpi(\mathrm{F})$. Note that, for every $t \in T$,

$$\begin{aligned}
\hat{\phi_1}(\top_A)(t) &= \bigvee_{B\in\phi_1(t)} S_X(B,\top_A) \\
&= \bigvee_{B\in\varpi(\Phi(t))} S_X(B,\top_A) = \bigvee_{C\in\Phi(t)} S_X(\top_C,\top_A) \\
&= \begin{cases} \top, & \exists C \in \Phi(t), \text{ s.t. } C \subseteq A, \text{ i.e., } A \in \Phi(t); \\ \bot, & \text{otherwise.} \end{cases}
\end{aligned}$$

Putting $C = \{t \in T | \hat{\phi_1}(\top_A)(t) = \top\}$, then $\top_C = \hat{\phi_1}(\top_A) \in \varpi(\mathrm{F})$, i.e., $C \in \mathrm{F}$ and $A \in \Phi(y)$ for any $y \in C$; it follows that $A \in K\Phi\mathrm{F}$. Therefore, $\eta(g\phi_1\varpi(\mathrm{F})) \subseteq K\Phi\mathrm{F}$.

Conversely, let $A \in K\Phi\mathrm{F}$. Then, there exists $B \in \mathrm{F}$ such that $A \in \Phi(y)$, i.e., $\top_A \in \varpi(\Phi(y)) = \phi_1(y)$ for any $y \in B$. That means $\hat{\phi_1}(\top_A)(y) = \top$ for any $y \in B$, and so $\top_B \leq \hat{\phi_1}(\top_A)$. Then, we get $\hat{\phi_1}(\top_A) \in \varpi(\mathrm{F})$, and hence $\top_A \in g\phi_1\varpi(\mathrm{F})$, i.e., $A \in \eta(g\phi_1\varpi(\mathrm{F}))$. Therefore, $K\Phi\mathrm{F} \subseteq \eta(g\phi_1\varpi(\mathrm{F}))$. □

Theorem 7. *Convergence space* (X, c) *fulfills* **(FD)** *iff* $(X, \pi(\mathrm{c}))$ *fulfills* **(TFDW)**.

Proof. $\Longrightarrow$. Let $\psi : T \longrightarrow X$, $\phi : T \longrightarrow \mathbf{F}_L^\top(X)$ with $\phi(t) \xrightarrow{\pi(\mathrm{c})} \psi(t)$ for every $t \in T$. Put $\Phi = \eta \circ \phi$; from $\phi(t) \xrightarrow{\pi(\mathrm{c})} \psi(t)$, we conclude

$$\Phi(t) = \eta(\phi(t)) \xrightarrow{\mathrm{c}} \psi(t).$$

Take any $\psi^{\Rightarrow}(\mathbf{F}) \xrightarrow{\pi(\mathrm{c})} x$; from Lemma 1(5), we obtain

$$\psi^{\Rightarrow}(\eta(\mathbf{F})) = \eta(\psi^{\Rightarrow}(\mathbf{F})) \xrightarrow{\mathrm{c}} x.$$

It follows by **(FD)** and Lemma 3(1) that $\eta(g\phi\mathbf{F}) \supseteq K\Phi\eta(\mathbf{F}) \xrightarrow{\mathrm{c}} x$, i.e., $g\phi\mathbf{F} \xrightarrow{\pi(\mathrm{c})} x$. Thus, the condition **(TFDW)** is satisfied.

$\Longleftarrow$. Let $\psi : T \longrightarrow X$, $\Phi : T \longrightarrow \mathrm{F}(X)$ with $\Phi(t) \xrightarrow{\mathrm{c}} \psi(t)$ for every $t \in T$. Putting $\phi = \varpi \circ \Phi$, then, from Lemma 1(1), we have

$$\eta \circ \phi(t) = \eta \circ \varpi \circ \Phi(t) = \Phi(t) \xrightarrow{\mathrm{c}} \psi(t), \text{ i.e., } \phi(t) \xrightarrow{\pi(\mathrm{c})} \psi(t).$$

Taking any $\psi^{\Rightarrow}(\mathrm{F}) \xrightarrow{\mathrm{c}} x$, from Lemma 1(5), we conclude

$$\eta\psi^{\Rightarrow}(\varpi(\mathbf{F})) = \psi^{\Rightarrow}((\eta \circ \varpi)(\mathbf{F})) = \psi^{\Rightarrow}(\mathrm{F}) \xrightarrow{\mathrm{c}} x, \text{ i.e., } \psi^{\Rightarrow}(\varpi(\mathbf{F})) \xrightarrow{\pi(\mathrm{c})} x.$$

Thus, $g\phi\varpi(\mathrm{F}) \xrightarrow{\pi(\mathrm{c})} x$ by **(TFDW)**. From Lemma 3(2), we further get $\eta(g\phi\varpi(\mathrm{F})) = K\Phi\mathrm{F} \xrightarrow{\mathrm{c}} x$. Hence, the condition **(FD)** is satisfied. □

From Theorems 2 and 7, we get that for $\top$-convergence spaces generated by convergence spaces, conditions **(TFD)** and **(TFDW)** are equivalent.

The following example shows that there is no **(TFD)**$\Leftrightarrow$**(TFDW)** for general $\top$-convergence space.

Example 2. Let $X = \{x, y\}$ and $L = ([0,1], \wedge)$. Define $A_x, A_y \in L^X$ as

$$A_x(z) = \begin{cases} 1, & z = x; \\ \frac{1}{2}, & z = y. \end{cases} ; A_y(z) = \begin{cases} \frac{1}{2}, & z = x; \\ 1, & z = y. \end{cases} ,$$

and take

$$\mathbf{F}_x = \{A \in L^X | A \geq A_x\}; \mathbf{F}_y = \{A \in L^X | A \geq A_y\}.$$

Then, it is easily seen that $\mathbf{F}_x, \mathbf{F}_y \in \mathbf{F}_L^\top(X)$. We define

$$\forall \mathbf{F} \in \mathbf{F}_L^\top(X), \forall z \in X, \mathbf{F} \xrightarrow{\mathbf{q}} z \Leftrightarrow \mathbf{F} \supseteq \mathbf{F}_z,$$

then $(X, \mathbf{q})$ is a $\top$-convergence space with $\top = 1$ and $\mathbf{U_q}(z) = \mathbf{F}_z$.

(1) $(X, \mathbf{q})$ satisfies **(TFDW)**. First, it is easily seen that $(X, \mathbf{q})$ satisfies **(TP3)**.

Second, let $\psi : T \longrightarrow X$ and $\phi : T \longrightarrow \mathbf{F}_L^\top(X)$ with $\phi(t) \xrightarrow{\mathbf{q}} \psi(t)$ for every $t \in T$. From **(TP3)**, we get that $\phi(t) \supseteq \mathbf{F}_{\psi(t)}$ for any $t \in T$.

Take any $\psi^\Rightarrow(\mathbf{F}) \xrightarrow{\mathbf{q}} x$; then, by **(TP3)**, we have $\mathbf{F}_x \subseteq \psi^\Rightarrow(\mathbf{F})$, and so $A_x \in \psi^\Rightarrow(\mathbf{F})$, i.e., $\psi^\leftarrow(A_x) \in \mathbf{F}$. Hence,

$$\begin{aligned} \hat{\phi}(A_x)(t) &= \bigvee_{B \in \phi(t)} S_X(B, A_x) \\ &\geq \bigvee_{B \in \mathbf{F}_{\psi(t)}} S_X(B, A_x) \\ &= S_X(A_{\psi(t)}, A_x) \\ &= A_{\psi(t)}(y) \rightarrow \frac{1}{2} \\ &= \begin{cases} 1, & \psi(t) = x; \\ \frac{1}{2}, & \psi(t) = y. \end{cases} \\ &= \psi^\leftarrow(A_x)(t), \end{aligned}$$

which means that $\hat{\phi}(A_x) \geq \psi^\leftarrow(A_x) \in \mathbf{F}$, i.e., $A_x \in g\phi\mathbf{F}$ and so $\mathbf{F}_x \subseteq g\phi\mathbf{F}$. Thus, $g\phi\mathbf{F} \xrightarrow{\mathbf{q}} x$.

(2) $(X, \mathbf{q})$ does not satisfy **(TU)**, and thus does not satisfy **(TFD)**. Indeed, it is easily observed that $\eta(\mathbf{U_q}(z)) = \{\{X\}\}$ for any $z \in X$ and so

$$\begin{aligned} k\mathbf{U_q}\mathbf{U_q}(z) &= \bigcup_{A \in \eta(\mathbf{U_q}(z))} \bigcap_{z \in A} \mathbf{U_q}(z) \\ &= \bigcap_{z \in X} \mathbf{U_q}(z) = \{\{\top_X\}\}. \end{aligned}$$

Obviously, $\mathbf{U_q}(z) \not\subseteq k\mathbf{U_q}\mathbf{U_q}(z)$. Thus, **(TU)** is not satisfied.

In ([14], Remark 3.3), Fang and Yue proved that a strong L-topological space (X, τ) corresponds uniquely to a $\top$-convergence space $(X, \mathbf{q})$ with the condition **(TFDW)** by taking that for each $x \in X$,

$$\mathbf{U_q}(x) = \mathbf{U}_\tau(x) := \{A \in L^X | \bigvee_{B \in \tau, B(x) = \top} S_X(B, A) = \top\}.$$

Then, they constructed a bijection between strong L-topological spaces and $\top$-convergence spaces with **(TFDW)**.

Let L be a continuous lattice and (X, δ) be a topological space. We prove below that, for any $x \in X$,

$$\mathbf{U}_{\pi(\delta)}(x) = \mathbf{U}_{\varrho(\delta)}(x).$$

That is, the $\top$-neighborhood system $\{\mathbf{U}_{\pi(\delta)}(x)\}_{x \in X}$ associated with topological generated $\top$-convergence space $(X, \pi(\delta))$ is equal to the $\top$-neighborhood system $\{\mathbf{U}_{\varrho(\delta)}(x)\}_{x \in X}$ associated with topological generated L-topological space $(X, \varrho(\delta))$.

Lemma 4. *Let L be a continuous lattice. Then, for a topological space (X, δ),*

$$A \in \varrho(\delta) \Longleftrightarrow \forall x \in X, A(x) \leq \bigvee_{x \in B \in \delta} S_X(\top_B, A).$$

Proof. Let $A \in \varrho(\delta)$ and $x \in X$. Taking any $\alpha \in L$ with $\alpha \ll A(x)$, then $x \in A_\alpha \in \delta$. It follows that

$$\alpha \leq \bigwedge_{y \in A_\alpha} A(y) = S_X(\top_{A_\alpha}, A) \leq \bigvee_{x \in B \in \delta} S_X(\top_B, A).$$

By the continuity of L and the arbitrariness of α, we obtain

$$A(x) \leq \bigvee_{x \in B \in \delta} S_X(\top_B, A).$$

Conversely, let $A(x) \leq \bigvee_{x \in B \in \delta} S_X(\top_B, A)$ for each $x \in X$. For each $\alpha \in L$, we prove below that $A_\alpha \in \delta$. Indeed, if $A_\alpha \neq \emptyset$, then taking any $x \in A_\alpha$ we have

$$\alpha \ll A(x) \leq \bigvee_{x \in B \in \delta} S_X(\top_B, A) = \bigvee_{x \in B \in \delta} \bigwedge_{y \in B} A(y).$$

Hence, there exists $B \in \delta$ such that $x \in B$ and $\alpha \ll A(y)$ for all $y \in B$, i.e., $B \subseteq A_\alpha$. Then, it follows that $A_\alpha \in \delta$ for any $\alpha \in L$. Therefore, $A \in \varrho(\delta)$. □

Theorem 8. *Let L be a continuous lattice. Then, for a topological space (X, δ), $\mathbf{U}_{\pi(\delta)}(x) = \mathbf{U}_{\varrho(\delta)}(x)$ for any $x \in X$.*

Proof. By the definition of $\pi(\delta)$, it is easily seen that $\mathbf{F} \xrightarrow{\pi(\delta)} x \Leftrightarrow \mathbf{F} \supseteq \varpi(\mathrm{U}_\delta(x))$, and so

$$\begin{aligned}
\mathbf{U}_{\pi(\delta)}(x) &= \varpi(\mathrm{U}_\delta(x)) \\
&= \{A \in L^X | \bigvee_{B \in \mathrm{U}_\delta(x)} S_X(\top_B, A) = \top\} \\
&= \{A \in L^X | \bigvee_{x \in B \in \delta} S_X(\top_B, A) = \top\}.
\end{aligned}$$

$$\mathbf{U}_{\varrho(\delta)}(x) = \{A \in L^X | \bigvee_{B \in \varrho(\delta), B(x) = \top} S_X(B, A) = \top\}.$$

Let $A \in \mathbf{U}_{\pi(\delta)}(x)$. Then,

$$\begin{aligned}\top &= \bigvee_{x\in B\in\delta} S_X(\top_B, A), \text{ by } \top_B \in \varrho(\delta), \top_B(x) = \top \\ &\le \bigvee_{C\in\varrho(\delta), C(x)=\top} S_X(C, A),\end{aligned}$$

i.e., $A \in \mathbf{U}_{\varrho(\delta)}(x)$. Conversely, let $A \in \mathbf{U}_{\varrho(\delta)}(x)$. Note that, by Lemma 4, we get that, for each $B \in \varrho(\delta), B(x) = \top$,

$$\top = B(x) = \bigvee_{x\in C\in\delta} S_X(\top_C, B).$$

Hence, from $A \in \mathbf{U}_{\varrho(\delta)}(x)$ it follows that

$$\begin{aligned}\top &= \bigvee_{B\in\varrho(\delta), B(x)=\top} S_X(B, A) \\ &= \bigvee_{B\in\varrho(\delta), B(x)=\top} (\top * S_X(B, A)) \\ &= \bigvee_{B\in\varrho(\delta), B(x)=\top} \Big(\bigvee_{x\in C\in\delta} S_X(\top_C, B) * S_X(B, A)\Big) \\ &\le \bigvee_{x\in C\in\delta} S_X(\top_C, A),\end{aligned}$$

i.e., $A \in \mathbf{U}_{\pi(\delta)}(x)$. $\square$

Remark 2. *Similar to ([14], Remark 3.3), we get from Theorem 8 that there is a bijection between topological generated $\top$-convergence spaces and topological generated L-topological spaces. Thus, $\top$-convergence spaces with our Fischer diagonal condition precisely characterizes topological generated L-topological spaces. Therefore, we establish the convergence theory associated with topological generated L-topological spaces.*

5. Conclusions

In this paper, we present a lattice-valued Fischer diagonal condition (**TFD**) through extending Kowalsky compression operator, and verify that there is a one-to-one correspondence between $\top$-convergence spaces with (**TFD**) and topological generated L-topological spaces. This shows that (**TFD**) can characterize topological generated L-topological spaces. That is to say, we establish the convergence theory associated with topological generated L-topological spaces. It is well-known that Fischer diagonal condition also plays an important role in uniform convergence spaces [1]. In the further work, we shall consider the fuzzy version of Fischer diagonal condition in $\top$-uniform convergence spaces defined in [39].

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