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Article

The Generalized Quadratic Gauss Sum and Its Fourth Power Mean

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Abstract: In this article, our main purpose is to introduce a new and generalized quadratic Gauss sum. By using analytic methods, the properties of classical Gauss sums, and character sums, we consider the calculating problem of its fourth power mean and give two interesting computational formulae for it.

Keywords: the generalized quadratic Gauss sums; the fourth power mean; analytic method; computational formula

MSC: 11L05, 11L40

1. Introduction

For any integer $q > 1$, let χ denote any Dirichlet character mod q. Then the classical Gauss sum $\tau(\chi, m; q)$ is defined as follows:

$$\tau(\chi, m; q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{am}{q}\right),$$

where m is any integer and $e(y) = e^{2\pi i y}$.

If $(m, q) = 1$ or χ is a primitive character mod q, then it is easy to prove the identity $\tau(\chi, m; q) = \overline{\chi}(m)\tau(\chi, 1; q)$, and $\tau(\chi, 1; q)$ is called the classical Gauss sum. Usually, we denote $\tau(\chi, 1; q)$ as $\tau(\chi)$.

The generalized quadratic Gauss $G(\chi, m; q)$ is defined as

$$G(\chi, m; q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{ma^2}{q}\right). \tag{1}$$

These sums have an important status in the research of analytic number theory. Many famous number theory problems are closely related to them. Therefore, these Gauss sums have been studied by many scholars, and they also have a series of interesting conclusions. For example, Zhang Wenpeng and Hu Jiayuan [1] proved that for any prime p with $p \equiv 1 \bmod 3$ and any third-order character ψ mod p, one has the equation

$$\tau^3(\psi) + \tau^3\left(\overline{\psi}\right) = dp,$$

where d is uniquely determined by $4p = d^2 + 27b^2$ and $d \equiv 1 \bmod 3$.

Chen Li [2] proved that for any prime p with $p \equiv 1 \bmod 6$ and any sixth-order character λ mod p, one gets the identity

$$\tau^3(\lambda) + \tau^3\left(\overline{\lambda}\right) = \begin{cases} p^{\frac{1}{2}}\left(d^2 - 2p\right) & \text{if } p = 12h+1, \\ -i \cdot p^{\frac{1}{2}}\left(d^2 - 2p\right) & \text{if } p = 12h+7, \end{cases}$$

where $i^2 = -1$, d is the same as in the above.

Recently, the classical Gauss sum is also researched by Wang Tingting and Chen Guohui [3]. They proved that

$$W_k(p,\chi) = \frac{\tau^{6k}(\chi)}{\tau^{6k}(\chi^5)} + \frac{\tau^{6k}(\overline{\chi})}{\tau^{6k}(\overline{\chi}^5)}$$

satisfying the second-order linear recurrence Formula

$$W_{k+1}(p,\chi) = \frac{2p^2 - 4pd^2 + d^4}{p^2} W_k(p,\chi) - W_{k-1}(p,\chi),$$

where $W_0(p,\chi) = 2$, $W_1(p,\chi) = \frac{2p^2-4pd^2+d^4}{p^2}$, p is a prime with $p \equiv 1 \bmod 12$, χ be any twelfth-order character $\bmod\ p$, and d is the same as before.

From the properties of the second-order linear recurrence sequence, we may easily calculate the general terms $W_k(\chi, p)$ for any positive integer k. That is,

$$W_k(\chi,p) = \left(\frac{\beta + \sqrt{\beta^2-4}}{2}\right)^k + \left(\frac{\beta - \sqrt{\beta^2-4}}{2}\right)^k,$$

where $\beta = \frac{2p^2-4pd^2+d^4}{p^2}$.

For any odd prime p and any integer n with $(n,p) = 1$, Zhang Wenpeng [4] obtained the identities

$$\sum_{\chi \bmod p} |G(\chi,n;p)|^4 = \begin{cases} (p-1)\left(3p^2 - 6p - 1\right) & \text{if } \ p \equiv 3 \bmod 4, \\ (p-1)\left(3p^2 - 6p - 1 + 4\left(\frac{n}{p}\right)\sqrt{p}\right) & \text{if } \ p \equiv 1 \bmod 4 \end{cases}$$

and

$$\frac{1}{p-1} \sum_{\chi \bmod p} |G(\chi,n;p)|^6 = 10p^3 - 25p^2 - 4p - 1, \text{ if } \ p \equiv 3 \bmod 4.$$

Many other papers related to classical Gauss sums, generalized quadratic Gauss sums, and character sums can also be found in references [5–18], we do not enumerate them one by one here.

In this paper, we introduce a generalized quadratic Gauss sum as follows:

$$\begin{aligned} & G(\chi_1, \chi_2, \cdots, \chi_k; q) \\ = & \sum_{a_1=1}^{q} \sum_{a_2=1}^{q} \cdots \sum_{a_k=1}^{q} \chi_1(a_1)\chi_2(a_2)\cdots\chi_k(a_k) e\left(\frac{(a_1+a_2+\cdots+a_k)^2}{q}\right), \end{aligned} \tag{2}$$

where k is any positive integer and $\chi_i \bmod q$, $1 \le i \le k$.

In fact, if we take $k = 1$, then (2) becomes (1) with $m = 1$. So (2) is also a generalized quadratic Gauss sum, and (1) is a special case of (2). Therefore, $G(\chi_1, \chi_2, \cdots, \chi_k; q)$ is a further promotion and extension of $G(\chi, 1; q)$.

We will consider the $2h$-th power mean of (2) in this paper. i.e.,

$$\sum_{\chi \bmod q} \left| \sum_{a_1=1}^{q} \sum_{a_2=1}^{q} \cdots \sum_{a_k=1}^{q} \chi_1(a_1)\chi_2(a_2)\cdots\chi_k(a_k) e\left(\frac{(a_1+a_2+\cdots+a_k)^2}{q}\right)\right|^{2h}. \tag{3}$$

If $q = p$ is an odd prime and $k = h = 2$, then we will use the analytic method and the properties of classical Gauss sums to give an exact computational formula for (3). That is, we shall prove the following results:

Theorem 1. *Let p be a prime with $p \equiv 3 \bmod 4$. Then for any character $\psi \bmod p$, we have the identity*

$$\begin{aligned} &\frac{1}{p-1}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\psi(b)e\left(\frac{(a+b)^2}{p}\right)\right|^4 \\ = &\begin{cases} p^5-7p^4+17p^3-10p^2-12p-1 & \text{if } \psi=\chi_0, \\ 3p^4-6p^3-p^2 & \text{if } \psi(-1)=-1, \\ 3p^4+E(\psi,p) & \text{if } \psi(-1)=1 \text{ and } \psi\neq\chi_0, \end{cases}\end{aligned}$$

where χ_0 denotes the principal character $\bmod p$, and $|E(\psi,p)| \le 23p^3$.

Theorem 2. *Let p be a prime with $p \equiv 1 \bmod 4$. Then for any character $\psi \bmod p$, we have the identity*

$$\begin{aligned} &\frac{1}{p-1}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)e\left(\frac{(a+b)^2}{p}\right)\right|^4 \\ = \; & p^5-7p^4+17p^3-6p^2-24p-1+4\sqrt{p}\left(p^3-5p^2+7p+1\right)\end{aligned}$$

and

$$\begin{aligned} &\frac{1}{p-1}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\psi(b)e\left(\frac{(a+b)^2}{p}\right)\right|^4 \\ = &\begin{cases} 3p^4-6p^3-p^2+4p^{\frac{5}{2}} & \text{if } \psi(-1)=-1, \\ 3p^4+H(\psi,p) & \text{if } \psi(-1)=1 \text{ and } \psi\neq\chi_0, \end{cases}\end{aligned}$$

where $H(\psi,p)$ satisfies the estimate $|H(\psi,p)| \le 22p^3$.

Theorem 3. *Let p be an odd prime, ψ be a fixed non-principal character $\bmod p$. Then for any character $\chi \bmod p$, we have the upper bound estimate*

$$|G(\chi,\psi;p)| \le 2p.$$

From Theorem 1 and Theorem 2 we may immediately deduce the following:

Corollary 1. *Let p be an odd prime. Then for any odd character $\psi \bmod p$, we have the identity*

$$\begin{aligned} &\frac{1}{p-1}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\psi(b)e\left(\frac{(a+b)^2}{p}\right)\right|^4 \\ = &\begin{cases} 3p^4-6p^3-p^2 & \text{if } p\equiv 3 \bmod 4, \\ 3p^4-6p^3-p^2+4p^{\frac{5}{2}} & \text{if } p\equiv 1 \bmod 4. \end{cases}\end{aligned}$$

Corollary 2. *Let p be an odd prime. Then for any non-principal character $\psi \bmod p$, we have the asymptotic formula*

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\psi(b)e\left(\frac{(a+b)^2}{p}\right)\right|^4 = 3p^5+K(\psi,p),$$

where $K(\psi,p)$ satisfies the estimate $|K(\psi,p)| \le 23p^4$.

Some notes: If ψ is a non-principal even character mod p, then we cannot get the exact value of the sum

$$\left|\sum_{a=1}^{p-1}\psi(a)e\left(\frac{a^2}{p}\right)\right|^4.$$

So, in this case, we can only get a sharp asymptotic formula for mean value (3).

It is clear that if $q = p$ is an odd prime and $k \geq 3$, maybe we can also give an accurate calculating formula for mean value (3). However, in this case, various discussions are required based on the different characters χ_i mod p with $1 \leq i \leq k$, and the situation is more complicated, so we do not discuss it further here.

If $p \equiv 3 \bmod 4$ and $\psi = \chi_0$ or $\psi(-1) = -1$, then from the result in [4] and the method of proving Theorem 1 we can also give an accurate calculating formula for (3) with $k = 3$ and $k = 2$.

For general integer q, does there exist a calculating formula similar to our theorems? These are open problems, which need to be further studied.

2. Several Simple Lemmas

To prove our theorems, we need two simple lemmas. In the process of proving our lemmas, we need to use some basic properties of classical Gaussian sums and character sums, all of which can be found in references [3,6,19], so there is no need to repeat them here.

Lemma 1. *Let p be a prime. For any character χ, ψ mod p with $\chi(-1)\psi(-1) = 1$, if $p \equiv 3 \bmod 4$, then we have the identity*

$$|G(\chi,\psi;p)|^2 = \begin{cases} p(p-2)^2+1 & \text{if } \chi=\psi=\chi_0 \\ \left(p^2+p\right) & \text{if } \chi\psi=\chi_0 \text{ and } \chi\neq\chi_0, \\ p\cdot\left|\sum\limits_{b=1}^{p-1}\chi(b)\psi(b)e\left(\frac{b^2}{p}\right)\right|^2 & \text{if } \chi\psi\neq\chi_0 \text{ and } \chi\neq\chi_0, \\ \left|\sum\limits_{b=1}^{p-1}\psi(b)e\left(\frac{b^2}{p}\right)\right|^2 & \text{if } \chi=\chi_0 \text{ and } \psi\neq\chi_0; \end{cases}$$

If $p \equiv 1 \bmod 4$, then we have the identity

$$|G(\chi,\psi;p)|^2 = \begin{cases} (1+\sqrt{p}(p-2))^2 & \text{if } \chi=\psi=\chi_0 \\ (p-\sqrt{p})^2 & \text{if } \chi\psi=\chi_0 \text{ and } \chi\neq\chi_0, \\ p\cdot\left|\sum\limits_{b=1}^{p-1}\chi(b)\psi(b)e\left(\frac{b^2}{p}\right)\right|^2 & \text{if } \chi\psi\neq\chi_0 \text{ and } \chi\neq\chi_0, \\ \left|\sum\limits_{b=1}^{p-1}\psi(b)e\left(\frac{b^2}{p}\right)\right|^2 & \text{if } \chi=\chi_0 \text{ and } \psi\neq\chi_0, \end{cases}$$

where χ_0 denotes the principal character mod p.

Proof. First, if $\chi(-1)\psi(-1) = -1$, then we have $G(\chi,\psi;p) = 0$. In fact, if $\chi(-1)\psi(-1) = -1$, then from the definition of $G(\chi,\psi;p)$ we have

$$\begin{aligned} G(\chi,\psi;p) &= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(-a)\psi(-b)e\left(\frac{(-a-b)^2}{p}\right) \\ &= \chi(-1)\psi(-1)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\psi(b)e\left(\frac{(a+b)^2}{p}\right) = -G(\chi,\psi;p), \end{aligned}$$

which implies the identity $G(\chi, \psi; p) = 0$.

On the other hand, for any integer k with $(k, p) = 1$, from the properties of the Legendre's symbol mod p (see Theorem 7.5.4 in [3]) we have

$$\begin{aligned} \sum_{a=0}^{p-1} e\left(\frac{ka^2}{p}\right) &= 1 + \sum_{a=1}^{p-1} (1 + \chi_2(a))\, e\left(\frac{ka}{p}\right) \\ &= \sum_{a=1}^{p-1} \chi_2(a) e\left(\frac{ka}{p}\right) = \chi_2(k) \cdot \tau(\chi_2), \end{aligned} \tag{4}$$

where $\chi_2 = \left(\frac{*}{p}\right)$ denotes the Legendre's symbol mod p.

Now if $\chi = \psi = \chi_0$, then we have

$$\begin{aligned} G(\chi, \psi; p) &= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} e\left(\frac{(a+b)^2}{p}\right) = \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} e\left(\frac{b^2(a+1)^2}{p}\right) \\ &= (p-1) + \sum_{a=1}^{p-2}\left(\sum_{b=0}^{p-1} e\left(\frac{b^2(a+1)^2}{p}\right) - 1\right) \\ &= 1 + \sum_{a=1}^{p-2}\sum_{b=0}^{p-1} e\left(\frac{b^2}{p}\right) = 1 + (p-2)\tau(\chi_2). \end{aligned} \tag{5}$$

If $\chi\psi = \chi_0$ and $\chi \neq \chi_0$, then applying the properties of the reduced residue system mod p and (4) we have

$$\begin{aligned} G(\chi, \psi; p) &= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi(a)\psi(b) e\left(\frac{(a+b)^2}{p}\right) = \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \chi(b)\psi(b) e\left(\frac{b^2(a+1)^2}{p}\right) \\ &= \chi(p-1)(p-1) + \sum_{a=1}^{p-2} \chi(a) \left(\sum_{b=0}^{p-1} e\left(\frac{b^2(a+1)^2}{p}\right) - 1\right) \\ &= \chi(-1)p + \sum_{a=1}^{p-2} \chi(a) \sum_{b=0}^{p-1} e\left(\frac{b^2}{p}\right) = \chi(-1)\left(p - \tau(\chi_2)\right). \end{aligned} \tag{6}$$

If $\chi \neq \chi_0$ and $\chi\psi \neq \chi_0$, then we have

$$\begin{aligned} G(\chi, \psi; p) &= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi(a)\psi(b) e\left(\frac{(a+b)^2}{p}\right) = \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \chi(b)\psi(b) e\left(\frac{b^2(a+1)^2}{p}\right) \\ &= \sum_{a=1}^{p-2} \chi(a)\overline{\chi}(a+1)\overline{\psi}(a+1) \sum_{b=1}^{p-1} \chi(b)\psi(b) e\left(\frac{b^2}{p}\right) \\ &= \sum_{a=1}^{p-1} \chi(1-a)\, \psi(a) \sum_{b=1}^{p-1} \chi(b)\psi(b) e\left(\frac{b^2}{p}\right). \end{aligned} \tag{7}$$

Please note that the identity

$$\begin{aligned} \sum_{a=1}^{p-1} \chi(1-a)\, \psi(a) &= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{p-1} \psi(a) \sum_{b=1}^{p-1} \overline{\chi}(b) e\left(\frac{b(1-a)}{p}\right) \\ &= \frac{1}{\tau(\overline{\chi})} \sum_{b=1}^{p-1} \overline{\chi}(b) e\left(\frac{b}{p}\right) \sum_{a=1}^{p-1} \psi(a) e\left(\frac{-ab}{p}\right) = \frac{\psi(-1)\tau(\psi) \cdot \tau\left(\overline{\chi}\overline{\psi}\right)}{\tau(\overline{\chi})}. \end{aligned} \tag{8}$$

Combining (7), (8) and $|\tau(\psi)| = \left|\tau\left(\overline{\chi}\overline{\psi}\right)\right| = |\tau\left(\overline{\chi}\right)| = \sqrt{p}$ we have

$$|G(\chi, \psi; p)|^2 = p \cdot \left|\sum_{b=1}^{p-1} \chi(b)\psi(b)e\left(\frac{b^2}{p}\right)\right|^2. \tag{9}$$

For any even character ψ mod p with $\psi \neq \chi_0$, we have

$$\begin{aligned} G(\chi_0, \psi; p) &= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \psi(b)e\left(\frac{(a+b)^2}{p}\right) = \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \psi(b)e\left(\frac{b^2(a+1)^2}{p}\right) \\ &= \sum_{a=1}^{p-2} \overline{\psi}\left(a+1\right) \sum_{b=1}^{p-1} \psi(b)e\left(\frac{b^2}{p}\right) = -\sum_{b=1}^{p-1} \psi(b)e\left(\frac{b^2}{p}\right). \end{aligned} \tag{10}$$

Please note that if $p \equiv 1 \bmod 4$, then $\tau\left(\chi_2\right) = \sqrt{p}$, and if $p \equiv 3 \bmod 4$, then $\tau\left(\chi_2\right) = i \cdot \sqrt{p}$. So, from (5), (6), (9) and (10) we may deduce Lemma 1 immediately. □

Lemma 2. *Let p be a prime. Then for any Dirichlet character ψ mod p, we have the identity*

$$\sum_{\chi \bmod p} \left|\sum_{a=1}^{p-1} \chi(a)\psi(a)e\left(\frac{a^2}{p}\right)\right|^4 = \begin{cases} (p-1)\left(3p^2 - 6p - 1\right) & \text{if } p \equiv 3 \bmod 4, \\ (p-1)\left(3p^2 - 6p - 1 + 4\sqrt{p}\right) & \text{if } p \equiv 1 \bmod 4. \end{cases}$$

Proof. It is clear that if $\chi(-1)\psi(-1) = -1$, then we have

$$\sum_{a=1}^{p-1} \chi(a)\psi(a)e\left(\frac{a^2}{p}\right) = 0.$$

If $\chi\psi \neq \chi_0$ and $\chi(-1)\psi(-1) = 1$, then from (4) and the properties of the reduced residue system mod p we have

$$\begin{aligned} &\left|\sum_{a=1}^{p-1} \chi(a)\psi(a)e\left(\frac{a^2}{p}\right)\right|^2 = \sum_{a=1}^{p-1} \chi(a)\psi(a) \sum_{b=1}^{p-1} e\left(\frac{b^2\left(a^2-1\right)}{p}\right) \\ &= \left(1 + \chi(-1)\psi(-1)\right)(p-1) + \sum_{a=2}^{p-2} \chi(a)\psi(a) \left(\sum_{b=0}^{p-1} e\left(\frac{b^2\left(a^2-1\right)}{p}\right) - 1\right) \\ &= 2p + \tau\left(\chi_2\right) \cdot \sum_{a=2}^{p-2} \chi(a)\psi(a)\chi_2\left(a^2-1\right) - \sum_{a=1}^{p-1} \chi(a)\psi(a) \\ &= 2p + \tau\left(\chi_2\right) \cdot \sum_{a=1}^{p-1} \chi(a)\psi(a)\chi_2\left(a^2-1\right). \end{aligned} \tag{11}$$

If $\psi = \chi_0$ and $p \equiv 3 \bmod 4$, then from (11) and the orthogonality of the characters mod p we have

$$\begin{aligned} &\sum_{\chi \bmod p} \left|\sum_{a=1}^{p-1} \chi(a)\psi(a)e\left(\frac{a^2}{p}\right)\right|^4 \\ &= \sum_{\substack{\chi \bmod p \\ \chi(-1)=1}} \left|2p + \tau\left(\chi_2\right) \cdot \sum_{a=1}^{p-1} \chi(a)\chi_2\left(a^2-1\right)\right|^2 + \left|\tau\left(\chi_2\right) - 1\right|^4 \\ &\quad - \left|2p + \tau\left(\chi_2\right) \cdot \sum_{a=1}^{p-1} \chi_2\left(a^2-1\right)\right|^2 \end{aligned}$$

$$
\begin{aligned}
&= 2p^2(p-1) + 4p\tau(\chi_2)\sum_{a=1}^{p-1}\chi_2\left(a^2-1\right)\sum_{\chi \bmod p}\chi(a) + \left|\tau\left(\chi_2\right)-1\right|^4 \\
&\quad +\tau^2(\chi_2)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_2\left(a^2-1\right)\chi_2\left(b^2-1\right)\sum_{\chi \bmod p}\chi(ab) - 4p^2 \\
&= 2p^2(p-1) + (p-1)\tau^2(\chi_2)\sum_{a=1}^{p-1}\chi_2\left(a^2-1\right)\chi_2\left(\overline{a}^2-1\right) - 3p^2+2p+1 \\
&= 2p^2(p-1) + p(p-1)(p-3) - 3p^2 + 2p + 1 \\
&= (p-1)\left(3p^2-6p-1\right). \qquad (12)
\end{aligned}
$$

If $\psi = \chi_0$ and $p \equiv 1 \bmod 4$, then note that the identity

$$
\sum_{a=1}^{p-1}\left(\frac{a^2-1}{p}\right) = \sum_{a=1}^{p-1}\left(\frac{a^2+2a}{p}\right) - 1 = \sum_{a=1}^{p-1}\left(\frac{1+2a}{p}\right) - 1 = -2,
$$

from the method of proving (12) we have

$$
\begin{aligned}
&\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)\psi(a)e\left(\frac{a^2}{p}\right)\right|^4 \\
&= 2p^2(p-1) + p(p-1)(p-3) + (\sqrt{p}-1)^4 - \left(2p+\sqrt{p}\sum_{a=1}^{p-1}\chi_2\left(a^2-1\right)\right)^2 \\
&= 2p^2(p-1) + p(p-1)(p-3) + (p+1-2\sqrt{p})^2 - (2p-2\sqrt{p})^2 \\
&= (p-1)\left(3p^2-6p-1+4\sqrt{p}\right). \qquad (13)
\end{aligned}
$$

If $\psi \neq \chi_0$ and $p \equiv 3 \bmod 4$, then from (11) and the method of proving (12) we also have the identity

$$
\begin{aligned}
&\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)\psi(a)e\left(\frac{a^2}{p}\right)\right|^4 \\
&= \sum_{\substack{\chi \bmod p\\ \chi(-1)=\psi(-1)}}\left|2p+\tau\left(\chi_2\right)\cdot\sum_{a=1}^{p-1}\chi(a)\psi(a)\chi_2\left(a^2-1\right)\right|^2 + \left|\tau\left(\chi_2\right)-1\right|^4 \\
&\quad - \left|2p+\tau\left(\chi_2\right)\cdot\sum_{a=1}^{p-1}\chi_2\left(a^2-1\right)\right|^2 \\
&= \sum_{\substack{\chi \bmod p\\ \chi(-1)=1}}\left|2p+\tau\left(\chi_2\right)\cdot\sum_{a=1}^{p-1}\chi(a)\chi_2\left(a^2-1\right)\right|^2 + \left|\tau\left(\chi_2\right)-1\right|^4 \\
&\quad - \left|2p+\tau\left(\chi_2\right)\cdot\sum_{a=1}^{p-1}\chi_2\left(a^2-1\right)\right|^2 = (p-1)\left(3p^2-6p-1\right). \qquad (14)
\end{aligned}
$$

Similarly, if $\psi \neq \chi_0$ and $p \equiv 1 \bmod 4$, then from (11) and the method of proving (13) we have the identity

$$\sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \chi(a)\psi(a) e\left(\frac{a^2}{p}\right) \right|^4 = (p-1)\left(3p^2 - 6p - 1 + 4\sqrt{p}\right). \tag{15}$$

Now Lemma 2 follows from (12), (13), (14) and (15). □

3. Proofs of the Theorems

In this section, we shall complete the proofs of our theorems. First we prove Theorem 1. If $p \equiv 3 \bmod 4$, let ψ be any character mod p, if $\psi = \chi_0$, then from Lemma 1 and Lemma 2 we have

$$\begin{aligned}
\sum_{\chi \bmod p} |G(\chi, \psi; p)|^4 &= \sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a) e\left(\frac{(a+b)^2}{p}\right) \right|^4 \\
&= \sum_{\substack{\chi \bmod p \\ \chi(-1)=1}} \left| \tau(\chi_2) \sum_{a=1}^{p-1} \chi(a) - \sum_{a=1}^{p-1} \chi(a) e\left(\frac{a^2}{p}\right) \right|^4 \\
&= \sum_{\chi \bmod p} \left| \sum_{b=1}^{p-1} \chi(b) e\left(\frac{b^2}{p}\right) \right|^4 + \left(1 + (p-2)^2 p\right)^2 - |\tau(\chi_2) - 1|^4 \\
&= (p-1)\left(3p^2 - 6p - 1\right) + \left(1 + (p-2)^2 p\right)^2 - (p+1)^2 \\
&= (p-1)\left(p^5 - 7p^4 + 17p^3 - 10p^2 - 12p - 1\right).
\end{aligned} \tag{16}$$

If $\psi \neq \chi_0$, then from Lemma 1 and Lemma 2 we have

$$\begin{aligned}
\sum_{\chi \bmod p} |G(\chi, \psi; p)|^4 &= \sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a)\psi(a) e\left(\frac{(a+b)^2}{p}\right) \right|^4 \\
&= p^2 \cdot \sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \chi(a)\psi(a) e\left(\frac{a^2}{p}\right) \right|^4 + \left(1 - p^2\right) \left| \sum_{b=1}^{p-1} \psi(b) e\left(\frac{b^2}{p}\right) \right|^4 \\
&= (p-1)\left(3p^4 - 6p^3 - p^2 - (p+1) \left| \sum_{b=1}^{p-1} \psi(b) e\left(\frac{b^2}{p}\right) \right|^4 \right).
\end{aligned} \tag{17}$$

Please note that if $\psi(-1) = -1$, then we have

$$\sum_{b=1}^{p-1} \psi(b) e\left(\frac{b^2}{p}\right) = 0. \tag{18}$$

If $\psi(-1) = 1$ and $\psi \neq \chi_0$, then we have the estimate

$$\left| \sum_{b=1}^{p-1} \psi(b) e\left(\frac{b^2}{p}\right) \right| \leq 2\sqrt{p}. \tag{19}$$

Combining (16)–(19) we may immediately deduce Theorem 1.

Similarly, we can use the method of proving Theorem 1 to prove Theorem 2. If $p \equiv 1 \bmod 4$ and $\psi = \chi_0$, then from Lemma 1 and Lemma 2 we have

$$\begin{aligned}
\sum_{\chi \bmod p} |G(\chi, m; p)|^4 &= \sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a) e\left(\frac{(a+b)^2}{p}\right) \right|^4 \\
&= \sum_{\substack{\chi \bmod p \\ \chi(-1)=1}} \left| \tau(\chi_2) \sum_{a=1}^{p-1} \chi(a) - \sum_{a=1}^{p-1} \chi(a) e\left(\frac{a^2}{p}\right) \right|^4 \\
&= \sum_{\chi \bmod p} \left| \sum_{b=1}^{p-1} \chi(b) e\left(\frac{b^2}{p}\right) \right|^4 + (1 + (p-2)\sqrt{p})^4 - (\sqrt{p} - 1)^4 \\
&= (p-1)\left(3p^2 - 6p - 1 + 4\sqrt{p}\right) + (1 + (p-2)\sqrt{p})^4 - (\sqrt{p}-1)^4 \\
&= (p-1)\left[p^5 - 7p^4 + 17p^3 - 6p^2 - 24p - 1 + 4\sqrt{p}\left(p^3 - 5p^2 + 7p + 1\right)\right]
\end{aligned}$$

or

$$\begin{aligned}
&\frac{1}{p-1} \sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a) e\left(\frac{(a+b)^2}{p}\right) \right|^4 \\
&\quad = p^5 - 7p^4 + 17p^3 - 6p^2 - 24p - 1 + 4\sqrt{p}\left(p^3 - 5p^2 + 7p + 1\right). \qquad (20)
\end{aligned}$$

If $p \equiv 1 \bmod 4$ and $\psi \neq \chi_0$, then form Lemma 1 and Lemma 2 we have

$$\begin{aligned}
\sum_{\chi \bmod p} |G(\chi, \psi; p)|^4 &= \sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a)\psi(a) e\left(\frac{(a+b)^2}{p}\right) \right|^4 \\
&= p^2 \cdot \sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \chi(a)\psi(a) e\left(\frac{a^2}{p}\right) \right|^4 + \left(1 - p^2\right) \left| \sum_{b=1}^{p-1} \psi(b) e\left(\frac{b^2}{p}\right) \right|^4 \\
&= (p-1)\left(3p^4 - 6p^3 - p^2 + 4p^{\frac{5}{2}} - (p+1)\left| \sum_{b=1}^{p-1} \psi(b) e\left(\frac{b^2}{p}\right) \right|^4\right). \qquad (21)
\end{aligned}$$

Combining (18)–(21) we complete the proof of Theorem 2.

For any characters χ and ψ mod p with $\psi \neq \chi_0$, applying (19) and Lemma 1 we may immediately deduce the upper bound estimate

$$|G(\chi, \psi; p)| \leq 2p.$$

This completes the proof of Theorem 3.

4. Conclusions

The main results of this paper are Theorem 1, Theorem 2, and Corollary 1. They obtain some exact expressions for the fourth power mean (3) with $k = 2$ and $q = p$, an odd prime. For Corollary 1 in particular, the result is very simple and beautiful. These works have good references for further research on generalized multivariate quadratic Gauss sums. In addition, these theorems also profoundly reveal the regularity of the value distribution of this kind of new Gauss sum. In other words, its value is mainly concentrated on $G(\chi_0, \chi_0; p)$, where χ_0 is the principal character mod p.

For the general integer $q > 1$ (or $q = p$ and $k \geq 3$), we also proposed two open problems. These will contribute to the further study of these contents.

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