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On Geometric Properties of Normalized Hyper-Bessel Functions

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Abstract: In this paper, the normalized hyper-Bessel functions are studied. Certain sufficient conditions are determined such that the hyper-Bessel functions are close-to-convex, starlike and convex in the open unit disc. We also study the Hardy spaces of hyper-Bessel functions.

Keywords: univalent functions; starlikeness; convexity; close-to-convexity; hyper-Bessel functions; Hardy space

MSC: Primary 30C45, 33C10; Secondary 30C20, 30C75

1. Introduction

Let $\mathcal{H}$ denote the class of functions that are analytic in $\mathcal{U} = \{z : |z| < 1\}$, and $\mathcal{A}$ denote the class of functions f that are analytic in $\mathcal{U}$ having the Taylor series form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in \mathcal{U}. \tag{1}$$

The class $\mathcal{S}$ of univalent functions f is the class of those functions in $\mathcal{A}$ that are one-to-one in $\mathcal{U}$. Let $\mathcal{S}^*$ denote the class of all functions f such that $f(\mathcal{U})$ is star-shaped domain with respect to origin while $\mathcal{C}$ denotes the class of functions f such that f maps $\mathcal{U}$ onto a domain which is convex. A function f in $\mathcal{A}$ belongs to the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α if and only if $Re\left(zf'(z)/f(z)\right) > \alpha$, $\alpha \in [0,1)$. For $\alpha \in [0,1)$, a function $f \in \mathcal{A}$ is convex of order α if and only if $Re\left(1 + zf''(z)/f'(z)\right) > \alpha$ in $\dot{\mathcal{U}}$. This class of functions is dented by $\mathcal{C}(\alpha)$. It is clear that $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{C}(0) = \mathcal{C}$ are the usual classes of starlike and convex functions respectively. A function f in $\mathcal{A}$ is said to be close-to-convex function in $\dot{\mathcal{U}}$, if $f(\mathcal{U})$ is close-to-convex. That is, the complement of $f(\mathcal{U})$ can be expressed as the union of non-intersecting half-lines. In other words a function f in $\mathcal{A}$ is said to be close-to-convex if and only if $Re\left(zf'(z)/g(z)\right) > 0$ for some starlike function g. In particular if $g(z) = z$, then $Re\left(f'(z)\right) > 0$. The class of close-to-convex functions is denoted by $\mathcal{K}$. The functions in class $\mathcal{K}$ are univalent in $\mathcal{U}$. For some details about these classes of functions one can refer to [1]. Consider the class $\mathcal{P}_{\delta}(\alpha)$ of functions p such that $p(0) = 1$ and

$$Re\left\{e^{i\delta}p(z)\right\} > \alpha, \ \ z \in \mathcal{U}, \alpha \in [0,1), \ \delta \in \mathbb{R}.$$

Also consider the class $\mathcal{R}_{\delta}(\alpha)$ of functions $f \in \mathcal{A}$ such that

$$Re\left\{e^{i\delta}f'(z)\right\} > \alpha, \ \ z \in \mathcal{U}, \alpha \in [0,1), \ \delta \in \mathbb{R}.$$

These classes were introduced and investigated by Baricz [2]. For $\delta = 0$, we have the classes $\mathcal{P}_0(\alpha)$ and $\mathcal{R}_0(\alpha)$. Also for $\delta = 0$ and $\alpha = 0$, we have the classes $\mathcal{P}$ and $\mathcal{R}$.

Special functions have great importance in pure and applied mathematics. The wide use of these functions have attracted many researchers to work on the different directions. Geometric properties of special functions such as Hypergeometric functions, Bessel functions, Struve functions, Mittag-Lefller functions, Wright functions and some other related functions is an ongoing part of research in geometric function theory. We refer for some geometric properties of these functions [2–6] and references therein.

We consider the hyper-Bessel function in the form of the hypergeometric functions defined as

$$J_{\gamma_c}(z) = \frac{\left(\frac{z}{c+1}\right)^{\gamma_1+\gamma_2+\ldots+\gamma_c}}{\prod\limits_{i=1}^{c}\Gamma(\gamma_i+1)}\ {}_0F_c\left(\overline{(\gamma_c+1)}; -\left(\frac{z}{c+1}\right)^{c+1}\right), \tag{2}$$

where the notation

$${}_mF_n\left(\begin{pmatrix}(\eta)_m\\(\delta)_n\end{pmatrix}; x\right) = \sum_{k=0}^{\infty}\frac{(\eta_1)_k(\eta_2)_k\ldots(\eta_m)_k}{(\delta_1)_k(\delta_2)_k\ldots(\delta_n)_k}\frac{x^k}{k!}, \tag{3}$$

represents the generalized Hypergeometric functions and γ_c represents the array of c parameters $\gamma_1, \gamma_2, \ldots, \gamma_c$. By combining Equations (2) and (3), we get the following infinite representation of the hyper-Bessel functions

$$J_{\gamma_c}(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\prod\limits_{i=1}^{c}\Gamma(\gamma_i+n+1)}\left(\frac{z}{c+1}\right)^{n(c+1)+\gamma_1+\gamma_2+\ldots+\gamma_c}, \tag{4}$$

since J_{γ_c} is not in class $\mathcal{A}$. Therefore, consider the hyper-Bessel function $\mathcal{J}_{\gamma_c}$ which is defined by

$$\mathcal{J}_{\gamma_c}(z) = 1 + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(n-1)!(c+1)^{(n-1)(c+1)}\prod\limits_{i=1}^{c}(\alpha_i+1)_{n-1}}z^{(n-1)(c+1)}. \tag{5}$$

It is observed that the function $\mathcal{J}_{\gamma_c}$ defined in (5) is not in the class $\mathcal{A}$. Here, we consider the following normalized form of the hyper-Bessel function for our own convenience.

$$\mathcal{H}_{\gamma_c}(z) = z\mathcal{J}_{\gamma_c}(z) = z + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(n-1)!(c+1)^{(n-1)(c+1)}\prod\limits_{i=1}^{c}(\gamma_i+1)_{n-1}}z^{(n-1)(c+1)+1}. \tag{6}$$

For some details about the hyper-Bessel functions one can refer to [7–9]. Recently Aktas et al. [8] studied some geometric properties of hyper-Bessel function. In particular, they studied radii of starlikeness, convexity, and uniform convexity of hyper-Bessel functions. Motivated by the above works, we study the geometric properties of hyper-Bessel function $\mathcal{H}_{\gamma_c}$ given by the power series (6). We determine the conditions on parameters that ensure the hyper-Bessel function to be starlike of order α, convex of order α, close-to-convex of order $(\frac{1+\alpha}{2})$. We also study the convexity and starlikeness in the domain $\mathcal{U}_{1/2} = \left\{z : |z| < \frac{1}{2}\right\}$. Sufficient conditions on univalency of an integral operator defined by hyper-Bessel function is also studied. We find the conditions on normalized hyper-Bessel function to belong to the Hardy space $\mathcal{H}^p$.

To prove our results, we require the following.

Lemma 1. *If $f \in \mathcal{A}$ satisfy $|f'(z) - 1| < 1$ for each $z \in \mathcal{U}$, then f is convex in $\mathcal{U}_{1/2} = \left\{z : |z| < \frac{1}{2}\right\}$* [10].

Lemma 2. *If $f \in \mathcal{A}$ satisfy $\left|\frac{f(z)}{z} - 1\right| < 1$ for each $z \in \mathcal{U}$, then f is starlike in $\mathcal{U}_{1/2} = \left\{z : |z| < \frac{1}{2}\right\}$* [11].

Lemma 3. *Let $\beta \in \mathbb{C}$ with $Re(\beta) > 0$, $c \in \mathbb{C}$ with $|c| \le 1$, $c \ne -1$ [12]. If $h \in \mathcal{A}$ satisfies*

$$\left| c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\,\frac{zh''(z)}{\beta h'(z)}\right)\right| \le 1, \quad z \in \mathcal{U},$$

then the integral operator

$$C_\beta(z) = \left\{\beta \int_0^z t^{\beta-1} h'(t)\,dt\right\}^{1/\beta}, \quad z \in \mathcal{U},$$

is analytic and univalent in $\mathcal{U}$.

Lemma 4. *If $f \in \mathcal{A}$ [13] satisfies the inequality*

$$|zf''(z)| < \frac{1-\alpha}{4}, \quad (z \in \mathcal{U}, 0 \le \alpha < 1),$$

then

$$Re f'(z) > \frac{1+\alpha}{4}, \quad (z \in \mathcal{U}, 0 \le \alpha < 1).$$

Lemma 5. *If $f \in \mathcal{A}$ satisfies $\left|\frac{zf''(z)}{f'(z)}\right| < \frac{1}{2}$[14], then $f \in \mathcal{UCV}$.*

2. Geometric Properties of Normalized Hyper-Bessel Function

Theorem 1. *Let $i \in \{1, 2, 3, ..., c\}$, $\gamma_i > -1$ with $\alpha \in [0, 1)$ and $z \in \mathcal{U}$. Then the following results are true:*

(i) If $\prod_{i=1}^{c} (\gamma_i + 1) > \frac{(2c+7-5\alpha)+\sqrt{(2c+7-5\alpha)^2 - 8(1-\alpha)(c+4-3\alpha)}}{4\varsigma(1-\alpha)}$, then $\mathcal{H}_{\gamma_c} \in \mathcal{S}^*(\alpha)$.

(ii) If $\prod_{i=1}^{c} (\gamma_i + 1) > \frac{\left\{(2c+7)(1-\alpha) - \left(2c^2+5c+3\right)\right\} + \Psi}{2\varsigma\left\{2(1-\alpha) - (4c^2+10c+6)\right\}}$, where

$$\Psi = \sqrt{\left\{(2c+7)(1-\alpha) - (2c^2+5c+3)\right\}^2 - 4\left\{2(1-\alpha) - (4c^2+10c+6)\right\}(c+4)(1-\alpha)},$$

then $\mathcal{H}_{\gamma_c} \in \mathcal{C}(\alpha)$.

(iii) If $\prod_{i=1}^{c} (\gamma_i + 1) > \frac{1-\alpha}{\varsigma(1-\alpha) - 4(c+1)(2c+3)}$, then $\mathcal{H}_{\gamma_c} \in \mathcal{K}(\frac{1+\alpha}{2})$.

(iv) If $\prod_{i=1}^{c} (\gamma_i + 1) > \frac{3-\alpha}{2\varsigma(1-\alpha)}$, then $\frac{\mathcal{H}_{\gamma_c}}{z} \in \mathcal{P}(\alpha)$.

Proof. (i) By using the inequalities

$$n! \geq n, \ \ (\gamma_i+1)_n \geq (\gamma_i+1)^n, \ \forall\, n \in \mathbb{N},$$

we obtain

$$\left|\mathcal{H}'_{\gamma_c}(z) - \frac{\mathcal{H}_{\gamma_c}(z)}{z}\right| \leq \frac{(c+1)}{\zeta\eta} \sum_{n\geq 1} \left(\frac{1}{\zeta\eta}\right)^{n-1},$$

where

$$\zeta = (c+1)^{c+1} \text{ and } \eta = \prod_{i=1}^{c} (\gamma_i + 1).$$

This implies that

$$\left|\mathcal{H}'_{\gamma_c}(z) - \frac{\mathcal{H}_{\gamma_c}(z)}{z}\right| \leq \frac{c+1}{\zeta\eta - 1}. \tag{7}$$

Furthermore, if we use the inequality

$$n! \geq 2^{n-1}, \ (\gamma_i+1)_n \geq (\gamma_i+1)^n, \ \forall\, n \in \mathbb{N},$$

then

$$\begin{aligned} \left|\frac{\mathcal{H}_{\gamma_c}(z)}{z}\right| &= \left|1 + \sum_{n\geq 1} \frac{(-1)^{n-1}}{(n-1)!(c+1)^{(n-1)(c+1)} \prod_{i=1}^{c} (\gamma_i+1)_{n-1}} z^{(n-1)(c+1)}\right| \\ &\geq 1 - \frac{1}{\zeta\eta} \sum_{n\geq 1} \left(\frac{1}{2\zeta\eta}\right)^{n-1} \\ &= \frac{2\zeta\eta - 3}{2\zeta\eta - 1}. \end{aligned} \tag{8}$$

By combining Equations (7) and (8), we obtain

$$\left|\frac{z\mathcal{H}'_{\gamma_c}(z)}{\mathcal{H}_{\gamma_c}(z)} - 1\right| \leq \frac{(c+1)(2\zeta\eta - 1)}{(\zeta\eta - 1)(2\zeta\eta - 3)}. \tag{9}$$

For $\mathcal{H}_{\gamma_c} \in \mathcal{S}^*(\alpha)$, we must have

$$\frac{(c+1)(2\zeta\eta - 1)}{(\zeta\eta - 1)(2\zeta\eta - 3)} < 1 - \alpha.$$

So, $\mathcal{H}_{\gamma_c} \in \mathcal{S}^*(\alpha)$, where $0 \leq \alpha < 1 - \frac{(c+1)(2\zeta\eta-1)}{(\zeta\eta-1)(2\zeta\eta-3)}$.

(ii) To prove that the function $\mathcal{H}_{\gamma_c} \in \mathcal{C}(\alpha)$, we have to show that $\left|\frac{z\mathcal{H}''_{\gamma_c}(z)}{\mathcal{H}'_{\gamma_c}(z)}\right| < 1-\alpha$. Consider

$$\begin{aligned}
\mathcal{H}_{\gamma_c}(z) &= \sum_{n\geq 0} \frac{(-1)^n}{n!(c+1)^{n(c+1)} \prod\limits_{i=1}^{c} (\gamma_i+1)_n} z^{n(c+1)+1}, \\
z\mathcal{H}''_{\gamma_c}(z) &= \sum_{n\geq 0} \frac{\left\{n^2(c+1)^2 + n(c+1)\right\}(-1)^n}{n!(c+1)^{n(c+1)} \prod\limits_{i=1}^{c} (\gamma_i+1)_n} z^{n(c+1)}, \\
&= \sum_{n\geq 1} \frac{(n-1)^2 (c+1)^2}{(n-1)!(c+1)^{(n-1)(c+1)} \prod\limits_{i=1}^{c} (\gamma_i+1)_{n-1}} z^{n(c+1)} \\
&\quad + \sum_{n\geq 1} \frac{(n-1)(c+1)}{(n-1)!(c+1)^{(n-1)(c+1)} \prod\limits_{i=1}^{c} (\gamma_i+1)_{n-1}} z^{n(c+1)}.
\end{aligned}$$

By using the inequalities

$$(n-1)! \geq \frac{(n-1)^2}{2},\ (n-1)! \geq n-1,\ (\alpha_i+1)_n \geq (\alpha_i+1)^n,\ \forall\, n \in \mathbb{N},$$

we have

$$\begin{aligned}
\left|z\mathcal{H}''_{\alpha_c}(z)\right| &= \left| \begin{array}{l} \sum\limits_{n\geq 1} \frac{(n-1)^2(c+1)^2}{(n-1)!(c+1)^{(n-1)(c+1)} \prod\limits_{i=1}^{c} (\alpha_i+1)_{n-1}} z^{n(c+1)} \\ + \sum\limits_{n\geq 1} \frac{(n-1)(c+1)}{(n-1)!(c+1)^{(n-1)(c+1)} \prod\limits_{i=1}^{c} (\alpha_i+1)_{n-1}} z^{n(c+1)} \end{array} \right| \\
&\leq 2(c+1)^2 \sum_{n\geq 1} \left(\frac{1}{\zeta\eta}\right)^{n-1} + (c+1) \sum_{n\geq 1} \left(\frac{1}{\zeta\eta}\right)^{n-1},
\end{aligned}$$

where

$$\zeta = (c+1)^{c+1} \text{ and } \eta = \prod_{i=1}^{c} (\gamma_i+1).$$

This implies that

$$\left|z\mathcal{H}''_{\gamma_c}(z)\right| \leq \frac{\zeta\eta\,(c+1)\,(2c+3)}{(\zeta\eta-1)} \tag{10}$$

Furthermore, if we use the inequalities

$$n! \geq n,\ n! \geq 2^{n-1},\ (\gamma_i+1)_n \geq (\gamma_i+1)^n,\ \forall\, n \in \mathbb{N},$$

then we get

$$\begin{aligned}\left|\mathcal{H}'_{\gamma_c}(z)\right| &\geq 1-\sum_{n\geq 1}\frac{n(c+1)+1}{n!\zeta^n\eta^n}\\ &\geq 1-\frac{c+1}{\zeta\eta}\sum_{n\geq 1}\left(\frac{1}{\zeta\eta}\right)^{n-1}+\frac{1}{\zeta\eta}\sum_{n\geq 1}\left(\frac{1}{2\zeta\eta}\right)^{n-1}\\ &= \frac{(\zeta\eta-1)(2\zeta\eta-3)-(2\zeta\eta-1)(c+1)}{(\zeta\eta-1)(2\zeta\eta-1)}.\end{aligned} \tag{11}$$

By combining Equations (10) and (11) , we get

$$\left|\frac{z\mathcal{H}''_{\gamma_c}(z)}{\mathcal{H}'_{\gamma_c}(z)}\right| \leq \frac{\zeta\eta(2\zeta\eta-1)(c+1)(2c+3)}{(\zeta\eta-1)(2\zeta\eta-3)-(2\zeta\eta-1)(c+1)} < 1-\alpha.$$

This implies that $\mathcal{H}_{\gamma_c} \in \mathcal{C}(\alpha)$, where $0 \leq \alpha < 1 - \frac{\zeta\eta(2\zeta\eta-1)(c+1)(2c+3)}{(\zeta\eta-1)(2\zeta\eta-3)-(2\zeta\eta-1)(c+1)}$.

(iii) Using the inequality (10) and Lemma 4, we have

$$\left|z\mathcal{H}''_{\gamma_c}(z)\right| \leq \frac{\zeta\eta(c+1)(2c+3)}{(\zeta\eta-1)} < \frac{1-\alpha}{4},$$

where $0 \leq \alpha < 1-4\frac{\zeta\eta(c+1)(2c+3)}{(\zeta\eta-1)}$. This shows that $\mathcal{H}_{\gamma_c} \in \mathcal{K}(\frac{1+\alpha}{2})$. Therefore $Re\left(\mathcal{H}'_{\gamma_c}(z)\right) > \frac{1+\alpha}{2}$.

(iv) To prove that $\frac{\mathcal{H}_{\gamma_c}}{z} \in \mathcal{P}(\alpha)$, we have to show that $|h(z)-1| < 1$, where $h(z) = \frac{\mathcal{H}_{\gamma_c}(z)/z-\alpha}{1-\alpha}$. By using the inequality

$$2^{n-1} \leq n!, n \in \mathbb{N},$$

we have

$$\begin{aligned}|h(z)-1| &= \left|\frac{1}{1-\alpha}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!\,(c+1)^{n(c+1)}\prod_{i=1}^{c}(\gamma_i+1)_n}z^{n(c+1)+1}\right|\\ &\leq \frac{1}{(1-\alpha)}\frac{1}{\varsigma\eta}\sum_{n=1}^{\infty}\left(\frac{1}{2\varsigma\eta}\right)^{n-1}\\ &= \frac{1}{(1-\alpha)}\frac{2}{2\varsigma\eta-1}.\end{aligned}$$

Therefore, $\frac{\mathcal{H}_{\gamma_c}}{z} \in \mathcal{P}(\alpha)$ for $0 < \alpha < 1 - \frac{2}{2(c+1)^{n(c+1)}\prod_{i=1}^{c}(\gamma_i+1)_n - 1}$. □

Putting $\alpha = 0$ in Theorem 1, we have the following results.

Corollary 1. *Let $i \in \{1,2,3,...,c\}, \gamma_i > -1$ and $z \in \mathcal{U}$. Then the followings are true:*

(i) *If* $\prod_{i=1}^{c}(\gamma_i+1) >, \frac{(2c+7)+\sqrt{(2c+7)^2-8(c+4)}}{4\varsigma}$, *then* $\mathcal{H}_{\gamma_c} \in \mathcal{S}^*$.

(ii) *If* $\prod_{i=1}^{c}(\gamma_i+1) > \frac{\left\{(2c+7)-\left(2c^2+5c+3\right)\right\}+\sqrt{\left\{(2c+7)-(2c^2+5c+3)\right\}^2-4(c+4)\left\{2-(4c^2+10c+6)\right\}}}{2\varsigma\left\{2-(4c^2+10c+6)\right\}}$, *then* $\mathcal{H}_{\gamma_c} \in \mathcal{C}$.

(iii) *If* $\prod_{i=1}^{c}(\gamma_i+1) > \frac{1}{\varsigma\{1-4(c+1)(2c+3)\}}$, *then* $\mathcal{H}_{\gamma_c} \in \mathcal{K}(\frac{1}{2})$.

(iv) *If* $\prod_{i=1}^{c}(\gamma_i+1) > \frac{3}{2\varsigma}$, *then* $\frac{\mathcal{H}_{\gamma_c}}{z} \in \mathcal{P}$.

3. Starlikeness and Convexity in $\mathcal{U}_{1/2}$

Theorem 2. *Let* $i \in \{1,2,3,...,c\}$, $\gamma_i > -1$ *and* $z \in \mathcal{U}$. *Then the following assertions are true:*

(i) *If* $\prod_{i=1}^{c}(\gamma_i+1) > \frac{3}{2\varsigma}$, *then* $\mathcal{H}_{\gamma_c}$ *is starlike in* $\mathcal{U}_{1/2}$.

(ii) *If* $\prod_{i=1}^{c}(\gamma_i+1) > \frac{(c+3)+\sqrt{c^2+4c+2}}{2\varsigma}$, *then* $\mathcal{H}_{\gamma_c}$ *is convex in* $\mathcal{U}_{1/2}$.

Proof. (i) By using the inequality $2^{n-1} \leq n!$, $n \in \mathbb{N}$, we obtain

$$\begin{aligned}\left|\frac{\mathcal{H}_{\gamma_c}(z)}{z}-1\right| &\leq \left|\sum_{n=1}^{\infty}\frac{(-1)^n}{n!\,(c+1)^{n(c+1)}\prod_{i=1}^{c}(\gamma_i+1)_n}z^{n(c+1)}\right| \\ &\leq \sum_{n=1}^{\infty}\left(\frac{1}{n!\left\{(c+1)^{(c+1)}\right\}^n\left\{\prod_{i=1}^{c}(\gamma_i+1)\right\}^n}\right) \\ &\leq \frac{1}{\varsigma\eta}\sum_{n=1}^{\infty}\left(\frac{1}{2\varsigma\eta}\right)^{n-1} = \frac{2}{2\varsigma\eta-1}.\end{aligned}$$

In view of Lemma 2, $\mathcal{H}_{\gamma_c}$ is starlike in $\mathcal{U}_{1/2}$, if $\frac{2}{2\varsigma\eta-1} < 1$, which is true under the given hypothesis. (ii) Consider,

$$\begin{aligned}\left|\mathcal{H}'_{\gamma_c}(z)-1\right| &\leq \sum_{n=1}^{\infty}\frac{n(c+1)+1}{n!\,(c+1)^{n(c+1)}\prod_{i=1}^{c}(\gamma_i+1)_n} \\ &\leq \sum_{n=1}^{\infty}\frac{n(c+1)}{n!\zeta^n\eta^n}+\sum_{n=1}^{\infty}\frac{1}{n!\zeta^n\eta^n}.\end{aligned}$$

Since, $n! \geq n$, for all $n \in \mathbb{N}$ and $n! \geq 2^{n-1}$, for all $n \in \mathbb{N}$. Therefore

$$\begin{aligned}\left|\mathcal{H}'_{\gamma_c}(z)-1\right| &\leq \frac{c+1}{\zeta\eta}\sum_{n=1}^{\infty}\left(\frac{1}{\zeta\eta}\right)^{n-1}+\frac{1}{\zeta\eta}\sum_{n=1}^{\infty}\left(\frac{1}{2\zeta\eta}\right)^{n-1} \\ &= \frac{(c+1)(2\zeta\eta-1)+2\,(\zeta\eta-1)}{(\zeta\eta-1)\,(2\zeta\eta-1)}.\end{aligned}$$

In view of Lemma 1, $\mathcal{H}_{\gamma_c}$ is convex in $\mathcal{U}_{1/2}$, if $\frac{(c+1)(2\zeta\eta-1)+2(\zeta\eta-1)}{(\zeta\eta-1)(2\zeta\eta-1)} < 1$, but this is true under the hypothesis. □

Consider the integral operator $\mathcal{F}_\beta : \mathcal{U} \to \mathbb{C}$, where $\beta \in \mathbb{C}$, $\beta \neq 0$,

$$\mathcal{F}_\beta(z) = \left\{ \beta \int_0^z t^{\beta-2} \mathcal{H}_{\gamma_c}(t)\, d(t) \right\}^{\frac{1}{\beta}}, \qquad z \in \mathcal{U}.$$

Here $\mathcal{F}_\beta \in \mathcal{A}$. In the next theorem, we obtain the conditions so that $\mathcal{F}_\beta$ is univalent in $\mathcal{U}$.

Theorem 3. *Let $i \in \{1, 2, 3, ..., c\}$, $\gamma_i > -1$ and $z \in \mathcal{U}$. Let $\prod_{i=1}^{c} (\gamma_i + 1) > \frac{(2c+7)+\sqrt{(7+2c)^2-8(2c+3)}}{4\varsigma}$ and suppose that $M \in \mathbb{R}^+$ such that $|\mathcal{H}_{\gamma_c}(z)| \leq M$ in the open unit disc. If*

$$|\beta - 1| + \frac{(c+1)(2\zeta\eta - 1)}{(\zeta\eta - 1)(2\zeta\eta - 3)} + \frac{M}{|\beta|} \leq 1,$$

then $\mathcal{F}_\beta$ is univalent in $\mathcal{U}$.

Proof. A calculations gives us

$$\frac{z\mathcal{F}_\beta''(z)}{\mathcal{F}_\beta'(z)} = \frac{z\mathcal{H}_{\gamma_c}'(z)}{\mathcal{H}_{\gamma_c}(z)} + \frac{z^{\beta-1}}{\beta}\mathcal{H}_{\gamma_c}(z) + \beta - 2,\ z \in \mathcal{U}.$$

Since $\mathcal{H}_{\gamma_c} \in \mathcal{A}$, then by the Schwarz Lemma, triangle inequality and Equation (9) , we obtain

$$\begin{aligned} \left(1 - |z|^2\right) \left| \frac{z\mathcal{F}_\beta''(z)}{\mathcal{F}_\beta'(z)} \right| &\leq \left(1 - |z|^2\right) \left\{ |\beta - 1| + \left| \frac{z\mathcal{H}_{\gamma_c}'(z)}{\mathcal{H}_{\gamma_c}(z)} - 1 \right| + \frac{|z|^{\mathcal{R}(\beta)}}{|\beta|} \left| \frac{\mathcal{H}_{\gamma_c}(z)}{z} \right| \right\} \\ &\leq \left(1 - |z|^2\right) \left\{ |\beta - 1| + \frac{(c+1)(2\zeta\eta - 1)}{(\zeta\eta - 1)(2\zeta\eta - 3)} + \frac{M}{|\beta|} \right\} \\ &\leq 1. \end{aligned}$$

This shows that the given integral operator satisfying the Becker's criterion for univalence [12], hence $\mathcal{F}_\beta$ is univalent in $\mathcal{U}$. □

4. Uniformly Convexity of Hyper-Bessel Functions

Theorem 4. *If $i \in \{1, 2, 3, ..., c\}$, $\gamma_i > -1$ and $z \in \mathcal{U}$. If $\prod_{i=1}^{c} (\gamma_i + 1) > \frac{(4c^2+8c-1)+\sqrt{(4c^2+8c-1)^2-4(c+4)(8c^2+20c+10)}}{2(8c^2+20c+10)\varsigma}$, then $\mathcal{H}_{\gamma_c} \in \mathcal{UCV}$.*

Proof. Since

$$\left| \frac{z\mathcal{H}_{\gamma_c}''(z)}{\mathcal{H}_{\gamma_c}'(z)} \right| \leq \frac{\zeta\eta(2\zeta\eta - 1)(c+1)(2c+3)}{(\zeta\eta - 1)(2\zeta\eta - 3) - (2\zeta\eta - 1)(c+1)}.$$

By using Lemma 5, we have

$$\left| \frac{z\mathcal{H}_{\gamma_c}''(z)}{\mathcal{H}_{\gamma_c}'(z)} \right| < \frac{1}{2},$$

if

$$\frac{\zeta\eta(2\zeta\eta - 1)(c+1)(2c+3)}{(\zeta\eta - 1)(2\zeta\eta - 3) - (2\zeta\eta - 1)(c+1)} < \frac{1}{2}.$$

This implies that

$$\prod_{i=1}^{c}(\gamma_i+1) > \frac{(4c^2+8c-1)+\sqrt{(4c^2+8c-1)^2-4(c+4)(8c^2+20c+10)}}{2(8c^2+20c+10)\varsigma}.$$

Hence, we obtain the required result. □

5. Hardy Spaces Of Hyper-Bessel Functions

Let $\mathcal{H}^\infty$ denote the space of all bounded functions on $\mathcal{H}$. Let $f \in \mathcal{H}$, set

$$M_p(r,f) = \begin{cases} \left(\frac{1}{2\pi}\int\limits_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta\right)^{1/p}, & 0 < p < \infty, \\ \sup\{|f(z)| : |z| \le r\}, & p = \infty. \end{cases}$$

Then the function $f \in \mathcal{H}^p$ if $M_p(r,f)$ is bounded for all $r \in [0,1)$. It is clear that

$$\mathcal{H}^\infty \subset \mathcal{H}^q \subset \mathcal{H}^p,\ 0 < q < p < \infty.$$

For some details, see [15] (page 2) . It is also known [16] (page 64, Section 4.5) (see also [15]) that for $Re\left(f'(z)\right) > 0$ in $\mathcal{U}$, then

$$\begin{cases} f' \in \mathcal{H}^q, & q < 1, \\ f \in \mathcal{H}^{q/(1-q)}, & 0 < q < 1. \end{cases}$$

We require the following results to prove our results.

Lemma 6. *$\mathcal{P}_0(\alpha) * P_0(\beta) \subset P_0(\gamma)$, where $\gamma = 1-2(1-\alpha)(1-\beta)$ with $\alpha, \beta < 1$ and the value of γ is best possible [17].*

Lemma 7. *For $\alpha, \beta < 1$ and $\gamma = 1-2(1-\alpha)(1-\beta)$, we have $\mathcal{R}_0(\alpha) * R_0(\beta) \subset R_0(\gamma)$ or equivalently $\mathcal{P}_0(\alpha) * P_0(\beta) \subset P_0(\gamma)$ [18].*

Lemma 8. *If the function $f \in \mathcal{C}(\alpha)$[19] , where $\alpha \in [0,1)$ is not of the form*

$$f(z) = \begin{cases} \theta + \eta(1-ze^{i\gamma})^{2\alpha-1}, & \alpha \neq \frac{1}{2}, \\ \theta + \eta \log(1-ze^{i\gamma}), & \alpha \neq \frac{1}{2}, \end{cases}$$

for $\zeta, \eta \in \mathbb{C}$ and $\gamma \in \mathbb{R}$, then the following statements hold:

(i) There exists $\delta = \delta(f) > 0$, such that $f' \in \mathcal{H}^{\delta + \frac{1}{2(1-\alpha)}}$.
(ii) If $\alpha \in [1, 1/2)$, then there exists $\tau = \tau(f) > 0$, such that $f \in \mathcal{H}^{\tau + 1/(1-2\alpha)}$.
(iii) If $\alpha \ge 1/2$, then $f \in \mathcal{H}^\infty$.

Theorem 5. *Let $i \in \{1,2,3,...,c\}$, $\gamma_i > -1$ with $\alpha \in [0,1)$ and $z \in \mathcal{U}$. Let*

$$\prod_{i=1}^{c}(\gamma_i+1) > \frac{\{(2c+7)(1-\alpha) - (2c^2+5c+3)\} + \Phi}{2\varsigma\{2(1-\alpha) - (4c^2+10c+6)\}},$$

where

$$\Phi = \sqrt{\{(2c+7)(1-\alpha) - (2c^2+5c+3)\}^2 - 4\{2(1-\alpha) - (4c^2+10c+6)\}(c+4)(1-\alpha)}.$$

Then

(i) $\mathcal{H}_{\gamma_c} \in \mathcal{H}^{1/1-2\alpha}$ *for* $\alpha \in [0, 1/2)$.
(ii) $\mathcal{H}_{\gamma_c} \in \mathcal{H}^{\infty}$ *for* $\alpha \geq 1/2$.

Proof. By using the definition of Hypergeometric function

$$ {}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

we have

$$\begin{aligned} \theta + \frac{\vartheta z}{(1 - ze^{i\psi})^{1-\alpha}} &= \theta + \vartheta z \, {}_2F_1(1, 1-2\alpha, 1; ze^{i\psi}) \\ &= \theta + \vartheta \sum_{n=0}^{\infty} \frac{(1-2\alpha)_n}{n!} e^{i\psi n} z^{n+1}, \end{aligned}$$

for $\theta, \vartheta \in \mathbb{C}$, $\alpha \neq 1/2$ and for $\psi \in \mathbb{R}$. On the other hand

$$\begin{aligned} \theta + \vartheta \log(1 - ze^{i\psi}) &= \theta - \vartheta z \, {}_2F_1(1,1,2; ze^{i\psi}) \\ &= \theta - \vartheta \sum_{n=0}^{\infty} \frac{1}{n+1} e^{i\psi n} z^{n+1}. \end{aligned}$$

This implies that $\mathcal{H}_{\gamma_c}$ is not of the form $\theta + \vartheta z(1 - ze^{i\psi})^{2\alpha-1}$ for $\alpha \neq 1/2$ and $\theta + \vartheta \log(1 - ze^{i\psi})$ for $\alpha = 1/2$ respectively. Also from part (ii) of Theorem 1, $\mathcal{H}_{\gamma_c}$ is convex of order α. Hence by using Lemma 8, we have required result. □

Theorem 6. *Let* $i \in \{1, 2, 3, ..., c\}, \gamma_i > -1$ *with* $\alpha \in [0, 1)$ *and* $z \in \mathcal{U}$. *If* $\prod_{i=1}^{c} (\gamma_i + 1) > \frac{3-\alpha}{2\varsigma(1-\alpha)}$, *then* $\frac{\mathcal{H}_{\gamma_c}}{z} \in \mathcal{P}(\alpha)$. *If* $f \in \mathcal{R}(\varrho)$, *with* $\varrho < 1$, *then* $\mathcal{H}_{\gamma_c} * f \in \mathcal{R}(\tau)$, *where* $\tau = 1 - 2(1-\alpha)(1-\varrho)$.

Proof. Let $h(z) = \mathcal{H}_{\gamma_c}(z) * f(z)$, then $h'(z) = \frac{\mathcal{H}_{\gamma_c}(z)}{z} * f'(z)$. Now from Theorem 1 of part (iv), we have $\frac{\mathcal{H}_{\gamma_c}}{z} \in \mathcal{P}(\alpha)$. By using Lemma 6 and the fact that $f' \in \mathcal{P}(\varrho)$, we have $h'(z) \in \mathcal{P}(\tau)$, where $\tau = 1 - 2(1-\alpha)(1-\varrho)$. Consequently, we have $h \in \mathcal{R}(\tau)$. □

Corollary 2. *Let* $i \in \{1, 2, 3, ..., c\}, \gamma_i > -1$ *with* $\alpha \in [0, 1)$ *and* $z \in \mathcal{U}$. *If* $\prod_{i=1}^{c} (\gamma_i + 1) > \frac{3-\alpha}{2\varsigma(1-\alpha)}$, *then* $\frac{\mathcal{H}_{\gamma_c}}{z} \in \mathcal{P}(\alpha)$. *If* $f \in \mathcal{R}(\varrho)$, $\varrho = (1-2\alpha)(2-2\alpha)$, *then* $\mathcal{H}_{\gamma_c} * f \in \mathcal{R}(0)$.

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