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On Some New Fixed Point Results in Complete Extended b-Metric Spaces

Quanita Kiran [1], Nayab Alamgir [2], Nabil Mlaiki [3,*] and Hassen Aydi [4,5,*]

1 School of Electrical Engineering and Computer Science (SEECS), National University of Sciences and Technology (NUST), Sector H-12, Islamabad 44000, Pakistan; quanita.kiran@seecs.edu.pk
2 School of Natural Sciences, National University of Sciences and Technology (NUST), Sector H-12, Islamabad 44000, Pakistan; nayab@sns.nust.edu.pk
3 Department of Mathematical Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia
4 Institut Supérieur d'Informatique et des Techniques de Communication, Université de Sousse, H. Sousse 4000, Tunisia
5 China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
* Correspondence: nmlaiki@psu.edu.sa (N.M.); hassen.aydi@isima.rnu.tn (H.A.)

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Abstract: In this paper, we specified a method that generalizes a number of fixed point results for single and multi-valued mappings in the structure of extended b-metric spaces. Our results extend several existing ones including the results of Aleksic et al. for single-valued mappings and the results of Nadler and Miculescu et al. for multi-valued mappings. Moreover, an example is given at the end to show the superiority of our results.

Keywords: extended b-metric space; set-valued functions; fixed point theorems

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1. Introduction and Preliminaries

Banach contraction principle [1] is a fundamental tool for providing the existence of solutions for many mathematical problems involving differential equations and integral equations. A mapping $T:\mathbf{U}\to\mathbf{U}$ on a metric space $(\mathbf{U},d)$ is called a contraction mapping, if there exists $\eta<1$ such that for all $u,v\in\mathbf{U}$,

$$d(Tu,Tv)\leq \eta d(u,v). \tag{1}$$

If the metric space is complete and T satisfies inequality (1), then T has a unique fixed point. Clearly, inequality (1) implies continuity of T. Naturally, a question arises as to whether we can find contractive conditions which will imply the existence of fixed points in a complete metric space, but will not imply continuity. In [2], Kannan derived the following result, which answers the said question. Let $T:\mathbf{U}\to\mathbf{U}$ be a mapping on a complete metric space $(\mathbf{U},d)$, which satisfies inequality:

$$d(Tu,Tv)\leq \eta[d(u,Tu)+d(v,Tv)], \tag{2}$$

where $\eta\in[0,\frac{1}{2})$ and $u,v\in\mathbf{U}$. The mapping satisfying inequality (2) is called a Kannan type mapping. There are number of generalizations of the contraction principle of Banach both for single-valued and multi-valued mappings, see ([3–13]). Chatterjea in [14] established the following alike co ntractive condition. Let $(\mathbf{U},d)$ be a complete metric space. A mapping $T:\mathbf{U}\to\mathbf{U}$ has a unique fixed point, if it satisfies the following inequality:

$$d(Tu,Tv)\leq \eta[d(u,Tv)+d(v,Tu)]. \tag{3}$$

where $\eta \in [0, \frac{1}{2})$ and $u, v \in \mathbf{U}$. The mapping satisfying inequality (3) is called a Chatterjea type mapping.

Due to the problem of the convergence of measurable functions with respect to a measure, Bakhtin [15], Bourbaki [16], and Czerwik [17,18] introduced the concept of b-metric spaces by weakening the triangle inequality of the metric space as follows:

Definition 1 ([17]). *Let $\mathbf{U}$ be a set and $s \geq 1$ a real number. A function $d : \mathbf{U} \times \mathbf{U} \to [0, \infty)$ is called a b-metric space, if it satisfies the following axioms for all $u_1,\ u_2,\ u_3 \in \mathbf{U}$:*

(1) $d(u_1, u_2) = 0$ *if and only if* $u_1 = u_2$;
(2) $d(u_1, u_2) = d(u_2, u_1)$;
(3) $d(u_1, u_3) \leq s[d(u_1, u_2) + d(u_2, u_3)]$.

The pair $(\mathbf{U}, d)$ is called a b-metric space.

Clearly, every metric space is a b-metric space with $s = 1$, but its converse is not true in general. After that, a number of research papers have been established that generalized the Banach fixed point result in the framework of b-metric spaces. In [19], Kir and Kiziltunc introduced the following results, which generalized Kannan and Chatterjea type mappings in b-metric spaces. Let $T : \mathbf{U} \to \mathbf{U}$ be a mapping on a complete b-metric space $(\mathbf{U}, d)$, which satisfies inequality:

$$d(Tu, Tv) \leq \eta[d(u, Tu) + d(v, Tv)]. \tag{4}$$

where $s\eta \in [0, \frac{1}{2})$ and $u, v \in \mathbf{U}$. Then T has a unique fixed point.

Let $(\mathbf{U}, d)$ be a complete b-metric space. A mapping $T : \mathbf{U} \to \mathbf{U}$ has a unique fixed point in $\mathbf{U}$, if it satisfies the following inequality:

$$d(Tu, Tv) \leq \eta[d(u, Tv) + d(v, Tu)], \tag{5}$$

for all $u, v \in \mathbf{U}$, where $\eta \in [0, \frac{1}{2})$. In [20], the given below results, which generalized Equation (4) for $\kappa_1 = \kappa_2 = \kappa_3 = 0$ and (5) for $\kappa_1 = \kappa_4 = 0$ and $\kappa_2 = \kappa_3$, have been derived.

Theorem 1 ([20]). *Let $(\mathbf{U}, d)$ be a complete b-metric space with constant $s \geq 1$. If $T : \mathbf{U} \to \mathbf{U}$ satisfies the inequality:*

$$d\ \ (Tu, Tv) \leq \kappa_1 d\ \ (u, v) + \kappa_2 d\ \ (u, Tu) + \kappa_3 d\ \ (v, Tv) + \kappa_4[d\ \ (v, Tu) + d\ \ (u, Tv)], \tag{6}$$

where,

$$\kappa_1 + 2s\kappa_2 + \kappa_3 + 2s\kappa_4 < 1,$$

then T has a unique fixed point.

Theorem 2 ([20]). *Let $(\mathbf{U}, d)$ be a complete b-metric space with constant $s \geq 1$. If $T : \mathbf{U} \to \mathbf{U}$ satisfies the inequality:*

$$d(Tu, Tv) \leq \kappa_1 d_\phi(u, v) + \kappa_2[d_\phi(u, Tu) + d_\phi(v, Tv)], \tag{7}$$

for all $u, v \in \mathbf{U}$, where $\kappa_1, \kappa_2 \in [0, \frac{1}{3})$, then T has a unique fixed point.

In [21], Koleva and Zlatanov proved the following result, which generalizes Chatterjea's type mappings in b-metric spaces and do not involve the b-metric constant.

Theorem 3 ([21]). *Let $(\mathbf{U}, d)$ be a complete b-metric space and d be a continuous function. If $T : \mathbf{U} \to \mathbf{U}$ is a Chatterjea's mapping, i.e., it satisfies inequality (3) such that $\sup_{n\in\mathbb{N}}\{d(T^n u, u)\} < \infty$ holds for every $u \in \mathbf{U}$. Then:*

(i) *There exists a unique fixed point of T, say ξ;*
(ii) *For any $u_0 \in \mathbf{U}$, the sequence $\{u_n\}_{n=1}^{\infty}$ converges to ξ, where $u_{n+1} = T^n u_n$, $n = 0, 1, 2, \ldots$;*
(iii) *There holds the priori error estimate.*

$$d(\xi, T^m u) \leq \left(\frac{\eta}{1-\eta}\right)^m \sup_{j\in\mathbb{N}}\{d(T^j u, u)\},$$

where $\eta \in [0, \frac{1}{2})$.

Ilchev and Zlatanov in [22] proved the following result generalizing Theorem 3 for $\kappa_1 = 0$.

Theorem 4 ([22]). *Let $(\mathbf{U}, d)$ be a complete b-metric space and d be a continuous function. If,*

(1) *$T : \mathbf{U} \to \mathbf{U}$ is a Reich mapping, i.e., there exist $\kappa_1, \kappa_2 \geq 0$, such that $\kappa_1 + 2\kappa_2 < 1$, so that the inequality*

$$d(Tu, Tv) \leq \kappa_1 d_\phi(u, v) + \kappa_2[d(u, Tv) + d(v, Tu)], \tag{8}$$

holds for every $u, v \in \mathbf{U}$;
(2) *the inequality $\sup_{n\in\mathbb{N}}\{d(T^n u, u)\} < \infty$ holds for every $u \in \mathbf{U}$,*

then:

(i) *There exists a unique fixed point of T, say ξ;*
(ii) *For any $u_0 \in \mathbf{U}$, the sequence $\{u_n\}_{n=1}^{\infty}$ converges to ξ, where $u_{n+1} = T^n u_n$, $n = 0, 1, 2, \ldots$;*
(iii) *There holds the priori error estimate.*

$$d(\xi, T^m u) \leq \left(\frac{\kappa_1 + \kappa_2}{1-\kappa_2}\right)^m \sup_{j\in\mathbb{N}}\{d(T^j u, u)\}.$$

In [23], the author introduced the following results, which improve Theorems 1 and 2 of [20].

Theorem 5 ([23]). *Let $(\mathbf{U}, d)$ be a complete b-metric space with a constant $s \geq 1$. If $T : \mathbf{U} \to \mathbf{U}$ satisfies the inequality:*

$$d(Tu, Tv) \leq \kappa_1 \ d(u, v) + \kappa_2 \ d(u, Tu) + \kappa_3 \ d(v, Tv) + \kappa_4[d(v, Tu) + d(u, Tv)], \tag{9}$$

where $\kappa_i \geq 0$, for $i = 1, 2, 3, 4$ and

$$\kappa_1 + \kappa_2 + \kappa_3 + 2s\kappa_4 < 1,$$

then T has a unique fixed point.

Theorem 6 ([23]). *Let $(\mathbf{U}, d)$ be a complete b-metric space with a constant $s \geq 1$. If $T : \mathbf{U} \to \mathbf{U}$ satisfies the inequality:*

$$d(Tu, Tv) \leq \kappa_1 \ d(u, v) + \kappa_2[\ d(u, Tu) + \ d(v, Tv)], \tag{10}$$

for all $u, v \in \mathbf{U}$, where $\kappa_1, \kappa_2 \in [0, \frac{1}{3})$ such that $\kappa_2 < \min\{\frac{1}{3}, \frac{1}{s}\}$, then T has a unique fixed point.

If $s = 1$, then $(\mathbf{U}, d)$ is a metric space and condition (9) implies:

$$d(Tu, Tv) \leq k \max\{d(u,v), d(u,Tu), d(v,Tv), \frac{d(v,Tu) + d(u,Tv)}{2}\}, \tag{11}$$

where $\kappa_1 + \kappa_2 + \kappa_3 + 2\kappa_4 < 1$. With Equation (11), we recover the well-known result for generalized Ciric's contraction mapping in the metric space and obtain a unique fixed point.

In 1969, Nadler [24] generalized the single-valued Banach contraction principle into a multi-valued contraction principle. This mapping has been carried out for a complete metric space $(\mathbf{U}, d)$ by using subsets of $\mathbf{U}$ that are nonempty closed and bounded. There are number of generalizations for Nadler's fixed point theorem (see [25–27]). In [28], the author introduced the given below quasi-contraction mapping and proved an existence and uniqueness fixed point theorem.

A mapping $T : \mathbf{U} \to \mathbf{U}$ on a metric space $(\mathbf{U}, d)$ is called a quasi-contraction, if there exists $q < 1$ such that for all $u, v \in \mathbf{U}$,

$$d(Tu, Tv) \leq q \max\{d(u,v), d(u,Tu), d(v,Tv), d(u,Tv), d(v,Tu)\}.$$

Amini-Harandi in [29] introduced the concept of q-multi-valued quasi-contractions and derived a fixed point theorem, which generalized Ciric's theorem [28].

A multi-valued map $T : \mathbf{U} \to \mathcal{CB}(\mathbf{U})$ on a metric space $(\mathbf{U}, d)$ is called a q-multi-valued quasi-contraction, if there exists $q < 1$ such that for all $u, v \in \mathbf{U}$,

$$d(Tu, Tv) \leq q \max\{d(u,v), d(u,Tu), d(v,Tv), d(u,Tv), d(v,Tu)\},$$

where $\mathcal{CB}(\mathbf{U})$ denotes the non-empty closed and bounded subsets of $\mathbf{U}$. In [30], Aydi et al. established the following result, which generalized Theorem 2.2 from [29] and Ciric's result [28].

Theorem 7 ([30])**.** *Let $(\mathbf{U}, d)$ be a complete b-metric space. Suppose that T is a q-multi-valued quasi-contraction and $q < \frac{1}{s^2+s}$, then T has a fixed point in $\mathbf{U}$.*

In 2017, Kamran et al. generalized the structure of a b-metric space and called it, an extended b-metric space. Thereafter, a number of research articles have appeared, which generalize the contraction principle of Banach in extended b-metric spaces for both single and multi-valued mappings (see [31–37]). In this paper, we illustrate a method (see Lemma 3), to generalize a number of fixed point results of single-valued and multi-valued mappings in the structure of extended b-metric spaces.

Definition 2 ([38])**.** *Let $\mathbf{U}$ be a nonempty set and $\phi : \mathbf{U} \times \mathbf{U} \to [1, \infty)$. A function $d_\phi : \mathbf{U} \times \mathbf{U} \to [0, \infty)$ is called an extended b-metric, if for all $u_1, u_2, u_3 \in \mathbf{U}$, it satisfies:*

(d_1) $d_\phi(u_1, u_2) = 0$ *iff* $u_1 = u_2$;
(d_2) $d_\phi(u_1, u_2) = d_\phi(u_2, u_1)$;
(d_3) $d_\phi(u_1, u_3) \leq \phi(u_1, u_3)[d_\phi(u_1, u_2) + d_\phi(u_2, u_3)]$.

The pair $(\mathbf{U}, d_\phi)$ is called an extended b-metric space.

Example 1. *Let $\mathbf{U} = [0, \infty)$. Define $d_\phi : \mathbf{U} \times \mathbf{U} \to [0, \infty)$ by:*

$$d_\phi(u, v) = \begin{cases} 0, & \text{if } u = v; \\ 3, & \text{if } u \text{ or } v \in \{1, 2\}, u \neq v; \\ 5, & \text{if } u \neq v \in \{1, 2\}; \\ 1, & \text{otherwise}. \end{cases}$$

Then $(\mathbf{U}, d_\phi)$ is an extended b-metric space, where $\phi : \mathbf{U} \times \mathbf{U} \to [1, \infty)$ is defined by:

$$\phi(u, v) = u + v + 1,$$

for all $u, v \in \mathbf{U}$.

Remark 1. *Every b-metric space is an extended b-metric space with constant function $\phi(u_1, u_2) = s$, for $s \geq 1$, but its converse is not true in general.*

Definition 3 ([35]). *Let $(\mathbf{U}, d_\phi)$ be an extended b-metric space, where $\phi : \mathbf{U} \times \mathbf{U} \to [1, \infty)$ is bounded. Then for all $\mathbf{A}, \mathbf{B} \in \mathcal{CB}(\mathbf{U})$, where $\mathcal{CB}(\mathbf{U})$ denotes the family of all nonempty closed and bounded subsets of $\mathbf{U}$, the Hausdorff–Pompieu metric on $\mathcal{CB}(\mathbf{U})$ induced by d_ϕ is defined by:*

$$H_\Phi(\mathbf{A}, \mathbf{B}) = \max\{\sup_{a \in \mathbf{A}} d_\phi(a, \mathbf{B}), \sup_{b \in \mathbf{B}} d_\phi(b, \mathbf{A})\},$$

where for every $a \in \mathbf{A}$, $d_\phi(a, \mathbf{B}) = \inf\{d_\phi(a, b) : b \in \mathbf{B}\}$ and $\Phi : \mathcal{CB}(\mathbf{U}) \times \mathcal{CB}(\mathbf{U}) \to [1, \infty)$ is such that:

$$\Phi(\mathbf{A}, \mathbf{B}) = \sup\{\phi(a, b) : a \in \mathbf{A}, b \in \mathbf{B}\}.$$

Theorem 8 ([31]). *Let $(\mathbf{U}, d_\phi)$ be an extended b-metric space. Then $(\mathcal{CB}(\mathbf{U}), H_\Phi)$ is an extended Hausdorff–Pompieu b-metric space.*

Lemma 1 ([39]). *Every sequence $\{u_n\}_{n \in \mathbb{N}}$ of elements from an extended b-metric space $(\mathbf{U}, d_\phi)$, having the property that for every $n \in \mathbb{N}$, there exists $\gamma \in [0, 1)$ such that:*

$$d_\phi(u_{n+1}, u_n) \leq \gamma d_\phi(u_n, u_{n-1}) \tag{12}$$

where for each $u_0 \in \mathbf{U}$, $\lim_{n,m \to \infty} \phi(u_n, u_m) < \frac{1}{\gamma}$. Then $\{u_n\}_{n=0}^{\infty}$ is a Cauchy sequence.

Definition 4. *Let $\mathbf{U}$ be any set and $T : \mathbf{U} \to \mathcal{CB}(\mathbf{U})$ be a multi-valued map. For any point $u_0 \in \mathbf{U}$, the sequence $\{u_n\}_{n=0}^{\infty}$ given by:*

$$u_{n+1} \in Tu_n, \quad n = 0, 1, 2, \ldots \tag{13}$$

is called an iterative sequence with initial point u_0.

2. Main Results

Definition 5. *Let $(\mathbf{U}, d_\phi)$ be an extended b-metric space. A function $T : \mathbf{U} \to \mathcal{CB}(\mathbf{U})$ is called continuous, if for every sequence $\{u_n\}_{n \in \mathbb{N}}$ and $\{v_n\}_{n \in \mathbb{N}}$ belongs to $\mathbf{U}$ and $u, v \in \mathbf{U}$ such that $\lim_{n \to \infty} u_n = u$, $\lim_{n \to \infty} v_n = v$ and $v_n \in Tu_n$. We have $v \in Tu$.*

Definition 6. *An extended b-metric space $(\mathbf{U}, d_\phi)$ is called $*$-continuous, if for every $A \in \mathcal{CB}(\mathbf{U})$, $\{u_n\}_{n \in \mathbb{N}} \in \mathbf{U}$ and $u \in \mathbf{U}$ such that $\lim_{n \to \infty} u_n = u$. We have $\lim_{n \to \infty} d_\phi(u_n, A) = d_\phi(u, A)$.*

Remark 2. *Note that $*$- continuity of d_ϕ is stronger than continuity of d_ϕ in first variable.*

Lemma 2. *For every sequence $\{u_n\}_{n \in \mathbb{N}}$ of elements from an extended b-metric space $(\mathbf{U}, d_\phi)$, the inequality*

$$d_\phi(u_0, u_k) \leq \sum_{i=0}^{k-1} d_\phi(u_i, u_{i+1}) \prod_{l=0}^{i} \phi(u_l, u_k), \tag{14}$$

is valid for every $k \in \mathbb{N}$.

Proof. From the triangle inequality for $k > 0$, we haveL

$$\begin{aligned} d_\phi(u_0,u_k) \quad \leq \quad & \phi(u_0,u_k)d_\phi(u_0,u_1)+\phi(u_0,u_k)\phi(u_1,u_k)d_\phi(u_1,u_2) \\ & +\cdots+\phi(u_0,u_k)\phi(u_1,u_k)\ldots\phi(u_{k-1},u_k)d_\phi(u_{k-1},u_k). \end{aligned}$$

This implies that:

$$d_\phi(u_0,u_k) \leq \sum_{i=0}^{k-1} d_\phi(u_i,u_{i+1}) \prod_{l=0}^{i} \phi(u_l,u_k).$$

□

Lemma 3. *Every sequence $\{u_n\}_{n\in\mathbb{N}}$ of elements from an extended b-metric space $(\mathbf{U},d_\phi)$, having the property that there exists $\gamma \in [0,1)$ such that:*

$$d_\phi(u_{n+1},u_n) \leq \gamma d_\phi(u_n,u_{n-1}) \tag{15}$$

for every $n \in \mathbb{N}$ is Cauchy.

Proof. First, by successively applying (15), we get:

$$d_\phi(u_n,u_{n+1}) \leq \gamma^n d_\phi(u_0,u_1), \tag{16}$$

for every $n \in \mathbb{N}$. Then by the Lemma 3, for all $m,k \in \mathbb{N}$, we have:

$$d_\phi(u_m,u_{m+k}) \leq \sum_{n=m}^{m+k-1} d_\phi(u_n,u_{n+1}) \prod_{l=0}^{n} \phi(u_l,u_{m+k})$$

$$d_\phi(u_m,u_{m+k}) \leq d_\phi(u_0,u_1) \sum_{n=m}^{m+k-1} \gamma^n \prod_{l=0}^{n} \phi(u_l,u_{m+k})$$

$$d_\phi(u_m,u_{m+k}) \leq d_\phi(u_0,u_1) \sum_{n=0}^{k-1} \gamma^{n+m} \prod_{l=0}^{n+m} \phi(u_l,u_{m+k})$$

$$d_\phi(u_m,u_{m+k}) \leq \gamma^m d_\phi(u_0,u_1) \sum_{n=0}^{k-1} \gamma^{n} \prod_{l=0}^{n+m} \phi(u_l,u_{m+k})$$

$$d_\phi(u_m,u_{m+k}) \leq \gamma^m d_\phi(u_0,u_1) \sum_{n=0}^{k-1} \gamma^{\log_\gamma \prod_{l=0}^{n+m} \phi(u_l,u_{m+k})+n}. \tag{17}$$

Now let us take two cases for $\log_\gamma \prod_{l=0}^{n+m} \phi(u_l,u_{m+k})+n$.

Case 1: If $\prod_{l=0}^{n+m} \phi(u_l,u_{m+k})$ is finite, let us say M, then $\lim_{n\to\infty} \log_\gamma M+n = \infty$. Hence the series $\sum_{n=0}^{k-1} \gamma^{\log_\gamma M+n}$ is convergent.

Case 2: If $\prod_{l=0}^{n+m} \phi(u_l,u_{m+k})$ is infinite, then $\lim_{n\to\infty} \log_\gamma \prod_{l=0}^{n+m} \phi(u_l,u_{m+k}) = \infty$, so there exist $n_0 \in \mathbb{N}$ such that $\log_\gamma \prod_{l=0}^{n+m} \phi(u_l,u_{m+k}) > M$, i.e.,

$$\gamma^{\log_\gamma \prod_{l=0}^{n+m} \phi(u_l,u_{m+k})+n} \leq \gamma^M.\gamma^n, \quad for\ each\ n \in \mathbb{N},\ n \geq n_0.$$

Hence the series $\sum_{n=0}^{k-1} \gamma^{\log_\gamma \prod_{l=0}^{n+m} \phi(u_l, u_{m+k})+n}$ is convergent. In both cases denoting by S the sum of this series, we come to the conclusion that:

$$d_\phi(u_m, u_{m+k}) \leq \gamma^m d_\phi(u_0, u_1) S,$$

for all $m, k \in \mathbb{N}$. Consequently, as $\lim_{m\to\infty} \gamma^m = 0$, we conclude that $\{u_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence. □

Remark 3. *Lemma 3 shows that the condition on ϕ in Lemma 1 corresponding to that for each $u_0 \in \mathbf{U}$, $\lim_{n,m\to\infty} \phi(u_n, u_m) < \frac{1}{\gamma}$, can be avoided. Therefore, Lemma 3 generalizes Lemma 1, which is the basis of the results from [36].*

Lemma 4. *Let $\mathbf{A}, \mathbf{B} \in \mathcal{CB}(\mathbf{U})$, then for every $\eta > 0$ and $b \in \mathbf{B}$ there exists $a \in \mathbf{A}$ such that:*

$$d_\phi(a, b) \leq H_\Phi(\mathbf{A}, \mathbf{B}) + \eta. \tag{18}$$

Proof. By definition of Hausdorff metric, for $\mathbf{A}, \mathbf{B} \in \mathcal{CB}(\mathbf{U})$ and for any $b \in \mathbf{Y}$, we have:

$$d_\phi(\mathbf{A}, b) \leq H_\Phi(\mathbf{A}, \mathbf{B}).$$

By the definition of infimum, we can let $\{a_n\}$ be a sequence in $\mathbf{A}$ such that:

$$d_\phi(b, a_n) < d_\phi(b, \mathbf{A}) + \eta, \quad where\ \eta > 0. \tag{19}$$

We know that $\mathbf{A}$ is closed and bounded, so there exists $a \in \mathbf{A}$ such that $a_n \to a$. Therefore, by (19), we have:

$$d_\phi(a, b) < d_\phi(\mathbf{A}, b) + \eta \leq H_\Phi(\mathbf{A}, \mathbf{B}) + \eta.$$

□

Theorem 9. *Let $(\mathbf{U}, d_\phi)$ be a complete extended b-metric space with $\phi : \mathbf{U} \times \mathbf{U} \to [1, \infty)$. If $T : \mathbf{U} \to \mathbf{U}$ satisfies the inequality:*

$$d_\phi(Tu, Tv) \leq \kappa_1 d_\phi(u, v) + \kappa_2 d_\phi(u, Tu) + \kappa_3 d_\phi(v, Tv) + \kappa_4 [d_\phi(v, Tu) + d_\phi(u, Tv)], \tag{20}$$

where $\kappa_i \geq 0$, for $i = 1, \ldots, 4$ and for each $u_0 \in \mathbf{U}$,

$$\kappa_1 + \kappa_2 + \kappa_3 + 2\kappa_4 \lim_{n,m\to\infty} \phi(u_n, u_m) < 1,$$

then T has a fixed point.

Proof. Let us choose an arbitrary $u_0 \in \mathbf{U}$ and define the iterative sequence $\{u_n\}_{n=0}^\infty$ by $u_n = Tu_{n-1} = T^{n-1}u_0$ for all $n \geq 1$. If $u_n = u_{n-1}$, then u_n is a fixed point of T and the proof holds. So we suppose $u_n \neq u_{n-1}$, $\forall\, n \geq 1$. Then from Equation (20), we have:

$$\begin{aligned} d_\phi(Tu_n, Tu_{n-1}) &\leq \kappa_1 d_\phi(u_n, u_{n-1}) + \kappa_2 d_\phi(u_n, Tu_n) + \kappa_3 d_\phi(u_{n-1}, Tu_{n-1}) \\ &\quad + \kappa_4 [d_\phi(u_{n-1}, Tu_n) + d_\phi(u_n, Tu_{n-1})]. \end{aligned}$$

From the triangle inequality, we get:

$$\begin{aligned} d_\phi(Tu_n, Tu_{n-1}) &\leq \kappa_1 d_\phi(u_n, u_{n-1}) + \kappa_2 d_\phi(u_n, Tu_n) + \kappa_3 d_\phi(u_{n-1}, Tu_{n-1}) \\ &\quad + \kappa_4 \phi(u_{n-1}, u_{n+1}) [d_\phi(u_{n-1}, u_n) + d_\phi(u_n, u_{n+1})]. \end{aligned}$$

This implies that:

$$\begin{aligned} d_\phi(u_{n+1}, u_n) &\le (\kappa_1 + \kappa_3 + \kappa_4\phi(u_{n-1}, u_{n+1}))d_\phi(u_n, u_{n-1}) \\ &\quad +(\kappa_2 + \kappa_4\phi(u_{n-1}, u_{n+1}))d_\phi(u_n, u_{n+1}). \end{aligned} \tag{21}$$

Similarly,

$$\begin{aligned} d_\phi(u_n, u_{n+1}) &\le (\kappa_1 + \kappa_2 + \kappa_4\phi(u_{n-1}, u_{n+1}))d_\phi(u_n, u_{n-1}) \\ &\quad +(\kappa_3 + \kappa_4\phi(u_{n-1}, u_{n+1}))d_\phi(u_n, u_{n+1}). \end{aligned} \tag{22}$$

By adding Equations (21) and (22), we get:

$$d_\phi(u_{n+1}, u_n) \le \eta d_\phi(u_n, u_{n-1}). \tag{23}$$

where,

$$\eta = \frac{2\kappa_1 + \kappa_2 + \kappa_3 + 2\kappa_4\phi(u_{n-1}, u_{n+1})}{2 - \kappa_2 - \kappa_3 - 2\kappa_4\phi(u_{n-1}, u_{n+1})}.$$

Since $\kappa_1 + \kappa_2 + \kappa_3 + 2\kappa_4 \lim_{n,m\to\infty} \phi(u_n, u_m) < 1$, multiply by 2,

$$2\kappa_1 + 2\kappa_2 + 2\kappa_3 + 4\kappa_4 \lim_{n,m\to\infty} \phi(u_n, u_m) < 2,$$

$$2\kappa_1 + 2\kappa_2 + 2\kappa_3 + (2\kappa_4 \lim_{n,m\to\infty} \phi(u_n, u_m) + 2\kappa_4 \lim_{n,m\to\infty} \phi(u_n, u_m)) < 2.$$

This implies that:

$$2\kappa_1 + \kappa_2 + \kappa_3 + 2\kappa_4 \lim_{n,m\to\infty} \phi(u_n, u_m) < 2 - \kappa_2 - \kappa_3 - 2\kappa_4 \lim_{n,m\to\infty} \phi(u_n, u_m).$$

$\Rightarrow \eta < 1$. Hence from Lemma 3, $\{u_n\}_{n=0}^{\infty}$ is a Cauchy sequence. As $\mathbf{U}$ is complete, therefore there exists $u \in \mathbf{U}$ such that $\lim_{n\to\infty} u_n = u$. Next, we will show that u is a fixed point of T. From the triangle inequality and Equation (20), we have:

$$\begin{aligned} d_\phi(u, Tu) &\le \phi(u, Tu)[d_\phi(u, u_{n+1}) + d_\phi(u_{n+1}, Tu)] \\ &\le \phi(u, Tu)[d_\phi(u, u_{n+1}) + \kappa_1 d_\phi(u_n, u) + \kappa_2 d_\phi(u_n, u_{n+1}) \\ &\quad +\kappa_3 d_\phi(u, Tu) + \kappa_4[d_\phi(u_n, Tu) + d_\phi(u, u_{n+1})] \\ &\le \phi(u, Tu)[d_\phi(u, u_{n+1}) + \kappa_1 d_\phi(u_n, u) + \kappa_2 d_\phi(u_n, u_{n+1}) \\ &\quad +\kappa_3 d_\phi(u, Tu) + \kappa_4 d_\phi(u, u_{n+1}) + \kappa_4\phi(u_n, Tu) \\ &\quad [d_\phi(u_n, u) + d_\phi(u, Tu)] \\ &\le \phi(u, Tu)[(1 + \kappa_4)d_\phi(u, u_{n+1}) + (\kappa_1 + \kappa_4\phi(u_n, Tu))d_\phi(u, u_n) \\ &\quad \kappa_2 d_\phi(u_n, u_{n+1}) + (\kappa_3 + \kappa_4\phi(u_n, Tu))d_\phi(u, Tu)].. \end{aligned}$$

So,

$$\begin{aligned} (1 - \kappa_3 - \kappa_4\phi(u_n, Tu))d_\phi(u, Tu) \le &\phi(u, Tu)[(1 + \kappa_4)d_\phi(u, u_{n+1}) + (\kappa_1 + \kappa_4\phi(u_n, Tu))d_\phi(u, u_n) \\ &+ \kappa_2 d_\phi(u_n, u_{n+1})]. \end{aligned} \tag{24}$$

Similarly,

$$\begin{aligned} (1 - \kappa_2 - \kappa_4\phi(u_n, Tu))d_\phi(u, Tu) \le &\phi(u, Tu)[(1 + \kappa_4)d_\phi(u, u_{n+1}) + (\kappa_1 + \kappa_4\phi(u_n, Tu))d_\phi(u, u_n) \\ &+ \kappa_3 d_\phi(u_n, u_{n+1})]. \end{aligned} \tag{25}$$

By adding Equations (24) and (25), we have:

$$(2-\kappa_2-\kappa_3-2\kappa_4\phi(u_n,Tu))d_\phi(u,Tu) \leq \phi(u,Tu)[2(1+\kappa_4)d_\phi(u,u_{n+1})+2(\kappa_1+\kappa_4\phi(u_n,Tu))d_\phi(u,u_n) + (\kappa_2+\kappa_3)d_\phi(u_n,u_{n+1})] \to 0,$$

as $n\to\infty$. This implies that:

$$(2-\kappa_2-\kappa_3-2\kappa_4\phi(u_n,Tu))d_\phi(u,Tu) \leq 0.$$

Since $(2-\kappa_2-\kappa_3-2\kappa_4\phi(u_n,Tu)) > 0$, we get $d_\phi(u,Tu)=0$, i.e., $Tu=u$. Now, we show that u is the unique fixed point of T. Assume that u' is another fixed point of T, then we have $Tu'=u'$. Also,

$$\begin{aligned} d_\phi(u,u') &= d_\phi(Tu,Tu') \\ &\leq \kappa_1 d_\phi(u,u')+\kappa_2 d_\phi(u,Tu')+\kappa_3 d_\phi(u',Tu)+\kappa_4[d_\phi(u,Tu')+d_\phi(u',Tu)] \\ &\leq \kappa_1 d_\phi(u,u')+\kappa_2 d_\phi(u,u')+\kappa_3 d_\phi(u',u)+\kappa_4[d_\phi(u,u')+d_\phi(u',u)] \\ &\leq (\kappa_1+2\kappa_4)d_\phi(u,u'). \end{aligned}$$

This implies that:

$$(1-\kappa_1-2\kappa_4)d_\phi(u,u') \leq 0.$$

As $\kappa_1+\kappa_2+\kappa_3+2\kappa_4 \leq \kappa_1+\kappa_2+\kappa_3+2\kappa_4 \lim_{n,m\to\infty}\phi(u_n,u_m) < 1$. Therefore $(1-\kappa_1-2\kappa_4)>0$, and $d_\phi(u,u')=0$, i.e., $u=u'$. Hence T has a unique fixed point in $\mathbf{U}$. □

Remark 4. *From the symmetry of the distance function d_ϕ, it is easy to prove similar to that done in [4,22] that $\kappa_2=\kappa_3$. Thus the inequality (20) is equivalent to the following inequality:*

$$d_\phi(Tu,Tv) \leq \kappa_1 d_\phi(u,v)+\kappa_2[d_\phi(u,Tu)+d_\phi(v,Tv)]+\kappa_4[d_\phi(v,Tu)+d_\phi(u,Tv)], \tag{26}$$

where $\kappa_1,\kappa_2,\kappa_4 \geq 0$ such that $\kappa_1+2\kappa_2+2\kappa_4\lim_{n,m\to\infty}\phi(u_n,u_m)<1$.
If $\kappa_1=\kappa_2=0$ and $\kappa_4\in[0,\frac{1}{2})$ in inequality (26), we obtain generalization of Chatterjea's map [14] in extended b-metric space.

Remark 5. *Theorem 9 generalizes and improves Theorem* 1.5 *of [23] and therefore Theorem* 2.1 *of [20]. Moreover, Theorem 9 generalizes and improves Theorem* 3.7 *from [40], that is, Theorem* 2.19 *from [41].*

Theorem 10. *Let $(\mathbf{U},d_\phi)$ be a complete extended b-metric space with $\phi:\mathbf{U}\times\mathbf{U}\to[1,\infty)$. If $T:\mathbf{U}\to\mathbf{U}$ satisfies the inequality:*

$$d_\phi(Tu,Tv) \leq \kappa_1 d_\phi(u,v)+\kappa_2[d_\phi(u,Tu)+d_\phi(v,Tv)], \tag{27}$$

for each $u,v\in\mathbf{U}$, where $\kappa_1,\kappa_2\in[0,\frac{1}{3})$. Moreover for each $u_0\in\mathbf{U}$,

$$\lim_{n,m\to\infty}\phi(u_n,u_m)\kappa_2 < 1,$$

then T has a unique fixed point.

Proof. Let us choose an arbitrary $u_0\in\mathbf{U}$ and define the iterative sequence $\{u_n\}_{n=0}^\infty$ by $u_n=Tu_{n-1}=T^{n-1}u_0$ for all $n\geq 1$. If $u_n=u_{n-1}$, then u_n is a fixed point of T and the proof holds. So we suppose $u_n\neq u_{n-1}$, $\forall\, n\geq 1$. Then from Equation (27), we have:

$$d_\phi(Tu_n,Tu_{n-1}) \leq \kappa_1 d_\phi(u_n,u_{n-1})+\kappa_2[d_\phi(u_{n-1},Tu_{n-1})+d_\phi(u_n,Tu_n)].$$

So,

$$(1-\kappa_2)d_\phi(u_{n+1},u_n) \leq (\kappa_1+\kappa_4)d_\phi(u_n,u_{n-1}).$$

$$d_\phi(u_n,u_{n+1}) \leq \frac{\kappa_1+\kappa_4}{1-\kappa_4}d_\phi(u_n,u_{n-1}).$$

This implies that:

$$d_\phi(u_{n+1},u_n) \leq \eta d_\phi(u_n,u_{n-1}). \tag{28}$$

where,

$$\eta = \frac{\kappa_1+\kappa_4}{1-\kappa_4}.$$

Since $\kappa_1,\kappa_2 \in [0,\frac{1}{3})$, so $\eta < 1$, from Lemma 3, $\{u_n\}_{n=0}^{\infty}$ is a Cauchy sequence. As $\mathbf{U}$ is complete, therefore there exists $u \in \mathbf{U}$ such that $\lim_{n\to\infty} u_n = u$. Next, we will show that u is a fixed point of T in $\mathbf{U}$. From the triangle inequality and Equation (27), we have:

$$\begin{aligned} d_\phi(u,Tu) &\leq \phi(u,Tu)[d_\phi(u,u_{n+1})+d_\phi(u_{n+1},Tu)] \\ &\leq \phi(u,Tu)[d_\phi(u,u_{n+1})+\kappa_1 d_\phi(u_n,u)+\kappa_2[d_\phi(u_n,u_{n+1})+d_\phi(u,Tu)].. \end{aligned}$$

So,

$$(1-\kappa_2\phi(u,Tu))d_\phi(u,Tu) \leq 0,$$

as $n\to\infty$. Since $\lim_{n,m\to\infty}\phi(u_n,u_m)\kappa_2 < 1$, we get $(1-\kappa_2\phi(u,Tu)) > 0$, and so $d_\phi(u,Tu) = 0$, i.e., $Tu = u$. We will show that u is the unique fixed point of T. Assume that $u^{'}$ is another fixed point of T, then we have $Tu^{'} = u^{'}$. Again,

$$\begin{aligned} d_\phi(u,u^{'}) &= d_\phi(Tu,Tu^{'}) \\ &\leq \kappa_1 d_\phi(u,u^{'})+\kappa_2[d_\phi(u,Tu)+d_\phi(u^{'},Tu^{'})] \\ &\quad +\kappa_1 d_\phi(u,u^{'}) < d_\phi(u,u^{'}), \end{aligned}$$

which is a contradiction. Hence T has a unique fixed point in $\mathbf{U}$. □

Remark 6. *Theorem 10 generalizes Theorem* 1.2 *of [20].*

For $u,v \in \mathbf{U}$ and $c,d \in [0,1]$, we will use the following notation:

$$N_{c_1,c_2}(u,v) = \max\{d_\phi(u,v), c_1 d_\phi(u,Tu), c_1 d_\phi(v,Tv), \tfrac{c_2}{2}(d_\phi(u,Tv)+d_\phi(v,Tu))\}.$$

Theorem 11. *Let $(\mathbf{U},d_\phi)$ be an extended b-metric space. Let $T:\mathbf{U}\to\mathcal{CB}(\mathbf{U})$ be a multi-valued mapping having the property that there exist $c_1,c_2 \in [0,1]$ and $\eta \in [0,1)$ such that:*

(*i*) *For each $u_0 \in \mathbf{U}$, $\lim_{n,m\to\infty}\eta c_2\phi(u_n,u_m) < 1$, here $u_n = T^n u_0$,*
(*ii*) *$H_\Phi(Tu,Tv) \leq \eta N_{c_1,c_2}(u,v)$ for all $u,v \in \mathbf{U}$.*

Then for every $u_0 \in \mathbf{U}$, there exist $\gamma \in [0,1)$ and a sequence $\{u_n\}_{n\in\mathbb{N}}$ of iterates from $\mathbf{U}$ such that for every $n \in \mathbb{N}$,

$$d_\phi(u_n,u_{n+1}) \leq \gamma d_\phi(u_{n-1},u_n). \tag{29}$$

Proof. Let us choose an arbitrary $u_0 \in \mathbf{U}$ and $u_1 \in Tu_0$. Consider:

$$\gamma = \max\{\eta, \frac{\eta c_2 \phi(u_{n-1}, u_{n+1})}{2 - \eta c_2 \phi(u_{n-1}, u_{n+1})}\}.$$

Clearly, $\gamma < 1$. If $u_1 = u_0$, then for every $n \in \mathbb{N}$, the sequence $\{u_n\}_{n\in\mathbb{N}}$ given by $u_n = u_0$ satisfies Equation(29). Since:

$$\begin{aligned} d_\phi(u_1, Tu_1)) &\leq d_\phi(Tu_0, Tu_1) \leq H_\Phi(Tu_0, Tu_1) \\ &\leq \eta N_{c_1,c_2}(u_0, u_1). \end{aligned}$$

there exists $u_2 \in Tu_1$ such that $d_\phi(u_1, u_2) \leq \eta N_{c_1,c_2}(u_0, u_1)$. If $u_2 = u_1$, then for every $n \in \mathbb{N}$, $n \geq 1$, the sequence $\{u_n\}_{n\in\mathbb{N}}$ given by $u_n = u_1$ satisfies Equation (29). By repeating this process, we obtain a sequence $\{u_n\}_{n\in\mathbb{N}}$ of elements from $\mathbf{U}$ such that $u_{n+1} \in Tu_n$ and $0 < d_\phi(u_n, u_{n+1}) \leq \eta N_{c_1,c_2}(u_{n-1}, u_n)$ for every $n \in \mathbb{N}$, $n \geq 1$. Then we have:

$$\begin{aligned} 0 &< d_\phi(u_n, u_{n+1}) \leq \eta N_{c_1,c_2}(u_{n-1}, u_n) \\ &\leq \eta \max\{d_\phi(u_{n-1}, u_n), c_1 d_\phi(u_{n-1}, Tu_{n-1}), c_1 d_\phi(u_n, Tu_n), \frac{c_2}{2} \\ &\quad (d_\phi(u_{n-1}, Tu_n) + d_\phi(u_n, Tu_{n-1}))\} \\ &\leq \eta \max\{d_\phi(u_{n-1}, u_n), c_1 d_\phi(u_{n-1}, u_n), c_1 d_\phi(u_n, u_{n+1}), \frac{c_2}{2}(d_\phi(u_{n-1}, u_{n+1}))\} \\ &\leq \eta \max\{d_\phi(u_{n-1}, u_n), c_1 d_\phi(u_{n-1}, u_n), c_1 d_\phi(u_n, u_{n+1}), \frac{c_2 \phi(u_{n-1}, u_{n+1})}{2} \qquad (30) \\ &\quad (d_\phi(u_{n-1}, u_n) + d_\phi(u_n, u_{n+1}))\}, \qquad (31) \end{aligned}$$

for every $n \in \mathbb{N}$. If we take:

$$\begin{aligned} &\max\{d_\phi(u_{n-1}, u_n), c_1 d_\phi(u_{n-1}, u_n), c_1 d_\phi(u_n, u_{n+1}), \frac{c_2 \phi(u_{n-1}, u_{n+1})}{2} \\ &(d_\phi(u_{n-1}, u_n) + d_\phi(u_n, u_{n+1}))\} = c_1 d_\phi(u_n, u_{n+1}), \end{aligned}$$

then from Equations (30) and (31), $0 < d(u_n, u_{n+1}) \leq \eta c_1 d_\phi(u_n, u_{n+1}) < \eta d_\phi(u_n, u_{n+1})$. As $\eta < 1$, so we obtain the contradiction. Therefore, we have:

$$\begin{aligned} d_\phi(u_n, u_{n+1}) &\leq \eta N_{c_1,c_2}(u_{n-1}, u_n) \\ &\leq \eta \max\{d_\phi(u_{n-1}, u_n), \frac{c_2 \phi(u_{n-1}, u_{n+1})}{2}(d_\phi(u_{n-1}, u_n) + d_\phi(u_n, u_{n+1}))\}. \end{aligned}$$

Consequently, $d_\phi(u_n, u_{n+1}) \leq \eta d_\phi(u_{n-1}, u_n)$ or

$$d_\phi(u_n, u_{n+1}) \leq \frac{\eta c_2 \phi(u_{n-1}, u_{n+1})}{2}(d_\phi(u_{n-1}, u_n) + d_\phi(u_n, u_{n+1})).$$

This implies that $d_\phi(u_n, u_{n+1}) \leq \eta d_\phi(u_{n-1}, u_n)$ or

$$d_\phi(u_n, u_{n+1}) \leq \frac{\eta c_2 \phi(u_{n-1}, u_{n+1})}{2 - \eta c_2 \phi(u_{n-1}, u_{n+1})} d_\phi(u_{n-1}, u_n),$$

for every $n \in \mathbb{N}$. Thus,

$$d_\phi(u_n, u_{n+1}) \leq \max\{\eta, \frac{\eta c_2 \phi(u_{n-1}, u_{n+1})}{2 - \eta c_2 \phi(u_{n-1}, u_{n+1})}\} d_\phi(u_{n-1}, u_n),$$

i.e.,

$$d_\phi(u_n, u_{n+1}) \leq \gamma d_\phi(u_{n-1}, u_n).$$

Thus, the sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfies Equation(29). Hence from Lemma 3, we conclude that $\{u_n\}_{n\in\mathbb{N}}$ is Cauchy sequence. □

Theorem 12. *Let $(\mathbf{U}, d_\phi)$ be a complete extended b-metric space. Let $T : \mathbf{U} \to \mathcal{CB}(\mathbf{U})$ be a multi-valued mapping having the property that there exist $c_1, c_2 \in [0,1]$ and $\eta \in [0,1)$ such that:*

(*i*) *For each $u_0 \in \mathbf{U}$, $\lim_{n,m\to\infty} \eta c_2\phi(u_n, u_m) < 1$, here $u_n = T^n u_0$,*
(*ii*) *$H_\Phi(Tu, Tv) \leq \eta N_{c_1,c_2}(u, v)$ for all $u, v \in \mathbf{U}$,*
(*iii*) *T is continuous.*

Then T has a fixed point in $\mathbf{U}$.

Proof. From Theorem 11, by taking in account condition (i) and (ii), we conclude that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence such that:

$$u_{n+1} \in Tu_n, \tag{32}$$

for every $n \in \mathbb{N}$. As $\mathbf{U}$ is complete, so there exists $u \in \mathbf{U}$ such that $\lim_{n\to\infty} u_n = u$. From inequality (3), by the continuity of T, it follows that:

$$u_{n+1} = Tu_n \to Tu, \quad \text{as } n \to \infty.$$

Therefore, $u \in Tu$. Hence T has a fixed point in $\mathbf{U}$. □

Theorem 13. *Let $(\mathbf{U}, d_\phi)$ be a complete extended b-metric space. Let $T : \mathbf{U} \to \mathcal{CB}(\mathbf{U})$ be a multi-valued mapping having the property that there exist $c_1, c_2 \in [0,1]$ and $\eta \in [0,1)$ such that:*

(*i*) *For each $u_0 \in \mathbf{U}$ $\lim_{n,m\to\infty} \eta c_2\phi(u_n, u_m) < 1$, here $u_n = T^n u_0$,*
(*ii*) *$H_\Phi(Tu, Tv) \leq \eta N_{c_1,c_2}(u, v)$ for all $u, v \in \mathbf{U}$,*
(*iii*) *T is $*$-continuous.*

Then T has a fixed point in $\mathbf{U}$.

Proof. From Theorem 3, by taking in account condition (i) and (ii), we conclude that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence such that:

$$u_{n+1} \in Tu_n, \tag{33}$$

for every $n \in \mathbb{N}$. As $\mathbf{U}$ is complete, so there exists $u \in \mathbf{U}$ such that $\lim_{n\to\infty} u_n = u$. Then we have:

$$\begin{aligned} d_\phi(u_{n+1}, Tu) &= d_\phi(Tu_n, Tu) \leq H_\Phi(Tu_n, Tu) \leq \eta N_{c_1,c_2}(u_n, u) \leq \eta \max\{d_\phi(u_n, u), c_1 \\ &\quad d_\phi(u_n, Tu_n), c_1 d_\phi(u, Tu), \frac{c_2}{2}(d_\phi(u_n, Tu) + d_\phi(u, Tu_n))\} \leq \eta \max\{ \\ &\quad d_\phi(u_n, u), c_1 d_\phi(u_n, u_{n+1}), c_1 d_\phi(u, Tu), \frac{c_2}{2}(d_\phi(u_n, Tu) + d_\phi(u, Tu_n))\} \\ &\leq \eta \max\{d_\phi(u_n, u), c_1 d_\phi(u_n, u_{n+1}), c_1 d_\phi(u, Tu), \frac{c_2}{2}(\phi(u_n, Tu) \qquad (34) \\ &\quad (d_\phi(u_n, u) + d_\phi(u, Tu))) + d_\phi(u, u_{n+1})\}, \qquad (35) \end{aligned}$$

for every $n \in \mathbb{N}$. Since $\lim\limits_{n\to\infty} u_n = u$, $\lim\limits_{n\to\infty} d_\phi(u_n, u_{n+1}) = 0$. Then $\lim\limits_{n\to\infty} d_\phi(u_{n+1}, Tu) = d_\phi(u, Tu)$. Therefore, by taking limit $n \to \infty$ in Equations (34) and (35), we obtain:

$$
\begin{aligned}
d_\phi(u,Tu) &\leq \eta N_{c_1,c_2}(u_n,u) \\
&\leq \eta \max\{0, c_1 d_\phi(u,Tu), \frac{c_2 \lim_{n\to\infty} \phi(u_n,Tu)}{2} d_\phi(u,Tu)\} \\
&\leq \max\{\eta c_1, \eta \frac{\eta c_2 \lim_{n\to\infty} \phi(u_n,Tu)}{2}\} d_\phi(u,Tu).
\end{aligned}
$$

As $\max\{\eta c_1, \eta \frac{\eta c_2 \lim_{n\to\infty} \phi(u_n,Tu)}{2}\} < 1$, so from above inequality $d_\phi(u,Tu) < d_\phi(u,Tu)$, which is impossible, therefore $d_\phi(u,Tu) = 0$ i.e., $u \in Tu$. Hence T has a fixed point in $\mathbf{U}$. $\square$

Theorem 14. *A multi-valued mapping $T : \mathbf{U} \to \mathcal{CB}(\mathbf{U})$ has a fixed point in a complete extended b-metric space $(\mathbf{U}, d_\phi)$, if it satisfies the following two axioms:*

(i) *There exist $c_1, c_2 \in [0,1]$ and $\eta \in [0,1)$ such that $H_\Phi(Tu,Tv) \leq \eta N_{c_1,c_2}(u,v)$ for all $u, v \in \mathbf{U}$,*

(ii) *For each $u_0 \in \mathbf{U}$, $\max\{\eta c_1 \lim_{n,m\to\infty} \phi(u_n,u_m), \eta c_2 \lim_{n,m\to\infty} \phi(u_n,u_m)\} < 1$, here $u_n = T^n u_0$.*

Proof. From Theorem 11, by taking in account condition (i) and (ii), we conclude that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence such that:

$$u_{n+1} \in Tu_n, \tag{36}$$

for every $n \in \mathbb{N}$. As $\mathbf{U}$ is complete, so there exists $u \in \mathbf{U}$ such that $\lim_{n\to\infty} u_n = u$. Then for every $n \in \mathbb{N}$, we have:

$$
\begin{aligned}
& d_\phi(u_{n+1},Tu) = d_\phi(Tu_n,Tu) \leq H_\Phi(Tu_n,Tu) \leq \eta N_{c_1,c_2}(u_n,u) \\
\leq\ & \eta \max\{d_\phi(u_n,u), c_1 d_\phi(u_n,Tu_n), c_1 d_\phi(u,Tu), \frac{c_2}{2}(d_\phi(u_n,Tu) + d_\phi(u,Tu_n))\} \\
\leq\ & \eta \max\{d_\phi(u_n,u), c_1 d_\phi(u_n,u_{n+1}), c_1 d_\phi(u,Tu), \frac{c_2}{2}(d_\phi(u_n,Tu) + d_\phi(u,Tu_n))\} \\
\leq\ & \eta \max\{d_\phi(u_n,u), c_1 d_\phi(u_n,u_{n+1}), c_1 d_\phi(u,Tu), \frac{c_2}{2}(\phi(u_n,Tu) \qquad (37)\\
& (d_\phi(u_n,u) + d_\phi(u,Tu))) + d_\phi(u,u_{n+1})\}. \qquad (38)
\end{aligned}
$$

Now, we will take two cases:

Case (i): If $d_\phi(u,Tu) \leq \lim_{n\to\infty} \sup d_\phi(u_n,Tu)$, then there exists a subsequence $\{u_{n_l}\}_{n\in\mathbb{N}}$ of $\{u_n\}$ such that $d_\phi(u,Tu) \leq \lim_{l\to\infty} d_\phi(u_{n_l+1},Tu)$, so for each $\epsilon > 0$, $\exists\, l_\epsilon \in \mathbb{N}$ such that for every $l \in \mathbb{N}$, $l \geq l_\epsilon$, we have:

$$
\begin{aligned}
& d_\phi(u,Tu) - \epsilon \leq d_\phi(u_{n_l+1},Tu) \\
\leq\ & \eta \max\{d_\phi(u_{n_l},u), c_1 d_\phi(u_{n_l},u_{n_l+1}), c_1 d_\phi(u,Tu), \frac{c_2}{2} \\
& (d_\phi(u_{n_l},Tu) + d_\phi(u,u_{n_l+1}))\} \\
\leq\ & \eta \max\{d_\phi(u_{n_l},u), c_1 d_\phi(u_{n_l},u_{n_l+1}), c_1 d_\phi(u,Tu), \frac{c_2}{2} \qquad (39)\\
& (\phi(u_{n_l},Tu)(d_\phi(u_{n_l},u) + d_\phi(u,Tu)) + d_\phi(u,u_{n_l+1})\}. \qquad (40)
\end{aligned}
$$

Since $\lim_{l\to\infty} u_{n_l} = u$, $\lim_{l\to\infty} d_\phi(u_{n_l},u_{n_l+1}) = 0$. Therefore, by taking limit $l \to \infty$ in Equations (39) and (40), we obtain:

$$
\begin{aligned}
d_\phi(u,Tu) - \epsilon &\leq \eta \max\{0, c_1 d_\phi(u,Tu), \frac{c_2 \lim_{l\to\infty} \phi(u_{n_l},Tu)}{2} d_\phi(u,Tu)\} \\
&\leq \eta \max\{c_1, \eta \frac{c_2 \lim_{l\to\infty} \phi(u_{n_l},Tu)}{2}\} d_\phi(u,Tu),
\end{aligned}
$$

for every $\epsilon > 0$. Thus,

$$d_\phi(u, Tu) \leq \max\{\eta c_1, \eta \frac{\eta c_2 \lim_{l\to\infty} \phi(u_{n_l}, Tu)}{2}\} d_\phi(u, Tu).$$

As $\max\{\eta c_1, \eta \frac{\eta c_2 \lim_{l\to\infty} \phi(u_{n_l}, Tu)}{2}\} < 1$, so from above inequality $d_\phi(u, Tu) < d_\phi(u, Tu)$, which is impossible, therefore $d_\phi(u, Tu) = 0$, i.e., $u \in Tu$. Hence T has a fixed point in $\mathbf{U}$.

Case (ii): If $d_\phi(u, Tu) > \lim_{n\to\infty} \sup d_\phi(u_n, Tu)$, then there exists $N_0 \in \mathbb{N}$ such that for every $n \geq N_0$, we have

$$d_\phi(u_{n_l}, Tu) \leq d_\phi(u, Tu).$$

From the triangle inequality, $d_\phi(u, Tu) \leq \phi(u, Tu)(d_\phi(u, u_{n+1}) + d_\phi(u_{n+1}, Tu))$, we obtain:

$$\begin{aligned}
& d_\phi(u, Tu) - \phi(u, Tu)(d_\phi(u, u_{n+1}) \leq \phi(u, Tu) d_\phi(u_{n+1}, Tu) \\
& \leq \phi(u, Tu)\eta \max\{d_\phi(u_n, u), c_1 d_\phi(u_n, u_{n+1}), c_1 d_\phi(u, Tu), \frac{c_2}{2}(d_\phi(u_n, Tu) + d_\phi(u, u_{n+1}))\} \\
& \leq \eta \max\{d_\phi(u_n, u), c_1 d_\phi(u_n, u_{n+1}), c_1 d_\phi(u, Tu), \frac{c_2}{2}(\phi(u_n, Tu) \qquad (41) \\
& (d_\phi(u_n, u) + d_\phi(u, Tu))) + d_\phi(u, u_{n+1})\}. \qquad (42)
\end{aligned}$$

Since $\lim_{n\to\infty} u_n = u$, $\lim_{n\to\infty} d_\phi(u_n, u_{n+1}) = 0$. Therefore by taking limit $n \to \infty$ in Equations (41) and (42), we obtain:

$$\begin{aligned}
& d_\phi(u, Tu) - \phi(u, Tu) d_\phi(u, u_{n+1}) \leq \\
& \phi(u, Tu)\eta \max\{0, c_1 d_\phi(u, Tu), \frac{c_2 \lim_{n\to\infty} \phi(u_n, Tu)}{2} d_\phi(u, Tu) \\
& \leq \phi(u, Tu) \max\{\eta c_1, \eta \frac{\eta c_2 \lim_{n\to\infty} \phi(u_n, Tu)}{2}\} d_\phi(u, Tu), \qquad (43)
\end{aligned}$$

from condition (ii), since $\phi(u, Tu) \max\{\eta c_1, \eta \frac{\eta c_2 \lim_{n\to\infty} \phi(u_n, Tu)}{2}\} < 1$, so from Equation (43), $d_\phi(u, Tu) < d_\phi(u, Tu)$, which is impossible, therefore $d_\phi(u, Tu) = 0$, i.e., $u \in Tu$. Hence T has a fixed point in $\mathbf{U}$.

□

Remark 7.

(i) *For $c_1, c_2 = 0$ in Theorem 12, we obtain Nadler's contraction principle for multi valued-mappings, i.e., Theorem 5 from [24].*

(ii) *Theorem 14 generalizes Theorems 12 and 13;*

(ii) *Theorem 14 generalizes Theorem 3.3 from [42], which generalizes Theorem 7 of [30]. Also, Theorem 7, which is a generalization of Theorem 2.2 from [29], improves Theorem 3.3 from [43], Corollary 3.3 from [5], and Theorem 1 from [28].*

Example 2. *Let $\mathbf{U} = \{\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots\} \cup \{0, 1\}$, $d_\phi(u_1, u_2) = (u_1 - u_2)^2$, for $u_1, u_2 \in \mathbf{U}$, where $\phi : \mathbf{U} \times \mathbf{U} \to [1, \infty)$ define by $\phi(u_1, u_2) = u_1 + u_2 + 1$. Then $\mathbf{U}$ is a complete extended b-metric space. Define mapping $T : \mathbf{U} \to \mathcal{CB}(\mathbf{U})$ as*

$$Tu = \begin{cases} \{\frac{1}{2^{n+1}}\}, & u = \frac{1}{2^n},\ n = 0, 1, 2, \ldots \\ u, & u = 0. \end{cases}$$

Hence T is continuous. Since $N_{c_1,c_2}(\frac{1}{2^n}, 0) = \frac{1}{2^{2n}}$*, for all* $c_1, c_2 \in [0,1]$*, we get:*

$$H_\Phi\Big(T\Big(\frac{1}{2^n}\Big), T(0)\Big) = \frac{1}{2^{2n+2}} \leq \frac{1}{2^{2n+1}} \leq \frac{1}{2} N_{c_1,c_2}\Big(\frac{1}{2^n}, 0\Big),$$

where $\eta = \frac{1}{2}$*. Also for each* $u_0 \in \mathbf{U}$, $\lim\limits_{n,m\to\infty} \eta c_2 \phi(u_n, u_m) < 1$*. Clearly, it satisfies all the conditions of Theorem 12, and so there exists a fixed point.*

Example 3. *Let* $\mathbf{U} = [0,\infty)$*. Define* $d_\phi(u_1, u_2) = (u_1 - u_2)^2$*, for* $u_1, u_2 \in \mathbf{U}$*, where* $\phi : \mathbf{U} \times \mathbf{U} \to [1, \infty)$*, where* $\phi(u_1, u_2) = u_1 + u_2 + 2$*. Then* $\mathbf{U}$ *is a complete extended b-metric space. Define mapping* $T : \mathbf{U} \to \mathcal{CB}(\mathbf{U})$ *as* $Tu = \{\frac{8}{9}u\}$ *for every* $u \in \mathbf{U}$*. Note that Theorem 14 is applicable by taking* $c_1 = c_2 = 0$ *and* $\eta = \frac{8}{9}$*.*

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References

1. Banach, S. Sur les operations dans les ensembles et leur application aux equation sitegrales. *Fundam. Math.* **1922**, *3*, 133–181. [CrossRef]
2. Kannan, R. Some results on fixed points. *Bull. Calc. Math. Soc.* **1972**, *25*, 727–730. [CrossRef]
3. Edelstein, M. An extension of Banach's contraction principle. *J. Lond. Math. Soc.* **1961**, *12*, 7–10.
4. Hardy, G.E.; Rogers, T.D. A generalization of a fixed point theorem of Reich. *Can. Math. Bull.* **1973**, *16*, 201–206. [CrossRef]
5. Rouhani, B.D.; Moradi, S. Common fixed point of multivalued generalized ϕ-weak contractive mappings. *Fixed Point Theory Appl.* **2010**, *2010*, 708984.
6. Aydi, H.; Shatanawi, W.; Vetro, C. On generalized weakly G-contraction mapping in G-metric spaces. *Comput. Math. Appl.* **2011**, *62*, 4222–4229. [CrossRef]
7. la Sen, M.D.; Singh, S.L.; Gordji, M.E.; Ibeas, A.; Agarwal, R.P. Fixed point-type results for a class of extended cyclic self-mappings under three general weak contractive conditions of rational type. *Fixed Point Theory Appl.* **2011**, *2011*, 102. [CrossRef]
8. Aydi, H.; Abbas, M.; Vetro, C. Partial Hausdorff Metric and Nadler's Fixed Point Theorem on Partial Metric Spaces. *Topol. Its Appl.* **2012**, *159*, 3234–3242. [CrossRef]
9. Kakutani, S. A generalization of Brouwer's fixed point Theorem. *Duke Math. J.* **1941**, *8*, 457–459. [CrossRef]
10. Aydi, H.; Bota, M.F.; Karapinar, E.; Moradi, S. A common fixed point for weak ϕ-contractions on b-metric spaces. *Fixed Point Theory* **2012**, *13*, 337–346.
11. Aydi, H.; Karapinar, E.; Samet, B. Fixed point theorems for various classes of cyclic mappings. *J. Appl. Math.* **2012**, *2012*, 867216. [CrossRef]
12. Mitrović, Z.D.; Aydi, H.; Hussain, N.; Mukheimer, A. Reich, Jungck, and Berinde common fixed point results on $\mathcal{F}$-metric spaces and an application. *Mathematics* **2019**, *7*, 387. [CrossRef]
13. Patle, P.R.; Patel, D.K.; Aydi, H.; Gopal, D.; Mlaiki, N. Nadler and Kannan type set valued mappings in M-metric spaces and an application. *Mathematics* **2019**, *7*, 373. [CrossRef]
14. Chatterjea, S.K. Fixed point theorems. *C. R. Acad. Bulgare Sci.* **1968**, *60*, 71–76. [CrossRef]
15. Bakhtin, I.A. The contraction mapping principle in almost metric spaces. *Funct. Anal.* **1989**, *30*, 26–37.
16. Bourbaki, N. *Topologie Generale*; Herman: Paris, France, 1974.
17. Czerwik, S. Nonlinear set-valued contraction mappings in b-metric spaces. *Atti Sem. Mat. Fis. Univ. Modena* **1998**, *46*, 263–276.

18. Czerwik, S. Contraction mappings in b-metric spaces. *Acta Math. Inform. Univ. Ostrav.* **1993**, *1*, 5–11.
19. Kir, M.; Kiziltunc, H. On some well known fixed point theorems in b-metric spaces. *Turk. Anal. Number Theory* **2013**, *1*, 13–16. [CrossRef]
20. Dubey, A.K.; Shukla, R.; Dubey, R.P. Some fixed point results in b-metric spaces. *Asian J. Math. Appl.* **2014**, *1*, 1–6.
21. Koleva, R.; Zlatanov, B. On fixed points for Chatterjeas maps in b-metric spaces. *Turk. J. Anal. Number Theory* **2016**, *4*, 31–34.
22. Ilchev, A.; Zlatanov, B. *On Fixed Points for Reich Maps in B-Metric Spaces*; Annual of Konstantin Preslavski University of Shumen, Faculty of Mathematics and Computer Science XVII C: Shumen, Bulgaria, 2016; pp. 77–88.
23. Aleksic, S.; Mitrovic, Z.D.; Radenovic, S. On some recent fixed point results for single and multi-valued mapping in b-metric space. *Fasc. Math.* **2018**, *61*, 5–16.
24. Nadler, S.B. Multi-valued contraction mappings. *Not. Am. Math. Soc.***1967**, *14*, 930. [CrossRef]
25. Singh, S.L.; Mishra, S.N.; Chugh, R.; Kamal, R. General Common Fixed Point Theorems and Applications. *J. Appl. Math.* **2012**, *2012*, 902312. [CrossRef]
26. la Sen, M.D.; Singh, S.L.; Gordji, M.E.; Ibeas, A.; Agarwal, R.P. Best proximity and fixed point results for cyclic multi-valued mappings under a generalized contractive condition. *Fixed Point Theory Appl.* **2013**, *2013*, 324. [CrossRef]
27. Petrusel, A.; Petrusel, G. On Reich's strict fixed point theorem for multi-valued operators in complete metric spaces. *J. Nonlinear Var. Anal.* **2018**, *2*, 103–112.
28. Ciric, L. A generalization of Banach's contraction principle. *Proc. Am. Math. Soc.* **1974**, *45*, 267–273. [CrossRef]
29. Amini-Harandi, A. Fixed point theory for set-valued quasi-contraction maps in metric spaces. *Appl. Math. Lett.* **2011**, *24*, 1791–1794. [CrossRef]
30. Aydi, H.; Bota, M.F.; Karapinar, E.; Mitrovic, S. A fixed point Theorem for set-valued quasi-contractions in b-metric spaces. *Fixed Point Theory Appl.* **2012**, *2012*, 88. [CrossRef]
31. Subashi, L. Some topological properties of extended b-metric space. In Proceedings of the 5th International Virtual Conference on Advanced Scientific Results, Zilina, Slovakia, 26–30 June 2017; Volume 5, pp. 164–167.
32. Subashi, L.; Gjini, N. Fractals in extended b-metric space. *J. Prog. Res. Math.* **2017**, *12*, 2057–2065.
33. Abdeljawad, T.; Mlaiki, N.; Aydi, H.; Souayah, N. Double Controlled Metric Type Spaces and Some Fixed Point Results. *Mathematics* **2018**, *6*, 320. [CrossRef]
34. Mlaiki, N.; Aydi, H.; Souayah, N.; Abdeljawad, T. Controlled metric type spaces and the related contraction principle. *Mathematics* **2018**, *6*, 194. [CrossRef]
35. Subashi, L.; Gjini, N. Some results on extended b-metric spaces and Pompeiu-Hausdorff metric. *J. Prog. Res. Math.* **2017**, *12*, 2021–2029.
36. Alqahtani, B.; Fulga, A.; Karapinar, E. Non-Unique fixed point results in extended b-metric space. *Mathematics* **2018**, *6*, 68. [CrossRef]
37. Shatanawi, W.; Mukheimer, A.; Abodayeh, K. Some fixed point Theorems in extended b-metric spaces. *Appl. Math. Phys.* **2018**, *80*, 71–78.
38. Kamran, T. A Generalization of b-metric space and some fixed point theorems. *Mathematics* **2017**, *5*, 19. [CrossRef]
39. Alqahtani, B.; Fulga, A.; Karapinar, E. Contractions with rational expression in the extended b-metric space. 2019, in press.
40. Jovanovic, M.; Kadelburg, Z.; Radenovic, S. Common fixed point results in metric-type spaces. *Fixed Point Theory Appl.* **2010**, *1*, 978121. [CrossRef]
41. Shah, M.H.; Simic, S.; Hussain, N.; Sretenovic, A.; Radenovic, S. Common fixed points theorems for occasionally weakly compatible pairs on cone metric type spaces. *J. Comput. Anal. Appl.* **2012**, *14*, 290–297.

42. Miculescu, R.; Mihail, A. New fixed point theorems for set-valued contractions in b-metric spaces. *J. Fixed Point Theory Appl.* **2017**, *19*, 2153–2163 [CrossRef]
43. Daffer, P.Z.; Kaneko, H. Fixed points of generalized contractive multi-valued mappings. *J. Math. Anal. Appl.* **1995**, *192*, 655–666. [CrossRef]