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# Traveling Wave Solutions of a Delayed Cooperative System

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**Abstract:** This paper deals with the dynamics of a delayed cooperative system without quasimonotonicity. Using the contracting rectangles, we obtain a sufficient condition on the stability of the unique positive steady state of the functional differential system. When the spatial domain is whole $\mathbb{R}$, the existence and nonexistence of traveling wave solutions are investigated, during which the asymptotic behavior is investigated by the contracting rectangles.

**Keywords:** contracting rectangle; large delay; minimal wave speed; population dynamics

## 1. Introduction

In population dynamics, there are many important cooperative systems modeling some natural phenomena. For example, some well-known mutualistic interactions are found in a wealth of different environments, such as diatom mats in the ocean, between mangroves and root borers, and between spiders and parasitic wasps [1,2]. In the literature, cooperative systems often lead to monotone dynamics [3–5]. One important cooperative system is the following Lotka–Volterra type system

$$\begin{cases} u_1'(t) = r_1u_1(t)[1 - u_1(t) + ru_2(t)], \\ u_2'(t) = r_2u_2(t)[1 - u_2(t) + su_1(t)], \end{cases} \tag{1}$$

in which all the parameters are positive. Because Equation (1) is cooperative 2-D ODE system, its dynamics has been fully understood. When the spatial variable is concerned, one model is the following reaction–diffusion system

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = d_1\Delta u_1(x,t) + r_1u_1(x,t)[1 - u_1(x,t) + ru_2(x,t)], \\ \frac{\partial u_2(x,t)}{\partial t} = d_2\Delta u_2(x,t) + r_2u_2(x,t)[1 - u_2(x,t) + su_1(x,t)], \end{cases} \tag{2}$$

in which $x \in \mathbb{R}, t > 0, d_1 > 0$ and $d_2 > 0$ are diffusive coefficients. We refer to the work of Li et al. [6] and Lin et al. [7] for some results on traveling wave solutions. Moreover, Li et al. [6] and Lin [8] studied the corresponding asymptotic spreading.

However, when it involves time delay, it is possible that it does not generate monotone semiflows, for example

$$\begin{cases} u_1'(t) = r_1u_1(t)[1 - u_1(t-\tau_1) + ru_2(t-\tau_2)], \\ u_2'(t) = r_2u_2(t)[1 - u_2(t-\tau_3) + su_1(t-\tau_4)], \end{cases} \tag{3}$$

in which $\tau_1, \tau_2, \tau_3, \tau_4$ are nonnegative constants. It is evident that Equation (3) is not monotone if $\tau_1 > 0$ or $\tau_3 > 0$. Huang and Zou [9], Li and Wang [10], Li and Lin [11], and Lin et al. [12] studied the existence of traveling wave solutions when $\tau_1$ and $\tau_3$ are small enough.

In this paper, we investigate the traveling wave solutions of

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = d_1 \Delta u_1(x,t) + r_1 u_1(x,t) F_1(u_1,u_2)(x,t), \\ \frac{\partial u_2(x,t)}{\partial t} = d_2 \Delta u_2(x,t) + r_2 u_2(x,t) F_2(u_1,u_2)(x,t), \end{cases} \tag{4}$$

where $x \in \mathbb{R}, t > 0$ and

$$\begin{cases} F_1(u_1,u_2)(x,t) = 1 - a_1 u_1(x,t) - b_1 \int_{-\tau}^{0} u_1(x,t+s) d\eta_{11}(s) + e_1 \int_{-\tau}^{0} u_2(x,t+s) d\eta_{12}(s), \\ F_2(u_1,u_2)(x,t) = 1 - a_2 u_2(x,t) - b_2 \int_{-\tau}^{0} u_2(x,t+s) d\eta_{22}(s) + e_2 \int_{-\tau}^{0} u_1(x,t+s) d\eta_{21}(s), \end{cases} \tag{5}$$

in which $r_1 > 0, r_2 > 0$, and $a_i, b_i, c_i, i = 1,2$, are nonnegative constants satisfying

$$a_1 a_2 > e_1 e_2 \tag{6}$$

and $\tau > 0$ such that

$$\eta_{ij}(s) \text{ is nondecreasing on } [-\tau, 0] \text{ and } \eta_{ij}(0) - \eta_{ij}(-\tau) = 1, \ i,j = 1,2.$$

Clearly, Equation (4) has a positive spatial homogeneous steady state formulated by

$$K = (k_1, k_2) = \left( \frac{e_2 + a_1 + b_1}{(a_1+b_1)(a_2+b_2) - e_1 e_2}, \frac{e_1 + a_2 + b_2}{(a_1+b_1)(a_2+b_2) - e_1 e_2} \right)$$

if $(a_1+b_1)(a_2+b_2) > e_1 e_2$ which is implied by Equation (6). Moreover, $0 = (0,0)$ is a trivial spatial homogeneous steady state.

It is clear that Equation (4) may not be a quasimonotone system, although its corresponding undelayed system is a cooperative system (see Li et al. [6]). At the same time, Equation (4) does not satisfy the local quasimonotonicity in [13,14]. Thus, it is difficult to study it by constructing two auxiliary quasimonotone systems. Of course, besides those in [13–16], there are also some results for delayed nonmonotone model with large delay. For example, if $e_1 < 0, e_2 < 0$ in Equation (4), then Martin and Smith [17] and Smith [5] gave some results on the stability of steady states. Lin and Ruan [18] further studied the existence and nonexistence of traveling wave solutions. Very recently, Meng et al. [19] investigated the monotone traveling wave solutions of Equation (4) if the intraspecific delay is small, which leads to the quasimonotonicity in the sense of exponential ordering [5]. Besides the traveling wave solutions, there are also some other features of entire solutions formulating by wave type solutions (see [20–22] for some examples of nonmonotone equations).

In this paper, by the ideas in [17,18], we study the dynamics of Equation (4). We first investigate the stability of the following initial value problem

$$\begin{cases} \frac{du_1(t)}{dt} = r_1 u_1(t) \left[ 1 - a_1 u_1(t) - b_1 \int_{-\tau}^{0} u_1(t+s) d\eta_{11}(s) + e_1 \int_{-\tau}^{0} u_2(t+s) d\eta_{12}(s) \right], \\ \frac{du_2(t)}{dt} = r_2 u_2(t) \left[ 1 - a_2 u_2(t) - b_2 \int_{-\tau}^{0} u_2(t+s) d\eta_{22}(s) + e_2 \int_{-\tau}^{0} u_1(t+s) d\eta_{21}(s) \right], \\ u_i(\theta) = \phi_i(\theta), \theta \in [-\tau, 0], i = 1,2, \end{cases} \tag{7}$$

in which all the parameters are the same as those in Equation (4) and $\phi_i(\theta), i = 1,2$, are continuous for $\theta \in [-\tau, 0]$. In fact, the stability was obtained by the authors of [23,24], and we present the result for the sake of verifying the asymptotic behavior of traveling wave solutions. Then, the existence and nonexistence of invasion traveling wave solutions of Equation (4) are considered in Section 3. More precisely, we give the existence of traveling wave solutions by constructing upper and lower solutions, investigate the asymptotic behavior by applying the contracting rectangles, and confirm the nonexistence of traveling wave solutions by utilizing the theory of asymptotic spreading and constructing an auxiliary equation.

## 2. Stability of Positive Steady States

In this paper, we use the standard partial ordering in $\mathbb{R}^2$. That is, for $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$, we write $u \geq v$ provided $u_i \geq v_i$ for $i = 1, 2$, $u > v$ provided $u \geq v$ but $u \neq v$, and $u \gg v$ provided $u_1 > v_1$ and $u_2 > v_2$.

To investigate Equation (7), we first introduce the following quasimonotone system

$$\begin{cases} \frac{d\overline{u}_1(t)}{dt} = r_1\overline{u}_1(t)\left[1 - a_1\overline{u}_1(t) + e_1\int_{-\tau}^{0}\overline{u}_2(t+s)d\eta_{12}(s)\right], \\ \frac{d\overline{u}_2(t)}{dt} = r_1\overline{u}_2(t)\left[1 - a_2\overline{u}_2(t) + e_2\int_{-\tau}^{0}\overline{u}_1(t+s)d\eta_{21}(s)\right], \\ \overline{u}_i(\theta) = \overline{\phi}_i(\theta), \theta \in [-\tau, 0], i = 1, 2, \end{cases} \tag{8}$$

where $\overline{\phi}_i(\theta), i = 1, 2$, are continuous for $\theta \in [-\tau, 0]$.

If Equation (6) holds, then Equation (8) admits a spatially homogeneous steady state as follows

$$\overline{K} = (\overline{k}_1, \overline{k}_2) = \left(\frac{e_2 + a_1}{a_1a_2 - e_1e_2}, \frac{e_1 + a_2}{a_1a_2 - e_1e_2}\right).$$

Evidently, $K|_{b_1=b_2=0} = \overline{K}$. Moreover, since

$$\begin{aligned} &\frac{d}{db_1}\left[\frac{e_2 + a_1 + b_1}{(a_1+b_1)(a_2+b_2) - e_1e_2}\right] \\ = &\frac{(a_1+b_1)(a_2+b_2) - e_1e_2 - (e_2+a_1+b_1)(a_2+b_2)}{((a_1+b_1)(a_2+b_2) - e_1e_2)^2} \\ = &\frac{-e_1e_2 - e_2(a_2+b_2)}{((a_1+b_1)(a_2+b_2) - e_1e_2)^2} \leq 0, \end{aligned}$$

and

$$\frac{d}{db_2}\left[\frac{e_1 + a_2 + b_2}{(a_1+b_1)(a_2+b_2) - e_1e_2}\right] \leq 0,$$

and $b_1 \geq 0, b_2 \geq 0$, we have proved $\overline{k}_i \geq k_i, i = 1, 2$.

The existence and uniqueness of mild solution of Equations (7) and (8) can be obtained by the theory of functional differential equations (see Hale and Verduyn Lunel [25]) and we omit it here. By the classical theory of classical monotone dynamics systems (see Smith [5]), the following result for Equation (8) is clear.

**Lemma 1.** *Assume that Equation (6) holds. If $\overline{\phi}_i(\theta) \geq 0, \theta \in [-\tau, 0]$ and $\overline{\phi}_i(0) > 0$ for $i = 1, 2$, then the unique mild solution $(u_1(t), u_2(t))$ to Equation (8) satisfies*

$$\overline{u}_i(t) \to \overline{k}_i,\ t \to \infty,\ i = 1, 2.$$

Using the quasipositivity (see Smith [5], Theorem 5.2.1), we can obtain the following conclusion on the positivity of mild solution to Equation (7).

**Lemma 2.** *Assume that $\phi_i(\theta) \geq 0$ for $\theta \in [-\tau, 0], i = 1, 2$. Then,*

$$u_i(t) \geq 0,\ i = 1, 2,\ t \in (0, t_1),$$

*in which $t_1$ (may be infinite) is the maximal interval of the existence of Equation (7).*

From Lemma 2, if $\overline{\phi}_i(\theta) \geq \phi_i(\theta) \geq 0, i = 1, 2, \theta \in [-\tau, 0]$, then $(u_1(t), u_2(t))$ becomes a sub solution of Equation (8) satisfying quasimonotonicity. By the standard comparison principle of functional differential equations with quasimonotonicity [17,26], the following conclusion can be obtained.

**Lemma 3.** *Assume that Equation (6) holds. If $\overline{\phi}_i(\theta) \geq 0$ for $\theta \in [-\tau, 0], i = 1, 2$, then Equation (8) admits a unique bounded mild solution $(\overline{u}_1(t), \overline{u}_2(t))$ for $t > 0$. Moreover,*

$$0 \leq \liminf_{t\to\infty} \overline{u}_i(t) \leq \limsup_{t\to\infty} \overline{u}_i(t) \leq \overline{k}_i, \; i = 1, 2.$$

*Let $(u_1(t), u_2(t))$ be the unique mild solution of Equation (7) with $\overline{\phi}_i(\theta) \geq \phi_i(\theta) \geq 0$ for $\theta \in [-\tau, 0], i = 1, 2$, then*

$$u_i(t) \leq \overline{u}_i(t), i = 1, 2, t \geq 0,$$

*and*

$$0 \leq \liminf_{t\to\infty} u_i(t) \leq \limsup_{t\to\infty} u_i(t) \leq \overline{k}_i, \; i = 1, 2.$$

We now present our main result of this section.

**Theorem 1.** *Assume that $\phi_i(\theta) \geq 0, \phi_i(0) > 0$ for all $\theta \in [-\tau, 0], i = 1, 2$. If $b_1\overline{k}_1 < 1$ and $b_2\overline{k}_2 < 1$ hold, then the unique mild solution $(u_1(t), u_2(t))$ of Equation (7) satisfies*

$$u_i(t) \to k_i, \; t \to \infty, \; i = 1, 2.$$

**Proof.** We prove it using the method by Smith [5]. For $s \in [0, 1]$, define $E(s)$ and $F(s)$ as follows

$$E(s) = (sk_1, sk_2), F(s) = (sk_1 + (1-s)(\overline{k}_1 + \epsilon_1), sk_2 + (1-s)(\overline{k}_2 + \epsilon_2)),$$

in which $\epsilon_1 >, \epsilon_2 > 0$ such that

$$b_1(\overline{k}_1 + \epsilon_1) < 1, \; b_2(\overline{k}_2 + \epsilon_2) < 1$$

and

$$e_1\epsilon_2 < a_1\epsilon_1, \; e_2\epsilon_1 < a_2\epsilon_2.$$

By Equation (6), they are admissible.

Assume that $(u_1(t+\theta), u_2(t+\theta)) \in [E(s), F(s)]$ for all $\theta \in [-\tau, 0], t \geq 0$ and some $s \in (0, 1)$. If $u_1(t) = sk_1$, then $\int_{-\tau}^{0} d\eta_{11}(s) = \int_{-\tau}^{0} d\eta_{12}(s) = 1$ and

$$a_1k_1 + b_1k_1 - e_1k_2 = 1$$

such that

$$\begin{aligned} & 1 - a_1u_1(t) - b_1\int_{-\tau}^{0} u_1(t+s)d\eta_{11}(s) + e_1\int_{-\tau}^{0} u_2(t+s)d\eta_{12}(s) \\ \geq\; & 1 - a_1u_1(t) - b_1\left(sk_1 + (1-s)(\overline{k}_1 + \epsilon_1)\right) + e_1sk_2 \\ =\; & 1 - a_1sk_1 - b_1\left(sk_1 + (1-s)(\overline{k}_1 + \epsilon_1)\right) + e_1sk_2 \\ =\; & 1 - s - b_1(1-s)(\overline{k}_1 + \epsilon_1) \\ =\; & (1-s)\left[1 - b_1(\overline{k}_1 + \epsilon_1)\right] \\ >\; & 0 \end{aligned} \tag{9}$$

from the definition of $\epsilon_1$. Similarly, if $(u_1(t+\theta), u_2(t+\theta)) \in [E(s), F(s)]$ and $u_2(t) = sk_2$ with $\theta \in [-\tau, 0], s \in (0,1)$ and $t \geq 0$, then

$$1 - a_2 u_2(t) - b_2 \int_{-\tau}^{0} u_2(t+s) d\eta_{22}(s) + e_2 \int_{-\tau}^{0} u_1(t+s) d\eta_{21}(s) > 0.$$

Moreover, when $(u_1(t+\theta), u_2(t+\theta)) \in [E(s), F(s)]$ with $\theta \in [-\tau, 0], t \geq 0$ and some $s \in [0,1)$, if $u_1(t) = sk_1 + (1-s)(\bar{k}_1 + \epsilon_1)$, then

$$\begin{aligned}
& 1 - a_1 u_1(t) - b_1 \int_{-\tau}^{0} u_1(t+s) d\eta_{11}(s) + e_1 \int_{-\tau}^{0} u_2(t+s) d\eta_{12}(s) \\
\leq \; & 1 - a_1 u_1(t) - b_1 s k_1 + e_1 \left( sk_2 + (1-s)(\bar{k}_2 + \epsilon_2) \right) \\
= \; & 1 - a_1 \left( sk_1 + (1-s)(\bar{k}_1 + \epsilon_1) \right) - b_1 s k_1 + e_1 \left( sk_2 + (1-s)(\bar{k}_2 + \epsilon_2) \right) \\
= \; & (1-s) \left[ 1 - a_1(\bar{k}_1 + \epsilon_1) + e_1(\bar{k}_2 + \epsilon_2) \right] \\
< \; & 0.
\end{aligned}$$

Similarly, we have

$$1 - a_2 u_2(t) - b_2 \int_{-\tau}^{0} u_2(t+s) d\eta_{22}(s) + e_2 \int_{-\tau}^{0} u_1(t+s) d\eta_{21}(s) < 0$$

when $(u_1(t+\theta), u_2(t+\theta)) \in [E(s), F(s)]$ with $\theta \in [-\tau, 0]$, and $u_2(t) = sk_2 + (1-s)(\bar{k}_2 + \epsilon_2)$ with $s \in [0,1), t \geq 0$.

Moreover, from Lemma 3, we also see that

$$\liminf_{t \to \infty} u_i(t) \geq (1 - b_i \bar{k}_i)/a_i > 0, \; i = 1, 2.$$

Let $s_0 > 0$ be small such that

$$s_0 k_i < (1 - b_i \bar{k}_i)/a_i, \; s_0 k_i + (1 - s_0)(\bar{k}_i + \epsilon_i) > \bar{k}_i.$$

Then, it implies that there exists $T > 0$ such that

$$E(s_0) < (u_1(t), u_2(t)) < F(s_0), \; t > T. \tag{10}$$

Define

$$\bar{u}_i = \limsup_{t \to \infty} u_i(t), \underline{u}_i = \liminf_{t \to \infty} u_i(t), i = 1, 2,$$

then they are positive constants by Equation (10). If $\bar{u}_i = \underline{u}_i = k_i$, then the conclusion is true. Otherwise, there exists $s' \in (s_0, 1)$ such that

$$s' = \sup \{ s : E(s) < (u_1(t), u_2(t)) < F(s) \},$$

which implies that at least one of the following is true

$$\underline{u}_i = s' k_i, \bar{u}_i = s' k_i + (1 - s')(\bar{k}_i + \epsilon_i), i = 1, 2.$$

If $\underline{u}_1 = s' k_1$, then there exists $\{t_n\}$ with $t_n \to \infty, n \to \infty$ such that

$$u(t_n) \to s' k_1, u'(t_n) \to 0, n \to \infty$$

and

$$\begin{aligned}
& \liminf_{t\to\infty}\left[1-a_1u_1(t)-b_1\int_{-\tau}^{0}u_1(t+s)d\eta_{11}(s)+e_1\int_{-\tau}^{0}u_2(t+s)d\eta_{12}(s)\right] \\
\geq\ & 1-a_1s'k_1-b_1\left(s'k_1+(1-s')(\bar{k}_1+\epsilon_1)\right)+e_1s'k_2 \\
=\ & 1-s'-b_1(1-s')(\bar{k}_1+\epsilon_1) \\
=\ & (1-s')\left[1-b_1(\bar{k}_1+\epsilon_1)\right] \\
>\ & 0
\end{aligned}$$

by Equation (9). A contradiction occurs. In a similar way, we can confirm that $s' < 1$ is impossible. The proof is complete. □

**Remark 1.** By the method of Smith [5], $[E(s), F(s)]$ is a contracting rectangle of Equation (7).

## 3. Traveling Wave Solutions

In this section, we consider the traveling wave solutions of Equation (4), throughout which Equation (6) holds. We first give the following definition.

**Definition 1.** *A traveling wave solution of Equation* (4) *is a special solution with the form*

$$u_i(x,t)=\varphi_i(x+ct),\ i=1,2,$$

*in which $c>0$ is the wave speed while $(\varphi_1(\xi),\varphi_2(\xi))\in C^2(\mathbb{R},\mathbb{R}^2)$ is the wave profile.*

By the definition, $(\varphi_1(\xi),\varphi_2(\xi))$ must satisfy the following functional differential system

$$\begin{cases} d_1\varphi_1''(\xi)-c\varphi_1'(\xi)+H_1(\varphi_1,\varphi_2)(\xi)=0, \\ d_2\varphi_2''(\xi)-c\varphi_2'(\xi)+H_2(\varphi_1,\varphi_2)(\xi)=0 \end{cases} \tag{11}$$

with

$$\begin{cases} H_1(\varphi_1,\varphi_2)(\xi)=r_1\varphi_1(\xi)\left[1-a_1\varphi_1(\xi)-b_1\int_{-\tau}^{0}\varphi_1(\xi+cs)d\eta_{11}(s)+e_1\int_{-\tau}^{0}\varphi_2(\xi+cs)d\eta_{12}(s)\right], \\ H_2(\varphi_1,\varphi_2)(\xi)=r_2\varphi_2(\xi)\left[1-a_2\varphi_2(\xi)-b_2\int_{-\tau}^{0}\varphi_2(\xi+cs)d\eta_{22}(s)+e_2\int_{-\tau}^{0}\varphi_1(\xi+cs)d\eta_{21}(s)\right]. \end{cases}$$

In particular, we also consider the positive traveling wave solutions of Equation (4) formulating the invasion of two cooperative species, which can be characterized by the following asymptotic boundary conditions

$$\lim_{\xi\to-\infty}\varphi_i(\xi)=0,\ \lim_{\xi\to\infty}\varphi_i(\xi)=k_i,\ i=1,2. \tag{12}$$

When the wave speed is small, we have the following result on the nonexistence of traveling wave solutions.

**Theorem 2.** *If $c<\max\{2\sqrt{d_1r_1},2\sqrt{d_2r_2}\}$, then Equation* (11) *has no bounded positive solution satisfying*

$$\lim_{\xi\to-\infty}(\varphi_1(\xi),\varphi_2(\xi))=(0,0),\ \liminf_{\xi\to\infty}\varphi_i(\xi)>0, i=1,2,\xi\in\mathbb{R}. \tag{13}$$

**Proof.** It suffices to study the case of $d_1r_1 \geq d_2r_2$. Were the statement false, then there exists $c' < 2\sqrt{d_1r_1}$ such that Equation (11) with $c=c'$ has a positive solution $(\varphi_1(\xi),\varphi_2(\xi))$ satisfying Equation (13). Let

$$2\sqrt{d_1r_1(1-4\epsilon)}=c',$$

then there exists $M > 0$ large (but finite) such that

$$d_1\varphi_1''(\xi) - c'\varphi_1'(\xi) + r_1\varphi_1(\xi)[1-\epsilon - M\varphi_1(\xi)] \le 0.$$

If fact, by Equation (13), there exists $\xi_0 < 0$ such that

$$b_1 \int_{-\tau}^{0} \varphi_1(\xi + cs)d\eta_{11}(s) < \epsilon, \xi < \xi_0.$$

Define $\varepsilon = \inf_{\xi \ge \xi_0} \varphi_1(\xi)$, then Equation (13) indicates $\varepsilon > 0$. Let

$$(M - a_1)\varepsilon = b_1 \sup_{\xi \in \mathbb{R}} \varphi_1(\xi),$$

and we have proved what we wanted.

By the definition $u_1(x,t) = \varphi_1(x + c't)$, then it satisfies

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} \ge d_1 \Delta u(x,t) + r_1 u_1(x,t)\left[1 - \epsilon - M u_1(x,t)\right], \\ u_1(x,0) = \varphi_1(x) > 0. \end{cases}$$

Using the theory of asymptotic spreading [27], we see that

$$\lim_{t\to\infty} \inf_{|x|\le c_1 t} u_1(x,t) > \frac{1-\epsilon}{M}, \tag{14}$$

in which $c_1 = 2\sqrt{d_1 r_1(1-2\epsilon)} > c'$. Let $-2x = (c_1 + c')t$, then a contradiction occurs between Equations (13) and (14) when $t \to \infty$. The proof is complete. □

When $c > \max\{2\sqrt{d_1r_1}, 2\sqrt{d_2r_2}\}$, we define

$$\gamma_i = \frac{c - \sqrt{c^2 - 4d_ir_i}}{2d_i}, \ \gamma_{2+i} = \frac{c + \sqrt{c^2 - 4d_ir_i}}{2d_i}, \ i = 1,2.$$

Further define

$$\Gamma = (\gamma_1, \gamma_3) \bigcap (\gamma_2, \gamma_4).$$

Then, there exists $c_0 \ge \max\{2\sqrt{d_1r_1}, 2\sqrt{d_2r_2}\}$ such that $c > c_0$ implies that $\Gamma$ is nonempty.

Assume that $c > c_0$, then we can choose $\gamma \in \Gamma$. Define continuous functions

$$\overline{\varphi}_i(\xi) = \min\{e^{\gamma_i\xi} + q\overline{k}_i e^{\gamma\xi}, \overline{k}_i\}, \ i = 1,2,$$

in which $q > 1$ is a positive constant clarified later. Moreover, select $\gamma_5, \gamma_6$ such that

$$\gamma_5 \in (\gamma_1, \min\{2\gamma_1, \gamma_3, \gamma_1 + \gamma_2\}), \ \gamma_6 \in (\gamma_2, \min\{2\gamma_2, \gamma_4, \gamma_1 + \gamma_2\})$$

Construct continuous functions as follows

$$\underline{\varphi}_i(\xi) = \max\{e^{\gamma_i\xi} - pe^{\gamma_{4+i}\xi}, 0\}, \ i = 1,2,$$

where $p > 1$ is a positive constant formulated later.

**Lemma 4.** *Assume that $c > c_0$. Then, there exist $p, q$ such that*

$$\begin{cases} d_i\overline{\varphi}_i''(\xi) - c\overline{\varphi}_i'(\xi) + \overline{H}_i(\xi) \le 0, e^{\gamma_i\xi} + q\overline{k}_i e^{\gamma\xi} \ne \overline{k}_i, \\ d_i\underline{\varphi}_i''(\xi) - c\underline{\varphi}_i'(\xi) + \underline{H}_i(\xi) \ge 0, e^{\gamma_i\xi} - pe^{\gamma_{4+i}\xi} \ne 0, \end{cases}$$

*in which*

$$
\begin{aligned}
\overline{H}_1(\xi) &= r_1\overline{\varphi}_1(\xi)\left[1 - a_1\overline{\varphi}_1(\xi) - b_1\int_{-\tau}^{0}\underline{\varphi}_1(\xi+cs)d\eta_{11}(s) + e_1\int_{-\tau}^{0}\overline{\varphi}_2(\xi+cs)d\eta_{12}(s)\right],\\
\overline{H}_2(\xi) &= r_2\overline{\varphi}_2(\xi)\left[1 - a_2\overline{\varphi}_2(\xi) - b_2\int_{-\tau}^{0}\underline{\varphi}_2(\xi+cs)d\eta_{22}(s) + e_2\int_{-\tau}^{0}\overline{\varphi}_1(\xi+cs)d\eta_{21}(s)\right],\\
\underline{H}_1(\xi) &= r_1\underline{\varphi}_1(\xi)\left[1 - a_1\underline{\varphi}_1(\xi) - b_1\int_{-\tau}^{0}\overline{\varphi}_1(\xi+cs)d\eta_{11}(s) + e_1\int_{-\tau}^{0}\underline{\varphi}_2(\xi+cs)d\eta_{12}(s)\right],\\
\underline{H}_2(\xi) &= r_2\underline{\varphi}_2(\xi)\left[1 - a_2\underline{\varphi}_2(\xi) - b_2\int_{-\tau}^{0}\overline{\varphi}_2(\xi+cs)d\eta_{22}(s) + e_2\int_{-\tau}^{0}\underline{\varphi}_1(\xi+cs)d\eta_{21}(s)\right].
\end{aligned}
$$

**Proof.** We first verify the inequality on $\overline{\varphi}_1(\xi)$. If $\overline{\varphi}_1(\xi) = \overline{k}_1$, then the result is clear. Otherwise, we have

$$
\begin{aligned}
\overline{H}_1(\xi) &\le r_1\overline{\varphi}_1(\xi)\left[1 - a_1\overline{\varphi}_1(\xi) + e_1\int_{-\tau}^{0}\overline{\varphi}_2(\xi+cs)d\eta_{12}(s)\right]\\
&\le r_1\overline{\varphi}_1(\xi)\left[1 - a_1\left(e^{\gamma_1\xi} + q\overline{k}_1e^{\gamma\xi}\right) + e_1\left(e^{\gamma_2\xi} + q\overline{k}_2e^{\gamma\xi}\right)\right]\\
&< r_1\overline{\varphi}_1(\xi)\left[1 - a_1e^{\gamma_1\xi} + e_1e^{\gamma_2\xi}\right]\\
&< r_1\overline{\varphi}_1(\xi)\left[1 + e_1e^{\gamma_2\xi}\right],
\end{aligned}
$$

which also implies that

$$
\begin{aligned}
& d_1\overline{\varphi}_1''(\xi) - c\overline{\varphi}_1'(\xi) + \overline{H}_1(\xi)\\
\le\ & (d_1\gamma_1^2 - c\gamma_1 + r_1)e^{\gamma_1\xi}\\
& + q\overline{k}_1e^{\gamma\xi}(d_1\gamma^2 - c\gamma + r_1 + e_1e^{\gamma_2\xi} + e_1e^{(\gamma_1+\gamma_2-\gamma)\xi}).
\end{aligned}
$$

Let $q > 1$ be large enough such that $e^{\gamma_2\xi} + q\overline{k}_1e^{\gamma\xi} > 1$ implies $-\xi > 0$ is large and

$$d_1\gamma^2 - c\gamma + r_1 + e_1e^{\gamma_2\xi} + e_1e^{(\gamma_1+\gamma_2-\gamma)\xi} < 0. \tag{15}$$

Then,

$$d_1\overline{\varphi}_1''(\xi) - c\overline{\varphi}_1'(\xi) + \overline{H}_1(\xi) \le 0.$$

Similarly, if $q > 1$ such that

$$d_2\gamma^2 - c\gamma + r_2 + e_2e^{\gamma_1\xi} + e_2e^{(\gamma_1+\gamma_2-\gamma)\xi} < 0, \tag{16}$$

then

$$d_2\overline{\varphi}_2''(\xi) - c\overline{\varphi}_2'(\xi) + \overline{H}_2(\xi) \le 0.$$

To continue our discussion, we fix $q > 1$ such that Equations (15) and (16) hold.

We now prove the following inequality

$$d_1\underline{\varphi}_1''(\xi) - c\underline{\varphi}_1'(\xi) + \underline{H}_1(\xi) \ge 0,$$

and the result is clear if $\underline{\varphi}_1(\xi) = 0$. Otherwise, we first choose $p_1 > 1$ such that $p > p_1$ indicates that

$$\overline{\varphi}_1(\xi) \le 2e^{\gamma_1\xi}, \overline{\varphi}_2(\xi) \le 2e^{\gamma_2\xi}$$

if $e^{\gamma_1\xi} \geq pe^{\gamma_5\xi}$ or $e^{\gamma_2\xi} \geq pe^{\gamma_6\xi}$. Then

$$\begin{aligned}
\underline{H}_1(\xi) &= r_1\underline{\varphi}_1(\xi)\left[1 - a_1\underline{\varphi}_1(\xi) - b_1\int_{-\tau}^0 \overline{\varphi}_1(\xi+cs)d\eta_{11}(s) + e_1\int_{-\tau}^0 \underline{\varphi}_2(\xi+cs)d\eta_{12}(s)\right] \\
&\geq r_1\underline{\varphi}_1(\xi)\left[1 - a_1\underline{\varphi}_1(\xi) - b_1\int_{-\tau}^0 \overline{\varphi}_1(\xi+cs)d\eta_{11}(s)\right] \\
&\geq r_1\underline{\varphi}_1(\xi)\left[1 - a_1\underline{\varphi}_1(\xi) - 2b_1e^{\gamma_1\xi}\right] \\
&\geq r_1\underline{\varphi}_1(\xi)\left[1 - (a_1+2b_1)e^{\gamma_1\xi}\right] \\
&\geq r_1\underline{\varphi}_1(\xi) - (a_1+2b_1)e^{2\gamma_1\xi}.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
& d_1\underline{\varphi}_1''(\xi) - c\underline{\varphi}_1'(\xi) + \underline{H}_1(\xi) \\
\geq\ & (d_1\gamma_5^2 - c\gamma_5 + r_1)pe^{\gamma_5\xi} - (a_1+2b_1)e^{2\gamma_1\xi}.
\end{aligned}$$

Choosing $p > p_1$ and

$$p > 1 - \frac{a_1+2b_1}{d_1\gamma_5^2 - c\gamma_5 + r_1} > 1,$$

we have obtained

$$d_1\underline{\varphi}_1''(\xi) - c\underline{\varphi}_1'(\xi) + \underline{H}_1(\xi) \geq 0.$$

Similarly, if

$$p > p_1 - \frac{a_1+2b_1}{d_1\gamma_5^2 - c\gamma_5 + r_1} - \frac{a_2+2b_2}{d_2\gamma_6^2 - c\gamma_6 + r_2} > 1,$$

then

$$d_2\underline{\varphi}_2''(\xi) - c\underline{\varphi}_2'(\xi) + \underline{H}_2(\xi) \geq 0.$$

The proof is complete. □

**Remark 2.** For any fixed $c$, we can first choose $q$, then $p$.

**Lemma 5.** *Assume that $c > c_0$. Then, Equation (11) has a strictly positive solution.*

**Proof.** We now prove the result by Schauder's fixed point theorem. Throughout the proof, we assume that $c$ is a fixed constant. Let $\beta > 0$ such that

$$\beta u + r_1u[1 - a_1u - b_1\overline{k}_1], \beta v + r_2v[1 - a_2v - b_2\overline{k}_2]$$

are monotone increasing in $u \in [0, \overline{k}_1], v \in [0, \overline{k}_2]$, respectively. Further, define

$$\lambda_1 = \frac{c - \sqrt{c^2 + 4\beta d_1}}{2d_1}, \qquad \lambda_2 = \frac{c + \sqrt{c^2 + 4\beta d_1}}{2d_1},$$
$$\lambda_3 = \frac{c - \sqrt{c^2 + 4\beta d_2}}{2d_2}, \qquad \lambda_4 = \frac{c + \sqrt{c^2 + 4\beta d_2}}{2d_2}.$$

Let $X$ be the following functional space

$$X = \{\mathbf{u} : \mathbf{u} \text{ is a bounded and uniformly continuous function from } \mathbb{R} \text{ to } \mathbb{R}^2\}.$$

If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ with $\mathbf{a} \leq \mathbf{b}$, then $X_{[\mathbf{a},\mathbf{b}]}$ is defined by

$$X_{[\mathbf{a},\mathbf{b}]} = \{\mathbf{u} \in X : \mathbf{a} \leq \mathbf{u}(\xi) \leq \mathbf{b}, \xi \in \mathbb{R}\}.$$

Let $4\mu = \min\{-\lambda_1, -\lambda_3\}$. Define

$$B_\mu\left(\mathbb{R}, \mathbb{R}^2\right) = \left\{\mathbf{u} \in X : \sup_{\xi \in \mathbb{R}} \|\mathbf{u}(\xi)\| e^{-\mu|\xi|} < \infty\right\}$$

and

$$|\mathbf{u}|_\mu = \sup_{\xi \in \mathbb{R}} \left\{\|\mathbf{u}(\xi)\| e^{-\mu|\xi|}\right\},$$

then $B_\mu\left(\mathbb{R}, \mathbb{R}^2\right)$ is a Banach space with the decay norm $|\cdot|_\mu$, where $\|\cdot\|$ denotes the standard supremum norm in $\mathbb{R}^2$.

We define

$$\Sigma = \{(\varphi_1, \varphi_2) \in X : (\underline{\varphi}_1, \underline{\varphi}_2) \leq (\varphi_1, \varphi_2) \leq (\overline{\varphi}_1, \overline{\varphi}_2)\}.$$

Then, $\Sigma$ is nonempty, convex. It is also bounded and closed in the sense of the decay norm $|\cdot|_\mu$. Moreover, if $(\varphi_1, \varphi_2) \in \Sigma$, we define $P = (P_1, P_2) : \Sigma \to X$ as follows

$$\begin{cases} P_1(\varphi_1, \varphi_2)(\xi) = \frac{1}{d_1(\lambda_2 - \lambda_1)} \left[\int_{-\infty}^{\xi} e^{\lambda_1(\xi - s)} + \int_{\xi}^{\infty} e^{\lambda_2(\xi - s)}\right] [\beta\varphi_1(s) + H_1(\varphi_1, \varphi_2)(s)] ds, \\ P_2(\varphi_1, \varphi_2)(\xi) = \frac{1}{d_2(\lambda_4 - \lambda_3)} \left[\int_{-\infty}^{\xi} e^{\lambda_3(\xi - s)} + \int_{\xi}^{\infty} e^{\lambda_4(\xi - s)}\right] [\beta\varphi_2(s) + H_2(\varphi_1, \varphi_2)(s)] ds, \end{cases} \tag{17}$$

where $(\varphi_1, \varphi_2) \in \Sigma$.

Similar to Ma [28] and Lin and Ruan [18], we can prove that $P : \Sigma \to \Sigma$ is completely continuous in the sense of the decay norm $|\cdot|_\mu$ (please see the Appendix A). Therefore, $P$ has a fixed point in $\Sigma$. Denote the fixed point by $(\varphi_1^*, \varphi_2^*)$, it is clear that $(\varphi_1^*, \varphi_2^*)$ satisfies Equation (11) and is strictly positive. The proof is complete. □

**Theorem 3.** *Assume that $c > c_0$, $b_1\overline{k}_1 < 1$ and $b_2\overline{k}_2 < 1$ hold. Then, Equations (11) and (12) have a strictly positive solution, which is a desired traveling wave solution of Equation (4).*

**Proof.** By what we have done, it suffices to verify the asymptotic behavior of Equation (12). We now prove it by the idea in Lin and Ruan ([18], Section 3). We first prove that

$$\liminf_{\xi \to \infty} \varphi_1(\xi) > 0, \liminf_{\xi \to \infty} \varphi_2(\xi) > 0.$$

In fact, $\varphi_1(\xi)$ satisfies

$$d_1\varphi_1''(\xi) - c\varphi_1'(\xi) + r_1\varphi_1(\xi)[1 - b_1\overline{k}_1 - a_1\varphi_1(\xi)] \leq 0.$$

Let $w(x,t) = \varphi_1(x + ct)$, then

$$\begin{cases} \dfrac{\partial w(x,t)}{\partial t} \geq d_1\Delta w(x,t) + r_1 w(x,t)\left[1 - b_1\overline{k}_1 - a_1 w(x,t)\right], \\ w(x,0) = \varphi_1(x). \end{cases} \tag{18}$$

By the theory of asymptotic spreading (see Aronson and Weinberger [27]) and the basic theory of reaction-diffusion equations (see Ye et al. [29]), if $\varphi_1(0) > 0$, then

$$\liminf_{t \to \infty} w(0,t) \geq (1 - b_1\overline{k}_1)/a_1.$$

By the invariant form of traveling wave solutions, we obtain

$$\liminf_{\xi\to\infty} \varphi_1(\xi) \geq (1 - b_1\bar{k}_1)/a_1 > 0. \tag{19}$$

Similarly, we have

$$\liminf_{\xi\to\infty} \varphi_2(\xi) \geq (1 - b_2\bar{k}_2)/a_2 > 0. \tag{20}$$

Define

$$\liminf_{\xi\to\infty} \varphi_i(\xi) = \varphi_i^-, \limsup_{\xi\to\infty} \varphi_i(\xi) = \varphi_i^+, i = 1,2.$$

Then, there exists $s_0 \in (0,1)$ such that

$$E(s_0) \leq (\varphi_1^-, \varphi_1^-) \leq (\varphi_1^+, \varphi_1^+) \leq F(s_0).$$

Because $E, F$ are continuous functions on bounded interval, if Equation (12) does not hold, then there exists $s_0 \in (0,1)$ such that at least one of the following is true

$$\varphi_i^- = s_0k_i, \varphi_i^+ = s_0k_i + (1-s_0)(\bar{k}_i + \epsilon_i), i = 1,2.$$

If $\varphi_1^- = s_0k_1$, then there exists $\{\xi_m\}_{m\in\mathbb{N}}$ with $\lim_{m\to\infty} \xi_m = \infty$ such that

$$\liminf_{\xi\to\infty} \varphi_1(\xi) = s_0k_1$$

and

$$\liminf_{m\to\infty} (d_1\varphi_1''(\xi_m) - c\varphi_1'(\xi_m)) \geq 0.$$

At the same time, the verification of contracting rectangle implies that

$$\liminf_{m\to\infty} H_1(\varphi_1, \varphi_2)(\xi_m) > 0$$

and a contradiction occurs because $(\varphi_1, \varphi_2)$ is a solution for all $\xi \in \mathbb{R}$. In a similar way, we can verify that

$$E(s) \ll (\varphi_1^-, \varphi_1^-) \leq (\varphi_1^+, \varphi_1^+) \ll F(s) \text{ for all } s \in (0,1),$$

and Equation (12) is true. The proof is complete. □

**Remark 3.** It is possible that $c_0 = \max\{2\sqrt{d_1r_1}, 2\sqrt{d_2r_2}\}$. If $c_0 = \max\{2\sqrt{d_1r_1}, 2\sqrt{d_2r_2}\}$ (e.g., $d_1 \geq d_2$ and $r_1 \geq r_2$), then it is the threshold such that Equations (11) and (12) have a positive solution. Very likely, $c_0$ is the spreading speed of some unknown functions (see [30,31] for a conclusion in predator–prey system).

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## Appendix A

**Lemma A1.** *$P : \Sigma \to \Sigma$ is completely continuous in the sense of the decay norm $|\cdot|_\mu$.*

**Proof.** The proof is similar to that in [32–35]. We first verify that $P : \Sigma \to \Sigma$. Let $\xi_1$ be a constant such that

$$e^{\gamma_1 \xi_1} + q\bar{k}_1 e^{\gamma \xi_1} = \bar{k}_1,$$

then

$$\varphi_1'(\xi_1^-) > \varphi_1'(\xi_1^+) = 0.$$

If $\xi \in (-\infty, \xi_1)$ and $(\varphi_1, \varphi_2) \in \Sigma$, then

$$\begin{aligned}
P_1(\varphi_1,\varphi_2)(\xi) &= \frac{1}{d_1(\lambda_2-\lambda_1)}\left[\int_{-\infty}^{\xi} e^{\lambda_1(\xi-s)} + \int_{\xi}^{\infty} e^{\lambda_2(\xi-s)}\right][\beta\varphi_1(s) + H_1(\varphi_1,\varphi_2)(s)]ds \\
&\le \frac{1}{d_1(\lambda_2-\lambda_1)}\left[\int_{-\infty}^{\xi} e^{\lambda_1(\xi-s)} + \int_{\xi}^{\infty} e^{\lambda_2(\xi-s)}\right][\beta\overline{\varphi}_1(s) + \overline{H}_1(s)]ds \\
&= \frac{1}{d_1(\lambda_2-\lambda_1)}\left[\int_{-\infty}^{\xi} e^{\lambda_1(\xi-s)} + \int_{\xi}^{\xi_1} e^{\lambda_2(\xi-s)}\right][\beta\overline{\varphi}_1(s) + \overline{H}_1(s)]ds \\
&\quad + \frac{1}{d_1(\lambda_2-\lambda_1)}\int_{\xi_1}^{\infty} e^{\lambda_2(\xi-s)}[\beta\overline{\varphi}_1(s) + \overline{H}_1(s)]ds \\
&\le \frac{1}{d_1(\lambda_2-\lambda_1)}\left[\int_{-\infty}^{\xi} e^{\lambda_1(\xi-s)} + \int_{\xi}^{\xi_1} e^{\lambda_2(\xi-s)}\right][\beta\overline{\varphi}_1(s) + c\overline{\varphi}_1'(s) - d_1\overline{\varphi}_1''(s)]ds \\
&\quad + \frac{1}{d_1(\lambda_2-\lambda_1)}\int_{\xi_1}^{\infty} e^{\lambda_2(\xi-s)}[\beta\overline{\varphi}_1(s) + c\overline{\varphi}_1'(s) - d_1\overline{\varphi}_1''(s)]ds \\
&= \overline{\varphi}_1(\xi) + \frac{e^{\lambda_2(\xi-\xi_1)}}{\lambda_2-\lambda_1}\left[\overline{\varphi}_1'(\xi_1^+) - \overline{\varphi}_1'(\xi_1^-)\right] \\
&\le \overline{\varphi}_1(\xi).
\end{aligned}$$

Here, we use the fact

$$\begin{aligned}
&\frac{d}{ds}\left[e^{-\lambda_i s}\left((c-d_1\lambda_i)\phi(s) - d_1\phi'(s)\right)\right] \\
={}& e^{-\lambda_i s}\left[-\lambda_i(c-d_1\lambda_i)\phi(s) + d_1\lambda_i\phi'(s)\right] \\
&+ e^{-\lambda_i s}\left((c-d_1\lambda_i)\phi'(s) - d_1\phi''(s)\right) \\
={}& e^{-\lambda_i s}\left[-d_1\phi''(s) + c\phi'(s) + \beta\phi(s)\right], i = 1, 2,
\end{aligned}$$

if $\phi(s)$ is twice differentiable.

By a similar recipe, we have

$$P_1(\varphi_1,\varphi_2)(\xi) \le \overline{\varphi}_1(\xi) + \frac{e^{\lambda_1(\xi-\xi_1)}}{\lambda_2-\lambda_1}\left[\overline{\varphi}_1'(\xi_1^+) - \overline{\varphi}_1'(\xi_1^-)\right] \le \overline{\varphi}_1(\xi)$$

if $\xi > \xi_1$ and $(\varphi_1, \varphi_2) \in \Sigma$. Due to the continuity of $P_1(\varphi_1, \varphi_2)(\xi), \varphi_1(\xi)$, we have

$$P_1(\varphi_1,\varphi_2)(\xi) \le \overline{\varphi}_1(\xi), \xi \in \mathbb{R}.$$

Similarly, we have

$$\underline{\varphi}_1(\xi) \le P_1(\varphi_1,\varphi_2)(\xi) \le \overline{\varphi}_1(\xi), \underline{\varphi}_2(\xi) \le P_2(\varphi_1,\varphi_2)(\xi) \le \overline{\varphi}_2(\xi)$$

if $\xi \in \mathbb{R}$ and $(\varphi_1, \varphi_2) \in \Sigma$, and so $P : \Sigma \to \Sigma$.

Assume that $(\varphi_1, \varphi_2), (\varphi_3, \varphi_4) \in \Sigma$, then

$$\begin{aligned}
& e^{-\mu|s|} \left|\beta\varphi_1(s) + H_1(\varphi_1, \varphi_2)(s) - [\beta\varphi_3(s) + H_1(\varphi_3, \varphi_4)(s)]\right| \\
\leq\ & (\beta + r_1 + 2r_1a_1\bar{k}_1)e^{-\mu|s|}|\varphi_1(s) - \varphi_3(s)| \\
& + e^{-\mu|s|}r_1b_1 \left|\varphi_1(s)\int_{-\tau}^{0} \varphi_1(s+ct)d\eta_{11}(t) - \varphi_3(s)\int_{-\tau}^{0} \varphi_3(s+ct)d\eta_{11}(t)\right| \\
& + e^{-\mu|s|}r_1e_1 \left|\varphi_1(s)\int_{-\tau}^{0} \varphi_2(s+ct)d\eta_{11}(t) - \varphi_3(s)\int_{-\tau}^{0} \varphi_4(s+ct)d\eta_{11}(t)\right| \\
\leq\ & (\beta + r_1 + 2r_1a_1\bar{k}_1)\left|(\varphi_1, \varphi_2) - (\varphi_3, \varphi_4)\right|_\mu \\
& + r_1b_1\bar{k}_1e^{-\mu|s|}\left\{|\varphi_1(s) - \varphi_3(s)| + \left|\int_{-\tau}^{0} \varphi_1(s+ct)d\eta_{11}(t) - \int_{-\tau}^{0} \varphi_3(s+ct)d\eta_{11}(t)\right|\right\} \\
& + r_1e_1\bar{k}_1e^{-\mu|s|}\left|\varphi_1(s) - \varphi_3(s)\right| \\
& + r_1e_1\bar{k}_2e^{-\mu|s|}\left|\int_{-\tau}^{0} \varphi_2(s+ct)d\eta_{11}(t) - \int_{-\tau}^{0} \varphi_4(s+ct)d\eta_{11}(t)\right| \\
\leq\ & (\beta + r_1 + 2r_1a_1\bar{k}_1)\left|(\varphi_1, \varphi_2) - (\varphi_3, \varphi_4)\right|_\mu \\
& + r_1b_1\bar{k}_1\left|(\varphi_1, \varphi_2) - (\varphi_3, \varphi_4)\right|_\mu \\
& + r_1b_1\bar{k}_1e^{\mu c\tau}\left|\int_{-\tau}^{0} e^{-\mu|s+ct|}\left[\varphi_1(s+ct) - \varphi_3(s+ct)\right]d\eta_{11}(t)\right| \\
& + r_1e_1\bar{k}_1\left|(\varphi_1, \varphi_2) - (\varphi_3, \varphi_4)\right|_\mu \\
& + r_1e_1\bar{k}_2e^{\mu c\tau}\left|\int_{-\tau}^{0} e^{-\mu|s+ct|}\left[\varphi_2(s+ct) - \varphi_4(s+ct)\right]d\eta_{11}(t)\right| \\
\leq\ & L\left|(\varphi_1, \varphi_2) - (\varphi_3, \varphi_4)\right|_\mu,
\end{aligned}$$

where $L > 0$ is defined by

$$L = \beta + r_1 + 2r_1a_1\bar{k}_1 + r_1b_1\bar{k}_1 + r_1b_1\bar{k}_1e^{\mu c\tau} + r_1e_1\bar{k}_1 + r_1e_1\bar{k}_2e^{\mu c\tau}.$$

By the above estimation, we have

$$\begin{aligned}
& \left|P_1(\varphi_1, \varphi_2)(\xi) - P_1(\varphi_3, \varphi_4)(\xi)\right| e^{-\mu|\xi|} \\
=\ & \frac{e^{-\mu|\xi|}}{d_1(\lambda_2 - \lambda_1)}\left|\left[\int_{-\infty}^{\xi} e^{\lambda_1(\xi-s)} + \int_{\xi}^{\infty} e^{\lambda_2(\xi-s)}\right]e^{\mu|s|}e^{-\mu|s|}[\beta\varphi_1(s) + H_1(\varphi_1, \varphi_2)(s)]ds\right. \\
& \left. - \left[\int_{-\infty}^{\xi} e^{\lambda_1(\xi-s)} + \int_{\xi}^{\infty} e^{\lambda_2(\xi-s)}\right]e^{\mu|s|}e^{-\mu|s|}[\beta\varphi_3(s) + H_1(\varphi_3, \varphi_4)(s)]ds\right| \\
\leq\ & \frac{Le^{-\mu|\xi|}\left|(\varphi_1, \varphi_2) - (\varphi_3, \varphi_4)\right|_\mu}{d_1(\lambda_2 - \lambda_1)}\left[\int_{-\infty}^{\xi} e^{\lambda_1(\xi-s)} + \int_{\xi}^{\infty} e^{\lambda_2(\xi-s)}\right]e^{\mu|s|}ds \\
\leq\ & \frac{L\left|(\varphi_1, \varphi_2) - (\varphi_3, \varphi_4)\right|_\mu}{d_1(\lambda_2 - \lambda_1)}\left[\int_{-\infty}^{\xi} e^{(\lambda_1+\mu)(\xi-s)} + \int_{\xi}^{\infty} e^{(\lambda_2-\mu)(\xi-s)}\right]e^{\mu|s|}ds \\
=\ & \frac{L\left|(\varphi_1, \varphi_2) - (\varphi_3, \varphi_4)\right|_\mu}{d_1(\lambda_2 - \lambda_1)}\left[\frac{1}{\lambda_2 - \mu} - \frac{1}{\lambda_1 + \mu}\right],
\end{aligned}$$

and so

$$\sup_{\xi\in\mathbb{R}}\left\{\left|P_1(\varphi_2, \varphi_2)(\xi) - P_1(\varphi_3, \varphi_4)(\xi)\right| e^{-\mu|\xi|}\right\} \leq \frac{L\left|(\varphi_1, \varphi_2) - (\varphi_3, \varphi_4)\right|_\mu}{d_1(\lambda_2 - \lambda_1)}\left[\frac{1}{\lambda_2 - \mu} - \frac{1}{\lambda_1 + \mu}\right].$$

By a similar argument on $P_2$, we see that $P : \Sigma \to \Sigma$ is continuous in the sense of $|\cdot|_\mu$.

We now prove that $P : \Sigma \to \Sigma$ is compact in the sense of $|\cdot|_\mu$. For any given $\epsilon > 0$, there exists a constant $N > 0$ such that

$$e^{-\mu N}(\bar{k}_1 + \bar{k}_2) < \epsilon. \tag{A1}$$

Since $\beta\varphi_i(s) + H_i(\varphi_1, \varphi_2)(s), i = 1, 2,$ are bounded and continuous in $s \in \mathbb{R}$, we see that

$$\begin{aligned} & \left| \frac{d}{d\xi} P_1(\varphi_1, \varphi_2)(\xi) \right| \\ = & \left| \frac{1}{d_1(\lambda_2 - \lambda_1)} \left[ \int_{-\infty}^{\xi} \lambda_1 e^{\lambda_1(\xi - s)} + \int_{\xi}^{\infty} \lambda_2 e^{\lambda_2(\xi - s)} \right] [\beta\varphi_1(s) + H_1(\varphi_1, \varphi_2)(s)] ds \right| \end{aligned}$$

and

$$\begin{aligned} & \left| \frac{d}{d\xi} P_2(\varphi_1, \varphi_2)(\xi) \right| \\ = & \left| \frac{1}{d_2(\lambda_4 - \lambda_3)} \left[ \int_{-\infty}^{\xi} \lambda_3 e^{\lambda_3(\xi - s)} + \int_{\xi}^{\infty} \lambda_4 e^{\lambda_4(\xi - s)} \right] [\beta\varphi_2(s) + H_2(\varphi_1, \varphi_2)(s)] ds \right| \end{aligned}$$

are uniformly bounded. By Ascoli–Arzela lemma, when we restrict $P(\Sigma)$ on $[-N, N]$, we have a finite $\epsilon-$net. By Equation (A1), the net is also a finite $\epsilon-$net of $P(\Sigma)$ in the sense of $|\cdot|_\mu$. The proof is complete. □

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