mathematics

MDPI

Article

Some Reciprocal Classes of Close-to-Convex and Quasi-Convex Analytic Functions

Shahid Mahmood [1], Janusz Sokół [2], Hari Mohan Srivastava [3,4] and Sarfraz Nawaz Malik [5,*]

1 Department of Mechanical Engineering, Sarhad University of Science & I.T, Peshawar 25000, Pakistan; shahidmahmood757@gmail.com

2 Department of Mathematics, Rzeszów University of Technology, Al. Powstańców, Warszawy 12, 35-959 Rzeszów, Poland; jsokol@prz.edu.pl

3 Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada; harimsri@math.uvic.ca

4 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

5 Department of Mathematics, COMSATS University Islamabad, Wah Campus 47040, Pakistan

* Correspondence: snmalik110@yahoo.com

Received: 18 December 2018; Accepted: 22 March 2019; Published: 27 March 2019

check for updates

Abstract: The present paper comprises the study of certain functions which are analytic and defined in terms of reciprocal function. The reciprocal classes of close-to-convex functions and quasi-convex functions are defined and studied. Various interesting properties, such as sufficiency criteria, coefficient estimates, distortion results, and a few others, are investigated for these newly defined sub-classes.

Keywords: subordination; functions with positive real part; reciprocals

MSC: 30C45; 30C50

1. Introduction

We denote by $\mathcal{A}$ the class of analytic functions on the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ having the following taylor series representation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

The analytic function f will be subordinate to an analytic function g, if there exists an analytic function w, known as a Schwarz function, with $w(0) = 0$ and $|w(z)| < |z|$, such that $f(z) = g(w(z))$. Moreover, if the function g is univalent in $\mathbb{U}$, then we have the following (see [1,2]):

$$f(z) \prec g(z), \quad z \in \mathbb{U} \quad \Longleftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Uralegaddi et al. [3] introduced the reciprocal classes $\mathcal{M}(\gamma)$ of starlike and $\mathcal{N}(\gamma)$ of convex functions for $1 \leq \gamma \leq \frac{4}{3}$, which were further studied by Owa et al. [4–6] for the values $\gamma \geq 1$.

The classes $\mathcal{M}(\gamma)$ of starlike functions and $\mathcal{N}(\gamma)$ of reciprocal order convex functions γ, $(\gamma > 1)$ are defined as follows:

$$\mathcal{M}(\gamma) = \left\{ f \in \mathcal{A} : \mathfrak{Re}\frac{zf'(z)}{f(z)} < \gamma,\ z \in \mathbb{U} \right\},$$

$$\mathcal{N}(\gamma) = \left\{ f \in \mathcal{A} : \mathfrak{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \gamma,\ z \in \mathbb{U} \right\}.$$

Using the same concept, together with the idea of k-uniformly starlike and γ ordered convex functions, Nishiwaki and Owa [7] defined the reciprocal classes of uniformly starlike $\mathcal{MD}(k,\gamma)$ and convex functions $\mathcal{ND}(k,\gamma)$. The class $\mathcal{MD}(k,\gamma)$ denotes the subclass of $\mathcal{A}$ consisting of functions f satisfying the inequality

$$\mathfrak{Re}\frac{zf'(z)}{f(z)} < k\left|\frac{zf'(z)}{f(z)} - 1\right| + \gamma,\ (z \in \mathbb{U}),$$

for some $\gamma\ (\gamma > 1)$ and $k\ (k \leq 0)$ and the class $\mathcal{ND}(k,\gamma)$ denotes the subclass of $\mathcal{A}$ consisting of functions $f(z)$ satisfying the inequality

$$\mathfrak{Re}\frac{(zf'(z))'}{f'(z)} < \gamma + k\left|\frac{(zf'(z))'}{f'(z)} - 1\right|,\ (z \in \mathbb{U}),$$

for some $\gamma\ (\gamma > 1)$ and $k\ (k \leq 0)$. They also proved that the well-known Alexander relation holds between $\mathcal{MD}(k,\gamma)$ and $\mathcal{ND}(k,\gamma)$. This means that

$$f \in \mathcal{ND}(k,\gamma) \quad \Leftrightarrow \quad zf' \in \mathcal{MD}(k,\gamma).$$

For a more detailed and recent study on uniformly convex and starlike functions, we refer the reader to [8–12].

Considering the above defined classes, we introduce the following classes.

Definition 1. *Let f belong to $\mathcal{A}$. Then, it will belong to the class $\mathcal{KD}(\beta,\gamma)$ if there exists $g \in \mathcal{MD}(\gamma)$ such that*

$$\mathfrak{Re}\left\{\frac{zf'(z)}{g(z)}\right\} < \beta, \quad (z \in \mathbb{U}), \tag{2}$$

for some $\beta, \gamma > 1$.

Definition 2. *Let f belong to $\mathcal{A}$. Then, it will belong to the class $\mathcal{QD}(\beta,\gamma)$ if there exists $g \in \mathcal{ND}(\gamma)$ such that*

$$\mathfrak{Re}\left\{\frac{(zf'(z))'}{g'(z)}\right\} < \beta, \quad (z \in \mathbb{U}), \tag{3}$$

for some $\beta, \gamma > 1$.

It is clear, from (2) and (3), that

$$f(z) \in \mathcal{QD}(\beta,\gamma) \quad \Leftrightarrow \quad zf'(z) \in \mathcal{KD}(\beta,\gamma).$$

Definition 3. *Let f belong to $\mathcal{A}$. Then, it will belong to the class $\mathcal{KD}(k,\beta,\gamma)$ if there exists $g \in \mathcal{MD}(k,\gamma)$ such that*

$$\mathfrak{Re}\left\{\frac{zf'(z)}{g(z)}\right\} < k\left|\frac{zf'(z)}{g(z)} - 1\right| + \beta, \quad (z \in \mathbb{U}), \tag{4}$$

for some $k \leq 0$ and $\beta, \gamma > 1$.

Definition 4. *Let f belong to $\mathcal{A}$. Then, it is said to be in the class $\mathcal{QD}(k,\beta,\gamma)$ if there exists $g \in \mathcal{ND}(k,\gamma)$ such that*

$$\Re\mathfrak{e}\left\{\frac{(zf'(z))'}{g'(z)}\right\} < k\left|\frac{(zf'(z))'}{g'(z)}-1\right|+\beta, \quad (z\in\mathbb{U}), \tag{5}$$

for some $k \le 0$ and $\beta,\gamma > 1$.

We can see, from (4) and (5), that the well-known relation of Alexander type holds between the classes $\mathcal{KD}(k,\beta,\gamma)$ and $\mathcal{QD}(k,\beta,\gamma)$, which means that

$$f(z) \in \mathcal{QD}(k,\beta,\gamma) \quad \Leftrightarrow \quad zf'(z) \in \mathcal{KD}(k,\beta,\gamma).$$

2. Preliminary Lemmas

Lemma 1. *For positive integers t and σ, we have*

$$\sigma\sum_{j=1}^{t}\frac{(\sigma)_{j-1}}{(j-1)!} = \frac{(\sigma)_t}{(t-1)!}, \tag{6}$$

where $(\sigma)_t$ is the Pochhammer symbol, defined by

$$(\sigma)_t = \frac{\Gamma(\sigma+t)}{\Gamma(\sigma)} = \sigma(\sigma+1)(\sigma+2)(\sigma+3)\cdots(\sigma+t-1).$$

Proof. Consider

$$\begin{aligned}
&\sigma\sum_{j=1}^{t}\frac{(\sigma)_{j-1}}{(j-1)!}\\
=\ &\sigma\left(1+\frac{\sigma}{1}+\frac{(\sigma)_2}{2!}+\frac{(\sigma)_3}{3!}+\frac{(\sigma)_4}{4!}+\cdots+\frac{(\sigma)_{t-1}}{(t-1)!}\right)\\
=\ &\sigma(1+\sigma)\left(1+\frac{\sigma}{2}+\frac{\sigma(\sigma+2)}{2\times 3}+\cdots+\frac{\sigma(\sigma+2)\cdots(\sigma+t-2)}{2\times\cdots\times(t-1)}\right)\\
=\ &\sigma(1+\sigma)\frac{(\sigma+2)}{2}\left(1+\frac{\sigma}{3}+\cdots+\frac{\sigma(\sigma+3)\cdots(\sigma+t-2)}{3\times 4\times\cdots\times(t-1)}\right)\\
=\ &\sigma(1+\sigma)\frac{(\sigma+2)}{2}\frac{(\sigma+3)}{3}\left(1+\frac{\sigma}{4}+\cdots+\frac{\sigma(\sigma+4)\cdots(\sigma+t-2)}{4\times\cdots\times(t-1)}\right)\\
=\ &\sigma(1+\sigma)\frac{(\sigma+2)}{2}\frac{(\sigma+3)}{3}\frac{(\sigma+4)}{4}\left(1+\frac{\sigma}{5}+\cdots+\frac{\sigma\cdots(\sigma+t-2)}{5\times 6\times\cdots\times(t-1)}\right)\\
=\ &\sigma(1+\sigma)\frac{(\sigma+2)}{2}\frac{(\sigma+3)}{3}\frac{(\sigma+4)}{4}\cdots\left(1+\frac{\sigma}{t-1}\right)\\
=\ &\sigma(1+\sigma)\frac{(\sigma+2)}{2}\frac{(\sigma+3)}{3}\frac{(\sigma+4)}{4}\cdots\left(\frac{\sigma+(t-1)}{t-1}\right)\\
=\ &\frac{(\sigma)_t}{(t-1)!}.
\end{aligned}$$

□

Lemma 2. *If $f(z) \in \mathcal{MD}(k,\gamma)$, then*

$$f(z) \in \mathcal{MD}\left(\frac{\gamma-k}{1-k}\right).$$

Proof. Using the definition , we write

$$\begin{aligned}\mathfrak{Re}\frac{zf'(z)}{f(z)} &< k\left|\frac{zf'(z)}{f(z)}-1\right|+\gamma \\ &\leq k\mathfrak{Re}\frac{zf'(z)}{f(z)}+\gamma-k,\end{aligned}$$

which implies that

$$(1-k)\,\mathfrak{Re}\frac{zf'(z)}{f(z)}<\gamma-k.$$

After simplification, we obtain

$$\mathfrak{Re}\frac{zf'(z)}{f(z)}<\frac{\gamma-k}{1-k},\;\;(k\leq 0,\;\gamma>1)\,.$$

As $\dfrac{\gamma-k}{1-k}>1$, we have $f(z)\in\mathcal{MD}\left(\dfrac{\gamma-k}{1-k}\right)$. With this, we obtain the required result. □

Lemma 3. *If f belongs to the class $\mathcal{MD}\,(k,\gamma)$, then*

$$|a_n|\leq\frac{(\delta_{k,\gamma})_{n-1}}{(n-1)!},\tag{7}$$

where

$$\delta_{k,\gamma}=\frac{2(\gamma-1)}{1-k}.\tag{8}$$

Proof. Let us define a function

$$p(z)=\frac{(\gamma-k)-(1-k)\left(\dfrac{zf'(z)}{f(z)}\right)}{\gamma-1},\tag{9}$$

where $p\in\mathcal{P}$, the class of Caratheodory functions (see [1]). One may write

$$\frac{zf'(z)}{f(z)}=\frac{(\gamma-k)-(\gamma-1)\,p(z)}{1-k},\tag{10}$$

or

$$zf'(z)=\left(\frac{\gamma-k}{1-k}-\frac{\gamma-1}{1-k}p(z)\right)f(z).\tag{11}$$

Let us write $p(z)$ as $p(z)=1+\sum\limits_{n=1}^{\infty}p_nz^n$ and let f have the series form, as in (1). Then, (11) can be written as

$$\sum_{n=1}^{\infty}na_nz^n=\left(\sum_{n=1}^{\infty}a_nz^n\right)\left(\frac{\gamma-k}{1-k}-\frac{\gamma-1}{1-k}\left(1+\sum_{n=1}^{\infty}p_nz^n\right)\right),\quad a_1=1$$

which reduces to

$$\begin{aligned}\sum_{n=1}^{\infty}na_nz^n &= \left(\sum_{n=1}^{\infty}a_nz^n\right)\left(1-\frac{\gamma-1}{1-k}\sum_{n=1}^{\infty}p_nz^n\right)\\ &= \sum_{n=1}^{\infty}a_nz^n-\frac{\gamma-1}{1-k}\left(\sum_{n=1}^{\infty}a_nz^n\right)\left(\sum_{n=1}^{\infty}p_nz^n\right).\end{aligned}$$

This implies that

$$\sum_{n=1}^{\infty}(n-1)\,a_n z^n = -\frac{\gamma-1}{1-k}\sum_{n=1}^{\infty}\left(\sum_{j=0}^{n-1}a_j p_{n-j}\right)z^n.$$

After comparing the n^{th} term's coefficients, appearing on both sides, combined with the fact that $a_0 = 0$, we obtain

$$a_n = \frac{-(\gamma-1)}{(n-1)(1-k)}\sum_{j=1}^{n-1}a_j p_{n-j}.$$

Now, we take the absolute value and then apply the triangle inequality to get

$$|a_n| \le \frac{\gamma-1}{(n-1)(1-k)}\sum_{j=1}^{n-1}|a_j|\,|p_{n-j}|\,.$$

Applying the coefficient estimates, such that $|p_n| \le 2\ (n \ge 1)$ for Caratheodory functions [1], we obtain

$$|a_n| \le \frac{2(\gamma-1)}{(n-1)(1-k)}\sum_{j=1}^{n-1}|a_j|\,.$$

$$|a_n| \le \frac{\delta_{k,\gamma}}{n-1}\sum_{j=1}^{n-1}|a_j|\,, \tag{12}$$

where $\delta_{k,\gamma} = \dfrac{2(\gamma-1)}{1-k}$. We prove (7) by induction on n. Thus, first for $n = 2$, we obtain the following from (12):

$$|a_2| \le \frac{\delta_{k,\gamma}}{1} = \frac{(\delta_{k,\gamma})_{2-1}}{(2-1)!}. \tag{13}$$

This proves that, for $n = 2$, (7) is true. For $n = 3$, we obtain

$$|a_3| \le \frac{\delta_{k,\gamma}}{2}(1+|a_2|) = \frac{\delta_{k,\gamma}(1+\delta_{k,\gamma})}{2} = \frac{(\delta_{k,\gamma})_{3-1}}{(3-1)!}.$$

This proves that when $n = 3$, (7) holds true. Now, we assume that for $t \le n$, (7) is true, that means

$$|a_t| \le \frac{(\delta_{k,\gamma})_{t-1}}{(t-1)!} \quad t = 1, 2, \ldots, n. \tag{14}$$

Using (12) and (14), we have

$$|a_{t+1}| \le \frac{\delta_{k,\gamma}}{t}\sum_{j=1}^{t}|a_j| \le \frac{\delta_{k,\gamma}}{t}\sum_{j=1}^{t}\frac{(\delta_{k,\gamma})_{j-1}}{(j-1)!}.$$

After applying (6), we obtain

$$|a_{t+1}| \le \frac{1}{t}\frac{(\delta_{k,\gamma})_t}{(t-1)!} = \frac{(\delta_{k,\gamma})_t}{t!}.$$

As a result of mathematical induction, it is shown that (7) is true for all $n \ge 2$. Hence, the required bound is obtained. □

Lemma 4 ([13])**.** *Let w be analytic in $\mathbb{U}$ with $w(0) = 0$. If there exists $z_0 \in \mathbb{U}$ such that*

$$\max_{|z|\le|z_0|}|w(z)| = |w(z_0)|\,,$$

then

$$z_0 w'(z_0) = c w(z_0),$$

where c is real and $c \geq 1$.

3. Main Results

Theorem 1. *If* $f(z) \in \mathcal{KD}\,(k, \beta, \gamma)$*, then*

$$f(z) \in \mathcal{KD}\left(\frac{\beta - k}{1 - k}, \gamma\right).$$

Proof. If $f(z) \in \mathcal{KD}\,(k, \beta, \gamma)$, then $k \leq 0$, $\beta > 1$, and so we obtain

$$\begin{aligned} \mathfrak{Re}\left\{\frac{zf'(z)}{g(z)}\right\} &< k\left|\frac{zf'(z)}{g(z)} - 1\right| + \beta \\ &\leq \beta + k\mathfrak{Re}\left\{\frac{zf'(z)}{g(z)} - 1\right\}, \end{aligned}$$

which leads to

$$\mathfrak{Re}\left\{\frac{zf'(z)}{g(z)}\right\} - k\mathfrak{Re}\left\{\frac{zf'(z)}{g(z)}\right\} < -k + \beta.$$

After simplification, we obtain

$$\mathfrak{Re}\left\{\frac{zf'(z)}{g(z)}\right\} < \frac{\beta - k}{1 - k}, \quad (k \leq 0,\ \beta > 1)\,. \tag{15}$$

This completes the proof. □

In a similar way, one can easily prove the following important result.

Theorem 2. *If* $f \in \mathcal{QD}\,(k, \beta, \gamma)$*, then*

$$f \in \mathcal{QD}\left(\frac{\beta - k}{1 - k}, \gamma\right).$$

Theorem 3. *If* $f(z) \in \mathcal{KD}\,(k, \beta, \gamma)$*, then*

$$|a_n| \leq \frac{(\delta_{k,\gamma})_{n-1}}{n!} + \frac{\left|\delta_{k,\beta}\right|}{n} \sum_{j=1}^{n-1} \frac{(\delta_{k,\gamma})_{j-1}}{(j-1)!},$$

where $\delta_{k,\gamma}$ *is given by* (8) *and*

$$\delta_{k,\beta} = \frac{2\,(\beta - 1)}{1 - k}. \tag{16}$$

Proof. If f is in the class $\mathcal{KD}(k, \beta, \gamma)$, then there exists $g(z) \in \mathcal{MD}\,(k, \gamma)$ such that the function

$$p(z) = \frac{(\beta - k) - (1 - k)\left(\dfrac{zf'(z)}{g(z)}\right)}{\beta - 1} \tag{17}$$

belongs to $\mathcal{P}$. Therefore, we write

$$zf'(z) = \frac{\beta - k}{1 - k} g(z) - \frac{\beta - 1}{1 - k} g(z) p(z). \tag{18}$$

Let us write $p(z)$ as $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, $g(z)$ as $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, and let $f(z)$ have the series form as in (1). Then, (18) can be written as

$$z + \sum_{n=2}^{\infty} n a_n z^n = \frac{\beta - k}{1 - k}\left(z + \sum_{n=2}^{\infty} b_n z^n\right) - \frac{\beta - 1}{1 - k}\left(1 + \sum_{n=1}^{\infty} p_n z^n\right)\left(z + \sum_{n=2}^{\infty} b_n z^n\right).$$

Comparing the nth term's coefficients on both sides, we obtain

$$n a_n = b_n - \frac{\beta - 1}{1 - k}\left[p_{n-1} + p_{n-2} b_2 + p_{n-3} b_3 + \ldots + p_1 b_{n-1}\right].$$

By taking the absolute value, we get

$$\begin{aligned} n|a_n| &= \left| b_n - \frac{\beta - 1}{1 - k}\left[p_{n-1} + p_{n-2} b_2 + p_{n-3} b_3 + \ldots + p_1 b_{n-1}\right] \right| \\ &\leq |b_n| + \frac{\beta - 1}{1 - k}\left|p_{n-1} + p_{n-2} b_2 + p_{n-3} b_3 + \ldots + p_1 b_{n-1}\right|. \end{aligned}$$

Applying the triangle inequality, we obtain

$$n|a_n| \leq |b_n| + \frac{\beta - 1}{1 - k}\left\{|p_{n-1}| + |p_{n-2} b_2| + |p_{n-3} b_3| + \ldots + |p_1 b_{n-1}|\right\}. \tag{19}$$

As $\mathfrak{Re}\left\{p(z)\right\} > 0$ in $\mathbb{U}$, we have $|p_n| \leq 2\ (n \geq 1)$ (see [1]). Then, from (19), we have

$$n|a_n| \leq |b_n| + \frac{2\,(\beta - 1)}{1 - k}\sum_{j=1}^{n-1} |b_j|,$$

where $b_1 = 1$. Using Lemma (3), we obtain

$$n|a_n| \leq \frac{\left(\delta_{k,\gamma}\right)_{n-1}}{(n-1)!} + \delta_{k,\beta}\sum_{j=1}^{n-1}\frac{\left(\delta_{k,\gamma}\right)_{j-1}}{(j-1)!},$$

where $\delta_{k,\beta} = \dfrac{2\,(\beta - 1)}{1 - k}$ and $\delta_{k,\gamma}$ is defined by (8). This can be written as

$$|a_n| \leq \frac{\left(\delta_{k,\gamma}\right)_{n-1}}{n!} + \frac{\delta_{k,\beta}}{n}\sum_{j=1}^{n-1}\frac{\left(\delta_{k,\gamma}\right)_{j-1}}{(j-1)!}.$$

This completes the proof. □

From Definition 4 and Theorem 2, we immediately get the following corollary.

Corollary 1. *If $f(z) \in \mathcal{QD}(k, \beta, \gamma)$, then*

$$|a_n| \leq \frac{1}{n}\left[\frac{\left(\delta_{k,\gamma}\right)_{n-1}}{n!} + \frac{\delta_{k,\beta}}{n}\sum_{j=1}^{n-1}\frac{\left(\delta_{k,\gamma}\right)_{j-1}}{(j-1)!}\right],$$

where $\delta_{k,\beta}$ and $\delta_{k,\gamma}$ are given by (16) *and* (8)*, respectively.*

By taking $k = 0$ in the above results, we obtain the coefficient inequality for the classes $\mathcal{KD}(\beta, \gamma)$ and $\mathcal{QD}(\beta, \gamma)$.

Theorem 4. *If a function $f \in \mathcal{KD}(k,\beta,\gamma)$, then there exists $g \in \mathcal{MD}(k,\gamma)$ such that*

$$\frac{zf'(z)}{g(z)} \prec 1 + 2(\beta_1 - 1) - \frac{2(\beta_1 - 1)}{1-z}, \quad (z \in \mathbb{U}), \tag{20}$$

where

$$\beta_1 = \frac{\beta - k}{1-k}. \tag{21}$$

Proof. Let $f(z) \in \mathcal{KD}(k,\beta,\gamma)$. Then, there exists $g(z)$ in $\mathcal{MD}(k,\gamma)$ and a Schwarz function $w(z)$ such that

$$\frac{\beta_1 - \left(\frac{zf'(z)}{g(z)}\right)}{\beta_1 - 1} = \frac{1+w(z)}{1-w(z)}, \tag{22}$$

as $w(z)$ is analytic $\mathbb{U}$ with $w(0) = 0$ and

$$\Re e\left(\frac{1+w(z)}{1-w(z)}\right) > 0, \quad (z \in \mathbb{U}).$$

So, from (22), we obtain

$$\begin{aligned}
\frac{zf'(z)}{g(z)} &= \beta_1 - (\beta_1 - 1)\left(\frac{1+w(z)}{1-w(z)}\right) \\
&= \frac{\beta_1(1-w(z)) - (\beta_1 - 1)(1+w(z))}{1-w(z)} \\
&= \frac{1 + w(z) - 2\beta_1 w(z)}{1-w(z)} \\
&= \frac{1 - w(z) - 2(\beta_1 - 1)w(z)}{1-w(z)} \\
&= \frac{1 - w(z) + 2(\beta_1 - 1) - 2(\beta_1 - 1)w(z) - 2(\beta_1 - 1)}{1-w(z)} \\
&= \frac{1 - w(z) + 2(\beta_1 - 1)(1 - w(z)) - 2(\beta_1 - 1)}{1-w(z)}.
\end{aligned}$$

This implies that

$$\frac{zf'(z)}{g(z)} = 1 + 2(\beta_1 - 1) - \frac{2(\beta_1 - 1)}{1-w(z)},$$

and hence

$$\frac{zf'(z)}{g(z)} \prec 1 + 2(\beta_1 - 1) - \frac{2(\beta_1 - 1)}{1-z}, \quad (z \in \mathbb{U}),$$

which is as required in (20). □

Corollary 2. *If $f \in \mathcal{QD}(k,\beta,\gamma)$, then there exists $g \in \mathcal{ND}(k,\gamma)$ such that*

$$\frac{(zf'(z))'}{g'(z)} \prec 1 + 2(\beta_1 - 1) - \frac{2(\beta_1 - 1)}{(1-z)}, \quad (z \in \mathbb{U}), \tag{23}$$

where β_1 is given by (21).

Theorem 5. *If $f \in \mathcal{KD}(k,\beta,\gamma)$, then there exists a function $g \in \mathcal{MD}(k,\gamma)$ such that*

$$\frac{1 - (2\beta_1 - 1)r}{1-r} \leq \Re\mathfrak{e}\frac{zf'(z)}{g(z)} \leq \frac{1 + (2\beta_1 - 1)r}{1+r}, \tag{24}$$

where $|z| = r < 1$ *and* β_1 *is given by* (21).

Proof. Using Theorem 4, we define the function ϕ as follows

$$\phi(z) = 1 + 2(\beta_1 - 1) + \frac{2(1-\beta_1)}{1-z}, (z \in \mathbb{U}).$$

Letting $z = re^{i\theta}(0 \le r < 1)$, we observe that

$$\mathfrak{Re}\phi(z) = 1 + 2(\beta_1 - 1) + \frac{2(1-\beta_1)(1 - r\cos\theta)}{1 + r^2 - 2r\cos\theta}.$$

Let us define

$$\psi(t) = \frac{1 - rt}{1 + r^2 - 2rt}, (t = \cos\theta).$$

As $\psi'(t) = \dfrac{r(1-r^2)}{(1+r^2-2rt)^2} \ge 0$ (since $r < 1$), we get

$$1 + 2(\beta_1 - 1) + \frac{2(1-\beta_1)}{1-r} \le \mathfrak{Re}\phi(z) \le 1 + 2(\beta_1 - 1) + \frac{2(1-\beta_1)}{1+r}.$$

After simplification, we have

$$\frac{1-(2\beta_1-1)r}{1-r} \le \mathfrak{Re}\phi(z) \le \frac{1+(2\beta_1-1)r}{1+r}.$$

With the fact that $\dfrac{zf'(z)}{g(z)} \prec \phi(z)$, $(z \in \mathbb{U})$ and as ϕ is univalent in $\mathbb{U}$, by using (22), we get the required result. □

Corollary 3. *If* $f \in \mathcal{QD}(k, \beta, \gamma)$*, then there exists* $g \in \mathcal{ND}(k, \gamma)$ *such that*

$$\frac{1-(2\beta_1-1)r}{1-r} \le \mathfrak{Re}\frac{(zf'(z))'}{g'(z)} \le \frac{1+(2\beta_1-1)r}{1+r}, \tag{25}$$

where $|z| = r < 1$ *and* β_1 *is given by* (21).

Theorem 6. *Assume that a function* $f \in \mathcal{A}$ *satisfies*

$$\mathfrak{Re}\left(\frac{zg'(z)}{g(z)} - \frac{zf''(z)}{f'(z)}\right) > \frac{\beta_1 + 1}{2\beta_1}, \quad (z \in \mathbb{U}), \tag{26}$$

for some $g(z) \in \mathcal{MD}(k, \gamma)$ *and for real* β_1 *given by* (21)*. If*

$$\phi(z) = \frac{zf'(z)}{g(z)}$$

is analytic in $\mathbb{U}$ *and* $\phi(z) \ne 0$ *and* $\phi(z) \ne 2\beta_1 - 1$ *in* $\mathbb{U}$*, then* $f \in \mathcal{KD}(k, \beta_1)$.

Proof. Let us define a function $w(z)$ by

$$w(z) = \frac{\phi(z) - 1}{\phi(z) + (1 - 2\beta_1)}, \quad z \in \mathbb{U}.$$

Then, $w(z)$ is analytic in $\mathbb{U}$ as $\phi(z) \neq 2\beta_1 - 1$ and

$$\phi(z) = \frac{zf'(z)}{g(z)} = \frac{1 + (1 - 2\beta_1)\, w(z)}{1 - w(z)}. \tag{27}$$

Because $\phi(z) \neq 0$, we use logarithmic differentiation to get

$$\frac{1}{z} + \frac{f''(z)}{f'(z)} - \frac{g'(z)}{g(z)} = \frac{(1 - 2\beta_1)\, w'(z)}{1 + (1 - 2\beta_1)\, w(z)} + \frac{w'(z)}{1 - w(z)},$$

which further yields

$$\frac{zg'(z)}{g(z)} - \frac{zf''(z)}{f'(z)} = 1 - \frac{(1 - 2\beta_1)\, zw'(z)}{1 + (1 - 2\beta_1)\, w(z)} - \frac{zw'(z)}{1 - w(z)}. \tag{28}$$

Then, we note that w is analytic in open unit disk and $w(0) = 0$. Therefore, from (28), we obtain

$$\begin{aligned} \mathfrak{Re}\left(\frac{zg'(z)}{g(z)} - \frac{zf''(z)}{f'(z)}\right) &= \mathfrak{Re}\left(1 - \frac{(1 - 2\beta_1)\, zw'(z)}{1 + (1 - 2\beta_1)\, w(z)} - \frac{zw'(z)}{1 - w(z)}\right) \\ &> \frac{\beta_1 + 1}{2\beta_1}. \end{aligned}$$

Suppose there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then, by Lemma 4, we can write $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = ce^{i\theta}$ for a point z_0, and we have

$$\begin{aligned} & \mathfrak{Re}\left(\frac{z_0 g'(z_0)}{g(z_0)} - \frac{z_0 f''(z_0)}{f'(z_0)}\right) \\ = \;& \mathfrak{Re}\left(1 - \frac{(1 - 2\beta_1)\, ce^{i\theta}}{1 + (1 - 2\beta_1)\, e^{i\theta}} - \frac{ce^{i\theta}}{1 - e^{i\theta}}\right) \\ = \;& \mathfrak{Re}\left(1 - \frac{c\,(1 - 2\beta_1)\left(e^{i\theta} + (1 - 2\beta_1)\right)}{1 + (1 - 2\beta_1)^2 + 2\,(1 - 2\beta_1)\cos\theta} + \frac{c\left(1 - e^{i\theta}\right)}{2\,(1 - \cos\theta)}\right) \\ = \;& 1 + \frac{c\,(2\beta_1 - 1)\left[\cos\theta + (1 - 2\beta_1)\right]}{1 + (1 - 2\beta_1)^2 + 2\,(1 - 2\beta_1)\cos\theta} + \frac{c}{2} \\ \leq \;& 1 - \frac{c\,(2\beta_1 - 1)}{2\beta_1} + \frac{c}{2} \\ = \;& 1 - \frac{c\,(\beta_1 - 1)}{2\beta_1} \\ \leq \;& 1 - \frac{\beta_1 - 1}{2\beta_1}, \text{ as } c < 1 \\ = \;& \frac{\beta_1 + 1}{2\beta_1}, \end{aligned}$$

which gives that

$$\mathfrak{Re}\left\{\frac{z_0 g'(z_0)}{g(z_0)} - \frac{z_0 f''(z_0)}{f'(z_0)}\right\} \leq \frac{\beta_1 + 1}{2\beta_1},$$

which is the contradiction to the supposed condition (26). Hence, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This implies that $|w(z)| < 1, (z \in \mathbb{U})$ and, therefore, by (27), we have

$$\frac{zf'(z)}{g(z)} \prec \frac{1-(2\beta_1-1)z}{1-z}$$

or

$$\Re\left\{\frac{zf'(z)}{g(z)}\right\} < \beta_1,\ z \in \mathbb{U}.$$

Hence, we conclude that $f(z) \in \mathcal{KD}(k, \beta_1)$. □

Theorem 7. *Assume that $k \le 0$ and $\beta > 1$. If $f \in \mathcal{A}$ and if there exists $g \in \mathcal{MD}\,(k,\gamma)$ such that*

$$\left|\frac{zf'(z)}{g(z)} - 1\right| < \frac{\beta-1}{1-k} \quad z \in \mathbb{U}, \tag{29}$$

then $f \in \mathcal{KD}(k, \beta, \gamma)$.

Proof. We have

$$\begin{aligned}
& \left|\frac{zf'(z)}{g(z)} - 1\right| < \frac{\beta-1}{1-k} \\
\Rightarrow\quad & (1-k)\left|\frac{zf'(z)}{g(z)} - 1\right| + 1 < \beta \\
\Rightarrow\quad & \left|\frac{zf'(z)}{g(z)} - 1\right| + 1 < k\left|\frac{zf'(z)}{g(z)} - 1\right| + \beta \\
\Rightarrow\quad & \Re\frac{zf'(z)}{g(z)} < k\left|\frac{zf'(z)}{g(z)} - 1\right| + \beta \\
\Rightarrow\quad & f \in \mathcal{KD}(k, \beta, \gamma).
\end{aligned}$$

□

Corollary 4. *Let $f \in \mathcal{A}$ have the form (1). Assume that $g = z + b_2z^2 + \cdots$ belongs to the class $\mathcal{MD}\,(k,\gamma)$ and satisfies*

$$\left|\frac{\sum_{n=2}^{\infty}(na_n - b_n)z^{n-1}}{1+\sum_{n=2}^{\infty} b_n z^{n-1}}\right| < \frac{\beta-1}{1-k} \quad z \in \mathbb{U}, \tag{30}$$

for some $k\,(k \le 0)$, $\beta\,(\beta > 1)$.

Then, $f(z) \in \mathcal{KD}(k, \beta, \gamma)$.

Proof. We have

$$\begin{aligned}
& \left|\frac{zf'(z)}{g(z)} - 1\right| \\
= \quad & \left|\frac{z + \sum\limits_{n=2}^{\infty} na_nz^n}{z + \sum\limits_{n=2}^{\infty} b_nz^n} - 1\right| \\
= \quad & \left|\frac{\sum_{n=2}^{\infty}(na_n - b_n)z^{n-1}}{1+\sum_{n=2}^{\infty} b_n z^{n-1}}\right| \\
< \quad & \frac{\beta-1}{1-k},
\end{aligned}$$

and hence (29) follows immediately from (30). □

Theorem 8. *Let $f \in \mathcal{A}$ have the form (1) and let $g = z + \sum_{n=2}^{\infty} b_n z^n$, belonging to the class $\mathcal{MD}(k, \gamma)$, satisfy*

$$1 + \sum_{n=2}^{\infty} (n|a_n| + y|b_n|) < y \quad z \in \mathbb{U}, \tag{31}$$

for some $k\,(k \leq 0)$, $\beta\,(\beta > 1)$ and where

$$y = \frac{(\beta - 1)}{(1 - k)} > 0.$$

Then, $f(z) \in \mathcal{KD}(k, \beta, \gamma)$.

Proof. Consider

$$\begin{aligned} & 1 + \sum_{n=2}^{\infty} (n|a_n| + y|b_n|) < y \qquad (32) \\ \Rightarrow\ & 1 + \sum_{n=2}^{\infty} n|a_n| < y - y\sum_{n=2}^{\infty} |b_n| \\ \Rightarrow\ & 0 < y - y\sum_{n=2}^{\infty} |b_n| \\ \Rightarrow\ & 0 < y - y\sum_{n=2}^{\infty} |b_n||z^{n-1}| \\ \Rightarrow\ & 0 < y\left|1 + \sum_{n=2}^{\infty} b_n z^{n-1}\right|. \qquad (33) \end{aligned}$$

We have

$$\begin{aligned} & 1 + \sum_{n=2}^{\infty} (n|a_n| + y|b_n|) < y \\ \Rightarrow\ & 1 + \sum_{n=2}^{\infty} n|a_n| < y - y\sum_{n=2}^{\infty} |b_n| \\ \Rightarrow\ & 1 + \sum_{n=2}^{\infty} n|a_n||z^{n-1}| < y - y\sum_{n=2}^{\infty} |b_n||z^{n-1}| \\ \Rightarrow\ & \left|1 + \sum_{n=2}^{\infty} n a_n z^{n-1}\right| < y\left|1 + \sum_{n=2}^{\infty} b_n z^{n-1}\right| \\ \Rightarrow\ & \left|\frac{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} b_n z^{n-1}}\right| < y, \end{aligned}$$

from (33). By (30), it follows that $f \in \mathcal{KD}(k, \beta, \gamma)$. □

Author Contributions: Conceptualization, S.M.; Formal analysis, S.N.M. and J.S.; Funding acquisition, S.M.; Investigation, S.M.; Methodology, S.M. and S.N.M.; Supervision, H.M.S. and J.S.; Validation, H.M.S.; Visualization, S.M.;Writing—original draft, S.M.;Writing—review and editing, S.M. and S.N.M.

Funding: This research is supported by Sarhad University of Science & I.T, Peshawar 25000, Pakistan.

Acknowledgments: The authors are grateful to referees for their valuable comments which improved the quality of work and presentation of paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Goodman, A.W. *Univalent Functions, Vol. I, II*; Polygnal Publishing House: Washington, NJ, USA, 1983.

2. Miller, S.S.; Mocanu, P.T. *Differential Subordinations Theory and Applications*; Marcel Decker Inc.: New York, NY, USA, 2000.
3. Uralegaddi, B.A.; Ganigi, M.D.; Sarangi, S.M. Univalent functions with positive coefficients. *Tamkang J. Math.* **1994**, *25*, 225–230.
4. Nishiwaki, J.; Owa, S. Coefficient estimates for certain classes of analytic functions. *J. Ineq. Pure Appl. Math.* **2002**, *3*, 1–5.
5. Nishiwaki, J.; Owa, S. Coefficient inequalities for analytic functions. *Int. J. Math. Math. Sci.* **2002**, *29*, 285–290. [CrossRef]
6. Owa, S.; Srivastava, H.M. Some generalized convolution properties associated with cerain subclasses of analytic functions. *J. Ineq. Pure Appl. Math.* **2002**, *3*, 1–13.
7. Nishiwaki, J.; Owa, S. Certain classes of analytic functions concerned with uniformly starlike and convex functions. *Appl. Math. Comput.* **2007**, *187*, 350–355. [CrossRef]
8. Arif, M.; Ayaz, M.; Sokół, J. On a subclass of multivalent close to convex functions. *Publ. Inst. Math.* **2017**, *101*, 161–168. [CrossRef]
9. Arif, M.; Umar, S.; Mahmood, S.; Sokół, J. New reciprocal class of analytic functions associated with linear operator. *Iran. J. Sci. Technol. Trans. Sci.* **2016**, *42*, 881–886. [CrossRef]
10. Mahmood, S.; Malik, S.N.; Farman, S.; Riaz, S.M.J.; Farwa, S. Uniformly alpha-quasi convex functions defined by Janowski functions. *J. Funct. Spaces* **2018** , *2018*, 6049512. [CrossRef]
11. Malik, S.N.; Mahmood, S.; Raza, M.; Farman, S.; Zainab, S.; Muhammad, N. Coefficient Inequalities of Functions Associated with Hyperbolic Domain. *Mathematics* **2019**, *7*, 88. [CrossRef]
12. Malik, S.N.; Mahmood, S.; Raza, M.; Farman, S.; Zainab, S. Coefficient Inequalities of Functions Associated with Petal Type Domain. *Mathematics* **2018**, *6*, 298. [CrossRef]
13. Jack, I.S. Functions starlike and convex of order α. *J. Lond. Math. Soc.* **1971**, *2*, 469–474. [CrossRef]