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The Application of Fractional Calculus in Chinese Economic Growth Models

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Abstract: In this paper, we apply Caputo-type fractional order calculus to simulate China's gross domestic product (GDP) growth based on R software, which is a free software environment for statistical computing and graphics. Moreover, we compare the results for the fractional model with the integer order model. In addition, we show the importance of variables according to the BIC criterion. The study shows that Caputo fractional order calculus can produce a better model and perform more accurately in predicting the GDP values from 2012–2016.

Keywords: Caputo fractional derivative; economic growth model; least squares method

MSC: 26A33

1. Introduction

As one of the most important macroeconomic statistics indicators, GDP is an effective tool for people to understand and grasp the macroeconomic operation of a country; it is also an important basis for formulating economic policies. However, the calculation of GDP is very complicated, so a good economic growth model (EGM) can effectively form the economic progress problem, and it can reduce the loss of human and material resources.

Derivatives and integrals are often used to describe the process of economic development. However, there are still some shortcomings in using classical calculus to model real data. In recent years, the existence of solutions to fractional order differential equations have been studied in [1–3]. In addition, fractional calculus is widely used to construct economic models; it incorporates the effects of memory in evolutionary processes; experimental results show that the fractional order model is superior to integer order model, such as [4–13].

Recently, Luo et al. [14] improved the fractional EGM model in [5] and adopted different computational methods to simulate GDP via MATLAB, SPSS, and R software. The simulation results showed that the newly-established fractional hybrid model had better performance than the classical model.

In this paper, we adopt the idea in [14] to apply Caputo fractional order EGM and integer order to study China's GDP growth, as well as the minimum mean-squared-error (MSE) to estimate the parameters in the model. In order to compare the fitting effect between the integer order and the fractional order model, we establish the minimum absolute error coefficient, determination, and the BIC index. Finally, we use the prediction effect of the absolute relative error evaluation model.

Summarizing, based on fractional calculus, this paper conducts modeling of China's economic growth. Through a case study, it shows that fractional calculus has a better effect than integral calculus in modeling. It would be possible to use Monte Carlo simulation to generate sample data (see [15,16]) and then conduct modeling comparison. In this paper, real data are used for modeling and then showing the advantage of fractional calculus. The purpose of the two methods is the same, but the case analysis is often more complex and difficult than simulation, so a simulation is not used in this paper.

2. Models Description

We select six explanatory variables in this paper, and they are land area (LA) (km^2), cultivated area (CL) (km^2), total population (TP) (million), total capital formation (TCF) (billion), exports of goods and services (EGS) (billion), and general government final consumer spending (GGFCS) (billion), and the explained variable is GDP (billion). The data used in this paper were all Chinese data from the world bank from 1961–2016.

In order to simplify the expression, we define the following symbols:

x_1	x_2	x_3	x_4	x_5	x_6	y	n	k	t
LA	CL	TP	TCF	EGS	GGFCS	GDP	NVM	NPM	year

The general expression of the EGM is $y = f(x_1, x_2, \cdots)$, where f is the given function. Thus, the integer order model (IOM) and Caputo fractional order model (CFOM) are considered as:

- IOM:
$$y(t) = \sum_{j=1,2,3,5,6} c_j x_j(t) + c_4 \int_{t_0}^{t} x_4(t)dt + c_7 \frac{dx_7(t)}{dt},$$
- CFOM:
$$y(t) = \sum_{j=1}^{7} c_k (D_{t_0,t}^{\alpha_k} x_k)(t),$$
where t_0 and α_k represent the starting year and order respectively; in addition, the Caputo derivative $D_{t_0,t}^{\alpha_k} x_k$ for a given function x_k is defined as (see [1]):
$$D_{t_0,t}^{\alpha_k} x_k(t) = \frac{1}{\Gamma(1-\alpha_k)} \int_{t_0}^{t} \frac{\frac{dx_k(s)}{ds}}{(t-s)^{\alpha_k}} ds, \; t > t_0, \; 0 < \alpha_k \leq 1.$$

In order to facilitate the comparison of GDP between different years, the GDP, TCF, EGS, and GGFCE used here were converted into unchangeable local currency. The data from 1961–2011 were selected as the training sample, and data from 2012–2016 were used as the test sample. Moreover, we used the average absolute deviation (MAD) and the coefficient of determination (R^2) to evaluate the model, and the absolute relative error criterion was used to compare the prediction effect of the model. Recall the following definitions:

$$MAD = \frac{\sum_{i=1}^{n} |y_i - \hat{y}_i|}{n},$$

and:

$$ARE_i = \left| \frac{y_i - \hat{y}_i}{y_i} \right|, \quad i = 1, 2, \cdots, n,$$

and:

$$R^2 = 1 - \frac{\sum_{i=1}^{n} (y_i - \hat{y}_i)^2}{\sum_{i=1}^{n} (y_i - \bar{y})^2}.$$

We often used the Akaike information criterion (AIC) and Bayesian information criterion (BIC) for the selection of variables in the model. Compared with the BIC criterion, the AIC criterion has the phenomenon of over-fitting. Therefore, we adopted the following BIC criterion:

$$BIC = \log\left(\frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2\right) + \frac{p\log n}{n},$$

and:

$$\omega_j = \frac{\exp\left(-\frac{(BIC_j - BIC_{min})}{2}\right)}{\sum_{j=1}^{p}\exp\left(-\frac{(BIC_j - BIC_{min})}{2}\right)}.$$

3. Main Results

3.1. Economic Data for China

By using the Chinese economic data from 1961–2016 in unchangeable local currency, we apply R software to get the following figure (see Figure 1).

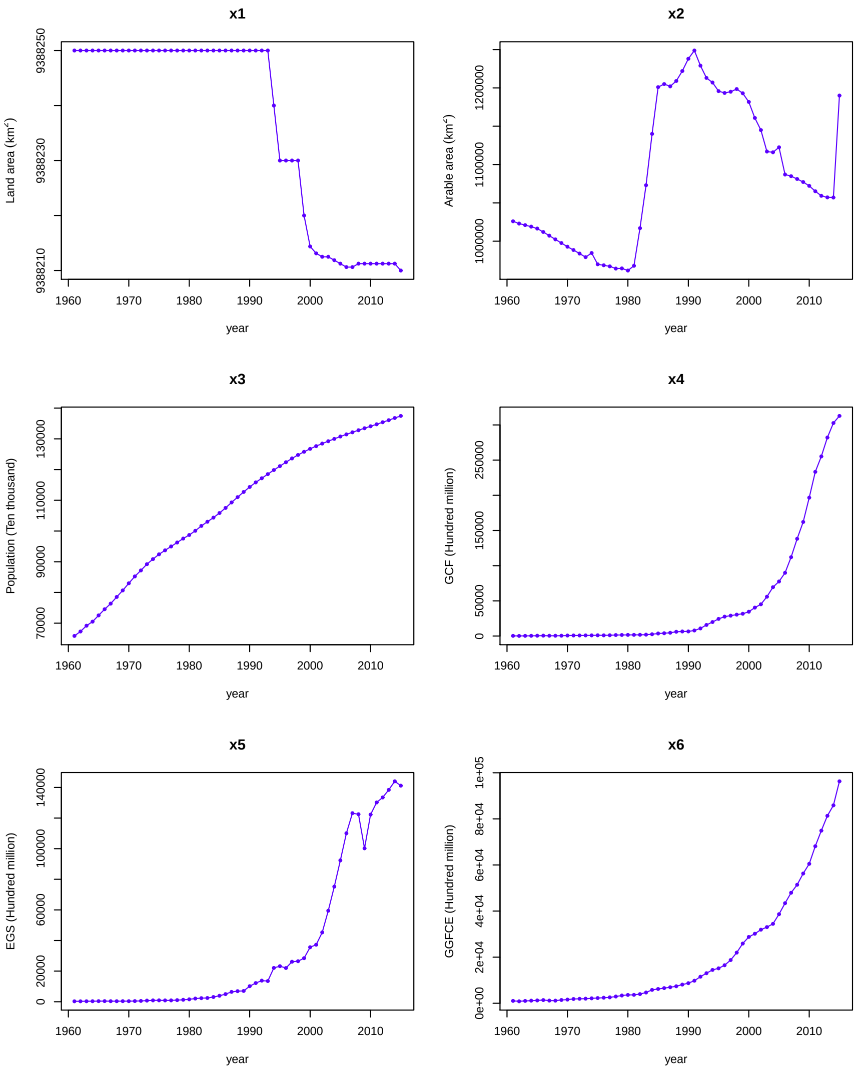

Figure 1. Data for China from 1961–2016. EGS, exports of goods and services.

3.2. Parameter Estimation

In this paper, we used R software and the least squares method to obtain the coefficient estimation in the integer order and Caputo fractional order models. Moreover, according to the MSE criteria, we gave the order of the Caputo fractional order model, and the following data were obtained (see Table 1).Table 2 shows the significance test results of the IOM and CFOM coefficients.

Table 1. The coefficients and orders of the integer order model (IOM) and the Caputo fractional order model (CFOM).

	IOM	CFOM		IOM	CFOM
α_1	0	−0.5389	c_1	−0.0051	0.0027
α_2	0	−1.3704	c_2	0.0286	0.0029
α_3	0	−0.6873	c_3	0.3220	−0.3619
α_4	−1	0.0960	c_4	0.1147	0.6058
α_5	0	−0.7777	c_5	0.4229	0.2329
α_6	0	0.0331	c_6	3.5943	1.3208
α_7	1	3.5251	c_7	0.7978	0.1087

Table 2. Significance level of the Caputo model.

	IOM		CFOM	
Variable	***t*-Value**	***p*-Value**	***t*-Value**	***p*-Value**
x_1	−5.066	7.76×10^{-6}	10.750	6.86×10^{-14}
x_2	2.853	6.58×10^{-3}	12.048	1.58×10^{-15}
x_3	4.127	1.61×10^{-4}	−10.697	8.04×10^{-14}
x_4	6.601	4.41×10^{-8}	16.408	2.00×10^{-16}
x_5	5.498	1.83×10^{-6}	19.996	2.00×10^{-16}
x_6	10.692	8.14×10^{-14}	6.645	3.80×10^{-8}
x_7	2.128	3.90×10^{-2}	2.304	2.60×10^{-2}

The results in Table 2 show that when the significance level was 0.05, the coefficients of IOM and CFOM passed the significance test.

3.3. Model Evaluation

In order to compare the performance of limited samples between IOM and CFOM, we present the values of MAD, R^2, and BIC index in the training sample set (see Table 3).

Table 3. Sample performance of IOM and CFOM.

Index	MSE	MAD	R^2	BIC
IOM	15,497,849	2,430.793	0.9991	17.0959
CFOM	1,906,429	1070.643	0.9999	15.0004

We adopted the BIC criterion to select variables in the model, and the importance of each variable was obtained, represented by ω (see Table 4).

Table 4. The importance of variables based on BIC.

	Variable	IOM	CFOM
BIC without one variable	x_1	17.47825	15.81924
	x_2	17.18849	16.06651
	x_3	17.34602	15.91584
	x_4	17.70702	15.02607
	x_5	17.54180	15.73556
	x_6	18.29923	16.06451
	x_7	17.11674	15.02607
ω found from the BIC without one variable	x_1	14.40%	12.92%
	x_2	16.64%	11.42%
	x_3	15.38%	12.31%
	x_4	12.84%	19.22%
	x_5	13.95%	13.48%
	x_6	9.55%	11.43%
	x_7	17.25%	19.22%

3.4. Fitting Results

Now, we give the fitting results of IOM and CFOM based on R software (see Figure 2).

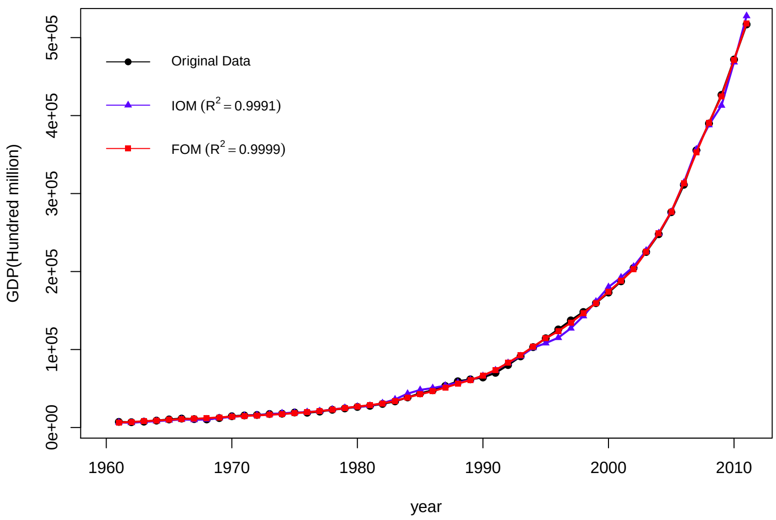

Figure 2. Data fitting.

3.5. Predicted Results

Finally, we present the forecast results of the IOM and CFOM models for China's GDP data from 2012–2016, and we calculate ARE index values, as shown in Table 5.

Table 5. Our results.

Year	Real Value	IOM		CFOM	
		Predict Value	ARE_i	Predict Value	ARE_i
2012	557,487.6	554,010.4	0.6237%	558,937.6	0.2601%
2013	600,735.4	611,769.9	1.8368%	601,514.2	0.1296%
2014	644,575.1	665,424.7	3.2346%	638,231.2	0.9842%
2015	689,052.1	741,602.5	7.6265%	675,637.3	1.9468%
2016	735,218.6	810,156.9	10.1927%	714,153.2	2.8652%

4. Conclusions

We selected six economic indicators in this paper and used IOM and CFOM to model China's GDP growth from 1961–2011. The fitting results showed that CFOM was significantly better than

IOM. To further illustrate the forecasting effect of the CFOM model, we presented the GDP forecast for China from 2012–2016 and compared it with the real value. It was found that the CFOM model not only had an advantage in fitting China's GDP growth, but also predicted it better. Finally, since all data were discrete, we intend to extend our study by applying the Caputo differences to create a fractional discrete time EGM.

Author Contributions: The contributions of all authors (H.M., J.W. and M.F.) are equal. All the main results were developed together.

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