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Reich, Jungck, and Berinde Common Fixed Point Results on $\mathcal{F}$ -Metric Spaces and an Application

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Abstract: Jleli and Samet (2018) introduced a new concept, named an $\mathcal{F}$ -metric space, as a generalization of the notion of a metric space. In this paper, we prove certain common fixed point theorems in $\mathcal{F}$ -metric spaces. As consequences of our results, we obtain results of Banach, Jungck, Reich, and Berinde in these spaces. An application in dynamic programming is also given.

Keywords: common fixed point; $\mathcal{F}$ -metric; dynamic programming

1. Introduction and Preliminaries

Fixed point theorems are tools in many fields in mathematics, physics, and computer science. The notion of metric spaces has been generalized by several authors, such as Czerwik [1], Khamsi and Hussain [2], Mlaiki et al. [3,4], Abdeljawad et al. [5], and so on. Very recently, Jleli and Samet [6] initiated the notion of $\mathcal{F}$ -metric spaces, where a generalization of the Banach contraction principle was provided.

We begin with a brief recollection of basic notions and the facts of $\mathcal{F}$ -metric spaces. First, denote by $\mathcal{F}$ the set of functions $f : (0, \infty) \rightarrow \mathbb{R}$ such that

($\mathcal{F}_1$) f is non-decreasing; that is, $0 < \xi < \eta$ implies $f(\xi) \leq f(\eta)$; and

($\mathcal{F}_2$) For each sequence $\{r_n\} \subset (0, +\infty)$,

$$\lim_{n \rightarrow +\infty} r_n = 0 \text{ if and only if } \lim_{n \rightarrow +\infty} f(r_n) = -\infty.$$

Definition 1 ([6]). Let X be a nonempty set and $D : X \times X \rightarrow [0, +\infty)$ be a function. If there exists $(f, a) \in \mathcal{F} \times [0, +\infty)$, such that

(D_1) $D(\xi, \eta) = 0 \Leftrightarrow \xi = \eta$;

(D_2) $D(\xi, \eta) = D(\eta, \xi)$; and

(D_3) For all $(\xi, \eta) \in X^2$, $p \in \mathbb{N}$ with $p \geq 2$, and for all $(v_i)_{i=1}^N \subset X$ with $(v_1, v_p) = (\xi, \eta)$, we have that

$$D(\xi, \eta) > 0 \text{ implies } f(D(\xi, \eta)) \leq f\left(\sum_{j=1}^{p-1} D(v_j, v_{j+1})\right) + a,$$

then D is called an $\mathcal{F}$ -metric on X . The pair (X, D) is called an $\mathcal{F}$ -metric space.

Definition 2 ([6]). Let $\{\theta_n\}$ be a sequence in an $\mathcal{F}$ -metric space (X, D) . Then:

- (i) $\{\theta_n\}$ is $\mathcal{F}$ -convergent to $\theta \in X$ if $\{\theta_n\}$ is convergent to θ with respect to the $\mathcal{F}$ -metric D ; that is, $\lim_{n \rightarrow \infty} D(\theta_n, \theta) = 0$.
- (ii) $\{\theta_n\}$ is $\mathcal{F}$ -Cauchy if $\lim_{n, m \rightarrow +\infty} D(\theta_n, \theta_m) = 0$.
- (iii) (X, D) is $\mathcal{F}$ -complete if each $\mathcal{F}$ -Cauchy sequence in X is $\mathcal{F}$ -convergent to some element in X .

The existence of common fixed points of maps verifying certain contractive conditions has been investigated extensively by many authors. In 1976, Jungck [7] proved a common fixed point theorem for commuting maps. Among the generalizations of the Banach contraction principle, a result of Reich [8] was notable. By combining Reich and Jungck type contractions, we establish a first common fixed point result of Reich and Jungck type in the class of $\mathcal{F}$ -metric spaces.

On the other hand, Berinde [9] initiated the concept of (α, L) weak contractions, and proved that a lot of the well-known contractive conditions imply (α, L) weak contractions. The concept of (α, L) weak contractions does not ask $(\alpha + L)$ to be less than 1, as happens in many kinds of fixed point theorems for contractive conditions that involve one or more of the displacements $d(x, y)$, $d(x, Tx)$, $d(y, Ty)$, $d(x, Ty)$, and $d(y, Tx)$. For more details on this concept, we refer the reader to [9–16], and the references therein. In the second part of paper, we will present a Berinde weak contraction type common fixed point theorem. Moreover, some illustrated consequences and examples are provided. Also, we notice a mistake in [17]. Finally, an application in dynamic programming is considered.

2. Main Results

The following lemma extends the corresponding result from metric, b-metric, and partial metric spaces, and it is known in the literature as the Jungck lemma. For the proof, we use the techniques of $\mathcal{F}$ -metric spaces.

Lemma 1. Let $\{y_n\}$ be a sequence in an $\mathcal{F}$ -metric space (X, D) . If there exists $\lambda \in (0, 1)$, such that

$$D(y_{n+1}, y_n) \leq \lambda D(y_n, y_{n-1}), \text{ for all } n \in \mathbb{N}, \quad (1)$$

then $\{y_n\}$ is an $\mathcal{F}$ -Cauchy sequence.

Proof. Consider $(f, a) \in \mathcal{F} \times [0, +\infty)$, such that (D_3) holds. If $D(y_0, y_1) = 0$, then we conclude that $y_n = y_0$ for all $n \in \mathbb{N}$, and so $\{y_n\}$ is $\mathcal{F}$ -Cauchy. So, we can assume $D(y_0, y_1) > 0$. From (1), we have

$$D(y_{n+1}, y_n) \leq \lambda^n D(y_1, y_0), \text{ for all } n \in \mathbb{N}.$$

It follows that

$$\sum_{i=n}^{m-1} D(y_{i+1}, y_i) \leq \frac{\lambda^n}{1-\lambda} D(y_1, y_0), \text{ for all } m > n. \quad (2)$$

Let $\epsilon > 0$. By $(\mathcal{F}_2)$, there exists $n_0 \in \mathbb{N}$, such that

$$f\left(\frac{\lambda^n}{1-\lambda} D(y_1, y_0)\right) \leq f(\epsilon) - a \text{ for all } n \geq n_0. \quad (3)$$

Hence, by (2), (3), and $(\mathcal{F}_1)$, we have

$$f\left(\sum_{i=n}^{m-1} D(y_{i+1}, y_i)\right) \leq f(\epsilon) - a \text{ for all } m > n \geq n_0. \quad (4)$$

Using (D_3) , we obtain that $D(y_n, y_m) > 0$ implies

$$f(D(y_n, y_m)) \leq f\left(\sum_{i=n}^{m-1} D(y_{i+1}, y_i)\right) + a. \quad (5)$$

From (4), we obtain

$$f(D(y_n, y_m)) \leq f(\epsilon) \text{ for all } m > n \geq n_0.$$

By $(\mathcal{F}_1)$, we have that

$$D(y_n, y_m) \leq \epsilon \text{ for all } m > n \geq n_0.$$

Therefore, $\{y_n\}$ is $\mathcal{F}$ -Cauchy. $\square$

2.1. A Generalization of the Results of Reich and Jungck

As an application of above lemma, we establish the following generalization of the results of Reich and Jungck in the class of $\mathcal{F}$ -metric spaces.

Theorem 1. Let T and I be two commuting self-maps of an $\mathcal{F}$ -complete $\mathcal{F}$ -metric space (X, D) , such that

$$D(Tx, Ty) \leq \alpha D(Ix, Iy) + \beta D(Ix, Tx) + \gamma D(Iy, Ty) \quad (6)$$

for all $x, y \in X$, where $\alpha \in (0, 1)$ and $\beta, \gamma \in [0, 1)$, such that $\alpha + \beta + \gamma < 1$. If $T(X) \subseteq I(X)$, and I, T are continuous, then there is a unique common fixed point of I and T .

Proof. Let $\theta_0 \in X$ be arbitrary. As $T\theta_0 \in I(X)$, there exists some $\theta_1 \in X$ so that $I\theta_1 = T\theta_0$. Generally, the sequence $\{\theta_n\}$ is defined by $I\theta_{n+1} = T\theta_n$. We show that $\{I\theta_n\}$ is an $\mathcal{F}$ -Cauchy sequence. By (6), we have

$$\begin{aligned} D(I\theta_{n+1}, I\theta_n) &= D(T\theta_n, T\theta_{n-1}) \\ &\leq \alpha D(I\theta_n, I\theta_{n-1}) + \beta D(I\theta_n, T\theta_n) + \gamma D(I\theta_{n-1}, T\theta_{n-1}) \\ &= \alpha D(I\theta_n, I\theta_{n-1}) + \beta D(I\theta_n, I\theta_{n+1}) + \gamma D(I\theta_{n-1}, I\theta_n). \end{aligned}$$

So,

$$D(I\theta_{n+1}, I\theta_n) \leq \lambda D(I\theta_n, I\theta_{n-1}) \text{ for all } n \in \mathbb{N}, \quad (7)$$

where $\lambda = \frac{\alpha+\gamma}{1-\beta} \in (0, 1)$.

Now, we distinguish the following two cases:

Case 1. If $I\theta_n = I\theta_{n+1}$ for some $n \geq 0$, then $T\theta_n = I\theta_n = \omega$. We claim that ω is the unique common fixed point of T and I . We have

$$T\omega = TI\theta_n = IT\theta_n = I\omega.$$

Let $D(\omega, T\omega) > 0$. Here, we get

$$\begin{aligned} D(\omega, T\omega) &= D(T\theta_n, T\omega) \\ &\leq \alpha D(I\theta_n, I\omega) + \beta D(I\theta_n, T\theta_n) + \gamma D(I\omega, T\omega) \\ &= \alpha D(\omega, T\omega) + \beta D(\omega, \omega) + \gamma D(T\omega, T\omega) \\ &= \alpha D(\omega, T\omega) \\ &< D(\omega, T\omega), \end{aligned}$$

which is a contradiction. Then, Equation (6) yields that $T\theta_n = I\theta_n = \omega$ is the unique common fixed point of T and I .

Case 2. If $I\theta_n \neq I\theta_{n+1}$ for all $n \geq 0$, from Lemma 1 and (7), we find that $\{I\theta_n\}$ is an $\mathcal{F}$ -Cauchy sequence in X , which is complete; hence, there exists $z \in X$ such that

$$\lim_{n \rightarrow \infty} I\theta_n = \lim_{n \rightarrow \infty} T\theta_{n-1} = z.$$

As the maps I and T are commuting, we obtain

$$Iz = I(\lim_{n \rightarrow \infty} T\theta_n) = \lim_{n \rightarrow \infty} IT\theta_n = \lim_{n \rightarrow \infty} TI\theta_n = T(\lim_{n \rightarrow \infty} I\theta_n) = Tz. \quad (8)$$

Let $v = Iz = Tz$. We get $Tv = TIz = ITz = Iv$. If $Tz \neq Tv$, by (6), we get

$$\begin{aligned} D(Tz, Tv) &\leq \alpha D(Iz, Iv) + \beta D(Iz, Tz) + \gamma D(Iv, Tv) \\ &= \alpha D(Tz, Tv) + \beta D(Tz, Tz) + \gamma D(Tv, Tv) \\ &= \alpha D(Tz, Tv) \\ &< D(Tz, Tv), \end{aligned}$$

a contradiction. Hence, $Tz = Tv$ and, finally, $Tv = Iv = v$. That is, I and T have a common fixed point. Further, (6) yields its uniqueness. $\square$

We present the following consequences of Theorem 1.

Theorem 2 ([6], Theorem 5.1). *Let T be a self-map of an $\mathcal{F}$ -complete $\mathcal{F}$ -metric space (X, D) , such that*

$$D(Tx, Ty) \leq \alpha D(x, y) \quad (9)$$

for all $x, y \in X$, where $\alpha \in (0, 1)$. Then, T has a unique fixed point.

Proof. Note that condition (9) implies that T is a continuous map. It suffices to take $\beta = \gamma = 0$ and $I = \text{Identity} = (Id)$ in Theorem 1. $\square$

Theorem 3 (Theorem of Reich in $\mathcal{F}$ -metric spaces, see [8]). *Let T be a self-map of an $\mathcal{F}$ -complete $\mathcal{F}$ -metric space (X, D) , such that*

$$D(Tx, Ty) \leq \alpha D(x, y) + \beta D(x, Tx) + \gamma D(y, Ty), \quad (10)$$

for all $x, y \in X$, where $\alpha \in (0, 1)$ and $\beta, \gamma \in [0, 1)$ such that $\alpha + \beta + \gamma < 1$. If T continuous, then T has a unique fixed point.

Proof. Putting $I = \text{Identity} = (Id)$ in Theorem 1, we get the result. $\square$

Theorem 4 (Theorem of Jungck in $\mathcal{F}$ -metric spaces, see [18]). *Let I and T be two commuting self-maps of an $\mathcal{F}$ -complete $\mathcal{F}$ -metric space (X, D) , such that*

$$D(Tx, Ty) \leq \alpha D(Ix, Iy), \quad (11)$$

for all $x, y \in X$, where $\alpha \in (0, 1)$. If $T(X) \subseteq I(X)$ and I is continuous, then there is a unique common fixed point of I and T .

Proof. We note that the condition (11) implies the continuity of the map T . Now, the proof follows directly from Theorem 1. $\square$

Remark 1. We may state the following open question: Is the continuity condition of the map T in Theorem 3 necessary?

2.2. A Weak Contraction Type Common Fixed Point Theorem in $\mathcal{F}$ -Metric Spaces

The aim of this section is to prove a Berinde weak contraction type common fixed point theorem, in the setting of $\mathcal{F}$ -metric spaces.

Theorem 5. Let (X, D) be an $\mathcal{F}$ -complete $\mathcal{F}$ -metric space. Suppose that T and S are two self-maps of X satisfying

$$D(Tx, Sy) \leq \alpha \max\{D(x, y), D(x, Tx), D(y, Sy)\} + L \min\{D(Tx, y), D(x, Sy)\}, \quad (12)$$

for all $x, y \in X$, where $\alpha \in (0, 1)$ and $L \geq 0$. Then, T and S have a common fixed point in X if at least one of the following conditions is satisfied:

- (i) T or S is continuous; and
- (ii) The function $f \in \mathcal{F}$ verifying (D_3) is assumed to be continuous. Additionally, α is chosen in order that $f(u) > f(\alpha u) + a$ for all $u \in (0, \infty)$, where a is also given by (D_3) .

Moreover, if $\alpha + L < 1$, then the common fixed point is unique.

Proof. First, note that if v is a fixed point of T (it will be the same when we consider the map S), then, from (12), we have

$$\begin{aligned} D(v, Sv) = D(Tv, Sv) &\leq \alpha \max\{D(v, v), D(v, Tv), D(v, Sv)\} \\ &+ L \min\{D(Tv, v), D(v, Sv)\} \\ &= \alpha D(v, Sv), \end{aligned}$$

which holds unless $D(v, Sv) = 0$; that is, $Sv = v$, so v is a fixed point of S . Hence, v is a common fixed point of T and S .

Let θ_0 be an arbitrary element in X . Define $\{\theta_n\}$ by $\theta_{2n+1} = T\theta_{2n}$ and $\theta_{2n+2} = S\theta_{2n+1}$, $n = 0, 1, 2, \dots$. Now,

$$\begin{aligned} D(\theta_{2n+1}, \theta_{2n+2}) &= D(T\theta_{2n}, S\theta_{2n+1}) \\ &\leq \alpha \max\{D(\theta_{2n}, \theta_{2n+1}), D(\theta_{2n}, T\theta_{2n}), D(\theta_{2n+1}, S\theta_{2n+1})\} \\ &+ L \min\{D(T\theta_{2n}, \theta_{2n+1}), D(\theta_{2n}, S\theta_{2n+1})\} \\ &\leq \alpha \max\{D(\theta_{2n}, \theta_{2n+1}), D(\theta_{2n+1}, \theta_{2n+2})\}. \end{aligned}$$

If, for some n , $\max\{D(\theta_{2n}, \theta_{2n+1}), D(\theta_{2n+1}, \theta_{2n+2})\} = D(\theta_{2n+1}, \theta_{2n+2})$, then

$$D(\theta_{2n+1}, \theta_{2n+2}) \leq \alpha D(\theta_{2n+1}, \theta_{2n+2}),$$

which is a contradiction, as $\alpha < 1$. So,

$$D(\theta_{2n+1}, \theta_{2n+2}) \leq \alpha D(\theta_{2n}, \theta_{2n+1}).$$

Similarly, it can be shown that

$$D(\theta_{2n+3}, \theta_{2n+2}) \leq \alpha D(\theta_{2n+1}, \theta_{2n+2}).$$

Now, from Lemma 1, we obtain that the sequence $\{\theta_n\}$ is $\mathcal{F}$ -Cauchy. As (X, D) is $\mathcal{F}$ -complete, the sequence $\{\theta_n\}$ $\mathcal{F}$ -converges to some point $x^* \in X$.

- (i) Suppose that T or S is a continuous map.

If T is continuous, we have that

$$x^* = \lim_{n \rightarrow \infty} \theta_{2n+1} = \lim_{n \rightarrow \infty} T\theta_{2n} = T(\lim_{n \rightarrow \infty} \theta_{2n}) = Tx^*.$$

From the beginning of the proof, we would have $x^* = Sx^*$.

- (ii) Suppose that the function $f \in \mathcal{F}$ verifying (D_3) is assumed to be continuous. Additionally, α is chosen in order that $f(u) > f(\alpha u) + a$ for all $u \in (0, \infty)$, where a is also given by (D_3) .

If $D(Tx^*, x^*) > 0$, we have

$$\begin{aligned} f(D(Tx^*, x^*)) &\leq f(D(Tx^*, S\theta_{2n+1}) + D(S\theta_{2n+1}, x^*)) + a \\ &\leq f(\alpha \max\{D(x^*, \theta_{2n+1}), D(x^*, Tx^*), D(\theta_{2n+1}, S\theta_{2n+1})\} \\ &\quad + D(\theta_{2n+2}, x^*)) \\ &\quad + L \min\{D(x^*, S\theta_{2n+1}), D(\theta_{2n+1}, Tx^*)\} + a. \end{aligned}$$

Letting $n \rightarrow \infty$ and using the continuity of f , we get

$$f(D(Tx^*, x^*)) \leq f(\alpha D(x^*, Tx^*)) + a,$$

which is a contradiction with respect condition (ii). Hence, we obtain $D(Tx^*, x^*) = 0$, so $x^* = Tx^*$. Therefore, x^* is a common fixed point of T and S .

For uniqueness, let y^* be another common fixed point of T and S . Then,

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Sy^*) \\ &\leq \alpha \max\{D(x^*, y^*), D(x^*, Tx^*), D(y^*, Sy^*)\} \\ &\quad + L \min\{D(x^*, Sy^*), D(y^*, Tx^*)\} \\ &= (\alpha + L)D(x^*, y^*). \end{aligned}$$

If $\alpha + L < 1$, it is clear that T and S have exactly one common fixed point. $\square$

Remark 2. 1. In ([6], Example 2.1), Jleli and Samet considered $D : X \times X \rightarrow [0, \infty)$, defined as

$$D(\xi, \eta) = \begin{cases} (\xi - \eta)^2 & \text{if } (\xi, \eta) \in [0, 3] \times [0, 3] \\ |\xi - \eta| & \text{if } (\xi, \eta) \notin [0, 3] \times [0, 3], \end{cases}$$

where $X = \{1, 2, \dots\}$. This D is an $\mathcal{F}$ -metric with $f(t) = \ln(t)$, $t > 0$, and $a = \ln(3)$. Note that f is continuous on $(0, +\infty)$ and the condition on α , which is $f(u) > f(\alpha u) + a$ for all $u > 0$, becomes $\ln(\alpha) + a < 0$, that is,

$$0 < \alpha < \frac{1}{3}.$$

This means that hypothesis (ii) in Theorem 5 is not superfluous.

2. If $S = T$ in Theorem 5, we obtain the main results of Berinde [9,12] in the new setting of $\mathcal{F}$ -metric spaces.
3. If $S = T$ and $L = 0$ in Theorem 5, we obtain a Ćirić type fixed point theorem in $\mathcal{F}$ -metric spaces; see [17].
4. Note that there is a gap in the proof of Theorem 2.1 in [17]. To be more clear, when proving that the map T has a fixed point x^* , Hussain and Kanwai [17] considered the limit as $n \rightarrow \infty$ in the three given cases, which is only true for some n . Our main result, corresponding to Theorem 5, is a correction of the above gap.

The following example illustrates Theorem 5.

Example 1. Let $X = [0, \infty)$ be endowed with the $\mathcal{F}$ -complete $\mathcal{F}$ -metric D given by

$$D(x, y) = \begin{cases} \exp(|x - y|), & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$$

Here, $f(t) = \frac{-1}{t}$ and $a = 1$. Define $T, S : X \rightarrow X$ by

$$Tx = \begin{cases} \frac{1}{2}x + 1, & \text{if } 0 \leq x \leq 2 \\ 2, & \text{otherwise} \end{cases} \quad \text{and} \quad Sx = 2 \quad \text{for all } x \in X.$$

Take $\alpha = \frac{1}{2}$ and $L = e^2 > 0$. Let $x, y \in X$. We have the following cases:

Case 1: Let $x \in [0, 2]$. If $y = x$, we have

$$D(Tx, Sy) = D\left(\frac{1}{2}x + 1, 2\right).$$

If $x = 2$, $D(Tx, Sy) = 0$. While, if $x \in [0, 2)$, we have

$$\begin{aligned} D(Tx, Sy) &= \exp\left(\left|\frac{1}{2}x - 1\right|\right) = \exp\left(1 - \frac{1}{2}x\right) \\ &\leq \frac{1}{2} \exp(|x - y|) + L \min\{D(x, Sy), D(y, Tx)\}. \end{aligned}$$

If $y \neq x$, we have $D(x, y) = \exp(|x - y|)$ and $D(Tx, Sy) = \exp\left(\left|\frac{1}{2}x - 1\right|\right)$. One writes

$$\exp\left(\left|\frac{1}{2}x - 1\right|\right) \leq \frac{1}{2} \exp(|x - y|) + L \min\{D(x, Sy), D(y, Tx)\},$$

which again implies that

$$D(Tx, Sy) \leq k \max\{D(x, y), D(x, Tx), D(y, Sy)\} + L \min\{D(x, Ty), D(y, Sx)\}.$$

Case II: Let $x \notin [0, 2]$. Here, (12) trivially holds. Additionally, condition (i) is satisfied. All the hypotheses of Theorem 5 are satisfied. Consequently, 2 is a common fixed point of T and S .

As a consequence of Theorem 5, we state the following corollaries:

Corollary 1. Let (X, D) be an $\mathcal{F}$ -complete $\mathcal{F}$ -metric space. Suppose that the map T is a self-map of X satisfying

$$D(Tx, Ty) \leq \alpha \max\{D(x, y), D(x, Tx), D(y, Ty)\} + L \min\{D(Tx, y), D(x, Ty)\}, \quad (13)$$

for all $x, y \in X$, where $\alpha \in (0, 1)$ and $L \geq 0$. Then, T has a fixed point in X if at least one of the following conditions is satisfied:

- (i) T is continuous;
- (ii) The function $f \in \mathcal{F}$ verifying (D_3) is assumed to be continuous. Additionally, α is chosen in order that $f(u) > f(\alpha u) + a$ for all $u \in (0, \infty)$, where a is also given by (D_3) .

Moreover, if $\alpha + L < 1$, then such a fixed point is unique.

In case that the function $f \in \mathcal{F}$ verifying (D_3) is assumed to be continuous, we may relax the condition of continuity of T , as follows:

Corollary 2. Let (X, D) be an $\mathcal{F}$ -complete $\mathcal{F}$ -metric space. Suppose that T is a self-map of X satisfying

$$D(Tx, Ty) \leq \alpha D(x, y) + L \min\{D(Tx, y), D(x, Ty)\}, \quad (14)$$

for all $x, y \in X$, where $\alpha \in (0, 1)$ and $L \geq 0$. Then, T has a fixed point in X .

Proof. If $D(Tx^*, x^*) > 0$, we have

$$\begin{aligned} f(D(Tx^*, x^*)) &\leq f(D(Tx^*, T\theta_{2n+1}) + D(T\theta_{2n+1}, x^*)) + a \\ &\leq f(\alpha D(x^*, \theta_{2n+1}) + D(\theta_{2n+2}, x^*)) \\ &\quad + L \min\{D(x^*, T\theta_{2n+1}), D(\theta_{2n+1}, Tx^*)\} + a. \end{aligned}$$

Letting $n \rightarrow \infty$ and using the continuity of f and $(\mathcal{F}_2)$, the right-hand side tends to $-\infty$, which is a contradiction. Hence, we must have $D(Tx^*, x^*) = 0$, and so $x^* = Tx^*$. $\square$

Example 2. Let $X = [0, 1]$ be endowed with the $\mathcal{F}$ -metric D and f be given as in Example 1. Consider $T : X \rightarrow X$ as

$$Tx = \begin{cases} 3x, & \text{if } 0 \leq x \leq \frac{1}{3} \\ 0, & \text{otherwise.} \end{cases}$$

Note that all the hypotheses of Corollary 2 are satisfied.

On the other hand, Theorem 5.1 of Jleli and Samet [6] is not applicable. Indeed, for $x = 0$ and $y = \frac{1}{3}$, we have

$$D(Tx, Ty) = D(0, 1) = \exp(1) > k \exp\left(\frac{1}{3}\right) = kD(x, y),$$

for each $k \in (0, 1)$.

3. Application

Applying our results, we give an application in dynamic programming. First, let W_1 and W_2 be two Banach spaces. Let $U \subset W_1$ be a state space and $E \subset W_2$ be a decision space. Consider

$$p(\xi) = \sup_{\eta \in E} \{f(\xi, \eta) + G_1(\xi, \eta, p(\delta(\xi, \eta)))\}, \quad x \in U, \quad (15)$$

and

$$p(\xi) = \sup_{\eta \in D} \{g(\xi, \eta) + G_2(\xi, \eta, p(\delta(\xi, \eta)))\}, \quad x \in U, \quad (16)$$

where $\delta : U \times E \rightarrow U$, $g : U \times E \rightarrow \mathbb{R}$, and $G_1, G_2 : U \times E \times \mathbb{R} \rightarrow \mathbb{R}$. Our aim is to resolve the system of functional Equations (15) and (16).

Denote, by $B(U)$, the set of all real bounded functions on U . For $q \in B(U)$, consider $\|q\| = \sup_{\xi \in U} |q(\xi)|$. Clearly, $(B(U), \|\cdot\|)$ is a Banach space.

We endow $B(U)$ with the $\mathcal{F}$ -metric (with $f(t) = \ln t$ and $a = 0$) defined by

$$D(h, k) = \sup_{t \in U} |h(t) - k(t)| = \|h - k\|.$$

We also define $T, S : B(U) \rightarrow B(U)$ by

$$T(h)(\xi) = \sup_{\eta \in E} \{g(\xi, \eta) + G_1(\xi, \eta, h(\delta(\xi, \eta)))\}, \quad (17)$$

and

$$S(k)(\xi) = \sup_{\eta \in E} \{g(\xi, \eta) + G_2(\xi, \eta, k(\delta(\xi, \eta)))\}, \quad (18)$$

for all $h, k \in B(U)$ and $\xi \in U$. Note that, if g, G_1 and G_2 are bounded functions, then T and S are well-defined. Our result is

Theorem 6. *If there exists $0 < r < 1$ such that, for all $(\xi, \eta) \in U \times E$,*

$$|G_1(\xi, \eta, h(\delta(\xi, \eta))) - G_2(\xi, \eta, k(\delta(\xi, \eta)))| \leq r \sup_{t \in W} |h(t) - k(t)|, \quad (19)$$

where the functions $G_1, G_2 : U \times E \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : U \times E \rightarrow \mathbb{R}$ are bounded, then the system given by Equations (15) and (16) has a unique bounded solution.

Proof. Let $\mu > 0$ be an arbitrary real number, $\xi \in U$, $h \in B(U)$, and $k \in B(U)$. Then, by (17), there exist $\eta_1, \eta_2 \in E$ such that

$$T(h)(\xi) < g(\xi, \eta_1) + G_1(\xi, \eta_1, h(\delta(\xi, \eta_1))) + \mu, \quad (20)$$

$$S(k)(\xi) < g(\xi, \eta_2) + G_2(\xi, \eta_2, k(\delta(\xi, \eta_2))) + \mu, \quad (21)$$

and

$$T(h)(\xi) \geq g(\xi, \eta_2) + G_1(\xi, \eta_2, h(\delta(\xi, \eta_2))), \quad (22)$$

$$S(k)(\xi) \geq g(\xi, \eta_1) + G_2(\xi, \eta_1, k(\delta(\xi, \eta_1))). \quad (23)$$

By (20) and (23), we get

$$\begin{aligned} T(h)(\xi) - S(k)(\xi) &\leq G_1(\xi, \eta_1, h(\delta(\xi, \eta_1))) - G_2(\xi, \eta_1, k(\delta(\xi, \eta_1))) + \mu \\ &\leq |G_1(\xi, \eta_1, h(\delta(\xi, \eta_1))) - G_2(\xi, \eta_1, k(\delta(\xi, \eta_1)))| + \mu. \end{aligned}$$

Furthermore, by (21) and (22), we have

$$\begin{aligned} S(k)(\xi) - T(h)(\xi) &\leq G_2(\xi, \eta_2, k(\delta(\xi, \eta_2))) - G_1(\xi, \eta_2, h(\delta(\xi, \eta_2))) + \mu \\ &\leq |G_1(\xi, \eta_2, h(\delta(\xi, \eta_2))) - G_2(\xi, \eta_2, k(\delta(\xi, \eta_2)))| + \mu. \end{aligned}$$

Thus, by (19), we have

$$|T(h)(\xi) - S(k)(\xi)| \leq r \sup_{t \in U} |h(t) - k(t)| + \mu.$$

Therefore, for $h, k \in B(U)$,

$$\begin{aligned} D(T(h), S(k)) &= r \sup_{\xi \in U} |T(h)(\xi) - T(k)(\xi)| \\ &\leq r \sup_{t \in U} |h(t) - k(t)| + \mu. \end{aligned}$$

As $\mu > 0$ is arbitrary, we get that

$$D(T(h), S(k)) \leq r \sup_{t \in U} |h(t) - k(t)| = rD(h, k).$$

On the other hand, condition (ii) in Theorem 5 holds, as $r \in (0, 1)$. Therefore, all conditions of Theorem 5 are verified. The operators T and S have a unique common fixed point (by taking $L = 0$, so $r + L < 1$). Then, there is a unique solution of the functional Equations (15) and (16). $\square$

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