

Article

Properties for ψ -Fractional Integrals Involving a General Function ψ and Applications

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Abstract: In this paper, we are concerned with the ψ -fractional integrals, which is a generalization of the well-known Riemann–Liouville fractional integrals and the Hadamard fractional integrals, and are useful in the study of various fractional integral equations, fractional differential equations, and fractional integrodifferential equations. Our main goal is to present some new properties for ψ -fractional integrals involving a general function ψ by establishing several new equalities for the ψ -fractional integrals. We also give two applications of our new equalities.

Keywords: fractional calculus; ψ -fractional integrals; fractional differential equations

1. Introduction

Fractional integrals and fractional derivatives are generalizations of classical integer-order integrals and integer-order derivatives, respectively, which have been found to be more adequate in the study of a lot of real world problems. In recent decades, various fractional-order models have been used in plasma physics, automatic control, robotics, and many other branches of science (cf., [1–24] and the references therein).

It is known that the ψ -fractional derivative operator, which was introduced in [22], extends the well-known Riemann–Liouville fractional derivative operator. Moreover, it is also easy to see that the ψ -fractional integral operator [14] extends the well-known Riemann–Liouville fractional integral operator and the Hadamard fractional integral operator (see Remark 1 below). Both the ψ -fractional derivative operator and the ψ -fractional integral operator are useful in the study of various fractional integral equations, fractional differential equations, and fractional integrodifferential equations.

The following known definitions about fractional integrals are used later.

Definition 1. [14] Let $[t_1, t_2] \in \mathbb{R}$ and $\alpha > 0$. The Riemann–Liouville fractional integrals (left-sided and right-sided) of order α are defined by

$$\mathcal{J}_{t_1+}^{\alpha} f(\mu) := \frac{1}{\Gamma(\alpha)} \int_{t_1}^{\mu} \frac{f(s)}{(\mu - s)^{1-\alpha}} ds, \quad \mu > t_1$$

and

$$\mathcal{J}_{t_2-}^{\alpha} f(\mu) := \frac{1}{\Gamma(\alpha)} \int_{\mu}^{t_2} \frac{f(s)}{(s - \mu)^{1-\alpha}} ds, \quad \mu < t_2,$$

respectively, where

$$\Gamma(t) = \int_0^{\infty} s^{t-1} e^{-s} ds,$$

is the Euler’s gamma function.

Definition 2. [23] Let $[t_1, t_2] \in \mathbb{R}$ and $\alpha > 0$. The Hadamard fractional integrals (left-sided and right-sided) of order α are defined by

$$H_{t_1+}^{\alpha} f(\mu) := \frac{1}{\Gamma(\alpha)} \int_{t_1}^{\mu} \left(\ln \frac{\mu}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds, \mu > t_1$$

and

$$H_{t_2-}^{\alpha} f(\mu) := \frac{1}{\Gamma(\alpha)} \int_{\mu}^{t_2} \left(\ln \frac{s}{\mu}\right)^{\alpha-1} \frac{f(s)}{s} ds, \mu < t_2,$$

respectively.

Definition 3. [14] Let $[t_1, t_2] \in \mathbb{R}$ and $\alpha > 0$. Suppose that $\psi(\mu) > 0$ is an increasing function on $(t_1, t_2]$, and $\psi'(\mu)$ is continuous on (t_1, t_2) . The ψ -fractional integrals (left-sided and right-sided) of order α are defined by

$$I_{t_1+}^{\alpha; \psi} f(\mu) = \frac{1}{\Gamma(\alpha)} \int_{t_1}^{\mu} \psi'(s) (\psi(\mu) - \psi(s))^{\alpha-1} f(s) ds, \mu > t_1 \quad (1)$$

and

$$I_{t_2-}^{\alpha; \psi} f(\mu) = \frac{1}{\Gamma(\alpha)} \int_{\mu}^{t_2} \psi'(s) (\psi(s) - \psi(\mu))^{\alpha-1} f(s) ds, \mu < t_2, \quad (2)$$

respectively.

Remark 1.

- (i) From [14], we know that, for a function f , the right-sided and left-sided Riemann–Liouville fractional integral of order α are defined by

$$\mathcal{J}_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, x > a$$

and

$$\mathcal{J}_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt, x < b,$$

respectively. If we take $\psi(x) = x$, then it follows from (1) that

$$I_{a+}^{\alpha; x} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt = \mathcal{J}_{a+}^{\alpha} f(x),$$

which is the right-sided Riemann–Liouville fractional integral.

- (ii) From [23], we know that, for a function f , the right-sided and left-sided Hadamard fractional integral of order α are defined by

$$H_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t} dt, x > a$$

and

$$H_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{f(t)}{t} dt, x < b,$$

respectively. Hence, taking $\psi(x) = \ln x$ in (1), we have

$$\begin{aligned} I_{a+}^{\alpha; \ln x} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x \frac{1}{t} (\ln x - \ln t)^{\alpha-1} f(t) dt \\ &= \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f(t) \frac{dt}{t} \\ &= H_{a+}^{\alpha} f(x), \end{aligned}$$

which is the right-sided Hadamard fractional integral.

Throughout this paper, we suppose that $\psi(\mu)$ is a strictly increasing function on $(0, \infty)$ and $\psi'(\mu)$ is continuous, $0 \leq t_1 < t_2$. $\zeta(\mu)$ is the inverse function of $\psi(\mu)$ and

$$\phi(\mu) := f(\mu) + f(t_1 + t_2 - \mu).$$

The rest of the paper is organized as follows. In Section 2, we give some new equalities for ψ -fractional integrals involving a general function ψ . To illustrate the applicability of our new equalities, we give two examples in Section 3 by introducing the ψ -means and presenting relationships between the arithmetic mean and the ψ -means, and by establishing a prior estimate for a class of fractional differential equations in view of the equalities established in Section 2.

2. Equalities for ψ -Fractional Integrals

Theorem 1. Let the function $f : [t_1, t_2] \rightarrow \mathbb{R}$ be differentiable. Then, for the ψ -fractional integrals in (1) and (2), we have

$$\begin{aligned} & \frac{f(t_1) + f(t_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(\psi(t_2) - \psi(t_1))^\alpha} [I_{t_1+}^{\alpha; \psi} f(t_2) + I_{t_2-}^{\alpha; \psi} f(t_1)] \\ &= \frac{\psi(t_2) - \psi(t_1)}{2} \int_0^1 [(1 - \mu)^\alpha - \mu^\alpha] f'(\zeta((1 - \mu)\psi(t_2) + \mu\psi(t_1))) \\ & \quad \cdot \zeta'((1 - \mu)\psi(t_2) + \mu\psi(t_1)) d\mu. \end{aligned} \quad (3)$$

Proof. Write

$$\begin{aligned} I &= \int_0^1 [(1 - \mu)^\alpha - \mu^\alpha] f'(\zeta((1 - \mu)\psi(t_2) + \mu\psi(t_1))) \zeta'((1 - \mu)\psi(t_2) + \mu\psi(t_1)) d\mu \\ &= I_1 + I_2, \end{aligned} \quad (4)$$

where

$$\begin{aligned} I_1 &= \int_0^1 (1 - \mu)^\alpha f'(\zeta((1 - \mu)\psi(t_2) + \mu\psi(t_1))) \zeta'((1 - \mu)\psi(t_2) + \mu\psi(t_1)) d\mu, \\ I_2 &= - \int_0^1 \mu^\alpha f'(\zeta((1 - \mu)\psi(t_2) + \mu\psi(t_1))) \zeta'((1 - \mu)\psi(t_2) + \mu\psi(t_1)) d\mu. \end{aligned}$$

Then, for I_1 , we have

$$\begin{aligned} I_1 &= \int_0^1 (1 - \mu)^\alpha f'(\zeta((1 - \mu)\psi(t_2) + \mu\psi(t_1))) \zeta'((1 - \mu)\psi(t_2) + \mu\psi(t_1)) d\mu \\ &= (1 - \mu)^\alpha \frac{f(\zeta(\mu\psi(t_1) + (1 - \mu)\psi(t_2)))}{\psi(t_1) - \psi(t_2)} \Big|_0^1 \\ &\quad - \frac{\alpha}{\psi(t_2) - \psi(t_1)} \int_0^1 (1 - \mu)^{\alpha-1} f(\zeta(\mu\psi(t_1) + (1 - \mu)\psi(t_2))) d\mu \\ &= \frac{f(t_2)}{\psi(t_2) - \psi(t_1)} - \frac{\alpha}{(\psi(t_2) - \psi(t_1))^{\alpha+1}} \int_{t_1}^{t_2} (\psi(\mu) - \psi(t_1))^{\alpha-1} f(\mu) \psi'(\mu) d\mu \\ &= \frac{f(t_2)}{\psi(t_2) - \psi(t_1)} - \frac{\Gamma(\alpha + 1)}{(\psi(t_2) - \psi(t_1))^{\alpha+1}} I_{t_2-}^{\alpha; \psi} f(t_1). \end{aligned} \quad (5)$$

For I_2 , we obtain

$$\begin{aligned}
 I_2 &= - \int_0^1 \mu^\alpha f'(\zeta((1-\mu)\psi(t_2) + \mu\psi(t_1))) \zeta'((1-\mu)\psi(t_2) + \mu\psi(t_1)) d\mu \\
 &= \mu^\alpha \frac{f(\zeta(\mu\psi(t_1) + (1-\mu)\psi(t_2)))}{\psi(t_2) - \psi(t_1)} \Big|_0^1 - \frac{\alpha}{\psi(t_2) - \psi(t_1)} \int_0^1 \mu^{\alpha-1} f(\zeta(\mu\psi(t_1) + (1-\mu)\psi(t_2))) d\mu \\
 &= \frac{f(t_1)}{\psi(t_2) - \psi(t_1)} - \frac{\alpha}{(\psi(t_2) - \psi(t_1))^{\alpha+1}} \int_{t_1}^{t_2} (\psi(t_2) - \psi(\mu))^{\alpha-1} f(\mu) \psi'(\mu) d\mu \\
 &= \frac{f(t_1)}{\psi(t_2) - \psi(t_1)} - \frac{\Gamma(\alpha+1)}{(\psi(t_2) - \psi(t_1))^{\alpha+1}} I_{t_1+}^{\alpha;\psi} f(t_2). \tag{6}
 \end{aligned}$$

Thus, by (4)–(6), we get

$$I = \frac{f(t_1) + f(t_2)}{\psi(t_2) - \psi(t_1)} - \frac{\Gamma(\alpha+1)}{(\psi(t_2) - \psi(t_1))^{\alpha+1}} [I_{t_1+}^{\alpha;\psi} f(t_2) + I_{t_2-}^{\alpha;\psi} f(t_1)]. \tag{7}$$

This implies that equality (3) is true. $\square$

Based on Theorem 1, we can obtain the following Theorems 2 and 3.

Theorem 2. If the function $f : [t_1, t_2] \rightarrow \mathbb{R}$ is differentiable, then for the ψ -fractional integrals in (1) and (2), we have

$$\begin{aligned}
 &\frac{\Gamma(\alpha+1)}{2(\psi(t_2) - \psi(t_1))^\alpha} [I_{t_1+}^{\alpha;\psi} f(t_2) + I_{t_2-}^{\alpha;\psi} f(t_1)] - f\left(\frac{t_1 + t_2}{2}\right) \\
 &= \frac{t_2 - t_1}{2} \int_0^1 g(\mu) f'(\mu t_1 + (1-\mu)t_2) d\mu \\
 &\quad - \frac{\psi(t_2) - \psi(t_1)}{2} \int_0^1 [(1-\mu)^\alpha - \mu^\alpha] f'(\zeta((1-\mu)\psi(t_2) + \mu\psi(t_1))) \\
 &\quad \cdot \zeta'((1-\mu)\psi(t_2) + \mu\psi(t_1)) d\mu, \tag{8}
 \end{aligned}$$

where

$$g(\mu) = \begin{cases} 1, & \mu \in [0, \frac{1}{2}), \\ -1, & \mu \in [\frac{1}{2}, 1]. \end{cases}$$

Proof. Notice that

$$\begin{aligned}
 &\frac{t_2 - t_1}{2} \int_0^1 g(\mu) f'(\mu t_1 + (1-\mu)t_2) d\mu \\
 &= \frac{t_2 - t_1}{2} \int_0^{\frac{1}{2}} f'(\mu t_1 + (1-\mu)t_2) d\mu - \frac{t_2 - t_1}{2} \int_{\frac{1}{2}}^1 f'(\mu t_1 + (1-\mu)t_2) d\mu \\
 &= -\frac{1}{2} f(\mu t_1 + (1-\mu)t_2) \Big|_0^{\frac{1}{2}} + \frac{1}{2} f(\mu t_1 + (1-\mu)t_2) \Big|_{\frac{1}{2}}^1 \\
 &= \frac{f(t_1) + f(t_2)}{2} - f\left(\frac{t_1 + t_2}{2}\right). \tag{9}
 \end{aligned}$$

Combining (3) from Theorem 1 and (9), we get (8). $\square$

Theorem 3. Let the function $f : [t_1, t_2] \rightarrow \mathbb{R}$ be differentiable. Then,

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(\psi(t_2) - \psi(t_1))^\alpha} [I_{t_1+}^{\alpha;\psi} f(t_2) + I_{t_2-}^{\alpha;\psi} f(t_1)] - f\left(\zeta\left(\frac{\psi(t_1) + \psi(t_2)}{2}\right)\right) \\ &= \frac{\psi(t_2) - \psi(t_1)}{2} \int_0^1 g(\mu) f'(\zeta((1-\mu)\psi(t_2) + \mu\psi(t_1))) \zeta'((1-\mu)\psi(t_2) + \mu\psi(t_1)) d\mu \\ & \quad - \int_0^1 [(1-\mu)^\alpha - \mu^\alpha] f'(\zeta((1-\mu)\psi(t_2) + \mu\psi(t_1))) \zeta'((1-\mu)\psi(t_2) + \mu\psi(t_1)) d\mu, \end{aligned} \quad (10)$$

where

$$g(\mu) = \begin{cases} 1, & \mu \in [0, \frac{1}{2}), \\ -1, & \mu \in [\frac{1}{2}, 1]. \end{cases}$$

Proof. Observe

$$\begin{aligned} & \frac{\psi(t_2) - \psi(t_1)}{2} \int_0^1 g(\mu) f'(\zeta((1-\mu)\psi(t_2) + \mu\psi(t_1))) \zeta'((1-\mu)\psi(t_2) + \mu\psi(t_1)) d\mu \\ &= \frac{\psi(t_2) - \psi(t_1)}{2} \int_0^{\frac{1}{2}} f'(\zeta((1-\mu)\psi(t_2) + \mu\psi(t_1))) \zeta'((1-\mu)\psi(t_2) + \mu\psi(t_1)) d\mu \\ & \quad - \frac{\psi(t_2) - \psi(t_1)}{2} \int_{\frac{1}{2}}^1 f'(\zeta((1-\mu)\psi(t_2) + \mu\psi(t_1))) \zeta'((1-\mu)\psi(t_2) + \mu\psi(t_1)) d\mu \\ &= -\frac{1}{2} f(\zeta(\mu\psi(t_1) + (1-\mu)\psi(t_2))) \Big|_0^{\frac{1}{2}} + \frac{1}{2} f(\zeta(\mu\psi(t_1) + (1-\mu)\psi(t_2))) \Big|_{\frac{1}{2}}^1 \\ &= \frac{f(t_1) + f(t_2)}{2} - f\left(\zeta\left(\frac{\psi(t_1) + \psi(t_2)}{2}\right)\right). \end{aligned} \quad (11)$$

Combining (11) of Theorem 1 and (3), we get the equality (10). $\square$

The following result involves a point μ between t_1 and t_2 .

Theorem 4. If the function $f : [t_1, t_2] \rightarrow \mathbb{R}$ is differentiable, then we have

$$\begin{aligned} & \Gamma(\alpha+1) [I_{t-}^{\alpha;\psi} f(t_1) + I_{t+}^{\alpha;\psi} f(t_2)] - [f(t_1)(\psi(\mu) - \psi(t_1))^\alpha + f(t_2)(\psi(t_2) - \psi(\mu))^\alpha] \\ &= (\psi(t_2) - \psi(\mu))^{\alpha+1} \int_0^1 (s^\alpha - 1) f'(\zeta((1-s)\psi(t_2) + s\psi(\mu))) \zeta'((1-s)\psi(t_2) + s\psi(\mu)) ds \\ & \quad - (\psi(\mu) - \psi(t_1))^{\alpha+1} \int_0^1 (s^\alpha - 1) f'(\zeta((1-s)\psi(t_1) + s\psi(\mu))) \\ & \quad \cdot \zeta'((1-s)\psi(t_1) + s\psi(\mu)) ds, \end{aligned} \quad (12)$$

where $\mu \in (t_1, t_2)$.

Proof. Observe

$$\begin{aligned}
 & \int_0^1 (s^\alpha - 1) f'(\zeta((1-s)\psi(t_2) + s\psi(\mu))) \zeta'((1-s)\psi(t_2) + s\psi(\mu)) ds \\
 = & \frac{1}{\psi(\mu) - \psi(t_2)} (s^\alpha - 1) f(\zeta(s\psi(\mu) + (1-s)\psi(t_2))) \Big|_0^1 \\
 & + \frac{\alpha}{\psi(t_2) - \psi(\mu)} \int_0^1 s^{\alpha-1} f(\zeta(s\psi(\mu) + (1-s)\psi(t_2))) ds \\
 = & -\frac{f(t_2)}{\psi(t_2) - \psi(\mu)} + \frac{\alpha}{(\psi(t_2) - \psi(\mu))^{\alpha+1}} \int_t^{t_2} (\psi(t_2) - \psi(s))^{\alpha-1} f(s) \psi'(s) ds \\
 = & -\frac{f(t_2)}{\psi(t_2) - \psi(\mu)} + \frac{\Gamma(\alpha+1)}{(\psi(t_2) - \psi(\mu))^{\alpha+1}} I_{t_+}^{\alpha; \psi} f(t_2),
 \end{aligned} \tag{13}$$

and

$$\begin{aligned}
 & \int_0^1 (s^\alpha - 1) f'(\zeta((1-s)\psi(t_1) + s\psi(\mu))) \zeta'((1-s)\psi(t_1) + s\psi(\mu)) ds \\
 = & \frac{1}{\psi(\mu) - \psi(t_1)} (s^\alpha - 1) f(\zeta(s\psi(\mu) + (1-s)\psi(t_1))) \Big|_0^1 \\
 & - \frac{\alpha}{\psi(\mu) - \psi(t_1)} \int_0^1 s^{\alpha-1} f(\zeta(s\psi(\mu) + (1-s)\psi(t_1))) ds \\
 = & \frac{f(t_1)}{\psi(\mu) - \psi(t_1)} - \frac{\alpha}{(\psi(\mu) - \psi(t_1))^{\alpha+1}} \int_{t_1}^t (\psi(s) - \psi(t_1))^{\alpha-1} f(s) \psi'(s) ds \\
 = & \frac{f(t_1)}{\psi(\mu) - \psi(t_1)} - \frac{\Gamma(\alpha+1)}{(\psi(\mu) - \psi(t_1))^{\alpha+1}} I_{t_-}^{\alpha; \psi} f(t_1).
 \end{aligned} \tag{14}$$

Combining (13) and (14), we get the result (12). $\square$

Next, we will give two equalities involving function ϕ .

Theorem 5. Let the function $f : [t_1, t_2] \rightarrow \mathbb{R}$ be differentiable. If $f \in L[t_1, t_2]$, then

$$\begin{aligned}
 & \frac{\phi(t_1) + \phi(t_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\psi(t_2) - \psi(t_1))^\alpha} [I_{t_1+}^{\alpha; \psi} \phi(t_2) + I_{t_2-}^{\alpha; \psi} \phi(t_1)] \\
 = & \frac{t_2 - t_1}{2(\psi(t_2) - \psi(t_1))^\alpha} \int_0^1 \mathfrak{g}(s) \phi'((1-s)t_1 + t_2s) ds,
 \end{aligned} \tag{15}$$

where

$$\mathfrak{g}(\mu) = (\psi((1-\mu)t_1 + t_2\mu) - \psi(t_1))^\alpha - (\psi(t_2) - \psi((1-\mu)t_1 + t_2\mu))^\alpha.$$

Proof. Write

$$\begin{aligned}
 I &= \int_0^1 \mathfrak{g}(s) \phi'((1-s)t_1 + t_2s) ds \\
 &= \int_0^1 (\psi((1-s)t_1 + t_2s) - \psi(t_1))^\alpha \phi'((1-s)t_1 + t_2s) ds \\
 &\quad - \int_0^1 (\psi(t_2) - \psi((1-s)t_1 + t_2s))^\alpha \phi'((1-s)t_1 + t_2s) ds \\
 &= I_1 + I_2.
 \end{aligned}$$

Then, for I_1 , we have

$$\begin{aligned}
 I_1 &= \int_0^1 (\psi((1-s)t_1 + t_2s) - \psi(t_1))^\alpha \phi'((1-s)t_1 + t_2s) ds \\
 &= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (\psi(s) - \psi(t_1))^\alpha d\phi(s) \\
 &= \left. \frac{(\psi(s) - \psi(t_1))^\alpha \phi(s)}{t_2 - t_1} \right|_{t_1}^{t_2} - \frac{\alpha}{t_2 - t_1} \int_{t_1}^{t_2} \frac{\psi'(s)}{(\psi(s) - \psi(t_1))^{1-\alpha}} \phi(s) ds \\
 &= \frac{(\psi(t_2) - \psi(t_1))^\alpha}{t_2 - t_1} \phi(t_2) - \frac{\Gamma(\alpha + 1)}{t_2 - t_1} I_{t_2-}^{\alpha; \psi} \phi(t_1).
 \end{aligned} \tag{16}$$

For I_2 , we obtain

$$\begin{aligned}
 I_2 &= - \int_0^1 (\psi(t_2) - \psi((1-s)t_1 + t_2s))^\alpha \phi'((1-s)t_1 + t_2s) ds \\
 &= - \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (\psi(t_2) - \psi(s))^\alpha d\phi(s) \\
 &= - \left. \frac{(\psi(t_2) - \psi(s))^\alpha \phi(s)}{t_2 - t_1} \right|_{t_1}^{t_2} - \frac{\alpha}{t_2 - t_1} \int_{t_1}^{t_2} \frac{\psi'(s)}{(\psi(t_2) - \psi(s))^{1-\alpha}} \phi(s) ds \\
 &= \frac{(\psi(t_2) - \psi(t_1))^\alpha}{t_2 - t_1} \phi(t_1) - \frac{\Gamma(\alpha + 1)}{t_2 - t_1} I_{t_1+}^{\alpha; \psi} \phi(t_2).
 \end{aligned} \tag{17}$$

By adding (16) and (17), we get

$$I = \frac{(\psi(t_2) - \psi(t_1))^\alpha}{t_2 - t_1} [\phi(t_1) + \phi(t_2)] - \frac{\Gamma(\alpha + 1)}{t_2 - t_1} [I_{t_2-}^{\alpha; \psi} \phi(t_1) + I_{t_1+}^{\alpha; \psi} \phi(t_2)].$$

This implies that the equality (15) is true. $\square$

Theorem 6. Let $f : [t_1, t_2] \rightarrow \mathbb{R}$ be a differentiable function and $f' \in L[t_1, t_2]$. If $h : [t_1, t_2] \rightarrow \mathbb{R}$ is integrable, then

$$\begin{aligned}
 &\frac{\phi(t_1) + \phi(t_2)}{2} [I_{t_1+}^{\alpha; \psi} h(t_2) + I_{t_2-}^{\alpha; \psi} h(t_1)] - [I_{t_1+}^{\alpha; \psi} (h\phi)(t_2) + I_{t_2-}^{\alpha; \psi} (h\phi)(t_1)] \\
 &= \frac{1}{2\Gamma(\alpha)} \int_{t_1}^{t_2} \left[\int_{t_1}^t \mathfrak{p}(s) h(s) ds - \int_t^{t_2} \mathfrak{p}(s) h(s) ds \right] \phi'(\mu) d\mu,
 \end{aligned} \tag{18}$$

where

$$\mathfrak{p}(\mu) = \frac{\psi'(\mu)}{(\psi(t_2) - \psi(\mu))^{1-\alpha}} + \frac{\psi'(\mu)}{(\psi(\mu) - \psi(t_1))^{1-\alpha}}.$$

Proof. Write

$$\begin{aligned}
 I &= \int_{t_1}^{t_2} \left[\int_{t_1}^t \mathfrak{p}(s) h(s) ds - \int_t^{t_2} \mathfrak{p}(s) h(s) ds \right] \phi'(\mu) d\mu \\
 &= \int_{t_1}^{t_2} \int_{t_1}^t \mathfrak{p}(s) h(s) ds \phi'(\mu) d\mu - \int_{t_1}^{t_2} \int_t^{t_2} \mathfrak{p}(s) h(s) ds \phi'(\mu) d\mu \\
 &= I_1 + I_2.
 \end{aligned}$$

Then, for I_1 , we have

$$\begin{aligned}
 I_1 &= \int_{t_1}^{t_2} \int_{t_1}^t \mathbf{p}(s)h(s)ds\phi'(\mu)d\mu \\
 &= \int_{t_1}^t \mathbf{p}(s)h(s)ds\phi(\mu)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \mathbf{p}(\mu)h(\mu)\phi(\mu)d\mu \\
 &= \int_{t_1}^{t_2} \mathbf{p}(s)h(s)ds\phi(t_2) - \int_{t_1}^{t_2} \mathbf{p}(\mu)h(\mu)\phi(\mu)d\mu \\
 &= \Gamma(\alpha)[I_{t_1+}^{\alpha;\psi}h(t_2) + I_{t_2-}^{\alpha;\psi}h(t_1)]\phi(t_2) \\
 &\quad - \Gamma(\alpha)[I_{t_1+}^{\alpha;\psi}(h\phi)(t_2) + I_{t_2-}^{\alpha;\psi}(h\phi)(t_1)].
 \end{aligned} \tag{19}$$

For I_2 , we obtain

$$\begin{aligned}
 I_2 &= - \int_{t_1}^{t_2} \int_t^{t_2} \mathbf{p}(s)h(s)ds\phi'(\mu)d\mu \\
 &= - \int_t^{t_2} \mathbf{p}(s)h(s)ds\phi(\mu)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \mathbf{p}(\mu)h(\mu)\phi(\mu)d\mu \\
 &= \int_{t_1}^{t_2} \mathbf{p}(s)h(s)ds\phi(t_1) - \int_{t_1}^{t_2} \mathbf{p}(\mu)h(\mu)\phi(\mu)d\mu \\
 &= \Gamma(\alpha)[I_{t_1+}^{\alpha;\psi}h(t_2) + I_{t_1-}^{\alpha;\psi}h(t_1)]\phi(t_1) \\
 &\quad - \Gamma(\alpha)[I_{t_1+}^{\alpha;\psi}(h\phi)(t_2) + I_{t_2-}^{\alpha;\psi}(h\phi)(t_1)].
 \end{aligned} \tag{20}$$

Combining (19) and (20), we get

$$I = \Gamma(\alpha)[I_{t_1+}^{\alpha;\psi}h(t_2) + I_{t_2-}^{\alpha;\psi}h(t_1)](\phi(t_1) + \phi(t_2)) - 2\Gamma(\alpha)[I_{t_1+}^{\alpha;\psi}(h\phi)(t_2) + I_{t_2-}^{\alpha;\psi}(h\phi)(t_1)].$$

This implies the equality (18). $\square$

For the last result of this section, we suppose that $\psi(0) = 0$ and $\psi(1) = 1$.

Theorem 7. Let the function $f : [\psi(t_1), \psi(t_2)] \rightarrow \mathbb{R}$ be differentiable. Then, the following equality holds:

$$\begin{aligned}
 &\frac{f(\psi(t_1)) + f(\psi(t_2))}{2} - \frac{\Gamma(\alpha + 1)}{2(\psi(t_2) - \psi(t_1))^\alpha} [I_{t_1+}^{\alpha;\psi}f \circ \psi(t_2) + I_{t_2-}^{\alpha;\psi}f \circ \psi(t_1)] \\
 &= \frac{\psi(t_2) - \psi(t_1)}{2} \int_0^1 [(1 - \psi(\mu))^\alpha - \psi^\alpha(\mu)]\psi'(\mu) \\
 &\quad \cdot f'((1 - \psi(\mu))\psi(t_2) + \psi(\mu)\psi(t_1))d\mu,
 \end{aligned} \tag{21}$$

where $f \circ \psi(\mu) = f(\psi(\mu))$.

Proof. Write

$$\begin{aligned}
 I &= \int_0^1 [(1 - \psi(\mu))^\alpha - \psi^\alpha(\mu)]\psi'(\mu)f'((1 - \psi(\mu))\psi(t_2) + \psi(\mu)\psi(t_1))d\mu \\
 &= \int_0^1 (1 - \psi(\mu))^\alpha\psi'(\mu)f'((1 - \psi(\mu))\psi(t_2) + \psi(\mu)\psi(t_1))d\mu \\
 &\quad - \int_0^1 \psi^\alpha(\mu)\psi'(\mu)f'((1 - \psi(\mu))\psi(t_2) + \psi(\mu)\psi(t_1))d\mu \\
 &= I_1 + I_2.
 \end{aligned}$$

Then, for I_1 , we get

$$\begin{aligned}
 I_1 &= \int_0^1 (1 - \psi(\mu))^\alpha \psi'(\mu) f((1 - \psi(\mu))\psi(t_2) + \psi(\mu)\psi(t_1)) d\mu \\
 &= \frac{(1 - \psi(\mu))^\alpha}{\psi(t_1) - \psi(t_2)} f((1 - \psi(\mu))\psi(t_2) + \psi(\mu)\psi(t_1)) \Big|_0^1 \\
 &\quad + \int_0^1 \frac{\alpha(1 - \psi(\mu))^{\alpha-1} \psi'(\mu)}{\psi(t_1) - \psi(t_2)} f((1 - \psi(\mu))\psi(t_2) + \psi(\mu)\psi(t_1)) d\mu \\
 &= \frac{f(\psi(t_2))}{\psi(t_2) - \psi(t_1)} - \frac{\alpha}{\psi(t_2) - \psi(t_1)} \int_{t_1}^{t_2} \left(\frac{\psi(\mu) - \psi(t_1)}{\psi(t_2) - \psi(t_1)} \right)^{\alpha-1} \frac{\psi'(\mu)}{\psi(t_2) - \psi(t_1)} f(\psi(\mu)) d\mu \\
 &= \frac{f(\psi(t_2))}{\psi(t_2) - \psi(t_1)} - \frac{\Gamma(\alpha + 1)}{(\psi(t_2) - \psi(t_1))^{\alpha+1}} I_{t_2-}^{\alpha; \psi} f \circ \psi(t_1). \tag{22}
 \end{aligned}$$

For I_2 , we have

$$\begin{aligned}
 I_2 &= - \int_0^1 \psi^\alpha(\mu) \psi'(\mu) f((1 - \psi(\mu))\psi(t_2) + \psi(\mu)\psi(t_1)) d\mu \\
 &= \frac{\psi^\alpha(\mu)}{\psi(t_2) - \psi(t_1)} f((1 - \psi(\mu))\psi(t_2) + \psi(\mu)\psi(t_1)) \Big|_0^1 \\
 &\quad - \int_0^1 \frac{\alpha \psi^{\alpha-1}(\mu) \psi'(\mu)}{\psi(t_2) - \psi(t_1)} f((1 - \psi(\mu))\psi(t_2) + \psi(\mu)\psi(t_1)) d\mu \\
 &= \frac{f(\psi(t_1))}{\psi(t_2) - \psi(t_1)} - \frac{\alpha}{\psi(t_2) - \psi(t_1)} \int_{t_1}^{t_2} \left(\frac{\psi(t_2) - \psi(\mu)}{\psi(t_2) - \psi(t_1)} \right)^{\alpha-1} \frac{\psi'(\mu)}{\psi(t_2) - \psi(t_1)} f(\psi(\mu)) d\mu \\
 &= \frac{f(\psi(t_1))}{\psi(t_2) - \psi(t_1)} - \frac{\Gamma(\alpha + 1)}{(\psi(t_2) - \psi(t_1))^{\alpha+1}} I_{t_1+}^{\alpha; \psi} f \circ \psi(t_2). \tag{23}
 \end{aligned}$$

By (22) and (23), we see that

$$I = \frac{f(\psi(t_1)) + f(\psi(t_2))}{\psi(t_2) - \psi(t_1)} - \frac{\Gamma(\alpha + 1)}{(\psi(t_2) - \psi(t_1))^{\alpha+1}} [I_{t_1+}^{\alpha; \psi} f \circ \psi(t_2) + I_{t_2-}^{\alpha; \psi} f \circ \psi(t_1)],$$

which means that equality (21) is true. $\square$

3. Applications

To illustrate the applicability of the new equalities established in previous section, we give two examples in this section.

Example 1. The arithmetic mean A is defined by

$$A(t_1, t_2) := \frac{t_1 + t_2}{2}, \quad t_1, t_2 > 0.$$

Now, we introduce the following ψ -means M_ψ and $\overline{M}_{\psi, n}$:

$$M_\psi(t_1, t_2) := \frac{\int_{t_1}^{t_2} \mu \psi'(\mu) d\mu}{\psi(t_2) - \psi(t_1)}, \quad t_1 \neq t_2, \tag{24}$$

and

$$\overline{M}_{\psi, n}(t_1, t_2) := \frac{\psi^{n+1}(t_2) - \psi^{n+1}(t_1)}{(n+1)(\psi(t_2) - \psi(t_1))}, \quad t_1 \neq t_2, \quad n \in \mathbb{N}.$$

As we can see from (24) that the ψ -mean $M_\psi(t_1, t_2)$ is just the following logarithmic mean [25] when $\psi(\mu) = \ln \mu$:

$$L(t_1, t_2) := \frac{t_2 - t_1}{\ln t_2 - \ln t_1}, \quad t_1 \neq t_2.$$

Moreover, we see that, when $\psi(\mu) = \mu$, the ψ -mean $M_\psi(t_1, t_2)$ is just the arithmetic mean $A(t_1, t_2)$.

The following two results, which are deduced by virtue of our new equalities in the last section, show new relationships between the arithmetic mean A and the two ψ -means above.

Theorem 8. Let $0 < t_1 < t_2$. Then,

$$|A(t_1, t_2) - M_\psi(t_1, t_2)| \leq \frac{\psi(t_2) - \psi(t_1)}{2(q+1)^{1/q}} \left(\int_0^1 [\zeta'(\mu\psi(t_1) + (1-\mu)\psi(t_2))]^{q'} d\mu \right)^{1/q'},$$

where $q > 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

Proof. Taking $\alpha = 1$ and $f(\mu) = \mu$ in Theorem 1 and using the Hölder inequality, we obtain

$$\begin{aligned} & |A(t_1, t_2) - M_\psi(t_1, t_2)| \\ & \leq \frac{\psi(t_2) - \psi(t_1)}{2} \int_0^1 |1 - 2\mu| \zeta'(\mu\psi(t_1) + (1-\mu)\psi(t_2)) d\mu \\ & \leq \frac{\psi(t_2) - \psi(t_1)}{2} \left(\int_0^1 |1 - 2\mu|^q d\mu \right)^{1/q} \left(\int_0^1 [\zeta'(\mu\psi(t_1) + (1-\mu)\psi(t_2))]^{q'} d\mu \right)^{1/q'}. \end{aligned}$$

Noticing that

$$\left(\int_0^1 |1 - 2\mu|^q d\mu \right)^{1/q} = \frac{1}{(q+1)^{1/q}},$$

we get the desired result. $\square$

Theorem 9. Let $0 < t_1 < t_2$, $\psi(0) = 0$ and $\psi(1) = 1$. Then,

$$\begin{aligned} & |A(\psi^n(t_1), \psi^n(t_2)) - \overline{M}_{\psi,n}(t_1, t_2)| \\ & \leq \frac{n(\psi(t_2) - \psi(t_1))^{1-1/q'}}{2(q+1)^{1/q}(q'(n-1)+1)^{1/q'}} \left(\psi^{q'(n-1)+1}(t_2) - \psi^{q'(n-1)+1}(t_1) \right)^{1/q'}, \end{aligned}$$

where $q > 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

Proof. Taking $\alpha = 1$ and $f(\mu) = \mu^n$ in Theorem 7 and using the Hölder inequality, we obtain

$$\begin{aligned} & |A(\psi^n(t_1), \psi^n(t_2)) - \overline{M}_{\psi,n}(t_1, t_2)| \\ & \leq \frac{n(\psi(t_2) - \psi(t_1))}{2} \int_0^1 |1 - 2\psi(\mu)| \psi'(\mu) (\psi(\mu)\psi(t_1) + (1-\psi(\mu))\psi(t_2))^{n-1} d\mu \\ & \leq \frac{n(\psi(t_2) - \psi(t_1))}{2} \left(\int_0^1 |1 - 2\psi(\mu)|^q \psi'(\mu) d\mu \right)^{1/q} \\ & \quad \cdot \left(\int_0^1 [\psi(\mu)\psi(t_1) + (1-\psi(\mu))\psi(t_2)]^{q'(n-1)} \psi'(\mu) d\mu \right)^{1/q'}. \end{aligned}$$

Observing that

$$\left(\int_0^1 [\psi(\mu)\psi(t_1) + (1 - \psi(\mu))\psi(t_2)]^{q'(n-1)} \psi'(\mu) d\mu \right)^{1/q'} = \frac{(\psi^{q'(n-1)+1}(t_2) - \psi^{q'(n-1)+1}(t_1))^{1/q'}}{(q'(n-1) + 1)^{1/q'} (\psi(t_2) - \psi(t_1))^{1/q'}},$$

we get the desired result. $\square$

Example 2. Consider the following fractional integrodifferential equations of Sobolev type with nonlocal conditions in $\mathbb{R}$:

$$\begin{cases} {}^c D^{\psi;\alpha} u(t) = f\left(t, u(t), \int_a^t \rho(t,s)h(t,s,u(s))ds\right), & t \in J := [a, T], \\ u(a) = u_a - g(u), \end{cases} \quad (25)$$

where ${}^c D^{\psi;\alpha}$, $\alpha \in (0, 1)$, is the ψ -Caputo fractional derivative of order α with the lower limit $a > 0$, $u_a \in \mathbb{R}$ and $g : C(J, \mathbb{R}) \rightarrow \mathbb{R}$, $f : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $\rho : \Delta \rightarrow \mathbb{R}$ and $h : \Delta \times \mathbb{R} \rightarrow \mathbb{R}$ ($\Delta = \{(t, s) \in [a, T] \times [a, T] : t \geq s\}$) are given functions.

Applying the operator $I_{a+}^{\alpha;\psi}$ to the first equation of the problem (25), we get for each $t \in (a, T]$,

$$\begin{aligned} u(t) &= u(a) + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f\left(s, u(s), \int_a^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau\right) ds \\ &= u_a - g(u) + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \\ &\quad f\left(s, u(s), \int_a^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau\right) ds. \end{aligned} \quad (26)$$

Substituting (26) into (3) of Theorem 1 with $f = u$, $t_1 = a$ and $t_2 = t$, we can obtain

$$\begin{aligned} &u_a - g(u) + \frac{1}{2\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f\left(s, u(s), \int_a^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau\right) ds \\ &- \frac{\alpha}{2(\psi(t) - \psi(a))^\alpha} \left[\int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} u(s) ds + \int_a^t \psi'(s)(\psi(s) - \psi(a))^{\alpha-1} u(s) ds \right] \\ &= \frac{\psi(t) - \psi(a)}{2} \int_0^1 [(1-s)^\alpha - s^\alpha] \zeta'(s\psi(a) + (1-s)\psi(t)) u'(\zeta(s\psi(a) + (1-s)\psi(t))) ds. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} u(s) ds + \int_a^t \psi'(s)(\psi(s) - \psi(a))^{\alpha-1} u(s) ds \\ &= \frac{2(\psi(t) - \psi(a))^\alpha}{\alpha} (u_a - g(u)) + \frac{(\psi(t) - \psi(a))^\alpha}{\Gamma(\alpha+1)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \\ &\quad f\left(s, u(s), \int_a^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau\right) ds - \frac{(\psi(t) - \psi(a))^{\alpha+1}}{\alpha} \int_0^1 [(1-s)^\alpha - s^\alpha] \\ &\quad \zeta'(s\psi(a) + (1-s)\psi(t)) u'(\zeta(s\psi(a) + (1-s)\psi(t))) ds. \end{aligned} \quad (27)$$

Using the fact that $|a^\alpha - b^\alpha| \leq |a - b|^\alpha$ ($a, b > 0$; $0 < \alpha < 1$) and Hölder inequality to (27), we obtain the following result.

Theorem 10. For each solution $u(t) \in C^1[a, T]$ of the problem (25), if $|u'(t)| \leq M$, then we have the following prior estimate:

$$\begin{aligned} & \left| \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} u(s) ds + \int_a^t \psi'(s)(\psi(s) - \psi(a))^{\alpha-1} u(s) ds \right| \\ & \leq \frac{2(\psi(t) - \psi(a))^\alpha}{\alpha} |u_a - g(u)| \\ & \quad + \frac{(\psi(t) - \psi(a))^\alpha}{\Gamma(\alpha + 1)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \left| f\left(s, u(s), \int_a^s \rho(s, \tau) h(s, \tau, u(\tau)) d\tau\right) \right| ds \\ & \quad + \frac{M(\psi(t) - \psi(a))^{\alpha+1}}{q^{1/q} \alpha^{1+1/q}} \left(\int_0^1 [\zeta'(s\psi(a) + (1-s)\psi(t))]^{q'} ds \right)^{1/q'}, \quad \forall t \in [a, T], \end{aligned}$$

where $q > 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

4. Conclusions

In this paper, we present new properties for ψ -fractional integrals involving a general function ψ by establishing several new equalities for the ψ -fractional integrals. The ψ -fractional integrals are generalizations of Riemann–Liouville fractional integrals and Hadamard fractional integrals, and our equalities are more general and new. To illustrate the applicability of our new equalities, we introduce the ψ -means and explore the relationships between the arithmetic mean and the ψ -means with the aid of our equalities. Moreover, we use our equalities to obtain an prior estimate for a class of fractional differential equations. How to study the properties of solutions to fractional equations involving ψ -Caputo fractional derivative? How to reveal other new properties about ψ -fractional integrals? How to find more applications of these properties? We will pay our attention to these problems in our future research.

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