mathematics

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Linguistic Spherical Fuzzy Aggregation Operators and Their Applications in Multi-Attribute Decision Making Problems

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Received: 6 April 2019; Accepted: 2 May 2019; Published: 8 May 2019

Abstract: The key objective of the proposed work in this paper is to introduce a generalized form of linguistic picture fuzzy set, so-called linguistic spherical fuzzy set (LSFS), combining the notion of linguistic fuzzy set and spherical fuzzy set. In LSFS we deal with the vague and defective information in decision making. LSFS is characterized by linguistic positive, linguistic neutral and linguistic negative membership degree which satisfies the conditions that the square sum of its linguistic membership degrees is less than or equal to 1. In this paper, we investigate the basic operations of linguistic spherical fuzzy sets and discuss some related results. We extend operational laws of aggregation operators and propose linguistic spherical fuzzy weighted averaging and geometric operators based on spherical fuzzy numbers. Further, the proposed aggregation operators of linguistic spherical fuzzy number are applied to multi-attribute group decision-making problems. To implement the proposed models, we provide some numerical applications of group decision-making problems. In addition, compared with the previous model, we conclude that the proposed technique is more effective and reliable.

Keywords: aggregation operators; spherical fuzzy set; linguistic spherical fuzzy set; decision making problems

1. Introduction

Fuzzy set was first defined by Zadeh in (1965) [1]. Membership function is the only characteristic of the fuzzy set, but it can be hard at times to characterize more fuzzy data. So, to solve this drawback, Atanassov [2] introduced the intuitionistic fuzzy set (IFS), which is the extended form of Fuzzy set (FS), that consists of positive and negative membership degree. After that, in 1994, the notion of interval-valued IFS was introduced by Atanassov [3–6]. Operational laws and comparison rules for the interval-valued IFS are defined by Atanassov. Recently, more multi-criteria decision making problems have been proposed, which depend on the IFS [7–13].

In (2013) Yager [14] defined Pythagorean fuzzy set (PyFS). The positive and negative membership degree of PFS satisfied that the sum of square of positive and negative membership degree is less than or equal to one. Yager and Abbasov [15] developed some aggregation operators for multi-attribute decision making (MADM) problems under the Pythagorean fuzzy information. Peng and Yang [16] explain their relationship among these aggregation operators and established the superiority and inferiority ranking of multi-attribute group decision making (MAGDM) method. Using Einstein operation Garg [17] defined the generalized Pythagorean fuzzy information aggregation. Gou et al. [18] discussed many properties of Pythagorean fuzzy set such as continuity, derivability, and differentiability. Zeng et al. [19] examined a hybrid method for Pythagorean fuzzy multiple-criteria decision making (MCDM). Zeng [20], applied the Pythagorean fuzzy probabilistic ordered weighted averaging operator on MAGDM problem. Sajjad et al. [21] defined Pythagorean hesitant fuzzy sets and also discussed their application to group decision making, where the weighting vector is not given. The operation of division and subtraction for the Pythagorean fuzzy set are defined by Peng and Yang [16], and explain their comparable properties. The notion of Pythagorean fuzzy linguistic sets is proposed by Peng and Yong [22], score function and operational laws for the Pythagorean fuzzy linguistic numbers are also developed. Khan et al. [23] proposed the pythagorean fuzzy Dombi aggregation operators based on PyFS information. Liu and Wang [24] proposed some q-rung orthopair fuzzy aggregation operators and discussed their applications to multiple-attribute decision making and in [25] proposed the q-Rung orthopai fuzzy Bonferroni mean aggregation operators.

Cuong [26,27] introduced a novel concept of picture fuzzy set (PFS), which dignified in three different functions presenting the positive, neutral and negative membership degrees. Cuong [28], studied some characteristics of PFSs and also approved their distance measures. Cuong and Hai [29] defined first time fuzzy logic operators and implications on PFSs, and also introduced principle operations for fuzzy derivation forms in the picture fuzzy logic. Cuong et al. [30] examined the characteristic of picture fuzzy t-norm and t-conorm. Phong et al. [31] explored certain configurations of picture fuzzy relations. Wei et al. [32–34] defined many procedures to compute the closeness between PFSs. Presently, many researchers have developed more models in the PFSs condition: Correlation coefficients of PFS were proposed by Sing [35], who applies them to clustering analysis. Son et al. [36,37] provided time arrangement calculation and temperature estimation on the basis of PFSs domain. Son et al. [38,39] defined picture fuzzy separation measures, generalized picture fuzzy distance measures and picture fuzzy association measures, and combined it to tackle grouping examination under PFSs condition. To improve the achievements of the classical fuzzy inference system, Son et al. [40] defined a novel fuzzy derivation structure on PFS. Thong et al. [41,42] utilized the picture fuzzy clustering method for complex data and particle clump optimization. Wei [43] exhibited PF aggregation operators and tested them to MADM problem for ranking EPR framework. Using the concept of picture fuzzy weighted cross-entropy, Wei [44] studied basic leadership technique and used this technique to rank the option. Based on PFSs, Yang et al. [45] defined adjustable soft discernibly matrix and tested it in decision making. Garg [46] designed aggregation operations on PFSs and applied them to MCDM problems. Peng et al. [47] proposed a PFS approach for the decision making problem. The readers can also see [48,49] for the PFS. To handle MAGDM problems, Ashraf et al. [50] give two techniques to aggregate the picture fuzzy information, one is picture fuzzy aggregation operators and the second one is using TOPSIS method. Bo and Zhang [51] studied more operations of picture fuzzy relations such as type-2 inclusion relation, type-2 union, type-2 intersection and type-2 complement operations and also defined the anti-reflexive kernel, symmetric kernel, reflexive closure and symmetric closure of a picture fuzzy relation. Ashraf et al. [52] developed the structure of cubic sets to the picture fuzzy sets. They also defined the notion of positive internal, neutral internal, negative internal and positive-external, neutral external and negative external cubic picture fuzzy sets. Ashraf [53] proposed the novel concept of picture fuzzy linguistic fuzzy set and discussed its applications. For further study, we refer to [54–57].

Sometimes in real life, we face many problems which cannot be handled by using PFS, for example when $P(x) + I(x) + N(x) > 1$. In such condition, PFS has no ability to obtain any satisfactory result. Ashraf et al. [58] proposed the new concept of spherical fuzzy set, which is a generalized form of PFSs and Pythgorean fuzzy set to resolve the issues in existing structures. Spherical fuzzy set gives more space to the decision maker to deal with uncertainty in decision making problems. After that, Ashraf and Saleem [59] proposed the aggregation operators for spherical fuzzy sets and in [60] Ashraf et al. proposed the GRA method for spherical fuzzy linguistic fuzzy set and discussed its applications. In [61] Zeng et al. proposed the covering-based spherical fuzzy rough set model hybrid with TOPSIS approach.

According to the analyses above, this study aims to propose the notion of linguistic spherical fuzzy set. In addition, due to the importance of aggregation operators in the decision making technique, we propose averaging and geometric aggregation operators for linguistic spherical fuzzy information. After that, we propose the multi-attribute decision making approach to deal with uncertainty in decision making problems based on defined aggregation operators.

The remainder of this article is arranged as: Section 2 briefly discuses the fundamental notations of linguistic fuzzy set and spherical fuzzy set. In Section 3, we define some operational laws of LSFSs and their proofs. Section 4 consists of linguistic spherical fuzzy aggregation operators. Section 5 consists of some discussion on the application of the defined approach. In Section 6, an algorithm is developed with the numerical example. In Section 7, we discuss the comparison and advantages of the proposed work and finally, conclusions are drawn in Section 8.

2. Preliminaries

Definition 1. *[27] Let $\underset{\cdot}{K}$ be a universal set, then a PFS $\mathbb{R}$ in $\underset{\cdot}{K}$ is defined as*

$$\mathbb{R} = \{(\underset{\cdot}{k}, \breve{a}_{\mathbb{R}}(\underset{\cdot}{k}), \breve{e}_{\mathbb{R}}(\underset{\cdot}{k}), \breve{\imath}_{\mathbb{R}}(\underset{\cdot}{k})) | \underset{\cdot}{k} \in \underset{\cdot}{K}\}, \tag{1}$$

where $\breve{a}_{\mathbb{R}}(\underset{\cdot}{k}), \breve{e}_{\mathbb{R}}(\underset{\cdot}{k}), \breve{\imath}_{\mathbb{R}}(\underset{\cdot}{k})$ $:\underset{\cdot}{K} \longrightarrow [0,1]$, and satisfy the condition that: $0 \leq \breve{a}_{\mathbb{R}}(\underset{\cdot}{k}), \breve{e}_{\mathbb{R}}(\underset{\cdot}{k}), \breve{\imath}_{\mathbb{R}}(\underset{\cdot}{k}) \leq 1$. Furthermore, $\breve{a}_{\mathbb{R}}(\underset{\cdot}{k})$, $\breve{e}_{\mathbb{R}}(\underset{\cdot}{k})$ and $\breve{\imath}_{\mathbb{R}}(\underset{\cdot}{k})$ indicate the positive, neutral and negative grads of the element $\underset{\cdot}{k} \in \underset{\cdot}{K}$ to the set $\mathbb{R}$, respectively. For each PFS $\mathbb{R} \subseteq \underset{\cdot}{K}$, $\pi_{\mathbb{R}}(\underset{\cdot}{k}) = 1 - \breve{a}_{\mathbb{R}}(\underset{\cdot}{k}) - \breve{e}_{\mathbb{R}}(\underset{\cdot}{k}) - \breve{\imath}_{\mathbb{R}}(\underset{\cdot}{k})$ is said to be the refusal degree of $\underset{\cdot}{K}$ to $\mathbb{R}$.

Definition 2. *[62,63] Let $\hat{S} = (\acute{s}_1, \acute{s}_2, ..., \acute{s}_g)$ be the finite and absolutely order distinct term set. Then, $\hat{S}$ is said to be a linguistic set, and the value of g is considered an odd number, e.g., $3, 5, ...,$ when $g = 3$, then $\hat{S}$ can be written as $\hat{S} = (\acute{s}_1, \acute{s}_2, \acute{s}_3) =$ (poor, fair, good)*

The following characteristics of the linguistic set $\hat{S}$ must be satisfied;

(1) Ordered: $\acute{s}_k \prec \acute{s}_l, \Leftrightarrow k \prec l$;

(2) Negation: $Neg(\acute{s}_k) = \acute{s}_{g-1-k}$;

(3) Max: $(\acute{s}_k, \acute{s}_l) = \acute{s}_k$, iff $k \geq l$;

(4) Min: $(\acute{s}_k, \acute{s}_l) = \acute{s}_k$, iff $k \leq l$.

The extended form of the discrete set $\hat{S}$ is called a continuous linguistic set and defined as $\hat{S}^* = \{\acute{s}_\psi | \acute{s}_0 \leq \acute{s}_\psi \leq \acute{s}_g, \psi \in [0, g]$,and if $\acute{s}_\psi \in \hat{S}^*$, then $\acute{s}_\psi$ is said to be original set otherwise, virtual set.

Definition 3. *[53] Let $\underset{\cdot}{K} \neq 0$, and $\hat{S}^* = \{\acute{s}_\psi | \acute{s}_0 \leq \acute{s}_\psi \leq \acute{s}_g, \psi \in [0, g]$, be a continues linguistic set. Then, a LPFS is defined as*

$$\mathbb{R} = \{\langle \underset{\cdot}{k}, \acute{s}_\alpha(\underset{\cdot}{k}), \acute{s}_\beta(\underset{\cdot}{k}), \acute{s}_\gamma(\underset{\cdot}{k}) \rangle | \underset{\cdot}{k} \in \underset{\cdot}{K}\}, \tag{2}$$

where $\langle \acute{s}_\alpha(\underset{\cdot}{k}), \acute{s}_\beta(\underset{\cdot}{k}), \acute{s}_\gamma(\underset{\cdot}{k}) \rangle \in \hat{S}^*$ represent the linguistic positive, linguistic neutral and linguistic negative degrees of the element $\underset{\cdot}{K}$ to $\mathbb{R}$. Simply, the triple of $\langle \acute{s}_\alpha(\underset{\cdot}{k}), \acute{s}_\beta(\underset{\cdot}{k}), \acute{s}_\gamma(\underset{\cdot}{k}) \rangle$ is denoted as $\mathbb{R} = \langle \acute{s}_\alpha, \acute{s}_\beta, \acute{s}_\gamma \rangle$ and referred to as linguistic picture fuzzy value (LPFV).

For any $\underset{\cdot}{k} \in \underset{\cdot}{K}$, the condition $\alpha + \beta + \gamma \leq g$ is always satisfied, and $\pi(\underset{\cdot}{k}) = \acute{s}_{g-\alpha-\beta-\gamma}$ is the linguistic refusal degree of $\underset{\cdot}{K}$ to $\mathbb{R}$. Obviously, if $\alpha - \beta - \gamma = g$, then LPFS has the minimum linguistic indeterminacy degree, that is,$\pi(\underset{\cdot}{k}) = \acute{s}_0$, which means that the membership degree of $\underset{\cdot}{k}$ to $\mathbb{R}$ can be precisely expressed with a single linguistic term and LPFS $\mathbb{R}$ is reduced to a linguistic variable. Oppositely, if $\alpha = \beta = \gamma = 0$, then LPFS $\mathbb{R}(\underset{\cdot}{k})$ has the maximum linguistic indeterminacy degree; that is, $\pi(\underset{\cdot}{k}) = \acute{s}_0$.

Definition 4. *[58] A SFS $\mathbb{R}$ on the universal set $\c{K}$ is defined as*

$$\mathbb{R} = \{\langle \c{k}, \breve{a}_{\mathbb{R}}(\c{k}), \breve{e}_{\mathbb{R}}(\c{k}), \breve{\imath}_{\mathbb{R}}(\c{k})\rangle | \c{k} \in \c{K}\}, \tag{3}$$

where the function $\breve{a}_{\mathbb{R}}(\c{k}), \breve{e}_{\mathbb{R}}(\c{k}), \breve{\imath}_{\mathbb{R}}(\c{k}) : \c{k} \to [0,1]$. For each $\c{k} \in \c{K}$ $\breve{a}_{\mathbb{R}}(\c{k}), \breve{e}_{\mathbb{R}}(\c{k})$ and $\breve{\imath}_{\mathbb{R}}(\c{k})$ are respectively called the positive, neutral and negative membership degree of $\c{k}$ in $\mathbb{R}$, it holds that $(\breve{a}_{\mathbb{R}}(\c{k}))^2 + (\breve{e}_{\mathbb{R}}(\c{k}))^2 + (\breve{\imath}_{\mathbb{R}}(\c{k}))^2 \leq 1$ for all $\c{k} \in \c{K}$, the degree of refusal is defined as $\pi_{\mathbb{R}}(\c{k}) = \sqrt{1 - (\breve{a}_{\mathbb{R}}(\c{k}))^2 + (\breve{e}_{\mathbb{R}}(\c{k}))^2 + (\breve{\imath}_{\mathbb{R}}(\c{k}))^2}$. For SFS $\{\langle \c{k}, \breve{a}_{\mathbb{R}}(\c{k}), \breve{e}_{\mathbb{R}}(\c{k}), \breve{\imath}_{\mathbb{R}}(\c{k})\rangle | \c{k} \in \c{k}\}$, a triple components $\langle \breve{a}_{\mathbb{R}}(\c{k}), \breve{e}_{\mathbb{R}}(\c{k}), \breve{\imath}_{\mathbb{R}}(\c{k})\rangle$ is called SFN and denoted as $\mathbb{R} = \langle \breve{a}_{\mathbb{R}}(\c{k}), \breve{e}_{\mathbb{R}}(\c{k}), \breve{\imath}_{\mathbb{R}}(\c{k})\rangle$, where $\breve{a}_{\mathbb{R}}(\c{k}), \breve{e}_{\mathbb{R}}(\c{k})$ and $\breve{\imath}_{\mathbb{R}}(\c{k}) \in [0,1]$, under the following condition;

$$0 \leq (\breve{a}_{\mathbb{R}}(\c{k}))^2 + (\breve{e}_{\mathbb{R}}(\c{k}))^2 + (\breve{\imath}_{\mathbb{R}}(\c{k}))^2 \leq 1.$$

3. Linguistic Spherical Fuzzy Set

Definition 5. *Let $\c{K}$ be a universe of discourse and $\hat{S}^* = \{\acute{s}_\alpha | \acute{s}_0 \leq \acute{s}_\alpha \leq \acute{s}_g, \alpha \in [0,g]$, be a continues linguistic term set. Then, a LSFS is defined as*

$$\mathbb{R} = \{\langle \c{k}, \acute{s}_\alpha(\c{k}), \acute{s}_\beta(\c{k}), \acute{s}_\gamma(\c{k})\rangle | \c{k} \in \c{K}\}, \tag{4}$$

where $\langle \acute{s}_\alpha(\c{k}), \acute{s}_\beta(\c{k}), \acute{s}_\gamma(\c{k})\rangle \in \hat{S}^*$ are the linguistic positive, neutral and negative membership degree of the element $\c{k}$ to $\mathbb{R}$. The triple $\langle \acute{s}_\alpha(\c{k}), \acute{s}_\beta(\c{k}), \acute{s}_\gamma(\c{k})\rangle$ is denoted as $\mathbb{R} = \langle \acute{s}_\alpha, \acute{s}_\beta, \acute{s}_\gamma\rangle$ and called is linguistic spherical fuzzy value (LSFV).

For any $\c{k} \in \c{K}$, the condition $\alpha^2 + \beta^2 + \gamma^2 \leq g^2$ is always satisfied, and $\pi(\c{k}) = \acute{s}_{\sqrt{g^2 - \alpha^2 - \beta^2 - \gamma^2}}$ is called linguistic refusal degree of $\c{k}$ to $\mathbb{R}$.

Definition 6. *Let $\mathbb{R} = \langle \acute{s}_{\alpha_1}, \acute{s}_{\beta_1}, \acute{s}_{\gamma_1}\rangle$ is LSFV with $\acute{s}_{\alpha_1}, \acute{s}_{\beta_1}, \acute{s}_{\gamma_1} \in \hat{S}^*$. Then, we defined the score function as*

$$\c{S}(\mathbb{R}) = \acute{s}_{\sqrt{(g^2 + \alpha^2 - \beta^2 - \gamma^2)/3}} \tag{5}$$

and defined the accuracy function as

$$\bar{E}(\mathbb{R}) = \acute{s}_{\sqrt{(\alpha^2 + \beta^2 + \gamma^2)/3}} \tag{6}$$

It can be easily verified that $0 \leq (g^2 + \alpha_1^2 - \beta_1^2 - \gamma_1^2)/3 \leq g^2$ and $\alpha_1^2 + \beta_1^2 + \gamma_1^2 \leq g^2$, which means that $\acute{s}_{\sqrt{g^2 + \alpha^2 - \beta^2 - \gamma^2/3}}, \acute{s}_{\sqrt{\alpha^2 + \beta^2 + \gamma^2/3}} \in \hat{S}^*$.

Now, we define the comparison rules for the two LSFNs $\mathbb{R}$ and $\mathbb{Z}$, based on the score and accuracy function.

(a) If $\c{S}(\mathbb{R}) > \c{S}(\mathbb{Z})$, then $\mathbb{R} > \mathbb{Z}$;
(b) If $\c{S}(\mathbb{R}) = \c{S}(\mathbb{Z})$ and
- *$\bar{E}(\mathbb{R}) > \bar{E}(\mathbb{Z})$, then $\mathbb{R} > \mathbb{Z}$;*
- *$\bar{E}(\mathbb{R}) = \bar{E}(\mathbb{Z})$, then, $\mathbb{R} = \mathbb{Z}$.*

Example 1. *Let $\mathbb{R} = \langle \acute{s}_3, \acute{s}_5, \acute{s}_2\rangle, \mathbb{Z} = \langle \acute{s}_4, \acute{s}_3, \acute{s}_1\rangle, \check{C} = \langle \acute{s}_6, \acute{s}_4, \acute{s}_3\rangle, D = \langle \acute{s}_2, \acute{s}_3, \acute{s}_5\rangle$ are the LSFNs, which are derived from $\hat{S}^* = \{\acute{s}_\alpha | \acute{s}_0 \leq \acute{s}_\alpha \leq \acute{s}_g, \alpha \in [0,7]\}$. Using Equation (5), we obtain*

$\c{S}(\mathbb{R}) = \acute{s}_{\sqrt{(49+9-25-4)/3}} = \acute{s}_{\sqrt{9.67}}$; $\c{S}(\mathbb{Z}) = \acute{s}_{\sqrt{(49+16-9-1)/3}} = \acute{s}_{\sqrt{18.34}}$;
$\c{S}(\check{C}) = \acute{s}_{\sqrt{(49+36-16-9)/3}} = \acute{s}_{\sqrt{20.00}}$; $\c{S}(D) = \acute{s}_{\sqrt{(49+4-9-25)/3}} = \acute{s}_{\sqrt{19.00}}$;
Thus, we obtain $\check{C} \succ D \succ \mathbb{Z} \succ \mathbb{R}$.

Definition 7. *Let $\mathbb{R} = \langle \acute{s}_{\alpha_1}, \acute{s}_{\beta_1}, \acute{s}_{\gamma_1}\rangle$ and $\mathbb{Z} = \langle \acute{s}_{\alpha_2}, \acute{s}_{\beta_2}, \acute{s}_{\gamma_2}\rangle$ are the two LSFNs, then*

(i) $\mathbb{R} = \mathbb{Z}$ if $\acute{s}_{\alpha_1} = \acute{s}_{\alpha_2}, \acute{s}_{\beta_1} = \acute{s}_{\beta_2}$ and $\acute{s}_{\gamma_1} = \acute{s}_{\gamma_2}$;
(ii) $\mathbb{R}^{\check{C}} = \langle \acute{s}_{\gamma_1}, \acute{s}_{\beta_1}, \acute{s}_{\alpha_1}\rangle$;
(iii) $\mathbb{R} \cap \mathbb{Z} = \langle \min(\acute{s}_{\alpha_1}, \acute{s}_{\alpha_2}), \min(\acute{s}_1, \acute{s}_{\beta_2}), \max(\acute{s}_{\gamma_1}, \acute{s}_{\gamma_2})\rangle$;
(iv) $\mathbb{R} \cup \mathbb{Z} = \langle \max(\acute{s}_{\alpha_1}, \acute{s}_{\alpha_2}), \min(\acute{s}_1, \acute{s}_{\beta_2}) \min(\acute{s}_{\gamma_1}, \acute{s}_{\gamma_2})\rangle$;

(v) $\mathbb{R} < \mathbb{Z}$ *if* $\acute{s}_{\alpha_1} < \acute{s}_{\alpha_2}, \acute{s}_{\beta_1} > \acute{s}_{\beta_2}$ *and* $\acute{s}_{\gamma_1} > \acute{s}_{\gamma_2}$.

Definition 8. *Let* $\mathbb{R} = \langle \acute{s}_{\alpha_1}, \acute{s}_{\beta_1}, \acute{s}_{\gamma_1} \rangle$ *and* $\mathbb{Z} = \langle \acute{s}_{\alpha_2}, \acute{s}_{\beta_2}, \acute{s}_{\gamma_2} \rangle$ *are the two LSFNs, where* $\acute{s}_{\alpha_1}, \acute{s}_{\beta_1}, \acute{s}_{\gamma_1}, \acute{s}_{\alpha_2}, \acute{s}_{\beta_2}, \acute{s}_{\gamma_2} \in \hat{S}^* = \{\acute{s}_\alpha | \acute{s}_0 \leq \acute{s}_\alpha \leq \acute{s}_g, \alpha \in [0, g]\}$ *with* $\lambda > 0$*(*λ *real number), then*

(1) $\mathbb{R} + \mathbb{Z} = \left(\acute{s}_{g\sqrt{\frac{\alpha_1^2}{g^2} + \frac{\alpha_2^2}{g^2} - \frac{\alpha_1^2\alpha_2^2}{g^4}}}, \acute{s}_{g\left(\frac{\beta_1\beta_2}{g^2}\right)}, \acute{s}_{g\left(\frac{\gamma_1\beta_2}{g^2}\right)} \right)$;

(2) $\mathbb{R} \times \mathbb{Z} = \left(\acute{s}_{g\left(\frac{\alpha_1\alpha_2}{g^2}\right)}, \acute{s}_{g\sqrt{\frac{\beta_1^2}{g^2} + \frac{\beta_2^2}{g^2} - \frac{\beta_1^2\beta_2^2}{g^4}}}, \acute{s}_{g\sqrt{\frac{\gamma_1^2}{g^2} + \frac{\gamma_2^2}{g^2} - \frac{\gamma_1^2\gamma_2^2}{g^4}}} \right)$;

(3) $\lambda\mathbb{R} = \left(\acute{s}_{g\sqrt{1-\left(1-\frac{\alpha_1^2}{g^2}\right)^\lambda}}, \acute{s}_{g\left(\frac{\beta_1}{g}\right)^\lambda}, \acute{s}_{g\left(\frac{\gamma_1}{g}\right)^\lambda} \right)$;

(4) $\mathbb{R}^\lambda = \left(\acute{s}_{g\left(\frac{\alpha_1}{g}\right)^\lambda}, \acute{s}_{g\sqrt{1-\left(1-\frac{\beta_1^2}{g^2}\right)^\lambda}}, \acute{s}_{g\sqrt{1-\left(1-\frac{\gamma_1^2}{g^2}\right)^\lambda}} \right)$.

Theorem 1. *Let* $\mathbb{R} = \langle \acute{s}_{\alpha_1}, \acute{s}_{\beta_1}, \acute{s}_{\gamma_1} \rangle$ *and* $\mathbb{Z} = \langle \acute{s}_{\alpha_2}, \acute{s}_{\beta_2}, \acute{s}_{\gamma_2} \rangle$ *are the two LSFNs, where* $\acute{s}_{\alpha_1}, \acute{s}_{\beta_1}, \acute{s}_{\gamma_1}, \acute{s}_{\alpha_2}, \acute{s}_{\beta_2}, \acute{s}_{\gamma_2} \in \hat{S}^* = \{\acute{s}_\alpha | \acute{s}_0 \leq \acute{s}_\alpha \leq \acute{s}_g, \alpha \in [0, g]\}$ *with* $\lambda, \lambda_1, \lambda_2 > 0$ *be a real number, then we have*

(1) $\lambda(\mathbb{R} + \mathbb{Z}) = \lambda\mathbb{R} + \lambda\mathbb{Z}$;
(2) $(\mathbb{R} \times \mathbb{Z})^\lambda = \mathbb{R}^\lambda \times \mathbb{Z}^\lambda$;
(3) $\lambda_1\mathbb{R} + \lambda_2\mathbb{R} = (\lambda_1 + \lambda_2)\mathbb{R}$;
(4) $\mathbb{R}^{\lambda_1} \times \mathbb{R}^{\lambda_2} = \mathbb{R}^{\lambda_1+\lambda_2}$;
(5) $\mathbb{R} + \mathbb{Z} = \mathbb{Z} + \mathbb{R}$;
(6) $\mathbb{R} \times \mathbb{Z} = \mathbb{Z} \times \mathbb{R}$.

Proof. From the following theorem, we shall only prove part (2), (3) and the proof of the rest are similar.

(2) Since $\mathbb{R}$ and $\mathbb{Z}$ are the LSFNs, then

$$\mathbb{R} \times \mathbb{Z} = \left(\acute{s}_{g(\alpha_1\alpha_2/g^2)}, \acute{s}_{g\sqrt{\beta_1^2/g^2 + \beta_2^2 - \beta_1^2\beta_2^2/g^4}}, \acute{s}_{g\sqrt{\gamma_1^2/g^2 + \gamma_2^2 - \gamma_1^2\gamma_2^2/g^4}} \right).$$

We have

$$\begin{aligned}
(\mathbb{R} \times \mathbb{Z})^\lambda &= \left(\acute{s}_{g(\alpha_1\alpha_2/g^2)^\lambda}, \acute{s}_{g\sqrt{1-(1-(\beta_1^2/g^2 + \beta_2^2/g^2 - \beta_1^2\beta_2^2/g^4))^\lambda}}, \acute{s}_{g\sqrt{1-(1-(\gamma_1^2/g^2 + \gamma_2^2/g^2 - \gamma_1^2\beta_2^2/g^4))^\lambda}} \right) \\
&= \left(\acute{s}_{g(\alpha_1/g)^\lambda(\alpha_2/g)^\lambda}, \acute{s}_{g\sqrt{1-(1-\beta_1^2/g^2)^\lambda(1-\beta_2^2/g^2)^\lambda}}, \acute{s}_{g\sqrt{1-(1-\gamma_1^2/g^2)^\lambda(1-\gamma_2^2/g^2)^\lambda}} \right) \\
&= \left(\begin{array}{c} \acute{s}_{g(\alpha_1/g)^\lambda g(\alpha_2/g)^\lambda}, \acute{s}_{g\sqrt{\{1-(1-\beta_1^2/g^2)^\lambda\} + \{1-(1-\beta_2^2/g^2)^\lambda\} - \{1-(1-\beta_1^2/g^2)^\lambda\}\{1-(1-\beta_2^2/g^2)^\lambda\}}}, \\ \acute{s}_{g\sqrt{\{1-(1-\gamma_1^2/g^2)^\lambda\} + \{1-(1-\gamma_2^2/g^2)^\lambda\} - \{1-(1-\gamma_1^2/g^2)^\lambda\}\{1-(1-\gamma_2^2/g^2)^\lambda\}}} \end{array} \right) \\
&= \left(\acute{s}_{g(\alpha_1/g^2)^\lambda}, \acute{s}_{g\sqrt{1-(1-\beta_1^2/g^2)^\lambda}}, \acute{s}_{g\sqrt{1-(1-\gamma_1^2/g^2)^\lambda}} \right) \\
&\quad \times \left(\acute{s}_{g(\alpha_2/g^2)^\lambda}, \acute{s}_{g\sqrt{1-(1-\beta_2^2/g^2)^\lambda}}, \acute{s}_{g\sqrt{1-(1-\gamma_2^2/g^2)^\lambda}} \right) \\
&= \mathbb{R}^\lambda \times \mathbb{Z}^\lambda.
\end{aligned}$$

(3) For a real number λ_1, λ_2 and LSFNs $\mathbb{R}$ and $\mathbb{Z}$, we have

$$\lambda_1\mathbb{R} = \left(\acute{s}_{g\sqrt{1-(1-\alpha_1^2/g^2)^{\lambda_1}}}, \acute{s}_{g(\beta_1/g^2)^{\lambda_1}}, \acute{s}_{g(\gamma_1/g^2)^{\lambda_1}} \right)$$

and

$$\lambda_2\mathbb{R} = \acute{s}_{g\sqrt{1-(1-\alpha_1^2/g^2)^{\lambda_1}}}, \acute{s}_{g(\beta_1/g^2)^{\lambda_1}}, \acute{s}_{g(\gamma_1/g^2)^{\lambda_1}}.$$

Thus, we have

$$\begin{aligned}
\lambda_1\mathbb{R}+\lambda_2\mathbb{Z} &= \begin{pmatrix} \acute{s}_{g\sqrt{\{1-(1-\alpha_1^2/g^2)^{\lambda_1}\}+\{1-(1-\alpha_1^2/g^2)^{\lambda_2}\}-\{1-(1-\alpha_1^2/g^2)^{\lambda_1}\}\{1-(1-\alpha_1^2/g^2)^{\lambda_2}\}}}, \\ \acute{s}_{g(\beta_1/g)^{\lambda_1}(\beta_1/g)^{\lambda_2}}, \acute{s}_{g(\gamma_1/g)^{\lambda_1}(\gamma_1/g)^{\lambda_2}} \end{pmatrix} \\
&= \left(\acute{s}_{g\sqrt{1-(1-\alpha_1^2/g^2)^{\lambda_1}(1-\alpha_1^2/g^2)^{\lambda_2}}}, \acute{s}_{g(\beta_1/g)^{\lambda_1+\lambda_2}}, \acute{s}_{g(\gamma_1/g)^{\lambda_1+\lambda_2}}\right) \\
&= (\lambda_1+\lambda_2)\mathbb{R}.
\end{aligned}$$

Hence, $\lambda_1\mathbb{R}+\lambda_2\mathbb{Z} = (\lambda_1+\lambda_2)\mathbb{R}$ □

Theorem 2. *Let* $\mathbb{R} = \langle \acute{s}_{\alpha_1}, \acute{s}_{\beta_1}, \acute{s}_{\gamma_1}\rangle$ *and* $\mathbb{Z} = \langle \acute{s}_{\alpha_2}, \acute{s}_{\beta_2}, \acute{s}_{\gamma_2}\rangle$ *are the two LSFNs, then*
(1) $(\mathbb{R}\cup\mathbb{Z})\times(\mathbb{R}\cap\mathbb{Z}) = \mathbb{R}\times\mathbb{Z}$;
(2) $(\mathbb{R}\cup\mathbb{Z})+(\mathbb{R}\cap\mathbb{Z}) = \mathbb{R}+\mathbb{Z}$.

Proof. Since, $\mathbb{R} = \langle \acute{s}_{\alpha_1}, \acute{s}_{\beta_1}, \acute{s}_{\gamma_1}\rangle$ and $\mathbb{Z} = \langle \acute{s}_{\alpha_2}, \acute{s}_{\beta_2}, \acute{s}_{\gamma_2}\rangle$ are the two LSFNs, then we have
$(\mathbb{R}\cup\mathbb{Z})\times(\mathbb{R}\cap\mathbb{Z})$
$= \max(\acute{s}_{\alpha_1}, \acute{s}_{\alpha_2}), \min(\acute{s}_{\beta_1}, \acute{s}_{\beta_2}), \min(\acute{s}_{\gamma_1}, \acute{s}_{\gamma_2}) \times \min(\acute{s}_{\alpha_1}, \acute{s}_{\alpha_2}), \min(\acute{s}_{\beta_1}, \acute{s}_{\beta_2}), \max(\acute{s}_{\gamma_1}, \acute{s}_{\gamma_2})$

$$\begin{aligned}
&= \begin{pmatrix} \acute{s}_{g\max(\acute{s}_{\alpha_1},\acute{s}_{\alpha_2})/g\min(\acute{s}_{\alpha_1},\acute{s}_{\alpha_2})/g}, \\ \acute{s}_{g\sqrt{\min(\acute{s}_{\beta_1^2},\acute{s}_{\beta_2^2})/g^2+\min(\acute{s}_{\beta_1^2},\acute{s}_{\beta_2^2})/g^2-\min(\acute{s}_{\beta_1^2},\acute{s}_{\beta_2^2})/g^2\min(\acute{s}_{\beta_1^2},\acute{s}_{\beta_2^2})/g^2}}, \\ \acute{s}_{g\sqrt{\min(\acute{s}_{\beta_1^2},\acute{s}_{\beta_2^2})/g^2+\max(\acute{s}_{\beta_1^2},\acute{s}_{\beta_2^2})/g^2-\min(\acute{s}_{\beta_1^2},\acute{s}_{\beta_2^2})/g^2\max(\acute{s}_{\beta_1^2},\acute{s}_{\beta_2^2})/g^2}}, \end{pmatrix} \\
&= \left(\acute{s}_{g(\alpha_1\alpha_2/g^2)}, \acute{s}_{g\sqrt{\beta_1^2/g^2+\beta_2^2/g^2-\beta_1^2\beta_2^2/g^4}}, \acute{s}_{g\sqrt{\gamma_1^2/g^2+\gamma_2^2/g^2-\gamma_1^2\gamma_2^2/g^4}}\right) \\
&= \mathbb{R}\times\mathbb{Z}.
\end{aligned}$$

Part (2) can be similarly proved. □

Theorem 3. *Let* $\mathbb{R}_1, \mathbb{R}_2$*, and* $\mathbb{R}_3$ *are three LSFNs, then*
(1) $(\mathbb{R}_1\cup\mathbb{R}_2)\cap\mathbb{R}_3 = (\mathbb{R}_1\cap\mathbb{R}_3)\cup(\mathbb{R}_2\cap\mathbb{R}_3)$;
(2) $(\mathbb{R}_1\cap\mathbb{R}_2)\cup\mathbb{R}_3 = (\mathbb{R}_1\cup\mathbb{R}_3\cap(\mathbb{R}_2\cup\mathbb{R}_3)$;
(3) $(\mathbb{R}_1\cup\mathbb{R}_2)+\mathbb{R}_3 = (\mathbb{R}_1+\mathbb{R}_3)\cup(\mathbb{R}_2+\mathbb{R}_3)$;
(4) $(\mathbb{R}_1\cap\mathbb{R}_2)+\mathbb{R}_3 = (\mathbb{R}_1+\mathbb{R}_3)\cap(\mathbb{R}_2+\mathbb{R}_3)$;
(5) $(\mathbb{R}_1\cup\mathbb{R}_2)\times\mathbb{R}_3 = (\mathbb{R}_1\times\mathbb{R}_3)\cup(\mathbb{R}_2\times\mathbb{R}_3)$;
(6) $(\mathbb{R}_1\cap\mathbb{R}_2)\times\mathbb{R}_3 = (\mathbb{R}_1\times\mathbb{R}_3)\cap(\mathbb{R}_2\times\mathbb{R}_3)$.

Proof. We shall only prove part (1), other parts can be similarly proved.
Let $\mathbb{R}_i = \langle \acute{s}_{\alpha_i}, \acute{s}_{\beta_i}, \acute{s}_{\gamma_i}\rangle, i = 1,2,3$ are the three LSFNs, then we have

$$\begin{aligned}
& (\mathbb{R}_1\cup\mathbb{R}_2)\cap\mathbb{R}_3 \\
={}& \left(\max(\acute{s}_{\alpha_1}, \acute{s}_{\alpha_2}), \min(\acute{s}_{\beta_1}, \acute{s}_{\beta_2}), \min(\acute{s}_{\gamma_1}, \acute{s}_{\gamma_2})\right) \cap (\acute{s}_{\alpha_3}, \acute{s}_{\beta_3}, \acute{s}_{\gamma_3}) \\
={}& (\min(\max(\acute{s}_{\alpha_1}, \acute{s}_{\alpha_2}), \acute{s}_{\alpha_3}), \min(\min(\acute{s}_{\beta_1}, \acute{s}_{\beta_2}), \acute{s}_{\beta_3}), \max(\min(\acute{s}_{\gamma_1}, \acute{s}_{\gamma_2}), \acute{s}_{\gamma_3}) \\
={}& (\min(\acute{s}_{\alpha_1}, \acute{s}_{\alpha_3}), \min(\acute{s}_{\beta_1}, \acute{s}_{\beta_3})\max(\acute{s}_{\gamma_1}, \acute{s}_{\gamma_3})) \cup (\min(\acute{s}_{\alpha_2}, \acute{s}_{\alpha_3}), \\
& \min(\acute{s}_{\beta_2}, \acute{s}_{\beta_3})\max(\acute{s}_{\gamma_2}, \acute{s}_{\gamma_3})) \\
={}& (\mathbb{R}_1\cap\mathbb{R}_3)\cup(\mathbb{R}_2\cap\mathbb{R}_3).
\end{aligned}$$

□

4. Linguistic Spherical Fuzzy Aggregation Operators

Some series of linguistic spherical fuzzy aggregation operators are defined in this section, and Λ represent the set of all LSFNs.

4.1. Averaging Aggregation Operators

Definition 9. *Let $\mathbb{R}_i = \left\langle \acute{s}_{\alpha_i}, \acute{s}_{\beta_i}, \acute{s}_{\gamma_i} \right\rangle$ $(i = 1, ..., n)$ be the set of LSFNs. Then, the LSFWA operator of dimension n is a function $LSFWA : \Lambda^n \to \Lambda$, and*

$$LSFWA(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) = \Re_1\mathbb{R}_1 + \Re_2\mathbb{R}_2 + ... + \Re_n\mathbb{R}_n \tag{7}$$

where $\Re = (\Re_1, \Re_2, ..., \Re_n)^g$ is the weighting vector of $\mathbb{R}_i (i = 1, ..., n)$ with $\Re_i > 0$, and $\sum_{i=1}^{n} \Re_i = 1$.

Theorem 4. *Let $\mathbb{R}_i = \left\langle \acute{s}_{\alpha_i}, \acute{s}_{\beta_i}, \acute{s}_{\gamma_i} \right\rangle$ $(i = 1, ..., n)$ be the set of LSFNs. Then, by using the equation of LSFWA operator, the aggregated value is still an LSFN and is given by*

$$LSFWA(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) = \left(\acute{s}_{g\sqrt{\left(1-\prod_{i=1}^{n}\left(1-\alpha_i^2/g^2\right)^{\Re_i}\right)}}, \acute{s}_{g\prod_{i=1}^{n}\left(\frac{\beta_i}{g}\right)^{\Re_i}}, \acute{s}_{g\prod_{i=1}^{n}\left(\frac{\gamma_i}{g}\right)^{\Re_i}} \right) \tag{8}$$

where $\Re = (\Re_1, \Re_2, ..., \Re_n)^g$ is the weighting vector of $\mathbb{R}_i (i = 1, ..., n)$ with $\Re_i > 0$, and $\sum_{i=1}^{n} \Re_i = 1$.

Proof. To prove this theorem, we utilized the mathematical induction principle on n.

Step 1: For $n = 2$, we have $\mathbb{R}_1 = \left\langle \acute{s}_{\alpha_1}, \acute{s}_{\beta_1}, \acute{s}_{\gamma_1} \right\rangle$ and $\mathbb{R}_2 = \left\langle \acute{s}_{\alpha_2}, \acute{s}_{\beta_2}, \acute{s}_{\gamma_2} \right\rangle$. Thus, by the LSFN operation, we have

$$\begin{aligned}
\Re_1\mathbb{R}_1 &= \left(\acute{s}_{g\sqrt{\left(1-\left(1-\alpha_1^2/g^2\right)^{\Re_1}\right)}}, \acute{s}_{g\left(\frac{\beta_1}{g}\right)^{\Re_1}}, \acute{s}_{g\left(\frac{\gamma_1}{g}\right)^{\Re_1}} \right), \\
\Re_2\mathbb{R}_2 &= \left(\acute{s}_{g\sqrt{\left(1-\left(1-\alpha_2^2/g^2\right)^{\Re_2}\right)}}, \acute{s}_{g\left(\frac{\beta_2}{g}\right)^{\Re_2}}, \acute{s}_{g\left(\frac{\gamma_2}{g}\right)^{\Re_2}} \right).
\end{aligned}$$

Adding these two equations, we obtain

$$\begin{aligned}
\text{LSFWA}(\alpha_1, \alpha_2) &= \Re_1\mathbb{R}_1 + \Re_2\mathbb{R}_2 \\
&= \left(\acute{s}_{g\sqrt{\left(1-\left(1-\alpha_1^2/g^2\right)^{\Re_1}\right)}}, \acute{s}_{g\left(\frac{\beta_1}{g}\right)^{\Re_1}}, \acute{s}_{g\left(\frac{\gamma_1}{g}\right)^{\Re_1}} \right) \\
&\quad + \left(\acute{s}_{g\sqrt{\left(1-\left(1-\alpha_2^2/g^2\right)^{\Re_2}\right)}}, \acute{s}_{g\left(\frac{\beta_2}{g}\right)^{\Re_2}}, \acute{s}_{g\left(\frac{\gamma_2}{g}\right)^{\Re_2}} \right) \\
&= \left(\acute{s}_{g\sqrt{\left(1-\left(1-\alpha_1^2/g^2\right)^{\Re_1}\left(1-\alpha_2^2/g^2\right)^{\Re_2}\right)}}, \acute{s}_{g\left(\frac{\beta_1}{g}\right)^{\Re_1}\left(\frac{\beta_2}{g}\right)^{\Re_2}}, \acute{s}_{g\left(\frac{\gamma_1}{g}\right)^{\Re_1}\left(\frac{\gamma_2}{g}\right)^{\Re_2}} \right) \\
&= \left(\acute{s}_{g\sqrt{\left(1-\prod_{i=1}^{2}\left(1-\alpha_i^2/g^2\right)^{\Re_i}\right)}}, \acute{s}_{g\prod_{i=1}^{2}\left(\frac{\beta_i}{g}\right)^{\Re_i}}, \acute{s}_{g\prod_{i=1}^{2}\left(\frac{\gamma_i}{g}\right)^{\Re_i}} \right).
\end{aligned}$$

Which shows the result holds for $n = 2$.
Step 2: Now, we suppose that Equation (8) is true for $n = k$, and prove for $n = k+1$, then we have

$$\begin{aligned}
\mathrm{LSFWA}(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_{k+1}) &= \sum_{i=1}^{k} \Re_i \mathbb{R}_i + \Re_{k+1} \mathbb{R}_{k+1} \\
&= \left(\acute{s}_{g\sqrt{\left(1-\prod_{i=1}^{k}\left(1-\alpha_i^2/g^2\right)^{\Re_i}\right)}}, \acute{s}_{g\prod_{i=1}^{k}\left(\frac{\beta_i}{g^2}\right)^{\Re_1}}, \acute{s}_{g\prod_{i=1}^{k}\left(\frac{\gamma_i}{g}\right)^{\Re_1}} \right) \\
&\quad + \left(\acute{s}_{g\sqrt{\left(1-\left(1-\alpha_{k+1}^2/g^2\right)^{\Re_{k+1}}\right)}}, \acute{s}, \acute{s}_{g\left(\frac{\beta_{k+1}}{g}\right)^{\Re_{k+1}}}, \acute{s}_{g\left(\frac{\gamma_{k+1}}{g}\right)^{\Re_{k+1}}} \right) \\
&= \left(\acute{s}_{g\sqrt{\left(1-\prod_{i=1}^{k+1}\left(1-\alpha_i^2/g^2\right)^{\Re_i}\right)}}, \acute{s}_{g\prod_{i=1}^{k+1}\left(\frac{\beta_i}{g}\right)^{\Re_1}}, \acute{s}_{g\prod_{i=1}^{k+1}\left(\frac{\gamma_i}{g}\right)^{\Re_1}} \right).
\end{aligned}$$

Hence, the result holds for $n = k+1$, and we proved by mathematical induction principle that the given result is true for all positive integers n. □

Example 2. *Let $\mathbb{R} = \langle \acute{s}_3, \acute{s}_5, \acute{s}_2 \rangle, \mathbb{Z} = \langle \acute{s}_4, \acute{s}_3, \acute{s}_1 \rangle, \check{C} = \langle \acute{s}_6, \acute{s}_4, \acute{s}_3 \rangle, D = \langle \acute{s}_2, \acute{s}_3, \acute{s}_5 \rangle$ are the LSFNs, which are derived from $\hat{S}^* = \{\acute{s}_\alpha | \acute{s}_0 \le \acute{s}_\alpha \le \acute{s}_g, \alpha \in [0,7]\}$. Assume that $\Re = (0.4, 0.2, 0.3, , 0.1)^T$ be the expert weight. Hence*

$$\begin{aligned}
\prod_{i=1}^{4}\left(1-\alpha_i^2/g^2\right)^{\Re_i} &= \left(1-3^2/7^2\right)^{0.4} \times \left(1-4^2/7^2\right)^{0.2} \times \left(1-6^2/7^2\right)^{0.3} \times \left(1-2^2/7^2\right)^{0.1} \\
&= 0.537; \\
\prod_{i=1}^{4}(\beta_i/g)^{\Re_i} &= (5/7)^{0.4} \times (3/7)^{0.2} \times (4/7)^{0.3} \times (3/7)^{0.1} \\
&= 0.573; \\
\prod_{i=1}^{4}(\beta_i/g)^{\Re_i} &= (2/7)^{0.4} \times (1/7)^{0.2} \times (3/7)^{0.3} \times (5/7)^{0.1} \\
&= 0.310.
\end{aligned}$$

So, using the LSFWA operator, we have

$$\begin{aligned}
LSFWA(\mathbb{R}_1, \mathbb{R}_2, \mathbb{R}_3, \mathbb{R}_4) &= \left(\acute{s}_{g\sqrt{\left(1-\prod_{i=1}^{4}\left(1-\alpha_i^2/g^2\right)^{\Re_i}\right)}}, \acute{s}_{g\prod_{i=1}^{4}\left(\frac{\beta_i}{g}\right)^{\Re_i}}, \acute{s}_{g\prod_{i=1}^{4}\left(\frac{\gamma_i}{g}\right)^{\Re_i}} \right) \\
&= \left(\acute{s}_{7\sqrt{1-0.5337}}, \acute{s}_{7\times 0.573}, \acute{s}_{7\times 0.310} \right) \\
&= \left(\acute{s}_{4.76}, \acute{s}_{3.99}, \acute{s}_{2.17} \right).
\end{aligned}$$

Theorem 5. *Let $\mathbb{R}_i = \left\langle \acute{s}_{\alpha_i}, \acute{s}_{\beta_i}, \acute{s}_{\gamma_i} \right\rangle (i = 1, ..., n)$ be the set of LSFNs and the weight vector of $\mathbb{R}_i (i = 1, ..., n)$ are $\Re = (\Re_1, \Re_2, ..., \Re_n)^g$, where $\Re_i > 0$ and $\sum_{i=1}^{n} \Re_i = 1$, then one has the following:*

(1) (Idempotency). If all $\mathbb{R}_i (i = 1, 2, ..., n)$ are equal i.e., $\mathbb{R}_i = \left\langle \acute{s}_{\alpha_i}, \acute{s}_{\beta_i}, \acute{s}_{\gamma_i} \right\rangle = \left\langle \acute{s}_\alpha, \acute{s}_\beta, \acute{s}_\gamma \right\rangle \forall i$, then

$$LSFWA(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) = \left\langle \acute{s}_\alpha, \acute{s}_\beta, \acute{s}_\gamma \right\rangle. \tag{9}$$

(2) (Monotonicity). Let $\mathbb{R}_i = \left\langle \acute{s}_{\alpha_i}, \acute{s}_{\beta_i}, \acute{s}_{\gamma_i} \right\rangle$ *and* $\mathbb{Z}_i = \left\langle \acute{s}_{\alpha_i^*}, \acute{s}_{\beta_i^*}, \acute{s}_{\gamma_i^*} \right\rangle$ *are the collection of LSFNs such that* $\acute{s}_{\alpha_i^*} \geq \acute{s}_{\alpha_i}, \acute{s}_{\beta_i^*} \leq \acute{s}_{\beta_i}$ *and* $\acute{s}_{\gamma_i^*} \leq \acute{s}_{\gamma_i}$*, then*

$$LSFWA(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) \leq LSFWA(\mathbb{Z}_1, \mathbb{Z}_2, ..., \mathbb{Z}_n). \tag{10}$$

(3) (Boundedness). Let $\mathbb{R}^- = \min_i(\acute{s}_{\alpha_i}), \min_i(\acute{s}_{\beta_i}), \max_i(\acute{s}_{\gamma_i})$ *and* $\mathbb{R}^+ = \max_i(\acute{s}_{\alpha_i}), \min_i(\acute{s}_{\beta_i}), \min_i(\acute{s}_{\gamma_i})$ *be the two LSFNs, then*

$$\mathbb{R} \leq LSFWA(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) \leq \mathbb{R}^+. \tag{11}$$

Proof. Let $\mathbb{R}_i = \left\langle \acute{s}_{\alpha_i}, \acute{s}_{\beta_i}, \acute{s}_{\gamma_i} \right\rangle$ $(i = 1, ..., n)$ be the set of LSFNs which implies that $\acute{s}_{\alpha_i}, \acute{s}_{\beta_i}, \acute{s}_{\gamma_i} \in \hat{S}^* = \{\acute{s}_\alpha | \acute{s}_0 \leq \acute{s}_\alpha \leq \acute{s}_g, \alpha \in [0, g]\}$ and $\alpha_i^2 + \beta_i^2 + \gamma_i^2 \leq g^2$. Then

(1) If $\mathbb{R}_i = \left\langle \acute{s}_{\alpha_i}, \acute{s}_{\beta_i}, \acute{s}_{\gamma_i} \right\rangle = \left\langle \acute{s}_\alpha, \acute{s}_\beta, \acute{s}_\gamma \right\rangle \forall i$, then

$$\begin{aligned}
\mathrm{LSFWA}(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) &= \left(\acute{s}_{g\sqrt{\left(1-\prod_{i=1}^{n}\left(1-\alpha_i^2/g^2\right)^{\Re_i}\right)}}, \acute{s}_{g\prod_{i=1}^{n}\left(\frac{\beta_i}{g}\right)^{\Re_i}}, \acute{s}_{g\prod_{i=1}^{n}\left(\frac{\gamma_i}{g}\right)^{\Re_i}} \right) \\
&= \left(\acute{s}_{g\sqrt{\left(1-\left(1-\alpha_i^2/g^2\right)^{\sum_{i=1}^{n}\Re_i}\right)}}, \acute{s}_{g\left(\frac{\beta_i}{g}\right)^{\sum_{i=1}^{n}\Re_i}}, \acute{s}_{g\left(\frac{\gamma_i}{g}\right)^{\sum_{i=1}^{n}\Re_i}} \right) \\
&= \left(\acute{s}_{g\sqrt{\left(1-\left(1-\alpha_i^2/g^2\right)^{\Re_i}\right)}}, \acute{s}_{g\left(\frac{\beta_i}{g}\right)^{\Re_i}}, \acute{s}_{g\left(\frac{\gamma_i}{g}\right)^{\Re_i}} \right) \\
&= \left\langle \acute{s}_\alpha, \acute{s}_\beta, \acute{s}_\gamma \right\rangle.
\end{aligned}$$

(2) If $\acute{s}_{\alpha_i^*} \geq \acute{s}_{\alpha_i}$ for all i, it implies that $\alpha_i^* \geq \alpha_i$, then we have

$$\begin{aligned}
\alpha_i^* \geq \alpha_i &\Longrightarrow 0 \leq 1 - \frac{\alpha_i^{*2}}{g^2} \leq 1 - \frac{\alpha_i^2}{g^2} \leq 1 \\
&\Longrightarrow \left(1 - \frac{\alpha_i^{*2}}{g^2}\right)^{\Re_i} \leq \left(1 - \frac{\alpha_i^2}{g^2}\right)^{\Re_i} \\
&\Longrightarrow \prod_{i=1}^{n}\left(1 - \frac{\alpha_i^{*2}}{g^2}\right)^{\Re_i} \leq \prod_{i=1}^{n}\left(1 - \frac{\alpha_i^2}{g^2}\right)^{\Re_i} \\
&\Longrightarrow 1 - \prod_{i=1}^{n}\left(1 - \frac{\alpha_i^{*2}}{g^2}\right)^{\Re_i} \geq 1 - \prod_{i=1}^{n}\left(1 - \frac{\alpha_i^2}{g^2}\right)^{\Re_i} \\
&\Longrightarrow g^2\left(1 - \prod_{i=1}^{n}\left(1 - \frac{\alpha_i^{*2}}{g^2}\right)^{\Re_i}\right) \geq g^2\left(1 - \prod_{i=1}^{n}\left(1 - \frac{\alpha_i^2}{g^2}\right)^{\Re_i}\right).
\end{aligned}$$

On the other hand, $\acute{s}_{\beta_i^*} \leq \acute{s}_{\beta_i}$ that is $\beta_i^* \leq \beta_i \ \forall i$, then, $\frac{\beta_i^*}{g} \leq \frac{\beta_i}{g}$ and hence $\prod_{i=1}^{n}\left(\frac{\beta_i^*}{g}\right)^{\Re_i} \leq \prod_{i=1}^{n}\left(\frac{\beta_i}{g}\right)^{\Re_i}$. and $\acute{s}_{\gamma_i^*} \leq \acute{s}_{\gamma_i}$ that is $\gamma_i^* \leq \gamma_i \ \forall i$, then, $\frac{\gamma_i^*}{g} \leq \frac{\gamma_i}{g}$ and hence $\prod_{i=1}^{n}\left(\frac{\gamma_i^*}{g}\right)^{\Re_i} \leq \prod_{i=1}^{n}\left(\frac{\gamma_i}{g}\right)^{\Re_i}$. Therefore, according to Definition 6, we get

$$
\begin{aligned}
&\left(\acute{s}_{g\sqrt{\left(1-\prod_{i=1}^{n}\left(1-\alpha_i^{*2}/g^2\right)^{\Re_i}\right)}}, \acute{s}_{g\prod_{i=1}^{n}\left(\frac{\beta_i^*}{g}\right)^{\Re_i}}, \acute{s}_{g\prod_{i=1}^{n}\left(\frac{\gamma_i^*}{g}\right)^{\Re_i}}\right) \\
\geq &\left(\acute{s}_{g\sqrt{\left(1-\prod_{i=1}^{n}\left(1-\alpha_i^{2}/g^2\right)^{\Re_i}\right)}}, \acute{s}_{g\prod_{i=1}^{n}\left(\frac{\beta_i}{g}\right)^{\Re_i}}, \acute{s}_{g\prod_{i=1}^{n}\left(\frac{\gamma_i}{g}\right)^{\Re_i}}\right).
\end{aligned}
$$

That is, $\text{LSFW}\mathbb{R}(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) \leq \text{LSFW}\mathbb{R}(\mathbb{Z}_1, \mathbb{Z}_2, ..., \mathbb{Z}_n)$.

(3) Since $\min_i(\acute{s}_{\alpha_i}) \leq \acute{s}_{\alpha_i} \leq \max_i(\acute{s}_{\alpha_i}), \min_i(\acute{s}_{\beta_i}) \leq \acute{s}_{\beta_i} \leq \max_i(\acute{s}_{\beta_i})$ and $\max_i(\acute{s}_{\gamma_i}) \leq \acute{s}_{\gamma_i} \leq \min_i(\acute{s}_{\gamma_i})$, $\forall\ i$, then, based on the properties of idempotency and monotonicity, we get

$$
\left(\begin{array}{c}
\min_i(\acute{s}_{\alpha_i}), \min_i(\acute{s}_{\beta_i}), \max_i(\acute{s}_{\gamma_i}) \leq \text{LSFWA}(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) \leq \max_i(\acute{s}_{\alpha_i}), \\
\min_i(\acute{s}_{\beta_i}), \min_i(\acute{s}_{\gamma_i})
\end{array}\right).
$$

That is, $\mathbb{R}^- \leq \text{LSFWA}(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) \leq \mathbb{R}^+$. □

Definition 10. *Let $\mathbb{R}_i = \langle \acute{s}_{\alpha_i}, \acute{s}_{\beta_i}, \acute{s}_{\gamma_i} \rangle$ $(i = 1, ..., n)$ be the set of LSFNs. Then, the LSFOWA operator of dimension n is a mapping $LSFOWA : \Lambda^n \rightarrow \Lambda$, and*

$$
\begin{aligned}
LSFOWA(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) &= \Re_1\mathbb{R}_{(1)} + \Re_2\mathbb{R}_{(2)} + ... + \Re_n\mathbb{R}_{(n)} \\
= &\left(\acute{s}_{g\sqrt{\left(1-\prod_{i=1}^{n}\left(1-\alpha_{(i)}^{2}/g^2\right)^{\Re_i}\right)}}, \acute{s}_{g\prod_{i=1}^{n}\left(\frac{\beta_{(i)}}{g}\right)^{\Re_i}}, \acute{s}_{g\prod_{i=1}^{n}\left(\frac{\gamma_{(i)}}{g}\right)^{\Re_i}}\right),
\end{aligned}
$$

where $\mathbb{R}_{(i)} = \left\langle \acute{s}_{\alpha_{(i)}}, \acute{s}_{\beta_{(i)}}, \acute{s}_{\gamma_{(i)}} \right\rangle$ is the ith largest of $\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n$ and $\Re = (\Re_1, \Re_2, ..., \Re_n)^T$ is the associated weight vector of $\mathbb{R}_{(i)} (i = 1, ..., n)$ with $\Re_i > 0$, and $\sum_{i=1}^{n} \Re_i = 1$.

Theorem 6. *Let $\mathbb{R}_i = \langle \acute{s}_{\alpha_i}, \acute{s}_{\beta_i}, \acute{s}_{\gamma_i} \rangle$ $(i = 1, ..., n)$ be the set of LSFNs and $\Re = (\Re_1, \Re_2, ..., \Re_n)^T$ be the associated weight vector of $\mathbb{R}_i (i = 1, ..., n)$ are , where $\Re_i > 0$ and $\sum_{i=1}^{n} \Re_i = 1$, then one has the following:*

(1) (Idempotency). If all $\mathbb{R}_i (i = 1, ..., n)$ are equal i.e., $\mathbb{R}_i = \langle \acute{s}_{\alpha_i}, \acute{s}_{\beta_i}, \acute{s}_{\gamma_i} \rangle = \langle \acute{s}_{\alpha}, \acute{s}_{\beta}, \acute{s}_{\gamma} \rangle \ \forall i$, then

$$
LSFOWA(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) = \langle \acute{s}_{\alpha}, \acute{s}_{\beta}, \acute{s}_{\gamma} \rangle. \tag{12}
$$

(2) (Monotonicity). Let $\mathbb{R}_i = \langle \acute{s}_{\alpha_i}, \acute{s}_{\beta_i}, \acute{s}_{\gamma_i} \rangle$ and $\mathbb{Z}_i = \left\langle \acute{s}_{\alpha_i^*}, \acute{s}_{\beta_i^*}, \acute{s}_{\gamma_i^*} \right\rangle$ are the set of LSFNs such that $\acute{s}_{\alpha_i^*} \geq \acute{s}_{\alpha_i}, \acute{s}_{\beta_i^*} \leq \acute{s}_{\beta_i}$ and $\acute{s}_{\gamma_i^*} \leq \acute{s}_{\gamma_i}$, then

$$
LSFOWA(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) \leq LSFOWA(\mathbb{Z}_1, \mathbb{Z}_2, ..., \mathbb{Z}_n). \tag{13}
$$

(3) (Boundedness). Let $\mathbb{R} = \min_i(\acute{s}_{\alpha_i}), \min_i(\acute{s}_{\beta_i}), \max_i(\acute{s}_{\gamma_i})$ and $\mathbb{R}^+ = \max_i(\acute{s}_{\alpha_i}), \min_i(\acute{s}_{\beta_i}), \min_i(\acute{s}_{\gamma_i})$ be the two LSFNs, then

$$
\mathbb{R}^- \leq LSFOWA(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) \leq \mathbb{R}^+. \tag{14}
$$

4.2. Geometric Aggregation Operators

Definition 11. *Let $\mathbb{R}_i = \left\langle \acute{s}_{\alpha_i}, \acute{s}_{\beta_i}, \acute{s}_{\gamma_i} \right\rangle$ $(i = 1, ..., n)$ be the set of LSFNs. Then, the LSFWG operator of dimension n is a mapping $LSFWG : \Lambda^n \rightarrow \Lambda$, and*

$$\begin{aligned} & LSFWG(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) \\ = & \prod_{i=1}^{n} (\mathbb{R}_i)^{\Re_i} \\ = & \left(\acute{s}_{g \prod_{i=1}^{n} \left(\frac{\alpha_i}{g}\right)^{\Re_i}}, \acute{s}_{g\sqrt{\left(1-\prod_{i=1}^{n}(1-\beta_i^2/g^2)^{\Re_i}\right)}}, \acute{s}_{g\sqrt{\left(1-\prod_{i=1}^{n}(1-\gamma_i^2/g^2)^{\Re_i}\right)}} \right), \end{aligned} \tag{15}$$

where Λ is the collection of all LSFNs, $\Re = (\Re_1, \Re_2, ..., \Re_n)^T$ is the weight vector of $\mathbb{R}_i (i = 1, ..., n)$, such that $\Re_i > 0$, and $\sum_{i=1}^{n} \Re_i = 1$.

Theorem 7. *Let $\mathbb{R}_i = \left\langle \acute{s}_{\alpha_i}, \acute{s}_{\beta_i}, \acute{s}_{\gamma_i} \right\rangle$ $(i = 1, ..., n)$ be the set of LSFNs and the weight vector of $\mathbb{R}_i (i = 1, ..., n)$ are $\Re = (\Re_1, \Re_2, ..., \Re_n)^T$, where $\Re_i > 0$ and $\sum_{i=1}^{n} \Re_i = 1$. Then we have the following properties:*

(1) (Idempotency). If all $\mathbb{R}_i (i = 1, ..., n)$ are equal i.e., $\mathbb{R}_i = \left\langle \acute{s}_{\alpha_i}, \acute{s}_{\beta_i}, \acute{s}_{\gamma_i} \right\rangle = \left\langle \acute{s}_{\alpha}, \acute{s}_{\beta}, \acute{s}_{\gamma} \right\rangle \forall i$, then

$$LSFWG(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) = \left\langle \acute{s}_{\alpha}, \acute{s}_{\beta}, \acute{s}_{\gamma} \right\rangle. \tag{16}$$

(2) (Monotonicity). Let $\mathbb{R}_i = \left\langle \acute{s}_{\alpha_i}, \acute{s}_{\beta_i}, \acute{s}_{\gamma_i} \right\rangle$ and $\mathbb{Z}_i = \left\langle \acute{s}_{\alpha_i^*}, \acute{s}_{\beta_i^*}, \acute{s}_{\gamma_i^*} \right\rangle$ are the collecttion of LSFNs such that $\acute{s}_{\alpha_i^*} \geq \acute{s}_{\alpha_i}, \acute{s}_{\beta_i^*} \leq \acute{s}_{\beta_i}$ and $\acute{s}_{\gamma_i^*} \leq \acute{s}_{\gamma_i}$, then

$$LSFWG(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) \leq LSFWG(\mathbb{Z}_1, \mathbb{Z}_2, ..., \mathbb{Z}_n). \tag{17}$$

(3) (Boundedness). Let $\mathbb{R} = (\min_i(\acute{s}_{\alpha_i}), \min_i(\acute{s}_{\beta_i}), \max_i(\acute{s}_{\gamma_i}))$ and $\mathbb{R}^+ = (\max_i(\acute{s}_{\alpha_i}), \min_i(\acute{s}_{\beta_i}), \min_i(\acute{s}_{\gamma_i}))$ be the two LSFNs, then

$$\mathbb{R}^- \leq LSFWG(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) \leq \mathbb{R}^+. \tag{18}$$

Proof. The proof of these properties are the same as the proof of Theorem 5. □

Definition 12. *Let $\mathbb{R}_i = \left\langle \acute{s}_{\alpha_i}, \acute{s}_{\beta_i}, \acute{s}_{\gamma_i} \right\rangle$ $(i = 1, ..., n)$ be the set of LSFNs. Then, the LSFOWG operator of dimension n is a mapping $LSFOWG : \Lambda^n \rightarrow \Lambda$, and*

$$\begin{aligned} LSFOWG(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) &= \prod_{i=1}^{n} \left(\mathbb{R}_{(i)}\right)^{\Re_i} \\ &= \left(\begin{array}{c} \acute{s}_{g \prod_{i=1}^{n} \left(\frac{\alpha_{(i)}}{g}\right)^{\Re_i}}, \acute{s}_{g\sqrt{\left(1-\prod_{i=1}^{n}\left(1-\beta_{(i)}^2/g^2\right)^{\Re_i}\right)}}, \\ \acute{s}_{g\sqrt{\left(1-\prod_{i=1}^{n}\left(1-\gamma_{(i)}^2/g^2\right)^{\Re_i}\right)}} \end{array} \right), \end{aligned} \tag{19}$$

where $\mathbb{R}_{(i)} = \left\langle \acute{s}_{\alpha_{(i)}}, \acute{s}_{\beta_{(i)}}, \acute{s}_{\gamma_{(i)}} \right\rangle$ is the ith largest of $\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n$ and $\Re = (\Re_1, \Re_2, ..., \Re_n)^T$ is the associated weight vector of $\mathbb{R}_{(i)} (i = 1, ..., n)$ with $\Re_i > 0$, and $\sum_{i=1}^{n} \Re_i = 1$.

The LSFOWG operator also satisfies the properties as given in Theorem 6.

Lemma 1. *Let* $k_i \geq 0, y_i > 0 (i = 1, 2, ..., n)$ *and* $\sum_{i=1}^{n} y_i = 1$, *then*

$$\prod_{i=1}^{n} (k_i)^{y_i} \leq \sum_{i=1}^{n} y_i k_i, \tag{20}$$

where equality holds only if $k_1 = k_2 = ... = k_n$.

We derived the following theorem, based on Lemma 1.

Theorem 8. *Let* $\mathbb{R}_i = \langle \acute{s}_{\alpha_i}, \acute{s}_{\beta_i}, \acute{s}_{\gamma_i} \rangle$ $(i = 1, ..., n)$ *be the set of LSFNs, then one has*

(i) $LSFWA(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) \geq LSFWG(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n)$*;*

(ii) $LSFOWA(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) \geq LSFOWG(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n)$

with equality if and only if $\mathbb{R}_1 = \mathbb{R}_2 ... = \mathbb{R}_n$

Proof. Let $\Re = (\Re_1, \Re_2, ..., \Re_n)^T$ be the weights of alternative $\mathbb{R}_i (i = 1, ..., n)$ with $\Re_i > 0$ and $\sum_{i=1}^{n} \Re_i = 1$. Then, part (i) can be proved as;

By the Lemma 1, we have

$$\begin{aligned}
\prod_{i=1}^{n} \left(1 - \alpha_i^2/g^2\right)^{\Re_i} \quad &\leq \quad \sum_{i=1}^{n} \Re_i \left(1 - \alpha_i^2/g^2\right) = \sum_{i=1}^{n} \Re_i - \sum_{i=1}^{n} \Re_i \alpha_i^2/g^2 \\
&\Longrightarrow \quad 1 - \prod_{i=1}^{n} \left(1 - \alpha_i^2/g^2\right)^{\Re_i} \geq \sum_{i=1}^{n} \Re_i \alpha_i^2/g^2 \geq \prod_{i=1}^{n} \left(\alpha_i^2/g^2\right)^{\Re_i}
\end{aligned}$$

with equality if and only if $\alpha_1 = \alpha_2 = ... = \alpha_n$; that is,

$$\begin{aligned}
g^2 \left(1 - \prod_{i=1}^{n} \left(1 - \alpha_i^2/g^2\right)^{\Re_i}\right) \quad &\geq \quad g^2 \prod_{i=1}^{n} \left(\alpha_i^2/g^2\right)^{\Re_i} \sqrt{g^2 \left(1 - \prod_{i=1}^{n} \left(1 - \alpha_i^2/g^2\right)^{\Re_i}\right)} \\
&\geq \quad \sqrt{g^2 \prod_{i=1}^{n} \left(\alpha_i^2/g^2\right)^{\Re_i}} = g \prod_{i=1}^{n} \left(\alpha_i/g\right)^{\Re_i} \\
&\Longrightarrow \quad \acute{s}_{g\sqrt{\left(1 - \prod_{i=1}^{n} \left(1 - \alpha_i^2/g^2\right)^{\Re_i}\right)}} \geq \acute{s}_{g \prod_{i=1}^{n} (\alpha_i/g)^{\Re_i}}
\end{aligned} \tag{21}$$

with equality if and only if $\alpha_1 = \alpha_2 = ... = \alpha_n$. On the other hand,

$$\begin{aligned}
\prod_{i=1}^{n} \left(\beta_i^2/g^2\right)^{\Re_i} \quad &\leq \quad \sum_{i=1}^{n} \Re_i (\beta_i^2/g^2) = 1 - \sum_{i=1}^{n} \Re_i \left(1 - \beta_i^2/g^2\right) \\
&\Longrightarrow \quad \prod_{i=1}^{n} \left(\beta_i^2/g^2\right)^{\Re_i} \leq 1 - \prod_{i=1}^{n} \Re_i \left(1 - \beta_i^2/g^2\right)^{\Re_i} \\
&\Longrightarrow \quad g^2 \prod_{i=1}^{n} \left(\beta_i^2/g^2\right)^{\Re_i} \leq g^2 \left(1 - \prod_{i=1}^{n} \Re_i \left(1 - \beta_i^2/g^2\right)^{\Re_i}\right) \\
&\Longrightarrow \quad g \prod_{i=1}^{n} \left(\beta_i/g\right)^{\Re_i} \leq g \sqrt{1 - \prod_{i=1}^{n} \Re_i \left(1 - \beta_i^2/g^2\right)^{\Re_i}} \\
&\Longrightarrow \quad \acute{s}_{g \prod_{i=1}^{n} (\beta_i/g)^{\Re_i}}, \acute{s}_{g\sqrt{1 - \prod_{i=1}^{n} \Re_i \left(1 - \beta_i^2/g^2\right)^{\Re_i}}}
\end{aligned} \tag{22}$$

with equality if and only if $\beta_1 = \beta_2 = ..., = \beta_n$.

And

$$
\begin{aligned}
\prod_{i=1}^{n}\left(\gamma_i^2/g^2\right)^{\Re_i} &\leq \sum_{i=1}^{n}\Re_i(\gamma_i^2/g^2) = 1-\sum_{i=1}^{n}\Re_i\left(1-\gamma_i^2/g^2\right)\\
&\Longrightarrow \prod_{i=1}^{n}\left(\gamma_i^2/g^2\right)^{\Re_i} \leq 1-\prod_{i=1}^{n}\Re_i\left(1-\gamma_i^2/g^2\right)^{\Re_i}\\
&\Longrightarrow g^2\prod_{i=1}^{n}\left(\gamma_i^2/g^2\right)^{\Re_i} \leq g^2\left(1-\prod_{i=1}^{n}\Re_i\left(1-\gamma_i^2/g^2\right)^{\Re_i}\right)\\
&\Longrightarrow g\prod_{i=1}^{n}\left(\gamma_i/g\right)^{\Re_i} \leq g\sqrt{1-\prod_{i=1}^{n}\Re_i\left(1-\gamma_i^2/g^2\right)^{\Re_i}}\\
&\Longrightarrow \acute{s}_{g\prod_{i=1}^{n}(\gamma_i/g)^{\Re_i}}, \acute{s}_{g\sqrt{1-\prod_{i=1}^{n}\Re_i\left(1-\gamma_i^2/g^2\right)^{\Re_i}}}
\end{aligned}
\tag{23}
$$

with equality if and only if $\gamma_1=\gamma_2=...=\gamma_n$.

Now, utilizing the score function of LSFNs, we have

$$
\begin{aligned}
&\left(\acute{s}_{g\sqrt{\left(1-\prod_{i=1}^{n}\left(1-\alpha_i^2/g^2\right)^{\Re_i}\right)}}, \acute{s}_{g\prod_{i=1}^{n}(\beta_i/g)^{\Re_i}}, \acute{s}_{g\prod_{i=1}^{n}(\gamma_i/g)^{\Re_i}}\right)\\
\geq\ &\left(\acute{s}_{g\prod_{i=1}^{n}(\alpha_i/g)^{\Re_i}}, \acute{s}_{g\sqrt{1-\prod_{i=1}^{n}\Re_i\left(1-\beta_i^2/g^2\right)^{\Re_i}}}, \acute{s}_{g\sqrt{1-\prod_{i=1}^{n}\Re_i\left(1-\gamma_i^2/g^2\right)^{\Re_i}}}\right)
\end{aligned}
$$

with equality if and only if $\alpha_1=\alpha_2=...=\alpha_n, \beta_1=\beta_2=...=\beta_n$ and $\gamma_1=\gamma_2=...=\gamma_n$; that as

$$\mathrm{LSFWA}(\mathbb{R}_1,\mathbb{R}_2,...,\mathbb{R}_n) \geq \ \mathrm{LSFWG}(\mathbb{R}_1,\mathbb{R}_2,...,\mathbb{R}_n).$$

□

Example 3. *Let* $\mathbb{R}_1=\langle\acute{s}_1,\acute{s}_3,\acute{s}_4\rangle, \mathbb{R}_2=\langle\acute{s}_3,\acute{s}_1,\acute{s}_5\rangle, \mathbb{R}_3=\langle\acute{s}_4,\acute{s}_2,\acute{s}_6\rangle, \mathbb{R}_4=\langle\acute{s}_2,\acute{s}_4,\acute{s}_3\rangle$ *are the LSFNs, which are derived from* $\hat{S}^*=\{\acute{s}_\alpha|\acute{s}_0\leq\acute{s}_\alpha\leq\acute{s}_g, \alpha\in[0,7]\}$. *Assume that* $\Re=(0.3,0.4,0.2,,0.1)^T$ *are the weighting vector of* $\mathbb{R}_i$.

We obtain the following aggregated LSFNs, by applying Equations (8), and (15):

$$
\begin{aligned}
LSFWA(\mathbb{R}_1,\mathbb{R}_2,\mathbb{R}_3,\mathbb{R}_4) &= \left(\acute{s}_{7\sqrt{\left(1-\prod_{i=1}^{4}\left(1-\alpha_i^2/7^2\right)^{\Re_i}\right)}}, \acute{s}_{7\prod_{i=1}^{4}\left(\frac{\beta_i}{7}\right)^{\Re_i}}, \acute{s}_{7\prod_{i=1}^{4}\left(\frac{\gamma_i}{7}\right)^{\Re_i}}\right)\\
&= \langle\acute{s}_{2.88},\acute{s}_{1.86},\acute{s}_{4.61}\rangle;\\
LSFWG(\mathbb{R}_1,\mathbb{R}_2,...,\mathbb{R}_n) &= \begin{pmatrix}\acute{s}_{7\prod_{i=1}^{4}\left(\frac{\alpha_i}{7}\right)^{\Re_i}}, \acute{s}_{7\sqrt{\left(1-\prod_{i=1}^{4}\left(1-\beta_i^2/7^2\right)^{\Re_i}\right)}},\\ \acute{s}_{g\sqrt{\left(1-\prod_{i=1}^{4}\left(1-\gamma_i^2/7^2\right)^{\Re_i}\right)}}\end{pmatrix}\\
&= \langle\acute{s}_{2.18},\acute{s}_{2.46},\acute{s}_{4.95}\rangle.
\end{aligned}
$$

From the computational results, we can write

$$
\begin{aligned}
LSFWA(\mathbb{R}_1,\mathbb{R}_2,...,\mathbb{R}_n) &= \langle\acute{s}_{2.88},\acute{s}_{1.86},\acute{s}_{4.61}\rangle \geq LSFWG(\mathbb{R}_1,\mathbb{R}_2,...,\mathbb{R}_n)\\
&= \langle\acute{s}_{2.18},\acute{s}_{2.46},\acute{s}_{4.95}\rangle.
\end{aligned}
$$

Now, to determine the aggregated LSFN, utilizing the LSFOWA and LSFOWG operators, first we find the score values of $\mathbb{R}_i (i = 1, 2, 3, 4)$ *as follows:*

$$\begin{aligned}
Ş(\mathbb{R}_1) &= \acute{s}_{\sqrt{(49+1-9-4)/3}} = \acute{s}_{\sqrt{12.33}} = \acute{s}_{3.51}, \\
Ş(\mathbb{R}_2) &= \acute{s}_{\sqrt{(49+9-1-25)/3}} = \acute{s}_{\sqrt{10.66}} = \acute{s}_{3.26}, \\
Ş(\mathbb{R}_3) &= \acute{s}_{\sqrt{(49+16-4-36)/3}} = \acute{s}_{\sqrt{8.33}} = \acute{s}_{2.88}, \\
Ş(\mathbb{R}_4) &= \acute{s}_{\sqrt{(49+4-16-9)/3}} = \acute{s}_{\sqrt{9.33}} = \acute{s}_{3.05}.
\end{aligned}$$

Since $Ş(\mathbb{R}_1) > Ş(\mathbb{R}_2) > Ş(\mathbb{R}_4) > Ş(\mathbb{R}_3)$*, then*

$$\mathbb{R}_{(1)} = \mathbb{R}_1,\ \mathbb{R}_{(2)} = \mathbb{R}_2,\ \mathbb{R}_{(3)} = \mathbb{R}_4,\ \mathbb{R}_{(4)} = \mathbb{R}_3.$$

The associated weight vector of $\mathbb{R}_{(i)}$ *are* $\Re = (0.155, 0.345, 0.345, 0.155)^T$*, and can be determined by the normal distribution method [64]. Then, by Equations (8) and (19), we have*

$$\begin{aligned}
LSFOWA(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) &= \left(\acute{s}_{7\sqrt{\left(1-\prod_{i=1}^{4}\left(1-\alpha_{(i)}^2/7^2\right)^{\Re_i}\right)}}, \acute{s}_{7\prod_{i=1}^{4}\left(\frac{\beta_{(i)}}{7}\right)^{\Re_i}}, \acute{s}_{7\prod_{i=1}^{4}\left(\frac{\gamma_{(i)}}{7}\right)^{\Re_i}} \right) \\
&= \left\langle \acute{s}_{2.79}, \acute{s}_{2.12}, \acute{s}_{4.14} \right\rangle; \\
LSFOWG(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) &= \left(\begin{array}{c} \acute{s}_{7\prod_{i=1}^{4}\left(\frac{\alpha_{(i)}}{7}\right)^{\Re_i}}, \acute{s}_{7\sqrt{\left(1-\prod_{i=1}^{4}\left(1-\beta_{(i)}^2/7^2\right)^{\Re_i}\right)}}, \\ \acute{s}_{7\sqrt{\left(1-\prod_{i=1}^{4}\left(1-\gamma_{(i)}^2/7^2\right)^{\Re_i}\right)}} \end{array} \right) \\
&= \left\langle \acute{s}_{2.32}, \acute{s}_{2.93}, \acute{s}_{4.64} \right\rangle.
\end{aligned}$$

From these results, we prove that

$$\begin{aligned}
LSFOWG(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) &= \left\langle \acute{s}_{2.79}, \acute{s}_{2.12}, \acute{s}_{4.14} \right\rangle > LSFOWG(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) \\
&= \left\langle \acute{s}_{2.32}, \acute{s}_{2.93}, \acute{s}_{4.64} \right\rangle.
\end{aligned}$$

4.3. Some Properties of Linguistic Spherical Fuzzy Weighted Aggregation Operators

Theorem 9. *Let* $\mathbb{R}_i = \left\langle \acute{s}_{\alpha_i}, \acute{s}_{\beta_i}, \acute{s}_{\gamma_i} \right\rangle (i = 1, ..., n)$ *be the set of LSFNs, and* $\mathbb{R} = \left\langle \acute{s}_{\alpha}, \acute{s}_{\beta}, \acute{s}_{\gamma} \right\rangle$ *is also LSFN, with the weighting vector* $\Re = (\Re_1, \Re_2, ..., \Re_n)^T$*, where* $\Re_i > 0$ *and* $\sum_{i=1}^{n} \Re_i = 1$*, then*

(1) $LSFWA(\mathbb{R}_1 + \mathbb{R}, \mathbb{R}_2 + \mathbb{R}, ..., \mathbb{R}_n + \mathbb{R}) \geq LSFWA(\mathbb{R}_1 \times \mathbb{R}, \mathbb{R}_2 \times \mathbb{R}, ..., \mathbb{R}_n \times \mathbb{R})$*,*
(2) $LSFWG(\mathbb{R}_1 + \mathbb{R}, \mathbb{R}_2 + \mathbb{R}, ..., \mathbb{R}_n + \mathbb{R}) \geq LSFWG(\mathbb{R}_1 \times \mathbb{R}, \mathbb{R}_2 \times \mathbb{R}, ..., \mathbb{R}_n \times \mathbb{R})$*,*
(3) $LSFWA(\mathbb{R}_1 + \mathbb{R}, \mathbb{R}_2 + \mathbb{R}, ..., \mathbb{R}_n + \mathbb{R}) \geq LSFWA(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) \times \mathbb{R}$*,*
(4) $LSFWG(\mathbb{R}_1 + \mathbb{R}, \mathbb{R}_2 + \mathbb{R}, ..., \mathbb{R}_n + \mathbb{R}) \geq LSFWG(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) \times \mathbb{R}$*,*
(5) $LSFWA((\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) + \mathbb{R} \geq LSFWA(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) \times \mathbb{R}$*,*
(6) $LSFWG((\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) + \mathbb{R} \geq LSFWG(\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n) \times \mathbb{R}$*.*

Proof. We shall prove only part (1), (3) and (5) and the proofs of the remaining parts are similarly proved.

(1) For $\mathbb{R}_i = \left\langle \acute{s}_{\alpha_i}, \acute{s}_{\beta_i}, \acute{s}_{\gamma_i} \right\rangle (i = 1, ..., n)$ and $\mathbb{R} = \left\langle \acute{s}_{\alpha}, \acute{s}_{\beta}, \acute{s}_{\gamma} \right\rangle$, we can write

$$\mathbb{R}_i + \mathbb{R} = \left(\acute{s}_{g\sqrt{\left(1-\left(1-\alpha_i^2/g^2\right)^{\Re_i}(1-\alpha^2/g^2)^{\Re}\right)}}, \acute{s}_{g\left(\frac{\beta_i}{g}\right)^{\Re_i}\left(\frac{\beta}{g}\right)^{\Re}}, \acute{s}_{g\left(\frac{\gamma_i}{g}\right)^{\Re_i}\left(\frac{\gamma}{g}\right)^{\Re}} \right).$$

Now,

$$\begin{aligned}
\frac{\alpha_i^2}{g^2}+\frac{\alpha^2}{g^2}-\frac{\alpha_i^2\alpha^2}{g^4} &\geq \frac{\alpha_i^2\alpha^2}{g^4} \\
&\Longrightarrow \left(1-\frac{\alpha_i^2}{g^2}\right)\left(1-\frac{\alpha^2}{g^2}\right)\leq 1-\frac{\alpha_i^2\alpha^2}{g^4} \\
&\Longrightarrow \prod_{i=1}^{n}\left(1-\frac{\alpha_i^2}{g^2}\right)^{\Re_i}\left(1-\frac{\alpha^2}{g^2}\right)\leq \prod_{i=1}^{n}\left(1-\frac{\alpha_i^2\alpha^2}{g^4}\right)^{\Re_i} \\
&\Longrightarrow \sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\alpha_i^2}{g^2}\right)^{\Re_i}\left(1-\frac{\alpha^2}{g^2}\right)}\geq \sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\alpha_i^2\alpha^2}{g^4}\right)^{\Re_i}} \\
&\Longrightarrow g\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\alpha_i^2}{g^2}\right)^{\Re_i}(1-\alpha^2/g^2)}\geq g\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\alpha_i^2\alpha^2}{g^4}\right)^{\Re_i}} \\
&\Longrightarrow \acute{s}_{g\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\alpha_i^2}{g^2}\right)^{\Re_i}(1-/g^2)}}\geq \acute{s}_{g\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\alpha_i^2\alpha^2}{g^4}\right)^{\Re_i}}}.
\end{aligned}$$

Similarly, we get

$$\acute{s}_{g\left(\prod_{i=1}^{n}\left(\frac{\beta_i}{g}\right)^{\Re_i}\left(\frac{\beta}{g}\right)\right)}\leq \acute{s}_{g\left(\prod_{i=1}^{n}\left(\sqrt{1-\left(1-\beta_i^2/g^2\right)}(1-\beta^2/g^2)\right)^{\Re_i}\right)}$$

and

$$\acute{s}_{g\left(\prod_{i=1}^{n}\left(\frac{\gamma_i}{g}\right)^{\Re_i}\left(\frac{\gamma}{g}\right)\right)}\leq \acute{s}_{g\left(\prod_{i=1}^{n}\left(\sqrt{1-\left(1-\gamma_i^2/g^2\right)}(1-\gamma^2/g^2)\right)^{\Re_i}\right)}.$$

Thus, by using Equation (8), we get

$$\begin{aligned}
&\mathrm{LSFWA}(\mathbb{R}_1+\mathbb{R},\mathbb{R}_2+\mathbb{R},...,\mathbb{R}_n+\mathbb{R}) \\
=&\left(\acute{s}_{g\sqrt{1-\prod_{i=1}^{n}\left(1-\alpha_i^2/g^2\right)^{\Re_i}(1-\alpha^2/g^2)}},\acute{s}_{g\left(\prod_{i=1}^{n}\left(\frac{\beta_i}{g}\right)^{\Re_i}\left(\frac{\beta}{g}\right)\right)},\acute{s}_{g\left(\prod_{i=1}^{n}\left(\frac{\gamma_i}{g}\right)^{\Re_i}\left(\frac{\gamma}{g}\right)\right)}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{LSFWA}(\mathbb{R}_1+\mathbb{R},\mathbb{R}_2+\mathbb{R},...,\mathbb{R}_n+\mathbb{R}) \\
=&\left(\begin{array}{c}\acute{s}_{g\sqrt{1-\prod_{i=1}^{n}\left(1-\alpha_i^2/g^2\right)^{\Re_i}(1-\alpha^2/g)}},\acute{s}_{g\left(\prod_{i=1}^{n}\left(\sqrt{1-\left(1-\beta_i^2/g^2\right)}(1-\beta^2/g^2)\right)^{\Re_i}\right)}, \\ \acute{s}_{g\left(\prod_{i=1}^{n}\left(\sqrt{1-\left(1-\gamma_i^2/g^2\right)}(1-\gamma^2/g^2)\right)^{\Re_i}\right)}\end{array}\right).
\end{aligned}$$

According to Definition 6, we get

$$\mathrm{LSFWA}(\mathbb{R}_1+\mathbb{R},\mathbb{R}_2+\mathbb{R},...,\mathbb{R}_n+\mathbb{R})\geq \mathrm{LSFWA}(\mathbb{R}_1\times\mathbb{R},\mathbb{R}_2\times\mathbb{R},...,\mathbb{R}_n\times\mathbb{R}).$$

(3) Since $1-\left(\frac{\alpha^2}{g^2}\right)\leq 1$, which implies that

$$\begin{aligned}
\prod_{i=1}^{n}\left(1-\alpha_i^2/g^2\right)^{\Re_i} &\geq \prod_{i=1}^{n}\left(1-\alpha_i^2/g^2\right)^{\Re_i}(1-\alpha^2/g^2)\\
&\Longrightarrow 1-\prod_{i=1}^{n}\left(1-\alpha_i^2/g^2\right)^{\Re_i}\leq 1-\prod_{i=1}^{n}\left(1-\alpha_i^2/g^2\right)^{\Re_i}(1-\alpha^2/g^2)\\
&\Longrightarrow \sqrt{1-\prod_{i=1}^{n}(1-\alpha_i^2/g^2)^{\Re_i}}\leq\sqrt{1-\prod_{i=1}^{n}(1-\alpha_i^2/g^2)^{\Re_i}(1-\alpha^2/g^2)}\\
&\Longrightarrow \frac{\alpha}{g}\sqrt{1-\prod_{i=1}^{n}(1-\alpha_i^2/g^2)^{\Re_i}}\leq\frac{\alpha}{g}\sqrt{1-\prod_{i=1}^{n}(1-\alpha_i^2/g^2)^{\Re_i}(1-\alpha^2/g^2)}\\
&\leq \sqrt{1-\prod_{i=1}^{n}(1-\alpha_i^2/g^2)^{\Re_i}(1-\alpha^2/g^2)}\\
&\Longrightarrow g(\alpha/g)\sqrt{1-\prod_{i=1}^{n}(1-\alpha_i^2/g^2)^{\Re_i}}\leq g\sqrt{1-\prod_{i=1}^{n}(1-\alpha_i^2/g^2)^{\Re_i}(1-\alpha^2/g^2)}\\
&\Longrightarrow \acute{s}_{g(\alpha/g)\sqrt{1-\prod_{i=1}^{n}(1-\alpha_i^2/g^2)^{\Re_i}}}\leq\acute{s}_{g\sqrt{1-\prod_{i=1}^{n}(1-\alpha_i^2/g^2)^{\Re_i}(1-\alpha^2/g^2)}}.
\end{aligned}$$

Similarly, we show that

$$\acute{s}_{g\prod_{i=1}^{n}\left(\frac{\beta_i}{g}\right)^{\Re_i}\left(\frac{\beta}{g}\right)}\leq\acute{s}_{g\sqrt{1-\left(1-\prod_{i=1}^{n}\left(\frac{\beta_i^2}{g^2}\right)^{\Re_i}\right)(1-\beta^2/g^2)}}$$

and

$$\acute{s}_{g\prod_{i=1}^{n}\left(\frac{\gamma_i}{g}\right)^{\Re_i}\left(\frac{\gamma}{g}\right)}\leq\acute{s}_{g\sqrt{1-\left(1-\prod_{i=1}^{n}\left(\frac{\gamma_i^2}{g^2}\right)^{\Re_i}\right)(1-\gamma^2/g^2)}}.$$

Since,

$$\begin{aligned}
&\mathrm{LSFWA}(\mathbb{R}_1+\mathbb{R},\mathbb{R}_2+\mathbb{R},...,\mathbb{R}_n+\mathbb{R})\\
=&\left(\acute{s}_{g\sqrt{1-\prod_{i=1}^{n}(1-\alpha_i^2/g^2)^{\Re_i}(1-\alpha^2/g^2)}},\acute{s}_{g\prod_{i=1}^{n}\left(\frac{\beta_i}{g}\right)^{\Re_i}\left(\frac{\beta}{g}\right)},\acute{s}_{g\prod_{i=1}^{n}\left(\frac{\gamma_i}{g}\right)^{\Re_i}\left(\frac{\gamma}{g}\right)}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{LSFWA}(\mathbb{R}_1,\mathbb{R}_2,...,\mathbb{R}_n)\times\mathbb{R}\\
=&\left(\acute{s}_{g\sqrt{1-\prod_{i=1}^{n}(1-\alpha_i^2/g^2)^{\Re_i}}},\acute{s}_{g\prod_{i=1}^{n}\left(\frac{\beta_i}{g}\right)^{\Re_i}},\acute{s}_{g\prod_{i=1}^{n}\left(\frac{\gamma_i}{g}\right)^{\Re_i}}\right)\times\left\langle\acute{s}_\alpha,\acute{s}_\beta,\acute{s}_\gamma\right\rangle\\
=&\left(\acute{s}_{g(\alpha/g)\sqrt{1-\prod_{i=1}^{n}(1-\alpha_i^2/g^2)^{\Re_i}}},\acute{s}_{g\sqrt{1-\left(1-\prod_{i=1}^{n}\left(\frac{\beta_i^2}{g^2}\right)^{\Re_i}\right)(1-\beta^2/g^2)}},\acute{s}_{g\sqrt{1-\left(1-\prod_{i=1}^{n}\left(\frac{\gamma_i^2}{g^2}\right)^{\Re_i}\right)(1-\gamma^2/g^2)}}\right).
\end{aligned}$$

Thus, according to definition 6, we get

$$\mathrm{LSFWA}(\mathbb{R}_1+\mathbb{R},\mathbb{R}_2+\mathbb{R},...,\mathbb{R}_n+\mathbb{R})\geq\mathrm{LSFWA}(\mathbb{R}_1,\mathbb{R}_2,...,\mathbb{R}_n)\times\mathbb{R}.$$

□

5. Algorithm for Decision Making Problem with Linguistic Spherical Fuzzy Information

Assume that we have m alternatives $\mathbb{R} = \{\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_m)$ and n attributes $\check{C} = \{\check{C}_1, \check{C}_2, ..., \check{C}_n)$, whose weighting vectors are $\Re = (\Re_1, \Re_2, ..., \Re_n)^T$, with $\Re_j > 0$ and $\sum_{j=1}^{n} \Re_j = 1$. Decision maker set is $D = \{d_1, d_2, ..., d_p)$, where v_k denotes their weights, such that $v_k > 0$ and $\sum_{k=1}^{p} v_k = 1$. The decision maker gives their preference values in LSFNs form, $\alpha_{ij}^k = \acute{s}_{\alpha_{ij}}^k, \acute{s}_{\beta_{ij}}^k, \acute{s}_{\gamma_{ij}}^k$, where $\acute{s}_{\alpha_{ij}}^k, \acute{s}_{\beta_{ij}}^k, \acute{s}_{\gamma_{ij}}^k \in \hat{S}^* = \{\acute{s}_h | h \in [0,1]\}$ for every alternative under the given attributes. The decision maker collective information is given in the form of the decision matrix $R_k = \left(\alpha_{ij}^k\right)_{m\times n}$.

Step 1. Developed the linguistic spherical fuzzy decision-matrix $R_k = \left(\alpha_{ij}^k\right)_{m\times n}$.

Step 2. Using the LSFOWA operator:

$$\begin{aligned} r_{ij} &= \left\langle \acute{s}_{\alpha_{ij}}, \acute{s}_{\beta_{ij}}, \acute{s}_{\gamma_{ij}} \right\rangle \\ &= \mathrm{LSFOWA}(\alpha_{ij}^1, \alpha_{ij}^2, ..., \alpha_{ij}^l) \\ &= \left(\acute{s}_{g\sqrt{1-\prod_{k=1}^{l}\left(1-\left(\alpha_{(ij)}^k\right)^2/g^2\right)^{v_k}}}, \acute{s}_{g\prod_{k=1}^{l}\left(\frac{\gamma_{(ij)}^k}{g}\right)^{v_k}}, \acute{s}_{g\prod_{k=1}^{l}\left(\frac{\gamma_{(ij)}^k}{g}\right)^{v_k}} \right) \end{aligned}$$

or using the LSFOWG operator:

$$\begin{aligned} r_{ij} &= \left\langle \acute{s}_{\alpha_{ij}}, \acute{s}_{\beta_{ij}}, \acute{s}_{\gamma_{ij}} \right\rangle \\ &= \mathrm{LSFOWG}(\alpha_{ij}^1, \alpha_{ij}^2, ..., \alpha_{ij}^l) \\ &= \left(\acute{s}_{g\prod_{k=1}^{l}\left(\frac{\alpha_{(ij)}^k}{g}\right)^{v_k}}, \acute{s}_{g\sqrt{1-\prod_{k=1}^{l}\left(1-\left(\beta_{(ij)}^k\right)^2/g^2\right)^{v_k}}}, \acute{s}_{g\sqrt{1-\prod_{k=1}^{l}\left(1-\left(\gamma_{(ij)}^k\right)^2/g^2\right)^{v_k}}} \right). \end{aligned}$$

Step 3. Accumulate all $r_{ij} (j = 1, ..., n)$ for every alternative $\mathbb{R}_i (i = 1, ..., m)$ by the LSFWA operator:

$$\begin{aligned} r_i &= \mathrm{LSFWA}(r_{i1}, r_{i2}, ..., r_{in}) \\ &= \left(\acute{s}_{g\sqrt{\left(1-\prod_{j=1}^{n}\left(1-\alpha_{ij}^2/g^2\right)^{\Re_i}\right)}}, \acute{s}_{g\prod_{j=1}^{n}\left(\frac{\beta_{ij}}{g}\right)^{\Re_i}}, \acute{s}_{g\prod_{j=1}^{n}\left(\frac{\gamma_{ij}}{g}\right)^{\Re_i}} \right) \end{aligned}$$

or using the LSFWG operator

$$\begin{aligned} r_i &= \mathrm{LSFWA}(r_{i1}, r_{i2}, ..., r_{in}) \\ &= \left(\acute{s}_{g\prod_{j=1}^{n}\left(\frac{\alpha_{ij}}{g}\right)^{\Re_i}}, \acute{s}_{g\sqrt{\left(1-\prod_{j=1}^{n}\left(1-\beta_{ij}^2/g^2\right)^{\Re_i}\right)}}, \acute{s}_{g\sqrt{\left(1-\prod_{j=1}^{n}\left(1-\gamma_{ij}^2/g^2\right)^{\Re_i}\right)}} \right). \end{aligned}$$

Step 4. Rank the alternative $\mathbb{R}_i$, according to the Definition 6.

6. Illustrative Example

In this section, we will present the proposed method of MADM based on linguistic spherical fuzzy information which relates the assessment and rank of heavy rainfall in the Lasbella district and adjoining areas of the Baluchistan, Pakistan. Then, the decision making approach will provide the desired ranking.

A recent storm caused a spell of heavy rainfall in the Lasbella district, and adjoining areas of Baluchistan, Pakistan were hit with unprecedented flash floods in February 2019. A large number of roads, which connect the Lasbella district with other parts of Baluchistan were destroyed in this flood. In this context, the Pakistan government has had to take on a considerable number of road building projects either to preserve the roads already built or to undertake new roads.

These projects have been carried out by a limited number of well-established contractors, and the selection process has been on the basis of bid price alone. In recent years, the increased project complexity, technical capability, higher performance, and safety and financial requirements have been demanding the use of multi-attribute decision making methods. For this, the Pakistan government has issued a notice in the newspapers, and three experts take the responsibility of selecting the best construction company out of a set of four possible alternatives, $X = \left\{ \begin{array}{c} \mathbb{R}_1 = \text{Ahmed Construction}, \mathbb{R}_2 = \text{Matracon Pakistan Private(Pvt) Limited(Ltd)}, \\ \mathbb{R}_3 = \text{Eastern Highway Company}, \mathbb{R}_4 = \text{Banu Mukhtar Concrete Pvt. Ltd.} \end{array} \right\}$

On the basis of the attributes, $\check{C}_1$ = technical capability, $\check{C}_2$ = higher performance, $\check{C}_3$ = saifiety, $\check{C}_4$ = financial requirements, bids for these projects will take place. The objective of the government is to choose the best construction company. In order to fulfill this, the three experts d_1, d_2 , and d_3 are call to give their preferences for every alternative under the given attributes with the linguistic term set $\hat{S} = (\acute{s}_1, \acute{s}_2, \acute{s}_3, \acute{s}_4, \acute{s}_5, \acute{s}_6, \acute{s}_7, \acute{s}_8, \acute{s}_9)$. Consider $\Re = (0.3, 0.1, 0.2, 0.4)^T$ the weighting vector of attributes.

Step 1: For each alternative, preferences of the experts are given in the form of decision-matrices $R_k = \left(\alpha_{ij}^k\right)_{4\times 4}$ $(k = 1, 2, 3)$ are shown in Tables 1–3.

Table 1. R_1 (Decision-matrix).

	$\check{C}_1$	$\check{C}_2$	$\check{C}_3$	$\check{C}_4$
$\mathbb{R}_1$	$\langle \acute{s}_3, \acute{s}_2, \acute{s}_5 \rangle$	$\langle \acute{s}_2, \acute{s}_4, \acute{s}_3 \rangle$	$\langle \acute{s}_7, \acute{s}_5, \acute{s}_4 \rangle$	$\langle \acute{s}_5, \acute{s}_4, \acute{s}_2 \rangle$
$\mathbb{R}_2$	$\langle \acute{s}_5, \acute{s}_7, \acute{s}_3 \rangle$	$\langle \acute{s}_6, \acute{s}_7, \acute{s}_8 \rangle$	$\langle \acute{s}_8, \acute{s}_3, \acute{s}_7 \rangle$	$\langle \acute{s}_4, \acute{s}_6, \acute{s}_4 \rangle$
$\mathbb{R}_3$	$\langle \acute{s}_4, \acute{s}_5, \acute{s}_2 \rangle$	$\langle \acute{s}_2, \acute{s}_3, \acute{s}_5 \rangle$	$\langle \acute{s}_6, \acute{s}_1, \acute{s}_4 \rangle$	$\langle \acute{s}_5, \acute{s}_7, \acute{s}_3 \rangle$
$\mathbb{R}_4$	$\langle \acute{s}_7, \acute{s}_1, \acute{s}_4 \rangle$	$\langle \acute{s}_3, \acute{s}_4, \acute{s}_8 \rangle$	$\langle \acute{s}_5, \acute{s}_6, \acute{s}_2 \rangle$	$\langle \acute{s}_6, \acute{s}_4, \acute{s}_2 \rangle$

Table 2. R_2 (Decision-matrix).

	$\check{C}_1$	$\check{C}_2$	$\check{C}_3$	$\check{C}_4$
$\mathbb{R}_1$	$\langle \acute{s}_4, \acute{s}_6, \acute{s}_2 \rangle$	$\langle \acute{s}_6, \acute{s}_5, \acute{s}_3 \rangle$	$\langle \acute{s}_4, \acute{s}_3, \acute{s}_4 \rangle$	$\langle \acute{s}_1, \acute{s}_4, \acute{s}_7 \rangle$
$\mathbb{R}_2$	$\langle \acute{s}_7, \acute{s}_3, \acute{s}_5 \rangle$	$\langle \acute{s}_1, \acute{s}_2, \acute{s}_6 \rangle$	$\langle \acute{s}_5, \acute{s}_3, \acute{s}_2 \rangle$	$\langle \acute{s}_2, \acute{s}_4, \acute{s}_3 \rangle$
$\mathbb{R}_3$	$\langle \acute{s}_5, \acute{s}_2, \acute{s}_4 \rangle$	$\langle \acute{s}_3, \acute{s}_6, \acute{s}_5 \rangle$	$\langle \acute{s}_2, \acute{s}_1, \acute{s}_4 \rangle$	$\langle \acute{s}_7, \acute{s}_5, \acute{s}_3 \rangle$
$\mathbb{R}_4$	$\langle \acute{s}_6, \acute{s}_1, \acute{s}_7 \rangle$	$\langle \acute{s}_3, \acute{s}_8, \acute{s}_3 \rangle$	$\langle \acute{s}_2, \acute{s}_2, \acute{s}_5 \rangle$	$\langle \acute{s}_4, \acute{s}_2, \acute{s}_6 \rangle$

Table 3. R_3 (Decision-matrix).

	$\check{C}_1$	$\check{C}_2$	$\check{C}_3$	$\check{C}_4$
$\mathbb{R}_1$	$\langle \acute{s}_8, \acute{s}_4, \acute{s}_3 \rangle$	$\langle \acute{s}_2, \acute{s}_2, \acute{s}_5 \rangle$	$\langle \acute{s}_1, \acute{s}_2, \acute{s}_6 \rangle$	$\langle \acute{s}_7, \acute{s}_3, \acute{s}_4 \rangle$
$\mathbb{R}_2$	$\langle \acute{s}_3, \acute{s}_7, \acute{s}_4 \rangle$	$\langle \acute{s}_5, \acute{s}_3, \acute{s}_2 \rangle$	$\langle \acute{s}_6, \acute{s}_2, \acute{s}_3 \rangle$	$\langle \acute{s}_2, \acute{s}_6, \acute{s}_3 \rangle$
$\mathbb{R}_3$	$\langle \acute{s}_6, \acute{s}_5, \acute{s}_1 \rangle$	$\langle \acute{s}_7, \acute{s}_6, \acute{s}_5 \rangle$	$\langle \acute{s}_2, \acute{s}_4, \acute{s}_8 \rangle$	$\langle \acute{s}_4, \acute{s}_3, \acute{s}_1 \rangle$
$\mathbb{R}_4$	$\langle \acute{s}_4, \acute{s}_6, \acute{s}_7 \rangle$	$\langle \acute{s}_6, \acute{s}_8, \acute{s}_4 \rangle$	$\langle \acute{s}_5, \acute{s}_7, \acute{s}_3 \rangle$	$\langle \acute{s}_6, \acute{s}_7, \acute{s}_8 \rangle$

Step 2: We utilized the normal distribution method [64] to compute the associated weighting vector of the decision maker and get $\upsilon = (0.243, 0.514, 0.243)^T$. Now, we utilize the LSFOWA and LSFOWG operators to aggregate the decision-matrices $R = (r_{ij})_{4\times 4}$ are shown in Tables 4 and 5.

Table 4. Aggregated value of R by utilizing the LSFOWA operator.

	$\check{C}_1$	$\check{C}_2$	$\check{C}_3$	$\check{C}_4$
$\mathbb{R}_1$	$\langle \acute{s}_{5.62}, \acute{s}_{3.07}, \acute{s}_{3.56} \rangle$	$\langle \acute{s}_{3.82}, \acute{s}_{3.58}, \acute{s}_{3.42} \rangle$	$\langle \acute{s}_{4.85}, \acute{s}_{3.10}, \acute{s}_{4.41} \rangle$	$\langle \acute{s}_{5.29}, \acute{s}_{3.76}, \acute{s}_{3.19} \rangle$
$\mathbb{R}_2$	$\langle \acute{s}_{5.32}, \acute{s}_{5.66}, \acute{s}_{3.65} \rangle$	$\langle \acute{s}_{4.22}, \acute{s}_{2.99}, \acute{s}_{4.88} \rangle$	$\langle \acute{s}_{6.43}, \acute{s}_{2.71}, \acute{s}_{2.99} \rangle$	$\langle \acute{s}_{3.24}, \acute{s}_{5.44}, \acute{s}_{3.50} \rangle$
$\mathbb{R}_3$	$\langle \acute{s}_{5.08}, \acute{s}_{3.13}, \acute{s}_{2.39} \rangle$	$\langle \acute{s}_{4.50}, \acute{s}_{5.02}, \acute{s}_{4.95} \rangle$	$\langle \acute{s}_{3.71}, \acute{s}_{1.37}, \acute{s}_{4.73} \rangle$	$\langle \acute{s}_{5.24}, \acute{s}_{4.13}, \acute{s}_{2.23} \rangle$
$\mathbb{R}_4$	$\langle \acute{s}_{5.97}, \acute{s}_{1.51}, \acute{s}_{6.04} \rangle$	$\langle \acute{s}_{4.12}, \acute{s}_{4.01}, \acute{s}_{5.59} \rangle$	$\langle \acute{s}_{4.50}, \acute{s}_{4.73}, \acute{s}_{2.73} \rangle$	$\langle \acute{s}_{5.17}, \acute{s}_{3.19}, \acute{s}_{4.88} \rangle$

Table 5. Aggregated value of R by utilizing the LSFOWG operator.

	$\check{C}_1$	$\check{C}_2$	$\check{C}_3$	$\check{C}_4$
$\mathbb{R}_1$	$\langle \acute{s}_{4.34}, \acute{s}_{4.93}, \acute{s}_{3.37} \rangle$	$\langle \acute{s}_{2.61}, \acute{s}_{4.02}, \acute{s}_{3.71} \rangle$	$\langle \acute{s}_{3.27}, \acute{s}_{2.85}, \acute{s}_{4.68} \rangle$	$\langle \acute{s}_{3.67}, \acute{s}_{3.82}, \acute{s}_{4.50} \rangle$
$\mathbb{R}_2$	$\langle \acute{s}_{4.79}, \acute{s}_{6.49}, \acute{s}_{3.82} \rangle$	$\langle \acute{s}_{2.27}, \acute{s}_{4.41}, \acute{s}_{6.36} \rangle$	$\langle \acute{s}_{5.89}, \acute{s}_{2.85}, \acute{s}_{4.41} \rangle$	$\langle \acute{s}_{2.82}, \acute{s}_{5.62}, \acute{s}_{3.60} \rangle$
$\mathbb{R}_3$	$\langle \acute{s}_{4.98}, \acute{s}_{3.82}, \acute{s}_{3.24} \rangle$	$\langle \acute{s}_{2.97}, \acute{s}_{5.47}, \acute{s}_{4.93} \rangle$	$\langle \acute{s}_{2.61}, \acute{s}_{2.38}, \acute{s}_{5.83} \rangle$	$\langle \acute{s}_{4.84}, \acute{s}_{5.01}, \acute{s}_{2.77} \rangle$
$\mathbb{R}_4$	$\langle \acute{s}_{5.62}, \acute{s}_{3.48}, \acute{s}_{6.55} \rangle$	$\langle \acute{s}_{3.54}, \acute{s}_{5.62}, \acute{s}_{5.90} \rangle$	$\langle \acute{s}_{3.98}, \acute{s}_{5.76}, \acute{s}_{4.76} \rangle$	$\langle \acute{s}_{4.84}, \acute{s}_{4.59}, \acute{s}_{6.36} \rangle$

Step 3: We used the LSFWA and LSFWG operators to aggregate $r_{ij} (j = 1, ..., 4)$ into the cumulative r_i for every alternative $\mathbb{R}_i (i = 1, ..., 4)$, under the attribute weighting vector $\Re = (0.3, 0.1, 0.2, 0.3)^T$. We summarized the corresponding results and their ranking in Table 6.

Step 4:. Rank of the alternatives are follows

Table 6. Overall preference value and the alternatives ranking.

	$\mathbb{R}_1$	$\mathbb{R}_2$	$\mathbb{R}_3$	$\mathbb{R}_4$	**Ranking**
LSFOWA LSFWA	$\langle \acute{s}_{5.24}, \acute{s}_{3.34}, \acute{s}_{3.58} \rangle$	$\langle \acute{s}_{4.93}, \acute{s}_{4.50}, \acute{s}_{3.58} \rangle$	$\langle \acute{s}_{4.93}, \acute{s}_{3.12}, \acute{s}_{2.85} \rangle$	$\langle \acute{s}_{5.24}, \acute{s}_{2.84}, \acute{s}_{4.65} \rangle$	$\mathbb{R}_3 > \mathbb{R}_1 > \mathbb{R}_4 > \mathbb{R}_2$
LSFOWG LSFWG	$\langle \acute{s}_{3.63}, \acute{s}_{4.13}, \acute{s}_{4.12} \rangle$	$\langle \acute{s}_{3.76}, \acute{s}_{5.25}, \acute{s}_{4.22} \rangle$	$\langle \acute{s}_{4.09}, \acute{s}_{4.50}, \acute{s}_{4.12} \rangle$	$\langle \acute{s}_{4.73}, \acute{s}_{4.85}, \acute{s}_{4.91} \rangle$	$\mathbb{R}_3 > \mathbb{R}_1 > \mathbb{R}_4 > \mathbb{R}_2$

7. Comparative Study and Discussion

In this comparison, we used the method proposed by S. Ashraf and S. Abdullah [58] to solve the example used in this paper. As the attribute information in [58] occurs in the form of SFNs, we then convert LSFNs to SFNs.

From Table 7, we observe that there is no difference in the ranking result between our approach and the approach developed in [58]. Thus, our method is more generalized than the method proposed in [58], and our approach has a wider range of applications than the approach proposed in [58]. From Table 7, it is clear that the ranking of alternatives is the same in the linguistic spherical fuzzy numbers and spherical fuzzy numbers [58].

Table 7. Ranking of different methods.

	$\mathbb{R}_1$	$\mathbb{R}_2$	$\mathbb{R}_3$	$\mathbb{R}_4$	**Ranking**
LSFWA operator	5.306	4.890	5.393	5.139	$\mathbb{R}_3 > \mathbb{R}_1 > \mathbb{R}_4 > \mathbb{R}_2$
LSFWG operator	4.477	4.073	4.489	4.310	$\mathbb{R}_3 > \mathbb{R}_1 > \mathbb{R}_4 > \mathbb{R}_2$
SPFWA operator [58]	0.614	0.612	0.640	0.610	$\mathbb{R}_3 > \mathbb{R}_1 > \mathbb{R}_4 > \mathbb{R}_2$

8. Conclusions

The objective of writing this article is to present the notion of "linguistic spherical fuzzy set", which is the combination of LFSs and SFSs. The basic properties of linguistic spherical fuzzy operators are discussed. Subsequently, we write a new algorithm for the decision-making based on the defined linguistic spherical fuzzy aggregation operators by analyzing the limitations and advantages in the existing literature. The proposed approach will yield an objective decision result based on information from the decision problem only. Finally, we include a descriptive example to show the appropriateness of the developed technique, and a comparison with the existing method.

Author Contributions: Conceptualization, S.A. (Shahzaib Ashraf) and S.A. (Saleem Abdullah); methodology, M.Q.; software, S.A. (Shahzaib Ashraf); validation, H.J., S.A. (Shahzaib Ashraf), S.A. (Saleem Abdullah) M.Q., M.B. and S.Z.; formal analysis, S.A. (Shahzaib Ashraf); investigation, S.A. (Shahzaib Ashraf); data curation, M.Q.; writing—original draft preparation, S.A. (Shahzaib Ashraf), M.Q.; writing—review and editing, H.J., S.A. (Saleem Abdullah), M.B. and S.Z.; visualization, S.A. (Saleem Abdullah), S.Z.; supervision, S.A. (Shahzaib Ashraf); project administration, S.A. (Shahzaib Ashraf); funding acquisition, S.Z.

Funding: This paper is supported by Philosophy and Social Science Planning Projects of Zhejiang (No. 16ZJQN022YB), Zhejiang Province Natural Science Foundation (No. LY18G010007), Zhejiang Public Technology Applied Research Projects (No. LGG18F020001), Major Humanities and Social Sciences Research Projects in Zhejiang Universities (No. 2018QN058) and K. C. Wong Magna Fund in Ningbo University.

Acknowledgments: The authors would like to thank the editor in chief, associate editor and the anonymous referees for detailed and valuable comments which helped to improve this manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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