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A Portfolio Choice Problem in the Framework of Expected Utility Operators †

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Abstract: Possibilistic risk theory starts from the hypothesis that risk is modeled by fuzzy numbers. In particular, in a possibilistic portfolio choice problem, the return of a risky asset will be a fuzzy number. The expected utility operators have been introduced in a previous paper to build an abstract theory of possibilistic risk aversion. To each expected utility operator, one can associate the notion of possibilistic expected utility. Using this notion, we will formulate in this very general context a possibilistic portfolio choice problem. The main results of the paper are two approximate calculation formulas for the corresponding optimization problem. The first formula approximates the optimal allocation with respect to risk aversion and investor's prudence, as well as the first three possibilistic moments. Besides these parameters, in the second formula, the temperance index of the utility function and the fourth possibilistic moment appear.

Keywords: expected utility operators; possibilistic moments; portfolio choice problem

1. Introduction

The portfolio choice problem becomes an important topic in the economic and financial literature by the mean-variance model of Markowitz [1] and by the immediate next contributions due to Merton [2], Samuelson [3], Fama [4], etc. Today, there is a wide variety of portfolio choice problems and a rich literature dedicated to them (see, e.g., [5–7]) (in the paper were mentioned those references that had a direct relationship with the models and the results obtained). For example, the work in [5] overviewed some econometric approaches to portfolio choice problems, among which were plug-in estimation and Bayesian decision theory. This paper follows the research stream generated by the investment model from [8,9] (see also [6], Chapter 4, or [7], Chapter 5).

An agent investing wealth w_0 in a risk-free asset and a risky asset is interested in determining the optimal proportion in which to divide w_0 between the two assets. The mathematical model is a decision problem (called the standard portfolio choice problem) formulated in the context of von Neumann–Morgenstern expected utility [5–9]. The agent will choose the optimal wealth allocation in the risky asset by determining the solution of an optimization problem. An exact solution of the optimization problem is difficult to obtain; therefore, approximate solutions have been searched. Using Taylor-type approximations, various forms of approximate solutions have been found with respect to two classes of parameters: indicators of the investor's risk performance (absolute risk aversion, prudence, temperance, etc.) and various moments of the return of the risky asset (see [10–14]).

Thereby, the solutions of the optimization problems from [6], Chapter 4, or [7], Chapter 5, are expressed only according to the Arrow–Pratt absolute risk aversion index [8,9] associated with the investor's utility function and the expected value and variance corresponding to the investment risk. In the works [10,12], solutions were obtained depending on the Arrow–Pratt index, the Kimball prudence index [15,16], and the first three moments of risk. A more complex form of the solution, depending on the four moments and the indices of absolute risk aversion, prudence, and temperance [16], was proven in [13]. Another form of the solution in which the first four moments appear was obtained in [11].

The investment moments in the mentioned papers were probabilistic, and the return of the risky asset was a random variable. In real life, one can consider a risky asset as a fuzzy number. Then, considering a mathematical context consisting of a weighting function f, a fuzzy number A, and a utility function u, one can define some notions of possibilistic expected utility, and by the particularization of u, one can obtain various indicators associated with the fuzzy number A (the expected value, variance, moments, etc.) (see [17–21], etc.). Depending on the choice of the triple (f, A, u), different themes of risk theory have been treated: risk aversion [22–24], optimal saving and prudence [25–27], precautionary saving [28], risk assessment in grid computing (Chapter 7 of [18]), and the fuzzy pay-off method [18,29]. In the work [30], a possibilistic portfolio problem was formulated using the notion of possibilistic expected utility from [24] based on the framework of Zadeh' s possibility theory [31]. We determined an approximate calculation formula of the optimal solution in terms of the first three possibilistic moments of the fuzzy numbers representing the return of the risky asset and the investor's risk aversion and prudence indicators.

On the other hand, in [23] or [20], the notion of possibilistic expected utility different from the one from [24] was proposed. Using this notion, we can formulate another possibilistic portfolio choice problem, with a different solution from [24]. The expected utility operators were introduced in [32] to elaborate a general theory of possibilistic risk aversion. Each expected utility operator defines the notion of possibilistic expected utility. By particularization, we obtained the possibilistic expected utilities of [23,24]. Each expected utility operator verifies a property of linearity with respect to the utility function. In [33], a new class of operators (called Jensen-type operators) was introduced, which no longer verify the linearity property, but which still allow for the development of a theory of risk aversion (including an Arrow–Pratt theorem). In addition, Tassak et al. [34] and Sadefo et al. [35] developed an approach based on credibility theory, which we do not consider in this paper.

In this paper, we will formulate a standard portfolio choice problem in the context where asset return is a triangular fuzzy number and by means of a remarkable class of expected utility operators, namely D-operators. The first result of the paper (Proposition 5) is an approximate calculation formula of the optimal solution depending on the risk aversion and prudence indicators, as well as the first three moments defined by means of the considered D-operator. The second result (Theorem 1) is a more refined approximation formula of the optimal solution: besides the above-mentioned parameters, in the solution component appear the investor's temperance and the fourth-order T-moment. In the proof of Theorem 1 is used the formula from Proposition 5, which explains their presentation as separate entities. Furthermore, the formula from Proposition 5 is much simpler than the one from Theorem 1 and can be preferred in some calculations. We noticed that forms of the optimal solution from the problems associated with the D-operators from [23,24] were obtained. The two results of the paper can produce useful formulas and in the case of other particular D-operators (e.g., for convex combinations of D-operators from [23,24]).

In Section 2, by analogy with the notion of (probabilistic) expected utility from classical von Neumann–Morgenstern utility theory, two different notions of possibilistic expected utilities are recalled. The definitions of expected utility operators and the two usual examples are also recalled. By a natural property of derivability, the D-operators are introduced, offering a general framework for the study of the possibilistic portfolio choice problem.

Section 3 states our portfolio choice problem and proposes the T-standard model (T is a D-operator) and solutions of the model. More precisely, the model is formulated, and the way to find two approximate optimal solutions for the optimization problem associated with such a model is analyzed. The first one is in terms of the risk aversion and prudence of the utility function of the investor and the first three T-moments associated with a fuzzy number A in the component of the excess return (in the case of a small portfolio risk). The second approximate formula is based on the above-mentioned indicators and on the investor's temperance and the fourth-order moment. Section 4 contains some concluding remarks.

2. Review of Possibilistic Expected Utility Theories and Introduction to D-Operators

2.1. Possibilistic Expected Utility and Possibilistic Variance of a Fuzzy Number

The von Neumann–Morgenstern expected utility theory (EU-theory), a natural framework to study risk parameters phenomena, is based on two elements ([6–8]):

- a utility function u representing an agent;
- a random variable X representing the risk.

Thus, the (probabilistic) expected utility $E(u(X))$, defined as the mean value of the random variable $u(X)$, is the fundamental notion of EU-theory providing the indicators associated with X such as expected value, variance, moments, covariance, etc. In addition, by means of u and its derivatives, notions describing various attitudes of the agent towards risk are defined, such as: risk aversion, prudence, temperance, etc. (see [6–9,15]).

In real life, there are many uncertain situations that are not described by probability theory, but by Zadeh's possibility theory [31]. Accordingly, it is necessary to develop a possibilistic EU-theory built starting from the following three elements:

- a utility function u representing an agent;
- a fuzzy number A representing the risk (with the level sets $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0,1]$);
- a weighting function $f : [0,1] \to \mathbb{R}$ (f is a non-negative and increasing function that satisfies $\int_0^1 f(\gamma)d\gamma = 1$).

Based on three elements, the two following concepts of possibilistic expected utilities were introduced [17,19,23,24]:

$$E_1(f, u(A)) = \frac{1}{2}\int_0^1 [u(a_1(\gamma)) + u(a_2(\gamma))]f(\gamma)d\gamma$$

and:

$$E_2(f, u(A)) = \int_0^1 [\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} u(x)dx]f(\gamma)d\gamma.$$

Note that all the integrals that appear in this paper will be assumed finite.

Let us specify two particular cases of these two concepts.

(1) Setting $u = 1_{\mathbb{R}}$ (the identity of $\mathbb{R}$) in these concepts, the two possibilistic expected utilities introduce the same notion of possibilistic expected value:

$$E_f(A) = E_1(f, 1_{\mathbb{R}}(A)) = E_2(f, 1_{\mathbb{R}}(A)) = \frac{1}{2}\int_0^1 [a_1(\gamma) + a_2(\gamma)]f(\gamma)d\gamma.$$

(2) Setting $u(x) = (x - E_f(A))^2$ in these concepts, two possibilistic variances are obtained (these two types of possibilistic variance were studied in several papers (see, e.g., [17–21]) and were applied in the study of different possibilistic models [18,22,25,29,30,36–38]):

$$Var_1(f, A) = \frac{1}{2}\int_0^1 [(a_1(\gamma) - E_f(A))^2 + (a_2(\gamma) - E_f(A))^2]f(\gamma)d\gamma,$$

and:

$$Var_2(f,A) = \int_0^1 [\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} (x - E_f(A))^2 dx] f(\gamma) d\gamma.$$

Let us end this subsection by recalling one type of fuzzy number useful in this paper ([20], Definition 2.3.3): A triangular fuzzy number $A = (a, \alpha, \beta)$ with $a \in \mathbb{R}$ and $\alpha, \beta \geq 0$ is defined by:

$$A(t) = \begin{cases} 1 - \frac{a-t}{\alpha} & a - \alpha \leq t \leq a, \\ 1 - \frac{t-a}{\beta} & a \leq t \leq a + \beta, \\ 0 & otherwise. \end{cases}$$

Then, the level sets of A are $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, with $a_1(\gamma) = a - (1-\gamma)\alpha$ and $a_2(\gamma) = a + (1-\gamma)\beta$ for any $\gamma \in [0,1]$, and the support of A is $supp(A) = (a - \alpha, a + \beta)$.

By Example 3.3.10 from [20], the possibilistic expected value $E_f(A)$ associated with the triangular fuzzy number $A = (a, \alpha, \beta)$ has the form: $E_f(A) = a + \frac{\beta - \alpha}{6}$.

In the following subsection, we will recall the definition of the expected utility operators and a few generalities on them [20,30]. We will introduce the D-operators by a property regulating the behavior of an expected utility operator towards the derivation of the utility function with respect to a parameter.

We will denote by $\mathcal{F}$ the set of fuzzy numbers and $\mathcal{C}(\mathbb{R})$ the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. For each $a \in \mathbb{R}$, we denote by $\bar{a} : \mathbb{R} \to \mathbb{R}$ the constant function $\bar{a}(x) = a$, for $x \in \mathbb{R}$. $1_\mathbb{R}$ will be the identity function of $\mathbb{R}$. We fix a weighting function $f : [0,1] \to \mathbb{R}$.

2.2. Expected Utility Operators and D-Operators

Let us recall the expected utility operators.

Definition 1 ([20,30]). *An (f-weighted) expected utility operator is a function $T : \mathcal{F} \times \mathcal{C}(\mathbb{R}) \to \mathbb{R}$ such that for any $a, b \in \mathbb{R}$, $g, h \in \mathcal{C}(\mathbb{R})$ and $A \in \mathcal{F}$, the following conditions are fulfilled:*

(a) $T(A, 1_\mathbb{R}) = E_f(A)$;
(b) $T(A, \bar{a}) = a$;
(c) $T(A, ag + bh) = aT(A,g) + bT(A,h)$;
(d) $g \leq h$ *implies* $T(A,g) \leq T(A,h)$.

The real number $T(A,g)$ will be called the generalized possibilistic expected utility of A w.r.t. f and g.

Several times in the paper, we will write $T(A, g(x))$ instead of $T(A,g)$.

For any integer $k \geq 1$, we define:

- the k^{th}-order T-moment of A: $T(A,g)$, where $g(x) = x^k$ for any $x \in \mathbb{R}$;
- the k^{th}-order central T-moment of A: $T(A,g)$, where $g(x) = (x - E_f(A))^k$ for any $x \in \mathbb{R}$.

In particular, we have the following notions:

- the T-variance of A: $Var_T(A) = T(A, (x - E_f(A))^2)$;
- the T-skewness of A: $Sk_T(A) = T(A, (x - E_f(A))^3)$;
- the T-kurtosis of A: $K_T(A) = T(A, (x - E_f(A))^4)$.

The most studied expected utility operators are defined in the following two examples.

Example 1 ([24,32]). *The expected utility operator $S_1 : \mathcal{F} \times \mathcal{C}(\mathbb{R}) \to \mathbb{R}$ is defined by:*

$$S_1(A,g) = \frac{1}{2} \int_0^1 [g(a_1(\gamma)) + g(a_2(\gamma))] f(\gamma) d\gamma.$$

for any fuzzy number A whose level sets are $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0,1]$ and for any $g \in \mathcal{C}(\mathbb{R})$.

Example 2 ([24,32]). *The expected utility operator* $S_2 : \mathcal{F} \times \mathcal{C}(\mathbb{R}) \rightarrow \mathbb{R}$ *is defined by* $S_2(A, g) = \int_0^1 [\frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} g(x)dx] f(\gamma)d\gamma$, *for any fuzzy number* A *whose level sets are* $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0, 1]$ *and for any* $g \in \mathcal{C}(\mathbb{R})$.

In the rest of this subsection, we will introduce and study D-operators useful throughout this paper. For that, we consider the set $\mathcal{U}$ of functions $g(x, \lambda) : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfying the following property: $g(x, \lambda)$ is continuous with respect to x and of class C^n with respect to λ.

Definition 2. *An expected utility operator* $T : \mathcal{F} \times \mathcal{U} \rightarrow \mathbb{R}$ *is called derivable with respect to parameter* λ *(D-*operator*) if for any fuzzy number* A *and for any function* $g(x, \lambda) \in \mathcal{U}$, *the following conditions are fulfilled:*

(D_1) The function $\lambda \longmapsto T(A, g(., \lambda))$ *is derivable (with respect to* λ*);*
(D_2) $T(A, \frac{\partial g(.,\lambda)}{\partial \lambda}) = \frac{d}{d\lambda} T(A, g(., \lambda))$.

Let us give two examples.

Proposition 1. *The expected utility operators* S_1 *and* S_2 *are D-operators.*

Let us establish some properties of D-operators.

Proposition 2. *Let* $a_1, a_2 \in \mathbb{R}$, $g, h \in \mathcal{U}$ *and the function* $u = a_1 g + a_2 h$. *Then:*

(a) $u \in \mathcal{U}$.
(b) For any fuzzy number A, *the following equality holds:*
$\frac{d}{d\lambda} T(A, u(., \lambda)) = a_1 \frac{d}{d\lambda} T(A, g(., \lambda)) + a_2 \frac{d}{d\lambda} T(A, h(., \lambda))$.

Proof. Property (a) is immediate. For (b), we will notice first that $\frac{\partial u(.,\lambda)}{\partial \lambda} = a_1 \frac{\partial g(.,\lambda)}{\partial \lambda} + a_2 \frac{\partial h(.,\lambda)}{\partial \lambda}$. Thus, applying Definition 1 (c) and Property (D_2), we obtain:

$$\begin{aligned} \frac{d}{d\lambda} T(A, u(., \lambda)) &= T(A, \frac{\partial u(.,\lambda)}{\partial \lambda}) \\ &= a_1 T(A, \frac{\partial g(.,\lambda)}{\partial \lambda}) + a_2 T(A, \frac{\partial h(.,\lambda)}{\partial \lambda}) \\ &= a_1 \frac{d}{d\lambda} T(A, g(., \lambda)) + a_2 \frac{d}{d\lambda} T(A, h(., \lambda)). \quad \square \end{aligned}$$

In rest of this paper, we will define the model, in the general framework of an EU-theory associated with a D-operator and a possibilistic investment model called the T-standard model. We will then examine the optimal solution of the obtained model.

3. T-Standard Model in the Possibilistic Portfolio Problem

We will begin the subsection with the short description of the probabilistic investment model from [6] (pp. 55–56), which will serve as the starting point in the construction of our T-standard model. Then, we state our model with its underlying portfolio optimization problem.

3.1. Portfolio Choice Problem and the T-Standard Model

An agent invests an initial wealth w_0 in a risk-free asset (bonds) and in a risky asset (stocks). We will assume that the agent has a utility function u of class $\mathcal{C}^2$, increasing and concave. The amount invested in the risky asset will be denoted by α; thus, $w_0 - \alpha$ will be the amount invested in the risk-free asset.

The return of the risky asset is a random variable X_0. Let r be the return of the risk-free asset and x a value of X_0. The future wealth of the risk-free strategy will be $(w_0 - \alpha)(1 + r)$, and the value of the portfolio $(w_0 - \alpha, \alpha)$ will be $w + \alpha(x - r)$ (by [6], pp. 65–66) where $w = w_0(1 + r)$. Then, the investor follows the determination of that α, which would be the solution of the optimization problem:

$$\max_\alpha E(u(w + \alpha(X_0 - r))). \tag{1}$$

Denoting $X = X_0 - r$ as the excess return, the problem (1) can be written as:

$$\max_{\alpha} E(u(w + \alpha X)). \tag{2}$$

In the monographs [6,7] and in the works [12–14], Problem (2) was studied when the portfolio risk was small, i.e., the excess return X had the form $X = k\mu + Y$ where $\mu > 0$ and Y was a random variable with $E(Y) = 0$. Then, Model (2) obtains the form:

$$\max_{\alpha} E(u(w + \alpha(\mu k + Y))) \tag{3}$$

called the probabilistic standard model.

In the following, the model (3) will inspire us in the choice of the hypotheses and in the formulation of the T-standard model.

We will fix a weighting function $f : [0,1] \to \mathbb{R}$ and a D-operator $T : \mathcal{F} \times \mathcal{U} \to \mathbb{R}$.

At the base of the construction of the T-standard model are the following assumptions:

(H_1) The return of the risky asset is a fuzzy number B_0;

(H_2) The formulation of the optimization problem will use the notion of generalized possibilistic expected utility associated with the D-operator T (see the previous section).

We will denote by B the possibilistic excess return $B_0 - r$. We will be in the case of a small possibilistic portfolio risk (see also [30]), i.e., $B = k\mu + A$, where $\mu > 0$ and A is a fuzzy number with $E_f(A) = 0$. In this case, $E_f(B) = k\mu$.

Inspired by Model (3), we will consider the optimization problem:

$$\max_{\alpha} T(A, u(w + \alpha(k\mu + x))). \tag{4}$$

having the total utility function:

$$V(\alpha) = T(A, u(w + \alpha(k\mu + x))). \tag{5}$$

Taking into account condition (D_2) from Definition 2, we can compute the derivatives of $V(\alpha)$ defined by Equation (7):

$$V'(\alpha) = \frac{d}{d\alpha} T(A, u(w + \alpha(k\mu + x))) = T(A, (k\mu + x)u'(w + \alpha(k\mu + x))) \tag{6}$$

$$V''(\alpha) = T(A, (k\mu + x)^2 u''(w + \alpha(k\mu + x))). \tag{7}$$

By the hypothesis, $u'' \leq 0$; thus, using Conditions (d) and (b) from Definition 1, from Equation (7), it will follow that $V''(\alpha) \leq 0$. We have proven that the function $V(\alpha)$ is concave.

Let $\alpha(k)$ be the solution of the optimization problem $\max_{\alpha} V(\alpha)$, where $V(\alpha)$ has the form of Equation (5). By Equation (6), the first-order condition $V'(\alpha(k)) = 0$ will be written as:

$$T(A, (k\mu + x)u'(w + \alpha(k)(k\mu + x))) = 0. \tag{8}$$

As in the case of the probabilistic model from [7], we assume $\alpha(0) = 0$.

Assuming that $\alpha(k)$ is of class C^n, we will look for Taylor-type approximations of the solution $\alpha(k)$ of Equation (8):

$$\alpha(k) \approx \sum_{j=0}^{n} \frac{k^j}{j!} \alpha^{(j)}(0). \tag{9}$$

We approximate the derivative $u'(w+\alpha(k\mu+x))$ by:

$$u'(w+\alpha(k\mu+x)) \approx \sum_{j=0}^{n} \frac{u^{(j+1)}(w)}{j!}\alpha^j(k\mu+x)^j.$$

from which, by multiplying by $k\mu+x$, one obtains:

$$(k\mu+x)u'(w+\alpha(k\mu+x)) \approx \sum_{j=0}^{n} \frac{u^{(j+1)}(w)}{j!}\alpha^j(k\mu+x)^{j+1}. \tag{10}$$

Taking into account Condition (c) of Definition 1, from Equation (10) it follows that:

$$T(A,(k\mu+x)u'(w+\alpha(k\mu+x))) \approx \sum_{j=0}^{n} \frac{u^{(j+1)}(w)}{j!}\alpha^j T(A,(k\mu+x)^{j+1})$$

By the previous approximation, the first-order condition defined by Equation (8) gets the form:

$$\sum_{j=0}^{n} \frac{u^{(j+1)}(w)}{j!}(\alpha(k))^j T(A,(k\mu+x)^{j+1}) \approx 0. \tag{11}$$

The approximation (11) is an n^{th}-order equation in the unknown $\alpha(k)$. In most cases, it is difficult to find the exact solution; thus, different forms of approximate calculation for $\alpha(k)$ will be searched.

In the following subsection, we establish approximate calculation formulas for $\alpha(k)$ in which appear the indicators of absolute risk aversion and prudence of the utility function u defined as follows: the Arrow–Pratt index of absolute risk aversion $r_u(w) = -\frac{u''(w)}{u'(w)}$ (see [8,9]) and the Kimball prudence index $P_u(w) = -\frac{u'''(w)}{u''(w)}$ (see [15]).

3.2. Optimal Allocation Based on Absolute Risk Aversion and Prudence

We will consider the optimization problem (5), keeping all the notations from Section 3.1.

We will search for an approximation of the optimal allocation $\alpha(k)$ of the form:

$$\alpha(k) \approx \alpha(0) + k\alpha'(0) + \frac{1}{2}k^2\alpha''(0) = k\alpha'(0) + \frac{1}{2}k^2\alpha''(0). \tag{12}$$

The two first key results of this paper determined the approximate values of $\alpha'(0)$ and $\alpha''(0)$ by means of risk aversion and prudence.

Proposition 3. $\alpha'(0) \approx \frac{\mu}{T(A,x^2)}\frac{1}{r_u(w)}$.

Proof. For $n=1$, Equation (11) gets the form:

$$u'(w)T(A,k\mu+x) + \alpha(k)u''(w)T(A,(k\mu+x)^2) \approx 0. \tag{13}$$

By Definition 1, we have $T(A,k\mu+x) = k\mu + T(A,x) = k\mu + E_f(A)$. Deriving Equation (13) with respect to k and taking into account (D_2), we will obtain:

$$u'(w)\mu + u''(w)[\alpha'(k)T(A,(k\mu+x)^2) + 2\alpha(k)\mu T(A,k\mu+x)] \approx 0.$$

Setting $k=0$ and considering that $\alpha(0)=0$, we have $u'(w)\mu + u''(w)T(A,x^2)\alpha'(0) \approx 0$, from which it follows immediately that $\alpha'(0) \approx -\frac{\mu}{T(A,x^2)}\frac{u'(w)}{u''(w)} = \frac{\mu}{T(A,x^2)}\frac{1}{r_u(w)}$. $\square$

Proposition 4. $\alpha''(0) \approx \frac{P_u(w)}{(r_u(w))^2}\frac{T(A,x^3)}{(T(A,x^2))^3}\mu^2$.

Proof. For $n = 2$, Equation (11) becomes:

$$u'(w)T(A, k\mu + x) + u''(w)\alpha(k)T(A, (k\mu + x)^2) + \frac{u'''(w)}{2}(\alpha(k))^2 T(A, (k\mu + x)^3) \approx 0.$$

We recall that $T(A, k\mu + x) = k\mu + E_f(A) = k\mu$. We derive the above relation with respect to k, considering (D_2):

$$u'(w)\mu + u''(w)[\alpha'(k)T(A, (k\mu + x)^2 + 2\alpha(k)\mu T(A, k\mu + x)]$$

$$+\frac{u'''(w)}{2}[2\alpha(k)\alpha'(k)T(A, (k\mu + x)^3) + 3(\alpha(k))^2\mu T(A, (k\mu + x)^2)] \approx 0.$$

Deriving one more time with respect to k, setting $k = 0$, and considering $\alpha(0) = 0$ and $E_f(A) = 0$, we will obtain:

$$u''(w)\alpha''(0)T(A, x^2) + u'''(w)(\alpha'(0))^2 T(A, x^3) \approx 0$$

from which one obtains:

$$\alpha''(0) \approx -\frac{u'''(w)}{u''(w)}\frac{T(A, x^3)}{T(A, x^2)}(\alpha'(0))^2.$$

Taking into account Proposition 3 and by using the Arrow–Pratt index of absolute risk aversion and the Kimball prudence index, it follows that:

$$\alpha''(0) \approx \frac{P_u(w)}{(r_u(w))^2}\frac{T(A, x^3)}{(T(A, x^2))^3}\mu^2.$$

□

The following first main result of this paper established the first approximate value of the solution of our T-model.

Proposition 5. *The optimal allocation $\alpha(k)$ can be approximated as:*

$$\alpha(k) \approx \frac{\mu}{r_u(w)}\frac{1}{T(A, x^2)}k + \frac{1}{2}\mu^2\frac{P_u(w)}{(r_u(w))^2}\frac{T(A, x^3)}{(T(A, x^2))^3}k^2.$$

Proof. $\alpha'(0)$ and $\alpha''(0)$ are replaced in Equation (12) with their approximate values from Propositions 3 and 4. □

Remark 1. *Since $E_f(A) = 0$, it follows that $T(A, x^2) = Var_T(A)$ and $T(A, x^3) = Sk_T(A)$. Thus, by Proposition 5, the optimal allocation $\alpha(k)$ can be approximated by:*

$$\alpha(k) \approx \frac{\mu}{r_u(w)}\frac{1}{Var_T(A)}k + \frac{1}{2}\mu^2\frac{P_u(w)}{(r_u(w))^2}\frac{Sk_T(A)}{(Var_T(A))^3}k^2. \tag{14}$$

As seen from Equation (14), $\alpha(k)$ is expressed according to:

- *the Arrow–Pratt index $r_u(w)$ and the prudence index $P_u(w)$*
- *the T-variance $Var_T(A)$ and the T-skewness $Sk_T(A)$.*

Let us display some examples of our solution.

Example 3. *We consider the utility function* $u(w) = \frac{w^a}{a}$ *with* $a \neq 0$. *A simple computation shows that* $r_u(w) = \frac{1-a}{w}$; $P_u(w) = \frac{2-a}{w}$ *and* $T_u(w) = \frac{3-a}{w}$, *from which it follows that* $\frac{1}{r_u(w)} = \frac{w}{1-a}$ *and* $\frac{P_u(w)}{(r_u(w))^2} = \frac{2-a}{(1-a)^2}w$.

Replacing $\frac{1}{r_u(w)}$ *and* $\frac{P_u(w)}{(r_u(w))^2}$ *in the expression of* $\alpha(k)$ *from Proposition 5, we find:*

$$\alpha(k) \approx \frac{k\mu w}{(1-a)T(A,x^2)}[1 + \frac{k\mu}{2}\frac{2-a}{1-a}\frac{T(A,x^3)}{(T(A,x^2))^3}].$$

We assume that the weighting function is $f(\gamma) = 2\gamma$, $A = (b, \alpha, \beta)$ *is a triangular fuzzy number with* $E_f(A) = 0$, *and* T *is the D-operator* S_1 *from Example 1.*

By [38], Remark 2.1 (a) and (b), we have:

$$S_1(A,x^2) = Var_{T_1}(A) = \frac{\alpha^2 + \beta^2 + \alpha\beta}{18}, \tag{15}$$

and:

$$S_1(A,x^3) = Sk_{T_1}(A) = \frac{19(\beta^2 - \alpha^2)}{1080} + \frac{\alpha\beta(\beta - \alpha)}{72}. \tag{16}$$

Then, Formula (3) becomes:

$$\alpha(k) \approx \frac{18k\mu w}{(1-a)(\alpha^2+\beta^2+\alpha\beta)}[1 + \frac{9k\mu(2-a)}{1-a}\frac{\frac{57(\beta^2-\alpha^2)}{10} + \frac{9\alpha\beta(\beta-\alpha)}{2}}{(\alpha^2+\beta^2+\alpha\beta)^3}].$$

In the case of a symmetric triangular fuzzy number $A = (b, \alpha)$, *we have* $b = 0$, $\alpha = \beta$, *and the previous formula becomes:*

$$\alpha(k) \approx \frac{6\mu w}{(1-a)\alpha^2}k.$$

Example 4. *Assume that the utility function* u *is HARA-type (see [7], Section 3.6):* $u(w) = \zeta(\delta + \frac{w}{\gamma})^{1-\gamma}$ *for* $\delta + \frac{w}{\gamma} > 0$.

According to [7], Section 3.6, we have: $r_u(w) = (\delta + \frac{w}{\gamma})^{-1}$ *and* $P_u(w) = \frac{\gamma+1}{\gamma}(\delta + \frac{w}{\gamma})^{-1}$, *from which it follows that:* $\frac{1}{r_u(w)} = \delta + \frac{w}{\gamma}$ *and* $\frac{P_u(w)}{(r_u(w))^2} = \frac{\gamma+1}{\gamma}(\delta + \frac{w}{\gamma})$.

Replacing in Equation (14) $\frac{1}{r_u(w)}$ *and* $\frac{P_u(w)}{(r_u(w))^2}$ *with the values obtained above, the optimal allocation* $\alpha(k)$ *will be approximated as:*

$$\alpha(k) \approx k\mu(\delta + \frac{w}{\gamma})\frac{1}{Var_T(A)} + \frac{1}{2}(k\mu)^2\frac{\gamma+1}{\gamma}(\delta + \frac{w}{\gamma})\frac{Sk_T(A)}{(Var_T(A))^3}.$$

Assume that the weighting function f *has the form* $f(t) = 2t$, $t \in [0,1]$. *The risk* A *is a triangular fuzzy number* $A = (b, \alpha, \beta)$ *with* $E_f(A) = 0$, *and* T *is the D-operator* S_1. *By Equations (15) and (16), in this case, the previous formula will get the form:*

$$\alpha(k) \approx 18\mu(\delta + \frac{w}{\gamma})\frac{1}{\alpha^2+\beta^2+\alpha\beta}k + \frac{18^3}{2}\mu^2\frac{\gamma+1}{\gamma}(\delta + \frac{w}{\gamma})\frac{\frac{19(\beta^2-\alpha^2)}{1080} + \frac{\alpha\beta(\beta-\alpha)}{72}}{(\alpha^2+\beta^2+\alpha\beta)^3}k^2.$$

In the following subsection, we shall prove an approximate calculation formula for the optimal allocation $\alpha(k)$ in terms of the following parameters:

- the indicators of risk aversion, prudence, and temperance associated with the utility function u;
- T-variance, T-skewness, and T-kurtosis associated with the fuzzy number A.

3.3. Optimal Allocation in Terms of Absolute Risk Aversion, Prudence, and Temperance

To find the way the temperance indicator $T_u(w) = -\frac{u^{iv}(w)}{u'''(w)}$ appears in the optimal solution $\alpha(k)$, we will write Formula (9) for $n = 3$:

$$\alpha(k) \approx k\alpha'(0) + \frac{1}{2}k^2\alpha''(0) + \frac{1}{3!}k^3\alpha'''(0). \tag{17}$$

The following third key result of this paper established an approximate calculation formula for $\alpha'''(0)$. The Proof is in Appendix A.

Proposition 6. *The real numbers $\alpha'(0)$, $\alpha''(0)$, and $\alpha'''(0)$ from Equation (17) verify the following dependence relation:*

$$\alpha'''(0)T(A,x^2) + 6\alpha'(0)\mu^2 - 3P_u(w)[\alpha'(0)\alpha''(0)T(A,x^3)+$$
$$3\mu(\alpha'(0))^2T(A,x^2)] + +\frac{T_u(w)}{P_u(w)}(\alpha'(0))^3T(A,x^4) \approx 0.$$

Using the dependence relation of Proposition 6, we will present below a more refined formula of the approximate calculation of the optimal solution $\alpha(k)$. Thus, the second main result of this paper established the approximate value of the optimal solution of our model.

Theorem 1. $\alpha(k) \approx \frac{\mu}{r_u(w)}\frac{1}{T(A,x^2)}k + \frac{1}{2}(\mu)^2\frac{P_u(w)}{(r_u(w))^2}\frac{T(A,x^3)}{(T(A,x^2))^3}k^2 + (-\frac{\mu^3}{r_u(w)}\frac{1}{(T(A,x^2))^2} + \frac{1}{2}\mu^3\frac{(P_u(w))^2}{(r_u(w))^3}\frac{(T(A,x^3))^2}{(T(A,x^2))^5} + \frac{3}{2}\mu^3\frac{P_u(w)}{(r_u(w))^2}\frac{1}{(T(A,x^2))^2} - \frac{1}{6}\mu^3\frac{T_u(w)}{P_u(w)(r_u(w))^3}\frac{T(A,x^4)}{(T(A,x^2))^4})k^3$.

Proof. According to Equation (12) and Proposition 5,

$$k\alpha'(0) + \frac{1}{2}k^2\alpha''(0) \approx \frac{k\mu}{r_u(w)}\frac{1}{T(A,x^2)} + \frac{1}{2}(k\mu)^2\frac{P_u(w)}{(r_u(w))^2}\frac{T(A,x^3)}{(T(A,x^2))^3}. \tag{18}$$

We remark that in the component of Equation (17), besides the expression of Equation (18), $\frac{1}{3!}k^3\alpha'''(0)$ appears. We will compute this term using the dependence relation between $\alpha'(0)$, $\alpha''(0)$, and $\alpha'''(0)$ from Proposition 6.

Taking into account the approximate values of $\alpha'(0)$ and $\alpha''(0)$ from Propositions 3 and 4, a simple computation shows that:

- $\frac{1}{3!}k^3 6\alpha'(0)\mu^2 = \frac{(k\mu)^3}{r_u(w)}\frac{1}{T(A,x^2)}$,
- $\frac{1}{3!}k^3 3P_u(w)\alpha'(0)\alpha''(0)T(A,x^3) = \frac{1}{2}(k\mu)^3\frac{(P_u(w))^2}{(r_u(w))^3}\frac{(T(A,x^3))^2}{T(A,x^2))^4}$,
- $\frac{1}{3!}k^3 9P_u(w)\mu(\alpha'(0))^2T(A,x^2) = \frac{3}{2}(k\mu)^3\frac{P_u(w)}{(r_u(w))^2}\frac{1}{(T(A,x^2)}$,
- $\frac{1}{3!}k^3(\alpha'(0))^3T(A,x^4)\frac{T_u(w)}{P_u(w)} = \frac{1}{6}(k\mu)^3\frac{T_u(w)}{P_u(w)(r_u(w))^3}\frac{T(A,x^4)}{(T(A,x^2))^3}$.

Multiplying the identity of Proposition 6 by $\frac{1}{3!}k^3$ and taking into account the four equalities above, it follows:

$$\frac{1}{3!}k^3\alpha'''(0)T(A,x^2) + \frac{(k\mu)^3}{r_u(w)}\frac{1}{T(A,x^2)} - \frac{1}{2}(k\mu)^3\frac{(P_u(w))^2}{(r_u(w))^3}\frac{(T(A,x^3))^2}{(T(A,x^2))^4}$$
$$-\frac{3}{2}(k\mu)^3\frac{P_u(w)}{(r_u(w))^2}\frac{1}{T(A,x^2)} + \frac{1}{6}(k\mu)^3\frac{T_u(w)}{P_u(w)(r_u(w))^3}\frac{T(A,x^4)}{(T(A,x^2))^3} \approx 0.$$

From this equation, we find the value of $\frac{1}{3!}k^3\alpha'''(0)$:

$$\frac{1}{3!}k^3\alpha'''(0) \approx -\frac{(k\mu)^3}{r_u(w)}\frac{1}{(T(A,x^2))^2} + \frac{1}{2}(k\mu)^3\frac{(P_u(w))^2}{(r_u(w))^3}\frac{(T(A,x^3))^2}{(T(A,x^2))^5}+$$

$$+\frac{3}{2}(k\mu)^3\frac{P_u(w)}{(r_u(w))^2}\frac{1}{(T(A,x^2))^2}-\frac{1}{6}(k\mu)^3\frac{T_u(w)}{P_u(w)(r_u(w))^3}\frac{T(A,x^4)}{(T(A,x^2))^4}.$$

Replacing in Equation (17), $k\alpha'(0)+\frac{1}{2}k^2\alpha''(0)$ with the value from Equation (18) and $\frac{1}{3!}k^3\alpha'''(0)$ with the above-computed value, it follows for $\alpha(k)$ the approximate value from the enunciation. □

We rewrite our second main result by means of the four first-order central moments of the risky asset.

Corollary 1.

$$\alpha(k)\approx\frac{\mu}{r_u(w)}\frac{1}{Var_T(A)}k+\frac{1}{2}(\mu)^2\frac{P_u(w)}{(r_u(w))^2}\frac{Sk_T(A)}{(Var_T(A))^3}k^2+$$
$$[-\frac{\mu^3}{r_u(w)}\frac{1}{(Var_T(A))^2}+\frac{1}{2}\mu^3\frac{(P_u(w))^2}{(r_u(w))^3}\frac{(Sk_T(A))^2}{(Var_T(A))^5}$$
$$+\frac{3}{2}\mu^3\frac{P_u(w)}{(r_u(w))^2}\frac{1}{(Var_T(A))^2}-\frac{1}{6}\mu^3\frac{T_u(w)}{P_u(w)(r_u(w))^3}\frac{K_T(A)}{(Var_T(A))^4}]k^3.$$

Proof. Since $E_f(A)=0$, we will have $T(A,x^2)=Var_T(A)$, $T(A,x^3)=Sk_T(A)$, and $T(A,x^4)=K_T(A)$. □

The approximate expression of $\alpha(k)$ from Corollary 1 is quite complicated. Therefore, we denote: $F_1=\frac{1}{r_u(w)}\frac{1}{Var_T(A)}$; $F_2=\frac{P_u(w)}{(r_u(w))^2}\frac{Sk_T(A)}{(Var_T(A))^3}$; $F_3=\frac{1}{r_u(w)}\frac{1}{(Var_T(A))^2}$; $F_4=\frac{(P_u(w))^2}{(r_u(w))^3}\frac{(Sk_T(A))^2}{(Var_T(A))^5}$; $F_5=\frac{P_u(w)}{(r_u(w))^2}\frac{1}{(Var_T(A))^2}$; and $F_6=\frac{T_u(w)}{P_u(w)(r_u(w))^3}\frac{K_T(A)}{(Var_T(A))^4}$.

From Corollary 1, we will obtain:

Remark 2. *The approximate value of the optimal allocation $\alpha(k)$ will be computed with the following formula:*

$$\alpha(k)\approx k\mu F_1+\frac{1}{2}(k\mu)^2F_2-(k\mu)^3[F_3-\frac{1}{2}F_4-\frac{3}{2}F_5+\frac{1}{6}F_6]. \tag{19}$$

To obtain the approximate value of $\alpha(k)$, we will compute first: $F_1,\ldots,F_6$ with the previous formulas, then these will be replaced in Equation (19).

One notices that the expression of the optimal solution from Proposition 5 appears in the component of the more general formula of Theorem 1 (in the form of the first two terms). The fact that the proof of Theorem 1 uses the expression from Proposition 5 imposes the separate presentation of the two forms of the approximate solution. Since the formula from Theorem 1 leads to quite complicated calculations, in some cases, the form of the solution from Remark 2 might be preferred.

Example 5. *We consider the possibilistic portfolio Problem 4 with the initial data:*

- *the weighting function is $f(t)=2t$, $t\in[0,1]$*
- *the agent's utility function is $u(w)=w^a$, $a>0$.*

Then, by Example 3, we have: $r_u(w)=\frac{1-a}{w}$; $P_u(w)=\frac{2-a}{w}$; $T_u(w)=\frac{3-a}{w}$; $\frac{1}{r_u(w)}=\frac{w}{1-a}$; and $\frac{P_u(w)}{(r_u(w))^2}=\frac{2-a}{(1-a)^2}w$.

Moreover, we will have: $\frac{(P_u(w))^2}{(r_u(w))^3}=\frac{(2-a)^2}{(1-a)^3}w$ and $\frac{T_u(w)}{P_u(w)(r_u(w))^3}=\frac{3-a}{(2-a)(1-a)^3}w^3$.

Replacing in $(F_i)_{i\in 1,2,3,4,5,6}$, it follows that: $F_1=\frac{w}{1-a}\frac{1}{Var_T(A)}$; $F_2=\frac{(2-a)w}{(1-a)^2}\frac{Sk_T(A)}{(Var_T(A))^3}$; $F_3=\frac{w}{1-a}\frac{1}{(Var_T(A))^3}$; $F_4=\frac{(2-a)^2w}{(1-a)^3}\frac{(S_kT(A))^2}{(Var_T(A))^5}$; $F_5=\frac{(2-a)w}{(1-a)^2}\frac{1}{(Var_T(A))^2}$; and $F_6=\frac{(3-a)w^3}{(2-a)(1-a)^3}\frac{K_T(A)}{(Var_T(A))^4}$.

Assume that the risk is represented by the triangular fuzzy number $A = (b, \alpha, \beta)$ and T is the D-operator S_1. Then, $Var_{S_1}(A)$ and $Sk_{S_1}(A)$ can be computed with Formulas (15) and (16). By [38], Remark 2.1 (2), for $K_{S_1}(A)$, we have the following value:

$$T(A, x^4) = K_{S_1}(A) = \frac{\beta^2\alpha^2}{72} + \frac{5(\alpha^4+\beta^4)}{432} + \frac{2\alpha\beta(\alpha^2+\beta^2)}{135}.$$

Replacing the values of $Var_{S_1}(A)$, $Sk_{S_1}(A)$, and $K_{S_1}(A)$ in the above expressions of F_1–F_6, we find the following forms of them:

$F_1 = \frac{w}{1-a}\frac{18}{\alpha^2+\beta^2+\alpha\beta}$,

$F_2 = \frac{(2-a)w}{(1-a)^2}\frac{18^3[\frac{19(\beta^2-\alpha^2)}{1080}+\frac{\alpha\beta(\beta-\alpha)}{72}]}{(\alpha^2+\beta^2+\alpha\beta)^3}$,

$F_3 = \frac{w}{1-a}\frac{18^3}{(\alpha^2+\beta^2+\alpha\beta)^3}$,

$F_4 = \frac{(2-a)^2 w}{(1-a)^3}\frac{18^5[\frac{19(\beta^2-\alpha^2)}{1080}+\frac{\alpha\beta(\beta-\alpha)}{72}]^2}{(\alpha^2+\beta^2+\alpha\beta)^5}$,

$F_5 = \frac{(2-a)w}{(1-a)^2}\frac{324}{(\alpha^2+\beta^2+\alpha\beta)^2}$, *and*

$F_6 = \frac{(3-a)w^3}{(2-a)(1-a)^3}\frac{1458\alpha^2\beta^2+1215(\alpha^4+\beta^4)+\frac{2\times 18^4}{135}\alpha\beta(\alpha^2+\beta^2)}{(\alpha^2+\beta^2+\alpha\beta)^4}$.

Replacing the obtained values of $F_1, \ldots, F_6$ in Equation (19), one obtains the approximate value of the optimal allocation $\alpha(k)$.

4. Concluding Remarks

In this paper, we addressed the optimization portfolio problem in the framework of a possibilistic EU-theory when the risky asset is a fuzzy number.

The first contribution of the paper was the introduction of special expected utility operators, called D-operators. These were defined by preserving the partial derivability of the utility function with respect to a parameter, which allowed the study of the first-order conditions of the optimization problems. The second contribution of the paper was the formulation of a possibilistic portfolio choice problem inside of EU-theory associated with a D-operator T. The third contribution was the proof of an approximate formula for the solution of the optimization problem associated with that portfolio problem based on the indicators of the investor's preferences (risk aversion, prudence, temperance) and the possibilistic moments associated with T. The form of the approximate solution depended on the fixed D-operator T.

In a future paper, we will analyze the way the probability-possibility transformation of [39] acts on the portfolio choice problems and on their solutions. Furthermore, we will research how by applying the Vercher et al. procedure [40] to a dataset one obtains a possibilistic choice problem having the form of the one studied in this paper. By applying the formulas from Proposition 5 and Theorem 1, we will be able to obtain approximate solutions of the resulting choice problem. Finding some procedures (theoretical or empirical) to choose the most appropriate D-operator depending on a consistent dataset remains an open problem. Another open problem is in the context of the D-operator, to study models with two types of risk, besides the investment risk from the standard model to appear as a background risk. Both the investment risk and the background risk can be either random variables or fuzzy numbers; thus, we would have four models per total. For each of the four background risk models, we should find approximations of the corresponding optimization problems, such that in the particular case $T = S_1$ are found the results from [30].

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Appendix A. Proof of Proposition 6

Proof. For $n = 3$, the first-order condition (11) gets the form

$u'(w)T(A, k\mu + x) + u''(w)\alpha(k)T(A,(k\mu + x)^2) + \frac{u'''(w)}{2}(\alpha(k))^2 T(A,(k\mu + x)^3) + \frac{u^{iv}(w)}{3!}(\alpha(k))^3 T(A,(k\mu + x)^4) \approx 0.$

From the previous section, we know that $T(A, k\mu + x) = k\mu$. For a better structure of the computations, we will denote

$T_1(k) = \alpha(k)T(A,(k\mu + x)^2)$, $T_2(k) = (\alpha(k))^2 T(A,(k\mu + x)^3)$ and $T_3(k) = (\alpha(k))^3 T(A,(k\mu + x)^4)$.

With these notations, the above equation will be written as:

$$u'(w)k\mu + u''(w)T_1(k) + \frac{u'''(w)}{2}T_2(k) + \frac{u^{iv}(w)}{3!}T_3(k) \approx 0.$$

From this, deriving three times, one will obtain:

$$u''(w)T_1'''(k) + \frac{u'''(w)}{2}T_2'''(k) + \frac{u^{iv}(w)}{3!}T_3'''(k) \approx 0.$$

For $k = 0$, it follows that:

$$u''(w)T_1'''(0) + \frac{u'''(w)}{2}T_2'''(0) + \frac{u^{iv}(w)}{3!}T_3'''(0) \approx 0. \tag{A1}$$

We recall from calculus that for the three-times derivable functions f, g, we have:

$$(fg)''' = f'''g + 3f''g' + 3f'g'' + fg'''. \tag{A2}$$

Formula (A2) will be used to determine $T_1'''(0)$, $T_2'''(0)$, and $T_3'''(0)$.

The computation of $T_1'''(0)$:

By Formula (A2), we have:

$$T_1'''(k) = \alpha'''(k)T(A,(k\mu + x)^2) + 3\alpha''(k)\frac{d}{dk}T(A,(k\mu + x)^2)$$

$$+3\alpha'(k)\frac{d^2}{dk^2}T(A,(k\mu + x)^2) + \alpha(k)\frac{d^3}{dk^3}T(A,(k\mu + x)^2).$$

Applying Condition (D_2), we will have: $\frac{d}{dk}T(A,(k\mu + x)^2) = 2\mu T(A, k\mu + x) = 2\mu^2 k$ and $\frac{d^2}{dk^2}T(A,(k\mu + x)^2) = 2\mu^2$, from which it follows that $\frac{d}{dk}T(A,(k\mu + x)^2)|_{k=0} = 0$ and $\frac{d^2}{dk^2}T(A,(k\mu + x)^2)|_{k=0} = 2\mu^2$.

Replacing these values in the above expression of $T_1'''(k)$ and knowing that $\alpha(0) = 0$, it will follow that:

$$T_1'''(0) = \alpha'''(0)T(A, x^2) + 6\alpha'(0)\mu^2.$$

The computation of $T_2'''(0)$:

If we denote $g(k) = (\alpha(k))^2$, then knowing Formula (A2), $T_2'''(k)$ is written as:

$$T_2'''(k) = g'''(k)T(A,(k\mu + x)^3) + 3g''(k)\frac{d}{dk}T(A,(k\mu + x)^3) +$$

$$3g'(k)\frac{d^2}{dk^2}T(A,(k\mu + x)^3) + g(k)\frac{d^3}{dk^3}T(A,(k\mu + x)^3).$$

To determine $T_2'''(0)$, we will compute the values of all terms in the component of $T_2'''(k)$ for $k = 0$.

We remark that: $g(k) = (\alpha(k))^2$; $g(0) = 0$; $g'(k) = 2\alpha'(k)\alpha(k)$; $g'(0) = 0$; $g''(k) = 2[\alpha''(k)\alpha(k) + (\alpha'(k))^2]$; $g''(0) = 2(\alpha'(0))^2$; $g'''(k) = 2[\alpha'''(k)\alpha(k) + 3\alpha'(k)\alpha''(k)]$; and $g'''(0) = 6\alpha'(0)\alpha''(0)$.

Applying Condition (D_2), we will compute the new derivatives: $\frac{d}{dk}T(A,(k\mu + x)^3) = 3\mu T(A,(k\mu + x)^2)$; $\frac{d}{dk}T(A,(k\mu + x)^3)|_{k=0} = 3\mu T(A,x^2)$; and $\frac{d^2}{dk^2}T(A,(k\mu + x)^3) = 6\mu^2 T(A,k\mu + x) = 6\mu^3 k$.

With these computations, we obtain the expression of $T_2'''(0)$:

$$T_2'''(0) = g'''(0)T(A,x^3) + 3g''(0)3\mu T(A,x^2)$$
$$= 6\alpha'(0)\alpha''(0)T(A,x^3) + 18\mu(\alpha'(0))^2 T(A,x^2).$$

The computation of $T_3'''(0)$:

We denote $h(k) = (\alpha(k))^3$. By the previous expression of $T_3(k)$ and formula (A2), we have:

$$T_3'''(k) = h'''(k)T(A,(k\mu + x)^4) + 3h''(k)\frac{d}{dk}T(A,(k\mu + x)^4)$$

$$+3h'(k)\frac{d^2}{dk^2}T(A,(k\mu + x)^4) + h(k)\frac{d^3}{dk^3}T(A,(k\mu + x)^4).$$

By a simple computation, one obtains $h(0) = h'(0) = h''(0) = 0$; and $h'''(0) = 6(\alpha'(0))^3$.

From this, setting $k = 0$ in the above expression of $T_3'''(k)$, it follows that:

$$T_3'''(k) = h'''(0)T(A,x^4) = (6\alpha'(0))^3 T(A,x^4).$$

Replacing the found values of $T_1'''(0)$, $T_2'''(0)$, and $T_3'''(0)$ in Equation (A1), one obtains:

$$u''(w)[\alpha'''(0)T(A,x^2) + 6\alpha'(0)\mu^2]$$

$$+\frac{u'''(w)}{2}[6\alpha'(0)\alpha''(0)T(A,x^3) + 18\mu(\alpha'(0))^2 T(A,x^2)]$$

$$+\frac{u^{iv}(w)}{6}[6(\alpha'(0))^3 T(A,x^4)] \approx 0.$$

Dividing by $u''(w)$ and taking into account that $\frac{u^{iv}(w)}{u''(w)} = \frac{T_u(w)}{P_u(w)}$, it follows the equation from Proposition 6. $\square$

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