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p-Topologicalness—A Relative Topologicalness in $\top$-Convergence Spaces

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Abstract: In this paper, p-topologicalness (a relative topologicalness) in $\top$-convergence spaces are studied through two equivalent approaches. One approach generalizes the Fischer's diagonal condition, the other approach extends the Gähler's neighborhood condition. Then the relationships between p-topologicalness in $\top$-convergence spaces and p-topologicalness in stratified L-generalized convergence spaces are established. Furthermore, the lower and upper p-topological modifications in $\top$-convergence spaces are also defined and discussed. In particular, it is proved that the lower (resp., upper) p-topological modification behaves reasonably well relative to final (resp., initial) structures.

Keywords: fuzzy topology; fuzzy convergence; lattice-valued convergence; $\top$-convergence space; relative topologicalness; p-topologcalness; diagonal condition; neighborhood condition

1. Introduction

The theory of convergence spaces [1] is natural extension of the theory of topological spaces. The topologicalness is important in the theory of convergence spaces since it mainly researches the condition of a convergence space to be a topological space. Generally, two equivalent approaches are used to characterize the topologicalness in convergence spaces. One approach is stated by the well-known Fischer's diagonal condition [2], the other approach is stated by Gähler's neighborhood condition [3]. In [4], by considering a pair of convergence spaces (X, p) and (X, q), Wilde and Kent investigated a kind of relative topologicalness, called p-topologicalness. When $p = q$, p-topologicalness is equivalent to topologicalness in convergence spaces. They also defined and discussed the lower and upper p-topological modifications in convergence spaces. Precisely, for a pair of convergence spaces (X, p) and (X, q), the lower (resp., upper) p-topological modification of (X, q) is defined as the finest (resp., coarsest) p-topological convergence space which is coarser (resp., finer) than (X, q). Similarly, a topological modification of (X, q) is defined as the finest topological convergence space which is coarser than (X, q).

Lattice-valued convergence spaces are common extension of convergence spaces and lattice-valued topological spaces. It should be pined out that lattice-valued convergence spaces are established on the basis of fuzzy sets. However, the lattice structure is used to replace the unit interval $[0, 1]$ as the truth table for membership degrees. In recent years, two kinds of lattice-valued convergence spaces received much attention: (1) the theory of stratified L-generalized convergence spaces based on L-filters, which is initiated by Jäger [5] and then developed by many researchers [6–25]; and (2) the theory of $\top$-convergence spaces based on $\top$-filters, which is investigated by Fang [26] in 2017. The topologicalness in stratified L-generalized convergence spaces was studied by Jäger [27–29] and Li [30,31], the p-topologicalness and p-topological modifications in stratified L-generalized convergence spaces were discussed by Li [32,33].

The topologicalness in $\top$-convergence spaces was researched by Fang [26] and Li [34]. In this paper, we shall consider the p-topologicalness and p-topological modifications in $\top$-convergence spaces.

The contents are arranged as follows. Section 2 recalls some basic notions as preliminary. Section 3 discusses the p-topologicalness in $\top$-convergence spaces by generalized Fischer's diagonal condition and generalized Gähler's neighborhood condition, respectively. Then the relationships between p-topologicalness in $\top$-convergence spaces and p-topologicalness in stratified L-generalized convergence spaces are established. Section 4 focuses on p-topological modifications in $\top$-convergence spaces. The lower and upper p-topological modifications in $\top$-convergence spaces are defined and discussed. Particularly, it is proved that the lower (resp., upper) p-topological modification behaves reasonably well relative to final (resp., initial) structures.

2. Preliminaries

Let L be a complete lattice with the top element $\top$ and the bottom element $\perp$. For a commutative quantale, we mean a pair $(L,*)$ such that $*$ is a commutative semigroup operation on L with the condition

$$\forall a \in L, \forall \{b_j\}_{j\in J} \subseteq L, a * \bigvee_{j\in J} b_j = \bigvee_{j\in J}(a * b_j).$$

$(L,*)$ is called integral if the top element $\top$ is the unique unit, i.e., $\forall a \in L, \top * a = a$. For any $a \in L$, each function $a*(-) : L \longrightarrow L$ has a right adjoint $a \to (-) : L \longrightarrow L$ defined as $a \to b = \bigvee\{c \in L : a * c \leq b\}$. In the following, we list the usual properties of $*$ and $\to$[35].

(1) $a \to b = \top \Leftrightarrow a \leq b$;
(2) $a * b \leq c \Leftrightarrow b \leq a \to c$;
(3) $a * (a \to b) \leq b$;
(4) $a \to (b \to c) = (a * b) \to c$;
(5) $(\bigvee\limits_{j\in J} a_j) \to b = \bigwedge\limits_{j\in J}(a_j \to b)$;
(6) $a \to (\bigwedge\limits_{j\in J} b_j) = \bigwedge\limits_{j\in J}(a \to b_j)$.

We call $(L,*)$ to be a meet continuous lattice if the complete lattice L is meet continuous [36], that is, $(L,\leq)$ satisfies the distributive law: $a \wedge (\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a \wedge b_i)$, for any $a \in L$ and any directed subsets $\{b_i | i \in I\}$ in L.

L is said to be continuous if $(L,\leq)$ is a continuous lattice [36], that is, for any nonempty family $\{a_{j,k} | j \in J, k \in K(j)\}$ in L with $\{a_{j,k} | k \in K(j)\}$ is directed for all $j \in J$, the identity

$$(\mathbf{DD}) \qquad \bigwedge_{j\in J} \bigvee_{k\in K(j)} a_{j,k} = \bigvee_{h\in N} \bigwedge_{j\in J} a_{j,h(j)}$$

holds, where N is the set of all choice functions on J with values $h(j) \in K(j)$ for all $j \in J$. Obviously, continuity implies meet-continuity for L.

In this article, unless otherwise stated, we always assume that $L = (L,*)$ is a commutative, integral, and meet continuous quantale.

A function $\mu : X \to L$ is called an L-fuzzy set in X, and all L-fuzzy sets in X is denoted as L^X. The operations $\vee, \wedge, *, \to$ on L can be translated pointwisely onto L^X. Said precisely, for any $\mu, \nu \in L^X$ and any $\{\mu_t | t \in T\} \subseteq L^X$,

$$\mu \leq \nu \text{ iff } \mu(x) \leq \nu(x) \text{ for any } x \in X,$$
$$(\bigvee_{i\in I} \mu_t)(x) = \bigvee_{t\in T} \mu_t(x), (\bigwedge_{t\in T} \mu_t)(x) = \bigwedge_{t\in T} \mu_t(x),$$
$$(\mu * \nu)(x) = \mu(x) * \nu(x), (\mu \to \nu)(x) = \mu(x) \to \nu(x).$$

We don't distinguish between a constant function and its value because no confusion will occur.

Let $f : X \longrightarrow Y$ be a function. We define $f^{\rightarrow} : L^X \longrightarrow L^Y$ and $f^{\leftarrow} : L^Y \longrightarrow L^X$ [35] by $f^{\rightarrow}(\mu)(y) = \bigvee_{f(x)=y} \mu(x)$ for $\mu \in L^X$ and $y \in Y$, and $f^{\leftarrow}(\nu)(x) = \nu(f(x))$ for $\nu \in L^Y$ and $x \in X$.

Let μ, ν be L-fuzzy sets in X. The subsethood degree [37–40] of μ, ν, denoted by $S_X(\mu, \nu)$, is defined by

$$S_X(\mu, \nu) = \bigwedge_{x \in X} (\mu(x) \rightarrow \nu(x)).$$

2.1. $\top$-Filters and Stratified L-Filters

A filter on a set X is an upper set of $(2^X, \subseteq)$ (2^X denotes the power set of X) wich is closed for finite meets and does not contain the empty set. The conception of filter has been generalized to the fuzzy setting in two methods; prefilters (or $\top$-filters more general) and L-filters. Both prefilters ($\top$-filters) and L-filters play important roles in the theory of fuzzy topology, see [26,27,34,35,41–44].

Definition 1 ([35]). *A nonempty subset $\mathbb{F} \subseteq L^X$ is called a $\top$-filter on the set X whenever:*
(TF1) $\bigvee_{x \in X} \lambda(x) = \top$ for all $\lambda \in \mathbb{F}$, (TF2) $\lambda \wedge \mu \in \mathbb{F}$ for all $\lambda, \mu \in \mathbb{F}$,
(TF3) if $\lambda \in L^X$ such that $\bigvee_{\mu \in \mathbb{F}} S_X(\mu, \lambda) = \top$, then $\lambda \in \mathbb{F}$.
The set of all $\top$-filters on X is denoted as $\mathbb{F}_L^{\top}(X)$.

Definition 2 ([35]). *A nonempty subset $\mathbb{B} \subseteq L^X$ is called a $\top$-filter base on the set X provided:*
(TB1) $\bigvee_{x \in X} \lambda(x) = \top$ for all $\lambda \in \mathbb{B}$, (TB2) if $\lambda, \mu \in \mathbb{B}$, then $\bigvee_{\nu \in \mathbb{B}} S_X(\nu, \lambda \wedge \mu) = \top$.

Each $\top$-filter base generates a $\top$-filter $\mathbb{F}_{\mathbb{B}}$ defined by

$$\mathbb{F}_{\mathbb{B}} := \{\lambda \in L^X | \bigvee_{\mu \in \mathbb{B}} S_X(\mu, \lambda) = \top\}.$$

Example 1 ([26,45]). *Let $f : X \longrightarrow Y$ be a function, $\mathbb{F} \in \mathbb{F}_L^{\top}(X)$ and $\mathbb{G} \in \mathbb{F}_L^{\top}(Y)$. Then*

(1) The family $\{f^{\rightarrow}(\lambda) | \lambda \in \mathbb{F}\}$ forms a $\top$-filter base on Y, and the $\top$-filter $f^{\Rightarrow}(\mathbb{F})$ generated by it is called the image of $\mathbb{F}$ under f. It is easily seen that $\mu \in f^{\Rightarrow}(\mathbb{F}) \Longleftrightarrow f^{\leftarrow}(\mu) \in \mathbb{F}$.

(2) The family $\{f^{\leftarrow}(\mu) | \mu \in \mathbb{G}\}$ forms a $\top$-filter base on Y if and only if $\bigvee_{y \in f(X)} \mu(y) = \top$ holds for all $\mu \in \mathbb{G}$, and the $\top$-filter $f^{\Leftarrow}(\mathbb{G})$ (if exists) generated by it is called the inverse image of $\mathbb{G}$ under f. Additionally, $\mathbb{G} \subseteq f^{\Rightarrow}(f^{\Leftarrow}(\mathbb{G}))$ holds whenever $f^{\Leftarrow}(\mathbb{G})$ exists. Particularly, $f^{\Leftarrow}(\mathbb{G})$ always exists and $\mathbb{G} = f^{\Rightarrow}(f^{\Leftarrow}(\mathbb{G}))$ if f is surjective.

(3) For any $x \in X$, the family $[x]_{\top} =: \{\lambda \in L^X | \lambda(x) = \top\}$ is a $\top$-filter on X, and $f^{\Rightarrow}([x]_{\top}) = [f(x)]_{\top}$.

A stratified L-filter [35] on a set X is a function $\mathcal{F} : L^X \longrightarrow L$ such that: $\forall \lambda, \mu \in L^X, \forall \alpha \in L$, (LF1) $\mathcal{F}(\bot) = \bot$, $\mathcal{F}(\top) = \top$; (LF2) $\mathcal{F}(\lambda) \wedge \mathcal{F}(\mu) = \mathcal{F}(\lambda \wedge \mu)$; (LFs) $\mathcal{F}(\alpha * \lambda) \geq \alpha * \mathcal{F}(\lambda)$.

The set of all stratified L-filters on X is denoted as $\mathcal{F}_L^s(X)$. A stratified L-filter $\mathcal{F}$ is called tight if $\mathcal{F}(\alpha) = \alpha$ for each $\alpha \in L$.

Example 2 ([35]). *Let $f : X \longrightarrow Y$ be a function, $\mathcal{F} \in \mathcal{F}_L^s(X)$ and $\mathcal{G} \in \mathcal{F}_L^s(Y)$. Then*

(1) The function $f^{\Rightarrow}(\mathcal{F}) : L^Y \longrightarrow L$ defined by $\mu \mapsto \mathcal{F}(\mu \circ f)$ is a stratified L-filter on Y called the image of $\mathcal{F}$ under f.

(2) For any $x \in X$, the function $[x] : L^X \longrightarrow L, [x](\lambda) = \lambda(x)$ is a stratified L-filter on X, and $f^{\Rightarrow}([x]) = [f(x)]$.

For each $\mathbb{F} \in \mathbb{F}_L^{\top}(X)$, define $\Lambda(\mathbb{F}) : L^X \longrightarrow L$ as

$$\forall \lambda \in L^X, \Lambda(\mathbb{F})(\lambda) = \bigvee_{\mu \in \mathbb{F}} S_X(\mu, \lambda),$$

then $\Lambda(\mathbb{F})$ is a tightly stratified L-filter on X [44].

Conversely, for each tightly stratified L-filter $\mathcal{F}$ on a set X, the family

$$\Gamma(\mathcal{F}) = \{\lambda \in L^X, \mathcal{F}(\lambda) = \top\}$$

is a $\top$-filter on X [44]. Given $\mathbb{F} \in \mathbb{F}_L^{\top}(X)$, we have $\Gamma\Lambda(\mathbb{F}) = \mathbb{F}$.

Lemma 1. *Let* $\{\mathbb{F}_j\}_{j\in J} \subseteq \mathbb{F}_L^{\top}(X)$. *If* L *is continuous then* $\Lambda(\bigcap_{j\in J} \mathbb{F}_j) = \bigwedge_{j\in J} \Lambda(\mathbb{F}_j)$.

Proof. For any $\lambda \in L^X$ and any $j \in J$, note that $\{S_X(\mu_j, \lambda) | \mu_j \in \mathbb{F}_j\}$ is a directed subset of L. Then

$$\begin{aligned} \bigwedge_{j\in J} \Lambda(\mathbb{F}_j)(\lambda) &= \bigwedge_{j\in J} \bigvee_{\mu_j \in \mathbb{F}_j} S_X(\mu_j, \lambda) \overset{(\mathbf{DD})}{=} \bigvee_{h\in N} \bigwedge_{j\in J} S_X(h(j), \lambda) \\ &= \bigvee_{h\in N} S_X(\bigvee_{j\in J} h(j), \lambda), \text{ by } \bigvee_{j\in J} h(j) \in \bigcap_{j\in J} \mathbb{F}_j \\ &\leq \bigvee_{\nu \in \bigcap_{j\in J} \mathbb{F}_j} S_X(\nu, \lambda) \leq \Lambda(\bigcap_{j\in J} \mathbb{F}_j)(\lambda). \end{aligned}$$

Thus $\Lambda(\bigcap_{j\in J} \mathbb{F}_j)(\lambda) = \bigwedge_{j\in J} \Lambda(\mathbb{F}_j)(\lambda)$ since $\Lambda(\bigcap_{j\in J} \mathbb{F}_j)(\lambda) \leq \bigwedge_{j\in J} \Lambda(\mathbb{F}_j)(\lambda)$ holds obviously. □

2.2. $\top$-Convergence Spaces and Stratified L-Generalized Convergence Spaces

Definition 3. *A* $\top$*-convergence structure* [26] *on a set* X *is a function* $q : \mathbb{F}_L^{\top}(X) \longrightarrow 2^X$ *satisfying*

(TC1) $[x]_{\top} \xrightarrow{q} x$ *for every* $x \in X$; **(TC2)** *if* $\mathbb{F} \xrightarrow{q} x$ *and* $\mathbb{F} \subseteq \mathbb{G}$, *then* $\mathbb{G} \xrightarrow{q} x$, *where* $\mathbb{F} \xrightarrow{q} x$ *is shorthand for* $x \in q(\mathbb{F})$. *The pair* (X, q) *is called a* $\top$*-convergence space.*

A function $f : X \longrightarrow X'$ between two $\top$-convergence spaces (X, q), (X', q') is called continuous if $f^{\Rightarrow}(\mathbb{F}) \xrightarrow{q'} f(x)$ whenever $\mathbb{F} \xrightarrow{q} x$.

The category whose objects are $\top$-convergence spaces and whose morphisms are continuous functions will be denoted by $\top$**-CS.** This category is topological over **SET** [26,46].

For a given source $(X \xrightarrow{f_i} (X_i, q_i))_{i\in I}$, the initial structure [47], q on X is defined by

$$\mathbb{F} \xrightarrow{q} x \Longleftrightarrow \forall i \in I, f_i^{\Rightarrow}(\mathbb{F}) \xrightarrow{q_i} f_i(x).$$

For a given sink $((X_i, q_i) \xrightarrow{f_i} X)_{i\in I}$, the final structure, q on X is defined as

$$\mathbb{F} \xrightarrow{q} x \Longleftrightarrow \begin{cases} \mathbb{F} \supseteq [x]_{\top}, & x \notin \cup_{i\in I} f_i(X_i); \\ \mathbb{F} \supseteq f_i^{\Rightarrow}(\mathbb{G}_i), & \exists i \in I, x_i \in X_i, \mathbb{G}_i \in \mathbb{F}_L^{\top}(X_i) \text{ s.t } f(x_i) = x, \mathbb{G}_i \xrightarrow{q_i} x_i. \end{cases}$$

Thus, when $X = \cup_{i\in I} f_i(X_i)$, the final structure q can be simplified as

$$\mathbb{F} \xrightarrow{q} x \Longleftrightarrow \mathbb{F} \supseteq f_i^{\Rightarrow}(\mathbb{G}_i) \text{ for some } \mathbb{G}_i \xrightarrow{q_i} x_i \text{ with } f(x_i) = x.$$

For a nonempty set X, we use $\top(X)$ to denote all $\top$-convergence structures on X. For $p, q \in \top(X)$, we say that q is finer than p, or p is coarser than q, denoted by $p \leq q$ for short, if the identity $\mathrm{id}_X : (X, q) \longrightarrow (X, p)$ is continuous, that is, $\mathbb{F} \xrightarrow{q} x \Longrightarrow \mathbb{F} \xrightarrow{p} x$. It is easily observed from [26,47] that $(\top(X), \leq)$ forms a completed lattice, and the discrete (resp., indiscrete) structure δ (resp., ι) is the top (resp., bottom) element of $(T(X), \leq)$, where δ is given by $\mathbb{F} \xrightarrow{\delta} x$ iff $\mathbb{F} \supseteq [x]_\top$; and ι is given by $\mathbb{F} \xrightarrow{\iota} x$ for all $\mathbb{F} \in \mathbb{F}_L^\top(X)$, $x \in X$.

Definition 4. *(Jäger [5] and Yao [25]) A stratified L-generalized convergence structure on a set X is a function* $\lim^q : \mathcal{F}_L^s(X) \longrightarrow L^X$ *satisfying*

(LC1) $\lim^q [x](x) = 1$ *for every* $x \in X$; **(LC2)** $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$, $\mathcal{F} \leq \mathcal{G} \Longrightarrow \lim^q \mathcal{F} \leq \lim^q \mathcal{G}$.

The pair $(X, \lim^q)$ *is called a stratified L-generalized convergence space.*

Let (X, q) be a $\top$-convergence space. We define $\lim^q : \mathcal{F}_L^s(X) \longrightarrow L^X$ as

$$\lim^q \mathcal{F}(x) = \begin{cases} \top, & \mathcal{F} \geq \Lambda(\mathbb{F}) \text{ for some } \mathbb{F} \xrightarrow{q} x; \\ \bot, & \text{otherwise.} \end{cases}$$

Note that $[x] = \Lambda([x]_\top)$. It follows that $(X, \lim^q)$ is a stratified L-generalized convergence space.

Remark 1. *When* $L = \{\bot, \top\}$, *both* $\top$*-convergence spaces and stratified L-generalized convergence spaces all reduce to convergence spaces. Therefore, these two kinds of lattice-valued convergence spaces are all natural extensions of convergence spaces.*

3. p-Topologicalness in $\top$-Convergence Spaces

In this section, we shall discuss the p-topologicalness in $\top$-convergence spaces by generalized Fischer's diagonal condition and generalized Gähler's neighborhood condition, respectively. We also try to establish the relationships between p-topologicalness in $\top$-convergence spaces and p-topologicalness in stratified L-generalized convergence spaces.

3.1. *p-Pretopologicalness in* $\top$*-Convergence Spaces*

Let (X, p) be a $\top$-convergence space. Then for any $x \in X$, the $\top$-filter

$$\mathbb{U}_p(x) = \cap\{\mathbb{F} \in \mathbb{F}_L^\top(X) | \mathbb{F} \xrightarrow{p} x\}$$

is called the $\top$-neighborhood with respect to p at x. Then the family $\mathbb{U}_p := \{\mathbb{U}_p(x)\}_{x \in X}$ is called the $\top$-neighborhood system generated by (X, p) [26]. It is easily seen that if $p, p' \in T(X)$ and $p \leq p'$ then $\mathbb{U}_p(x) \subseteq \mathbb{U}_{p'}(x)$ for any $x \in X$.

In the following, we shorten a pair of $\top$-convergence spaces (X, p) and (X, q) as (X, p, q). It is easy to check that the following conditions are equivalent:

p**-(TP1)**: $\forall \{\mathbb{F}_j\}_{j \in J} \subseteq \mathbb{F}_L^\top(X), \forall x \in X, \forall j \in J, \mathbb{F}_j \xrightarrow{p} x \Longrightarrow \bigcap_{j \in J} \mathbb{F}_j \xrightarrow{q} x$.

p**-(TP2)**: $\forall \mathbb{F} \in \mathbb{F}_L^\top(X), \forall x \in X, \mathbb{F} \supseteq \mathbb{U}_p(x) \Longrightarrow \mathbb{F} \xrightarrow{q} x$.

p**-(TP3)**: $\forall x \in X$, $\mathbb{U}_p(x) \xrightarrow{q} x$.

Definition 5. *Assume that* (X, p, q) *is a pair of* $\top$*-convergence spaces. Then q is said to be p-pretopological if it fulfills either of the above three conditions.*

Remark 2. *When $p = q$, p-pretopologicalness is precise the pretopologicalness in [26]. In this case, it is observed easily that the "$\Longrightarrow$" in p-***(TP2)** *can be replaced with "$\Longleftrightarrow$". In the following, when $p = q$, we omit the prefix "p" in symbols p-***(TP1)***–p-***(TP3)***. This simplification is also used for the subsequent p-topological conditions.*

Proposition 1. *A $\top$-convergence structure q on X is pretopological iff it is p-pretopological for any $q \in \top(X)$ with $q \leq p$.*

Proof. Let (X, q) be pretopological and $q \leq p$. Then by $q \leq p$ we have $\mathbb{U}_p(x) \supseteq \mathbb{U}_q(x)$ for any $x \in X$. By pretopologicalness of q we get that $\mathbb{U}_q(x) \xrightarrow{q} x$. It follows that $\mathbb{U}_p(x) \xrightarrow{q} x$. Thus q is p-pretopological. The converse implication is obvious. □

The following example shows there is no p-pretopologicalness implies pretopologicalness in general.

Example 3. *Let L be the linearly ordered frame $(\{\bot, \alpha, \top\}, \wedge, \top)$ with $\bot < \alpha < \top$, and $X = \{x, y\}$. For each $\mathbb{F} \in \mathbb{F}_L^\top(X)$ and $z \in X$, let $\mathbb{F} \xrightarrow{p} z \Longleftrightarrow \mathbb{F} \supseteq [z]$. In [26], it is proved that (X, p) is a $\top$-convergence space and for each $z \in X, \mathbb{U}_p(z) = [z]$.*

For $x, y \in X$, it is easily seen that the subsets $\mathbb{F}_x, \mathbb{F}_y$ of L^X defined by

$$\mathbb{F}_x = \{\lambda \in L^X : \lambda(x) \geq \alpha, \lambda(y) = \top\};\ \mathbb{F}_y(\lambda) = \{\lambda \in L^X : \lambda(y) \geq \alpha, \lambda(x) = \top\}$$

are all $\top$-filters on X. For each $\mathbb{F} \in \mathbb{F}_L^\top(X)$ and each $z \in X$, let $\mathbb{F} \xrightarrow{q} z \Longleftrightarrow \mathbb{F} \supseteq [z]$ or *$\mathbb{F} \supseteq \mathbb{F}_z$. Then (X, q) is a $\top$-convergence space. For each $z \in X$, $\mathbb{U}_q(z) = [z] \cap \mathbb{F}_z = \{\top_X\}$ and so $[z] \cap \mathbb{F}_z \neq [z], \mathbb{F}_z$.*

*Obviously, q satisfies p-***(TP3)***. But q is not pretopological since we have no $\mathbb{U}_q(z) \xrightarrow{q} z$.*

3.2. p-Topologicalness in $\top$-Convergence Spaces

At first, we fix the notions of diagonal $\top$-filter and neighborhood $\top$-filter to state p-topologicalness. Let J, X be any sets and $\phi : J \longrightarrow \mathbb{F}_L^\top(X)$ be any function. Then a function $\hat{\phi} : L^X \to L^J$ is defined as

$$\forall \lambda \in L^X, \forall j \in J, \hat{\phi}(\lambda)(j) = \Lambda(\phi(j))(\lambda) = \bigvee_{\mu \in \phi(j)} S_X(\mu, \lambda).$$

For all $\mathbb{F} \in \mathbb{F}_L^\top(J)$, it is proved that a subset of L^X defined by

$$k\phi\mathbb{F} := \{\lambda \in L^X | \hat{\phi}(\lambda) \in \mathbb{F}\}$$

is a $\top$-filter, called diagonal $\top$-filter of $\mathbb{F}$ under ϕ [26]. In addition, for any $\lambda, \mu \in L^X$, it was proved in [26] that $S_X(\lambda, \mu) \leq S_J(\hat{\phi}(\lambda), \hat{\phi}(\mu))$.

Definition 6 ([34])**.** *Let (X, p) be a $\top$-convergence space and $\mathbb{U}_p : X \longrightarrow \mathbb{F}_L^\top(X)$ be the $\top$-neighborhood system generated by (X, p). Then for each $\mathbb{F} \in \mathbb{F}_L^\top(X)$, the $\top$-filter $\mathbb{U}_p(\mathbb{F}) := k\mathbb{U}_p\mathbb{F}$, is called neighborhood $\top$-filter of $\mathbb{F}$ w.r.t. p.*

Let $\mathbb{N}$ be the set of natural numbers including 0. Let (X, p) be a $\top$- convergence space and $\mathbb{F} \in \mathbb{F}_L^\top(X)$. For any $n \in \mathbb{N}$, we define $\mathbb{U}_p^0(\mathbb{F}) = \mathbb{F}$, and if $\mathbb{U}_p^n(\mathbb{F})$ has been defined, then we define the $n + 1$ th iteration of the neighborhood $\top$-filter of $\mathbb{F}$ inductive by $\mathbb{U}_p^{n+1}(\mathbb{F}) = \mathbb{U}_p(\mathbb{U}_p^n(\mathbb{F}))$.

Proposition 2. *Let (X, p) be a $\top$- convergence space, $n \in \mathbb{N}$ and $\mathbb{F}, \mathbb{G} \in \mathbb{F}_L^\top(X)$. Then*

(1) $\mathbb{U}_p^n(\mathbb{F}) \subseteq \mathbb{F}$,

(2) *if* $\mathbb{F} \subseteq \mathbb{G}$, *then* $\mathbb{U}_p^n(\mathbb{F}) \subseteq \mathbb{U}_p^n(\mathbb{G})$,
(3) *if* $p' \in T(X)$ *and* $p \leq p'$, *then* $\mathbb{U}_p^n(\mathbb{F}) \subseteq \mathbb{U}_{p'}^n(\mathbb{F})$.

Proof. It is obvious. □

Definition 7. *Let* $f : (X, q) \longrightarrow (Y, p)$ *be a function between* $\top$*-convergence spaces. Then* f *is said to be an interior function if* $f^{\rightarrow}(\widehat{\mathbb{U}_q}(\lambda)) \leq \widehat{\mathbb{U}_p}(f^{\rightarrow}(\lambda))$ *for all* $\lambda \in L^X$.

Proposition 3. *Let* $f : (X, q) \longrightarrow (Y, p)$ *be a function between* $\top$*- convergence spaces and* $\mathbb{F} \in \mathbb{F}_L^{\top}(X)$.
(1) *If* f *is continuous, then* $f^{\Rightarrow}(\mathbb{U}_q^n(\mathbb{F})) \supseteq \mathbb{U}_p^n(f^{\Rightarrow}(\mathbb{F}))$.
(2) *If* f *is an interior function, then* $f^{\Rightarrow}(\mathbb{U}_q^n(\mathbb{F})) \subseteq \mathbb{U}_p^n(f^{\Rightarrow}(\mathbb{F}))$.

Proof. (1) We prove $f^{\Rightarrow}(\mathbb{U}_q^n(\mathbb{F})) \supseteq \mathbb{U}_p^n(f^{\Rightarrow}(\mathbb{F}))$ inductively.

For each $\mathbb{F} \xrightarrow{q} x$ and each $\lambda \in \mathbb{U}_p(f(x))$ we have $\lambda \in f^{\Rightarrow}(\mathbb{F})$, i.e., $f^{\leftarrow}(\lambda) \in \mathbb{F}$ and then

$$f^{\leftarrow}(\lambda) \in \cap\{\mathbb{F} | \mathbb{F} \xrightarrow{q} x\} = \mathbb{U}_q(x),$$

i.e., $\lambda \in f^{\Rightarrow}(\mathbb{U}_q(x))$. Thus $\mathbb{U}_p(f(x)) \subseteq f^{\Rightarrow}(\mathbb{U}_q(x))$.

Fixing $\lambda \in L^Y$, we get

$$\begin{aligned}\widehat{\mathbb{U}_p}(\lambda)(f(x)) = \bigvee_{\mu \in \mathbb{U}_p(f(x))} S_Y(\mu, \lambda) \leq \bigvee_{f^{\leftarrow}(\mu) \in \mathbb{U}_q(x)} S_X(f^{\leftarrow}(\mu), f^{\leftarrow}(\lambda)) \leq \\ \bigvee_{\nu \in \mathbb{U}_q(x)} S_X(\nu, f^{\leftarrow}(\lambda)) = \widehat{\mathbb{U}_q}(f^{\leftarrow}(\lambda))(x).\end{aligned}$$

It follows that $f^{\leftarrow}(\widehat{\mathbb{U}_p}(\lambda)) = \widehat{\mathbb{U}_q}(f^{\leftarrow}(\lambda))$. Thus

$$\begin{aligned}\lambda \in \mathbb{U}_p(f^{\Rightarrow}(\mathbb{F})) &\implies \widehat{\mathbb{U}_p}(\lambda) \in f^{\Rightarrow}(\mathbb{F}) \Longrightarrow f^{\leftarrow}(\widehat{\mathbb{U}_p}(\lambda)) \in \mathbb{F} \Longrightarrow \widehat{\mathbb{U}_q}(f^{\leftarrow}(\lambda)) \in \mathbb{F} \\ &\implies f^{\leftarrow}(\lambda) \in \mathbb{U}_q(\mathbb{F}) \Longrightarrow \lambda \in f^{\Rightarrow}(\mathbb{U}_q(\mathbb{F})).\end{aligned}$$

So, $f^{\Rightarrow}(\mathbb{U}_q^n(\mathbb{F})) \supseteq \mathbb{U}_p^n(f^{\Rightarrow}(\mathbb{F}))$ when $n = 1$.

We assume that $f^{\Rightarrow}(\mathbb{U}_q^n(\mathbb{F})) \supseteq \mathbb{U}_p^n(f^{\Rightarrow}(\mathbb{F}))$ when $n = k$. Then we need to check that $f^{\Rightarrow}(\mathbb{U}_q^n(\mathbb{F})) \supseteq \mathbb{U}_p^n(f^{\Rightarrow}(\mathbb{F}))$ when $n = k + 1$. Indeed,

$$f^{\Rightarrow}(\mathbb{U}_q^{k+1}(\mathbb{F})) = f^{\Rightarrow}(\mathbb{U}_q(\mathbb{U}_q^k(\mathbb{F}))) \supseteq \mathbb{U}_p(f^{\Rightarrow}(\mathbb{U}_q^k(\mathbb{F}))) \supseteq \mathbb{U}_p(\mathbb{U}_p^k(f^{\Rightarrow}(\mathbb{F}))) = \mathbb{U}_p^{k+1}(f^{\Rightarrow}(\mathbb{F})).$$

(2) We check only the inequalities for $n = 1$.

Let f be an interior function. For each $\lambda \in \mathbb{U}_q(\mathbb{F})$, i.e., $\widehat{\mathbb{U}_q}(\lambda) \in \mathbb{F}$ we have $f^{\rightarrow}(\widehat{\mathbb{U}_q}(\lambda)) \in f^{\Rightarrow}(\mathbb{F})$ and then $\widehat{\mathbb{U}_p}(f^{\rightarrow}(\lambda)) \in f^{\Rightarrow}(\mathbb{F})$ by f is an interior function. That means $f^{\leftarrow}(\lambda) \in \mathbb{U}_p(f^{\Rightarrow}(\mathbb{F}))$. Thus $f^{\Rightarrow}(\mathbb{U}_q(\mathbb{F})) \subseteq \mathbb{U}_p(f^{\Rightarrow}(\mathbb{F}))$. □

Now, we tend our attention to p-topologicalness.

We say a pair of $\top$-convergence spaces (X, p, q) satisfy the Gähler $\top$-neighborhood condition if p**-(TG)**: $\forall \mathbb{F} \in \mathbb{F}_L^{\top}(X), \forall x \in X, \mathbb{F} \xrightarrow{q} x \Longrightarrow \mathbb{U}_p(\mathbb{F}) \xrightarrow{q} x$.

Definition 8. *Let* (X, p, q) *be a pair of* $\top$*-convergence spaces. Then* q *is called* p*-topological if the condition* p*-(TG) is satisfied.*

Remark 3. *When $L = \{\perp, \top\}$, the condition p-(TG) is precise the Gähler neighborhood condition in [4], which is used to define p-topological convergence spaces. Therefore, our p-topologicalness is a natural extension of crisp p-topologicalness.*

We say a pair of $\top$-convergence spaces (X, p, q) satisfy the Fischer $\top$-diagonal condition if

p**-(TF)**: Let J, X be any sets, $\psi : J \longrightarrow X$, and $\phi : J \longrightarrow \mathbb{F}_L^\top(X)$ such that $\phi(j) \xrightarrow{p} \psi(j)$, for each $j \in J$. Then for each $\mathbb{F} \in \mathbb{F}_L^\top(J)$ and each $x \in X$, $\psi^\Rightarrow(\mathbb{F}) \xrightarrow{q} x$ implies $k\phi\mathbb{F} \xrightarrow{q} x$.

Restricting $J = X$ and $\psi =$id in p**-(TF)**, we obtain a weaker condition p**-(TK)**. When $p = q$, p**-(TF)** is precise the Fischer $\top$-diagonal condition **(TF)**, and p**-(TK)** is precise the Kowalsky $\top$-diagonal condition **(TK)** in [26].

Proposition 4. *Let (X, p, q) be a pair of $\top$-convergence spaces. Then (1) p-(TF)$\Longrightarrow$p-(TP1)+p-(TK), and (2) p-(TK)$\Longrightarrow$p-(TF) if p satisfies (TP1).*

Proof. (1) Obviously, p-(TF)$\Longrightarrow$ p-(TK). Now, we check p-(TF)$\Longrightarrow$ p-(TP1). Let $\{\mathbb{F}_j\}_{j\in J} \subseteq \mathbb{F}_L^\top(X)$ and $x \in X$ satisfy $\forall j \in J, \mathbb{F}_j \xrightarrow{p} x$. Take $\psi(j) \equiv x$, $\phi(j) = \mathbb{F}_j$ and $\mathbb{F} = \mathbb{F}_\perp$ (i.e., $\mathbb{F}_\perp = \{\top_J\}$, the smallest $\top$-filter on J) in p-(TF), then it is easily seen that $\psi^\Rightarrow(\mathbb{F}_\perp) = [x]_\top$ and $k\phi\mathbb{F}_\perp = \bigcap_{j\in J}\mathbb{F}_j$. Because $\psi^\Rightarrow(\mathbb{F}_\perp) = [x]_\top \xrightarrow{q} x$ we have $k\phi\mathbb{F}_\perp = \bigcap_{j\in J}\mathbb{F}_j \xrightarrow{q} x$ by p-(TF).

(2) Let J, X, ψ, ϕ satisfy the condition of p-(TF). Then we define a function $\widetilde{\phi} : X \longrightarrow \mathbb{F}_L^\top(X)$ as $\widetilde{\phi}(x) = \bigcap\{\phi(j) : j \in J, \psi(j) = x\}$ if there exists $j \in J$ such that $\psi(j) = x$ and $\widetilde{\phi}(x) = [x]_\top$ if not so. For each $x \in X$, if $\widetilde{\phi}(x) = [x]_\top$ then $\widetilde{\phi}(x) \xrightarrow{p} x$. If $\widetilde{\phi}(x) = \bigcap\{\phi(j) : j \in J, \psi(j) = x\}$ then by $\phi(j) \xrightarrow{p} x$ and (TP1) we have $\widetilde{\phi}(x) \xrightarrow{p} x$. Let $\mathbb{F} \in \mathbb{F}_L^\top(J)$ and $\psi^\Rightarrow(\mathbb{F}) \xrightarrow{q} x$. Then by p-(TK) we obtain $k\widetilde{\phi}\psi^\Rightarrow(\mathbb{F}) \xrightarrow{q} x$. One can prove that $k\phi\mathbb{F} \supseteq k\widetilde{\phi}\psi^\Rightarrow(\mathbb{F})$. Thus $k\phi\mathbb{F} \xrightarrow{q} x$. □

Corollary 1. *Let (X, p, q) be a pair of $\top$-convergence spaces. If p satisfies (TP1) then p-(TF)$\Leftrightarrow$p-(TK)+p-(TP1). In particular, when $p = q$ we have (TF)$\Longleftrightarrow$(TK)+(TP1) [26].*

Remark 4. *Let L, X and $\mathbb{F}_z(z \in X)$ be defined as in Example 3. Let q be defined as $\mathbb{F} \xrightarrow{q} z$ for any $\mathbb{F} \in \mathbb{F}_L^\top(X)$ and any $z \in X$, and let p be defined as $\mathbb{F} \xrightarrow{p} z \Longleftrightarrow \mathbb{F} \supset \mathbb{F}_\perp$. Then (X, p, q) is a pair of $\top$-convergence spaces. Obviously, the axiom p-(TF) is satisfied. But p does not fulfill the axiom (TP1) since $\mathbb{U}_p(z) = \mathbb{F}_\perp \overset{p}{\not\longrightarrow} z$. Thus this example shows that p-(TF) does not imply (TP1) of (X, p) generally. Therefore, we guess that the additional condition (TP1) in the above corollary can not be removed.*

The following theorem shows that if we restricting the lattice-context slightly, p-topologicalness can be described by Fischer $\top$-diagonal condition p**-(TF)**.

Theorem 1. *Let (X, p, q) be a pair of $\top$-convergence spaces. Then p-(TG)$\Longrightarrow$p-(TF), and the converse inclusion holds if L is continuous.*

Proof. p-**(TG)**$\Longrightarrow$$p$-**(TF)**. Let J, X, ϕ, ψ satisfy the condition of p-**(TF)**. For any $\mathbb{F} \in \mathbb{F}_L^{\top}(J)$, we prove below that $\mathbb{U}_p(\psi^{\Rightarrow}(\mathbb{F})) \subseteq k\phi\mathbb{F}$. Let $\lambda \in L^X$,

$$\begin{aligned}
\bigvee_{\mu\in\mathbb{F}} S_X(\psi^{\rightarrow}(\mu), \widehat{\mathbb{U}_p}(\lambda)) &= \bigvee_{\mu\in\mathbb{F}} \bigwedge_{x\in X} ((\bigvee_{\psi(j)=x} \mu(j)) \rightarrow \bigvee_{\nu\in\mathbb{U}_p(x)} S_X(\nu, \lambda)) \\
&= \bigvee_{\mu\in\mathbb{F}} \bigwedge_{j\in J} (\mu(j) \rightarrow \bigvee_{\nu\in\mathbb{U}_p(\psi(j))} S_X(\nu, \lambda)), \text{ by } \phi(j) \xrightarrow{p} \psi(j) \\
&\leq \bigvee_{\mu\in\mathbb{F}} \bigwedge_{j\in J} (\mu(j) \rightarrow \bigvee_{\nu\in\phi(j)} S_X(\nu, \lambda)) \\
&= \bigvee_{\mu\in\mathbb{F}} \bigwedge_{j\in J} (\mu(j) \rightarrow \hat{\phi}(\lambda)(j)) = \bigvee_{\mu\in\mathbb{F}} S_J(\mu, \hat{\phi}(\lambda)).
\end{aligned}$$

It follows that

$$\lambda \in \mathbb{U}_p(\psi^{\Rightarrow}(\mathbb{F})) \quad \Longrightarrow \quad \bigvee_{\mu\in\mathbb{F}} S_X(\psi^{\rightarrow}(\mu), \widehat{\mathbb{U}_p}(\lambda)) = \top \Longrightarrow \bigvee_{\mu\in\mathbb{F}} S_J(\mu, \hat{\phi}(\lambda)) = \top \Longrightarrow \lambda \in k\phi\mathbb{F}.$$

Thus $\mathbb{U}_p(\psi^{\Rightarrow}(\mathbb{F})) \subseteq k\phi\mathbb{F}$.

If $\psi^{\Rightarrow}(\mathbb{F}) \xrightarrow{q} x$ then it follows by p-**(TG)** that $\mathbb{U}_p(\psi^{\Rightarrow}(\mathbb{F})) \xrightarrow{q} x$, and so $k\phi\mathbb{F} \xrightarrow{q} x$. That is, p-**(TF)** is satisfied.

p-**(TF)**$\Longrightarrow$$p$-**(TG)**. Note that Lemma 1 holds since L is continuous. Take

$$J = \{(\mathbb{G}, y) \in \mathbb{F}_L^{\top}(X) \times X | \mathbb{G} \xrightarrow{p} y\}; \psi : J \longrightarrow X, (\mathbb{G}, y) \mapsto y; \phi : J \longrightarrow \mathbb{F}_L^{\top}(X), (\mathbb{G}, y) \mapsto \mathbb{G}.$$

Then $\forall j \in J$, $\phi(j) \xrightarrow{p} \psi(j)$. Because $[y] \xrightarrow{p} y$ we have that ψ is a surjective function. Thus for each $\mathbb{F} \in \mathbb{F}_L^{\top}(X)$, $\mathbb{H} = \psi^{\Leftarrow}(\mathbb{F}) \in \mathbb{F}_L^{\top}(J)$ exists and $\psi^{\Rightarrow}(\mathbb{H}) = \mathbb{F}$.

We prove below that $k\phi\mathbb{H} = \mathbb{U}_p(\mathbb{F})$. For any $y \in X$, denote $I_y = \{\mathbb{G} \in \mathbb{F}_L^{\top}(X) | \mathbb{G} \xrightarrow{p} y\}$. Then for any $\lambda \in L^X$,

$$\begin{aligned}
\bigvee_{\mu\in\mathbb{F}} S_J(\psi^{\leftarrow}(\mu), \hat{\phi}(\lambda)) &= \bigvee_{\mu\in\mathbb{F}} \bigwedge_{(\mathbb{G},y)\in J} (\psi^{\leftarrow}(\mu)(\mathbb{G}, y) \rightarrow \hat{\phi}(\lambda)(\mathbb{G}, y)) \\
&= \bigvee_{\mu\in\mathbb{F}} \bigwedge_{y\in X} \bigwedge_{\mathbb{G}\in I_y} (\mu(y) \rightarrow \Lambda(\mathbb{G})(\lambda)) \\
&= \bigvee_{\mu\in\mathbb{F}} \bigwedge_{y\in X} (\mu(y) \rightarrow \bigwedge_{\mathbb{G}\in I_y} \Lambda(\mathbb{G})(\lambda)), \text{ by Lemma 1} \\
&= \bigvee_{\mu\in\mathbb{F}} \bigwedge_{y\in X} (\mu(y) \rightarrow \Lambda(\bigcap_{\mathbb{G}\in I_y} \mathbb{G})(\lambda)) \\
&= \bigvee_{\mu\in\mathbb{F}} \bigwedge_{y\in X} (\mu(y) \rightarrow \Lambda(\mathbb{U}_p(y))(\lambda)) \\
&= \bigvee_{\mu\in\mathbb{F}} \bigwedge_{y\in X} (\mu(y) \rightarrow \widehat{\mathbb{U}_p}(\lambda)(y)) = \bigvee_{\mu\in\mathbb{F}} S_X(\mu, \widehat{\mathbb{U}_p}(\lambda)).
\end{aligned}$$

It follows that

$$\begin{aligned}
\lambda \in k\phi\mathbb{H} \quad &\Longleftrightarrow \quad \hat{\phi}(\lambda) \in \psi^{\Leftarrow}(\mathbb{F}) \Longleftrightarrow \bigvee_{\mu\in\mathbb{F}} S_J(\psi^{\leftarrow}(\mu), \hat{\phi}(\lambda)) = \top \Longleftrightarrow \bigvee_{\mu\in\mathbb{F}} S_X(\mu, \widehat{\mathbb{U}_p}(\lambda)) = \top \\
&\Longleftrightarrow \quad \widehat{\mathbb{U}_p}(\lambda) \in \mathbb{F} \Longleftrightarrow \lambda \in \mathbb{U}_p(\mathbb{F}).
\end{aligned}$$

Thus $k\phi\mathbb{H} = \mathbb{U}_p(\mathbb{F})$.

Let $\mathbb{F} = \psi^{\Rightarrow}(\mathbb{H}) \xrightarrow{q} x$. Then by p-**(TF)** we have $k\phi\mathbb{H} = \mathbb{U}_p(\mathbb{F}) \xrightarrow{q} x$. That is, p-**(TG)** holds. □

The following theorem shows that for pretopological $\top$-convergence spaces, p-topologicalness can be described by Fischer $\top$-diagonal condition p-**(TF)**.

Theorem 2. *Let (X, p, q) be a pair of $\top$-convergence spaces and (X, p) be pretopological. Then p-(TF)$\Longleftrightarrow$p-(TG).*

Proof. Most of the proof can copy that of Theorem 1. We only check that

$$\bigvee_{\mu\in\mathbb{F}} S_J(\psi^{\leftarrow}(\mu), \hat{\phi}(\lambda)) = \bigvee_{\mu\in\mathbb{F}} S_X(\mu, \widehat{\mathbb{U}_p}(\lambda)),$$

for any $\lambda \in L^X$ in p-**(TF)**$\Longrightarrow$$p$-**(TG)**. Indeed, since p is pretopological then $\mathbb{U}_p(y) \in I_y$ for any $y \in X$. Thus

$$\begin{aligned}
\bigvee_{\mu\in\mathbb{F}} S_J(\psi^{\leftarrow}(\mu), \hat{\phi}(\lambda)) &= \bigvee_{\mu\in\mathbb{F}} \bigwedge_{(\mathbb{G},y)\in J} (\psi^{\leftarrow}(\mu)(\mathbb{G}, y) \to \hat{\phi}(\lambda)(\mathbb{G}, y)) \\
&= \bigvee_{\mu\in\mathbb{F}} \bigwedge_{y\in X} \bigwedge_{\mathbb{G}\in I_y} (\mu(y) \to \Lambda(\mathbb{G})(\lambda)) \\
&= \bigvee_{\mu\in\mathbb{F}} \bigwedge_{y\in X} (\mu(y) \to \bigwedge_{\mathbb{G}\in I_y} \Lambda(\mathbb{G})(\lambda)), \text{ by } \mathbb{U}_p(y) \in I_y \\
&= \bigvee_{\mu\in\mathbb{F}} \bigwedge_{y\in X} (\mu(y) \to \Lambda(\mathbb{U}_p(y))(\lambda)) \\
&= \bigvee_{\mu\in\mathbb{F}} S_X(\mu, \widehat{\mathbb{U}_p}(\lambda)). \quad \square
\end{aligned}$$

By Corollary 1 and Theorem 2 we get the following corollary.

Corollary 2. *[34] Let (X, p) be a $\top$-convergence space. Then (TF)$\Longleftrightarrow$(TG).*

Remark 5. *The above corollary is one of the main results in [34]. Based on this equivalence, it was proved that $\top$-convergence spaces with (TF) or (TG) characterize precisely the conical L-topological spaces in [44].*

The following theorem shows that p-topologicalness is preserved under initial constructions.

Theorem 3. *Let $\{(X_i, q_i, p_i)\}_{i\in I}$ be pairs of $\top$-convergence spaces with each q_i being p_i-topological. If q (resp., p) is the initial structure on X relative to the source $(X \xrightarrow{f_i} (X_i, q_i))_{i\in I}$ (resp., $(X \xrightarrow{f_i} (X_i, p_i))_{i\in I}$), then (X, q) is p-topological.*

Proof. Let $\mathbb{F} \xrightarrow{q} x$. Then by definition of q, we have $f_i^{\Rightarrow}(\mathbb{F}) \xrightarrow{q_i} f_i(x)$ for any $i \in I$. Because q_i is p_i-topological we have $\mathbb{U}_{p_i}(f_i^{\Rightarrow}(\mathbb{F})) \xrightarrow{q_i} f_i(x)$. Then by Proposition 3 (1) we have $f_i^{\Rightarrow}(\mathbb{U}_p(\mathbb{F})) \supseteq \mathbb{U}_{p_i}(f_i^{\Rightarrow}(\mathbb{F}))$ and so $f_i^{\Rightarrow}(\mathbb{U}_p(\mathbb{F})) \xrightarrow{q_i} f_i(x)$ for all $i \in I$. That is, $\mathbb{U}_p(\mathbb{F}) \xrightarrow{q} x$. Thus q is p-topological. □

The next theorem shows that p-topologicalness is preserved under final constructions with some additional conditions.

Theorem 4. *Let $\{(X_i, q_i, p_i)\}_{i\in I}$ be pairs of $\top$- convergence spaces with each q_i being p_i-topological. Let q (resp., p) be the final structure on X w.r.t. The sink $((X_i, q_i) \xrightarrow{f_i} X)_{i\in I}$ (resp., $((X_i, p_i) \xrightarrow{f_i} X)_{i\in I}$). If $X = \cup_{i\in I} f_i(X_i)$ and each $f_i : (X_i, p_i) \longrightarrow (X, p)$ is an interior function, then (X, q) is p-topological.*

Proof. Let $\mathbb{F} \xrightarrow{q} x$. Then by definition of q, there exists $i \in I, x_i \in X_i, \mathbb{G}_i \in \mathbb{F}_L^{\top}(X_i)$ such that $f_i(x_i) = x, f_i^{\Rightarrow}(\mathbb{G}_i) \subseteq \mathbb{F}$ and $\mathbb{G}_i \xrightarrow{q_i} x_i$.

By $f_i^{\Rightarrow}(\mathbb{G}_i) \subseteq \mathbb{F}$ and f_i is a interior function we have $f_i^{\Rightarrow}(\mathbb{U}_{p_i}(\mathbb{G}_i)) \subseteq \mathbb{U}_p(f_i^{\Rightarrow}(\mathbb{G}_i)) \subseteq \mathbb{U}_p(\mathbb{F})$.

By $\mathbb{G}_i \xrightarrow{q_i} x_i$ and q_i is p_i-topological we have $\mathbb{U}_{p_i}(\mathbb{G}_i) \xrightarrow{q_i} x_i$, and then $f_i^{\Rightarrow}(\mathbb{U}_{p_i}(\mathbb{G}_i)) \xrightarrow{q} f_i(x_i) = x$.

Then it follows that $\mathbb{U}_p(\mathbb{F}) \xrightarrow{q} x$. By Theorem 1 we get that q is p-topological. □

From Theorem 3 and Theorem 4, we conclude easily the following corollary. It will tell us that p-topologicalness is preserved under supremum and infimum in the lattice $\top(X)$.

Corollary 3. *Let $\{q_i | i \in I\} \subseteq \top(X)$ and $p \in \top(X)$ such that each (X, q_i) is p-topological. Then both $(X, \inf\{q_i\}_{i\in I})$ and $(X, \sup\{q_i\}_{i\in I})$ are all p-topological.*

3.3. On the Relationship between p-Topologicalness in ⊤-Convergence Spaces and in Stratified L-Generalized Convergence Spaces

Let J, X be any set and $\Phi : J \longrightarrow \mathcal{F}_L^s(X)$ be any function. Then a function $\hat{\Phi} : L^X \to L^J$ is defined as $\forall \lambda \in L^X, \forall j \in J, \hat{\Phi}(\lambda)(j) = \Phi(j)(\lambda)$. For all $\mathcal{F} \in \mathcal{F}_L^s(J)$, it is proved that the function $K\Phi\mathcal{F} : L^X \longrightarrow L$ defined by $\forall \lambda \in L^X, K\Phi\mathcal{F}(\lambda) = \mathcal{F}(\hat{\Phi}(\lambda))$ is a stratified L-filter, which is called the diagonal L-filter of $\mathcal{F}$ under Φ [27,30].

Let $(X, \lim^p)$ be a stratified L-generalized convergence space. For any $\alpha \in L, x \in X$, let $\mathcal{U}_p^{\alpha}(x) = \bigwedge\{\mathcal{F} : \lim^p \mathcal{F}(x) \geq \alpha\}$. Take $\Phi = \mathcal{U}_p^{\alpha} : X \longrightarrow \mathcal{F}_L^s(X)$, then for each $\mathcal{F} \in \mathcal{F}_L^s(X)$, the stratified L-filter $\mathcal{U}_p^{\alpha}(\mathcal{F}) := k\mathcal{U}_p^{\alpha}\mathcal{F}$ is called α-level neighborhood L-filter of $\mathcal{F}$ w.r.t. $\lim^p$ [29].

We say a pair of stratified L-generalized convergence spaces $(X, \lim^p, \lim^q)$ satisfy the Fischer L-diagonal condition if

p-**(LF)**: Let J, X be any sets, $\Psi : J \longrightarrow X$ and $\Phi : J \longrightarrow \mathcal{F}_L^s(X)$ be functions.

$$\forall \mathcal{F} \in \mathcal{F}_L^s(J), \forall x \in X, \lim{}^q \Psi^{\Rightarrow}(\mathcal{F})(x) \wedge \bigwedge_{j\in J} \lim{}^p \Phi(j)(\Psi(j)) \leq \lim{}^q K\Phi\mathcal{F}(x).$$

We say a pair of stratified L-generalized convergence spaces $(X, \lim^p, \lim^q)$ satisfy the Gähler L-neighborhood condition if p-(LG): $\forall \alpha \in L, \forall \mathcal{F} \in \mathcal{F}_L^s(X), \alpha * \lim^q \mathcal{F} \leq \lim^q \mathcal{U}_p^{\alpha}(\mathcal{F})$.

It was proved in [32] that p-(LF)$\Longleftrightarrow$ p-(LG).

Definition 9 ([32]). *Let $(X, \lim^p, \lim^q)$ be a pair of stratified L-generalized convergence spaces. Then $\lim^q$ is called p-topological if the condition p-(LF) or p-(LG) is satisfied.*

Lemma 2. *Let $\phi : J \longrightarrow \mathbb{F}_L^{\top}(X)$ be any function and $\mathbb{F} \in \mathbb{F}_L^{\top}(J)$. Then*

(1) $\Lambda(k\phi\mathbb{F}) \leq K(\Lambda \circ \phi)\Lambda(\mathbb{F})$;

(2) $k\phi\mathbb{F} = \Gamma(K(\Lambda \circ \phi)\Lambda(\mathbb{F}))$.

Proof. (1) Let $\lambda \in L^X$. Then for any $j \in J, \widehat{\phi}(\lambda)(j) = \Lambda(\phi(j))(\lambda) = (\Lambda \circ \phi)(j)(\lambda) = \widehat{\Lambda \circ \phi}(\lambda)(j)$. It follows

$$\begin{aligned}
\Lambda(k\phi\mathbb{F})(\lambda) &= \bigvee_{\mu \in k\phi\mathbb{F}} S_X(\mu, \lambda) = \bigvee_{\widehat{\phi}(\mu) \in \mathbb{F}} S_X(\mu, \lambda) \\
&\leq \bigvee_{\widehat{\phi}(\mu) \in \mathbb{F}} S_J(\widehat{\phi}(\mu), \widehat{\phi}(\lambda)) \leq \bigvee_{\nu \in \mathbb{F}} S_J(\nu, \widehat{\phi}(\lambda)) \\
&= \bigvee_{\nu \in \mathbb{F}} S_J(\nu, \widehat{\Lambda \circ \phi}(\lambda)) = \Lambda(\mathbb{F})(\widehat{\Lambda \circ \phi}(\lambda)) = K(\Lambda \circ \phi)\Lambda(\mathbb{F})(\lambda).
\end{aligned}$$

(2) Let $\lambda \in L^X$. Then

$$\begin{aligned}\lambda \in k\phi\mathbb{F} &\iff \widehat{\Lambda\circ\phi}(\lambda) \in \mathbb{F} \iff \widehat{\Lambda\circ\phi}(\lambda) \in \mathbb{F} \iff \Lambda(\mathbb{F})(\widehat{\Lambda\circ\phi}(\lambda)) = \top \\ &\iff K(\Lambda\circ\phi)\Lambda(\mathbb{F})(\lambda) = \top \iff \lambda \in \Gamma(K(\Lambda\circ\phi)\Lambda(\mathbb{F})). \quad \square\end{aligned}$$

Theorem 5. *Let (X, p, q) be pair of $\top$-convergence spaces and L be continuous. Then $\lim^q$ is p-topological iff q is p-topological.*

Proof. Let q be p-topological. We check that $(X, \lim^p, \lim^q)$ satisfies p-**(LG)**. Obviously, we need only prove that $\lim^q \mathcal{F}(x) = \top$ implies $\lim^q \mathcal{U}_p^\alpha(\mathcal{F})(x) = \top$ for any $\alpha \neq \perp$.

Note that for any $\alpha \neq \perp$ and any $x \in X$ we have

$$\begin{aligned}\mathcal{U}_p^\alpha(x) &= \bigwedge\{\mathcal{F}|\lim^p \mathcal{F}(x) = \top\} = \bigwedge\{\mathcal{F}|\mathcal{F} \geq \Lambda(\mathbb{F}), \mathbb{F} \xrightarrow{p} x\} \\ &= \bigwedge\{\Lambda(\mathbb{F})|\mathbb{F} \xrightarrow{p} x\} \overset{\text{Lemma1}}{=} \Lambda(\bigcap \mathbb{F}\{|\mathbb{F} \xrightarrow{p} x\}) = \Lambda(\mathbb{U}_p(x)).\end{aligned}$$

Let $\lim^q \mathcal{F}(x) = \top$ then $\mathcal{F} \geq \Lambda(\mathbb{F})$ for some $\mathbb{F} \xrightarrow{q} x$. It follows by p-(TG) that $\mathbb{U}_p(\mathbb{F}) \xrightarrow{q} x$ and

$$\mathcal{U}_p^\alpha(\mathcal{F}) \geq \mathcal{U}_p^\alpha(\Lambda(\mathbb{F})) = K\mathcal{U}_p^\alpha\Lambda(\mathbb{F}) = K(\Lambda\circ\mathbb{U}_p)\Lambda(\mathbb{F}) \overset{\text{Lemma2(1)}}{\geq} \Lambda(k\mathbb{U}_p\mathbb{F}) = \Lambda(\mathbb{U}_p(\mathbb{F})),$$

and so $\lim^q \mathcal{U}_p^\alpha(\mathcal{F})(x) = \top$ as desired.

Conversely, let $\lim^q$ be p-topological. We check that (X, p, q) satisfies p-**(TG)**.

Assume that $\mathbb{F} \xrightarrow{q} x$. It follows by p-**(LG)** that

$$\lim{}^q\mathcal{U}_p^\top(\Lambda(\mathbb{F}))(x) = \lim{}^q K\mathcal{U}_p^\top\Lambda(\mathbb{F}) = \lim{}^q K(\Lambda\circ\mathbb{U}_p)\Lambda(\mathbb{F}) = \top,$$

and then $K(\Lambda\circ\mathbb{U}_p)\Lambda(\mathbb{F}) \geq \Lambda(\mathbb{G})$ for some $\mathbb{G} \xrightarrow{q} x$. By Lemma 2(2) we have

$$k\mathbb{U}_p\mathbb{F} = \Gamma(K(\Lambda\circ\mathbb{U}_p)\Lambda(\mathbb{F})) \supseteq \Gamma\Lambda(\mathbb{G}) = \mathbb{G}.$$

So, $\mathbb{U}_p(\mathbb{F}) = k\mathbb{U}_p\mathbb{F} \xrightarrow{q} x$ as desired. $\square$

4. Lower and Upper p-Topological Modifications in $\top$-Convergence Spaces

In this section, we shall discuss the p-topological modification in $\top$-convergence spaces.

At first, we fix a lemma for later use. The proof is obvious, so we omit it.

Lemma 3. (1) *If (X, q) is p-topological, then $\mathbb{F} \xrightarrow{q} x$ implies $\mathbb{U}_p^n(\mathbb{F}) \xrightarrow{q} x$ for any $n \in \mathbb{N}$.*

(2) *If (X, q) is p-topological, then (X, q) is p'-topological for any $p \leq p'$.*

(3) *(X, ι) is p-topological for any $p \in \top(X)$.*

4.1. Lower p-Topological Modification

Corollary 3 shows that p-topologicalness is preserved under supremum in the lattice $\top(X)$. Lemma 3(3) shows that the indiscrete space (X, ι) is p-topological for any $p \in \top(X)$. These two results make the following definition available.

Definition 10. *Let (X, p, q) be a pair of $\top$-convergence spaces. Then there is a finest p-topological $\top$-convergence structure $\tau_p q$ on X which is coarser than q. The structure $\tau_p q$ is called the lower p-topological modification of q.*

The next theorem gives a direct characterization on lower p-topological modification.

Theorem 6. *Let $p, q \in \top(X)$. Then $\mathbb{F} \xrightarrow{\tau_p q} x \Longleftrightarrow \exists n \in \mathbb{N}, \mathbb{G} \xrightarrow{q} x$ s.t. $\mathbb{F} \supseteq \mathbb{U}_p^n(\mathbb{G})$.*

Proof. Let q' be defined as $\mathbb{F} \xrightarrow{q'} x \Longleftrightarrow \exists n \in \mathbb{N}, \mathbb{G} \xrightarrow{q} x$ s.t. $\mathbb{F} \supseteq \mathbb{U}_p^n(\mathbb{G})$. We need only check that $\tau_p q = q'$.

It is obvious that $q' \in \top(X)$ and $q' \leq q$. We prove that q' is p-topological. Indeed, let $\mathbb{F} \xrightarrow{q'} x$. Then there exists $n \in \mathbb{N}, \mathbb{G} \xrightarrow{q} x$ such that $\mathbb{F} \supseteq \mathbb{U}_p^n(\mathbb{G})$. It follows that $\mathbb{U}_p(\mathbb{F}) \supseteq \mathbb{U}_p(\mathbb{U}_p^n(\mathbb{G})) = \mathbb{U}_p^{n+1}(\mathbb{G})$ and so $\mathbb{U}_p(\mathbb{F}) \xrightarrow{q'} x$, as desired. Thus q' is p-topological.

Let (X, r) be p-topological and $r \leq q$. We prove below $r \leq q'$. Indeed, let $\mathbb{F} \xrightarrow{q'} x$. Then there exists $n \in \mathbb{N}, \mathbb{G} \xrightarrow{q} x$ such that $\mathbb{F} \supseteq \mathbb{U}_p^n(\mathbb{G})$, and then $\mathbb{G} \xrightarrow{r} x$ by $q \leq r$. Since r is p-topological it follows by Lemma 3(1) we have $\mathbb{F} \supseteq \mathbb{U}_p^n(\mathbb{G}) \xrightarrow{r} x$. Thus $r \leq q'$. $\square$

Theorem 7. *Let $f : (X, q) \longrightarrow (X', q')$ and $f : (X, p) \longrightarrow (X', p')$ be continuous function between $\top$-convergence spaces. Then $f : (X, \tau_p q) \longrightarrow (X', \tau_{p'} q')$ is also continuous.*

Proof. For any $\mathbb{F} \in \mathbb{F}_L^{\top}(X)$ and $x \in X$.

$$\begin{aligned}
\mathbb{F} \xrightarrow{\tau_p q} x \quad &\Longrightarrow \quad \exists n \in \mathbb{N}, \mathbb{G} \xrightarrow{q} x \text{ s.t. } \mathbb{F} \supseteq \mathbb{U}_p^n(\mathbb{G}) \\
&\Longrightarrow \quad \exists n \in \mathbb{N}, f^{\Rightarrow}(\mathbb{G}) \xrightarrow{q'} f(x) \text{ s.t. } f^{\Rightarrow}(\mathbb{F}) \supseteq f^{\Rightarrow}(\mathbb{U}_p^n(\mathbb{G})) \\
&\Longrightarrow \quad \exists n \in \mathbb{N}, f^{\Rightarrow}(\mathbb{G}) \xrightarrow{q'} f(x) \text{ s.t. } f^{\Rightarrow}(\mathbb{F}) \supseteq \mathbb{U}_{p'}^n(f^{\Rightarrow}(\mathbb{G})) \\
&\Longrightarrow \quad f^{\Rightarrow}(\mathbb{F}) \xrightarrow{\tau_{p'} q'} (f(x)),
\end{aligned}$$

where the second implication holds for $f : (X, q) \longrightarrow (X', q')$ being continuous, and the third implication holds by $f : (X, p) \longrightarrow (X', p')$ being continuous and Proposition 3(1). $\square$

The next theorem shows that lower p-topological modification behaves reasonably well relative to final structures.

Theorem 8. *Let $\{(X_i, q_i, p_i)\}_{i \in I}$ be pairs of spaces in $\top$-CS and let q be the final structure w.r.t. The sink $((X_i, q_i) \xrightarrow{f_i} X)_{i \in I}$ with $X = \cup_{i \in I} f_i(X_i)$. If (X, p) is in $\top$-CS such that each $f_i : (X_i, p_i) \longrightarrow (X, p)$ is a continuous interior function, then $\tau_p q$ is the final structure w.r.t. the sink $((X_i, \tau_{p_i} q_i) \xrightarrow{f_i} X)_{i \in I}$.*

Proof. Let s denote the final structure w.r.t. The sink $((X_i, \tau_{p_i} q_i) \xrightarrow{f_i} X)_{i \in I}$. Let $\mathbb{F} \in \mathbb{F}_L^{\top}(X)$ and $x \in X$. Then

$$\begin{aligned}
\mathbb{F} \xrightarrow{s} x \quad &\Longrightarrow \quad \exists i \in I, x_i \in X_i, f_i(x_i) = x, \mathbb{G}_i \xrightarrow{\tau_{p_i} q_i} x_i \text{ s.t. } f_i^{\Rightarrow}(\mathbb{G}_i) \subseteq \mathbb{F}, \text{by Theorem 6} \\
&\Longrightarrow \quad \exists i \in I, x_i \in X_i, n \in \mathbb{N}, f_i(x_i) = x, \mathbb{H}_i \xrightarrow{q_i} x_i \text{ s.t. } \mathbb{U}_{p_i}^n(\mathbb{H}_i) \subseteq \mathbb{G}_i, f_i^{\Rightarrow}(\mathbb{G}_i) \subseteq \mathbb{F}, \text{Proposition 3(1)} \\
&\Longrightarrow \quad \exists i \in I, x_i \in X_i, n \in \mathbb{N}, f_i^{\Rightarrow}(\mathbb{H}_i) \xrightarrow{q} x \text{ s.t. } \mathbb{U}_p^n(f_i^{\Rightarrow}(\mathbb{H}_i)) \subseteq f_i^{\Rightarrow}(\mathbb{U}_{p_i}^n(\mathbb{H}_i)) \subseteq f_i^{\Rightarrow}(\mathbb{G}_i), f_i^{\Rightarrow}(\mathbb{G}_i) \subseteq \mathbb{F} \\
&\Longrightarrow \quad \exists i \in I, x_i \in X_i, n \in \mathbb{N}, f_i^{\Rightarrow}(\mathbb{H}_i) \xrightarrow{q} x \text{ s.t. } \mathbb{U}_p^n(f_i^{\Rightarrow}(\mathbb{H}_i)) \subseteq \mathbb{F} \\
&\Longrightarrow \quad \mathbb{F} \xrightarrow{\tau_p q} x.
\end{aligned}$$

Conversely,

$$\begin{aligned}
\mathbb{F} \xrightarrow{\tau_p q} x &\Longrightarrow \exists n \in \mathbb{N}, \mathbb{G} \xrightarrow{q} x \text{ s.t. } \mathbb{U}_p^n(\mathbb{G}) \subseteq \mathbb{F} \\
&\Longrightarrow \exists i \in I, x_i \in X_i, f_i(x_i) = x, n \in \mathbb{N}, \mathbb{H}_i \xrightarrow{q_i} x_i \text{ s.t. } f_i^{\Rightarrow}(\mathbb{H}_i) \subseteq \mathbb{G}, \mathbb{U}_p^n(\mathbb{G}) \subseteq \mathbb{F} \\
&\Longrightarrow \exists i \in I, x_i \in X_i, f_i(x_i) = x, n \in \mathbb{N}, \mathbb{H}_i \xrightarrow{q_i} x_i \text{ s.t. } \mathbb{U}_p^n(f_i^{\Rightarrow}(\mathbb{H}_i)) \subseteq \mathbb{U}_p^n(\mathbb{G}), \mathbb{U}_p^n(\mathbb{G}) \subseteq \mathbb{F} \\
&\Longrightarrow \exists i \in I, x_i \in X_i, f_i(x_i) = x, n \in \mathbb{N}, \mathbb{H}_i \xrightarrow{q_i} x_i \text{ s.t. } f_i^{\Rightarrow}(\mathbb{U}_{p_i}^n(\mathbb{H}_i)) \subseteq \mathbb{U}_p^n(f_i^{\Rightarrow}(\mathbb{H}_i)) \subseteq \mathbb{F} \\
&\Longrightarrow \exists i \in I, x_i \in X_i, f_i(x_i) = x, n \in \mathbb{N}, \mathbb{U}_{p_i}^n(\mathbb{H}_i) \xrightarrow{\tau_{p_i} q_i} x_i \text{ s.t. } f_i^{\Rightarrow}(\mathbb{U}_{p_i}^n(\mathbb{H}_i)) \subseteq \mathbb{F} \\
&\Longrightarrow \mathbb{F} \xrightarrow{s} x,
\end{aligned}$$

where the fourth implication uses Proposition 3(2). □

The following corollary shows that lower p-topological modification behaves reasonably well relative to infimum in the lattice $\top(X)$.

Corollary 4. *Let $\{q_i | i \in I\} \subseteq \top(X)$, $p \in \top(X)$ and $q = \inf\{q_i | i \in I\}$. Then $\tau_p q = \inf\{\tau_p q_i | i \in I\}$.*

At last, we give the notion of topological modification. By Corollary 3, it is observed that topologicalness is preserved under supremum in the lattice $\top(X)$. Since the indiscrete space is topological, the following notion is available.

Definition 11. *Let (X, q) be a $\top$-convergence space. Then there exists a finest topological $\top$-convergence structure τq which is coarser than q. The structure τq is called the topological modification of (X, q). Indeed, $\tau q = \sup\{p | p \leq q$ and p is topological$\}$.*

4.2. Upper p-Topological Modification

Note that for an arbitrary $p \in \top(X)$, the discrete space (X, δ) is generally not p-topological. Thus for a given $q \in \top(X)$, there may not exist p-topological $\top$-convergence structure on X which is finer than q.

Definition 12. *Let (X, p, q) be a pair of $\top$-convergence spaces. If there exists a coarsest p-topological $\top$-convergence structure $\tau^p q$ on X which is finer than q, then it is called the upper p-topological modification of q.*

From Corollary 3 we easily conclude that the existence of $\tau^p q$ depends on the existence of a p-topological $\top$-convergence structure on X which is finer than q. Additionally, note that $\tau_p \delta$ is the finest p-topological $\top$-convergence structure on X. Then it follows immediately that $\tau^p q$ exists if and only if $q \leq \tau_p \delta$. Using Theorem 6, this result can be stated as below.

Theorem 9. *Let (X, p, q) be a pair of $\top$-convergence spaces. Then $\tau^p q$ exists if and only if $\mathbb{U}_p^n([x]_\top) \xrightarrow{q} x$ for all $x \in X$, $n \in \mathbb{N}$.*

Proof. For each $\mathbb{F} \in \mathbb{F}_L^\top(X)$ and each $x \in X$, by Theorem 6 we have

$$\mathbb{F} \xrightarrow{\tau_p \delta} x \Longleftrightarrow \exists n \in \mathbb{N}, \mathbb{G} \xrightarrow{\delta} x \text{ s.t. } \mathbb{U}_p^n(\mathbb{G}) \subseteq \mathbb{F}.$$

Necessity. Let $\tau^p q$ exist. Then $q \leq \tau_p \delta$. It follows that for all $x \in X$, $n \in \mathbb{N}$

$$[x]_\top \xrightarrow{\delta} x \Longrightarrow \mathbb{U}_p^n([x]_\top) \xrightarrow{\tau_p\delta} x \Longrightarrow \mathbb{U}_p^n([x]_\top) \xrightarrow{q} x.$$

Sufficiency. Let $\mathbb{U}_p^n([x]_\top) \xrightarrow{q} x$ for all $x \in X$, $n \in \mathbb{N}$. Then for all $\mathbb{F} \in \mathbb{F}_L^\top(X)$ we have

$$\begin{aligned}
\mathbb{F} \xrightarrow{\tau_p\delta} x \quad & \Longrightarrow \quad \exists n \in \mathbb{N}, \mathbb{G} \xrightarrow{\delta} x \text{ s.t. } \mathbb{U}_p^n(\mathbb{G}) \subseteq \mathbb{F} \\
& \Longrightarrow \quad \exists n \in \mathbb{N}, [x]_\top \subseteq \mathbb{G} \text{ s.t. } \mathbb{U}_p^n(\mathbb{G}) \subseteq \mathbb{F} \\
& \overset{\text{Proposition 2(2)}}{\Longrightarrow} \quad \exists n \in \mathbb{N}, \mathbb{U}_p^n(([x]_\top) \subseteq \mathbb{U}_p^n(\mathbb{G}) \text{ s.t. } \mathbb{U}_p^n(\mathbb{G}) \subseteq \mathbb{F} \\
& \Longrightarrow \quad \exists n \in \mathbb{N} \text{ s.t. } \mathbb{U}_p^n(([x]_\top) \subseteq \mathbb{F} \\
& \Longrightarrow \quad \mathbb{F} \xrightarrow{q} x.
\end{aligned}$$

It follows that $q \leq \tau_p \delta$, which means that $\tau^p q$ exists. $\square$

The next theorem gives a direct characterization on upper p-topological modification whenever it exists.

Theorem 10. *Let (X, p, q) be a pair of $\top$-convergence spaces. If $\tau^p q$ exists, then $\mathbb{F} \xrightarrow{\tau^p q} x \iff \forall n \in \mathbb{N}, \mathbb{U}_p^n(\mathbb{F}) \xrightarrow{q} x$.*

Proof. Let q' be defined as $\mathbb{F} \xrightarrow{q'} x \iff \forall n \in \mathbb{N}, \mathbb{U}_p^n(\mathbb{F}) \xrightarrow{q} x$.

(1) $q' \in \top(X)$.

(TC1) Let $x \in X$. Then by Theorem 9 we have $\mathbb{U}_p^n([x]_\top) \xrightarrow{q} x$ for all $n \in \mathbb{N}$, which means $[x]_\top \xrightarrow{q'} x$.
(TC2) It is obvious.

(2) $q \leq q'$. Indeed, let $\mathbb{F} \xrightarrow{q'} x$ then $\mathbb{F} = \mathbb{U}_p^0(\mathbb{F}) \xrightarrow{q} x$.

(3) (X, q') is p-topological. Indeed, let $\mathbb{F} \xrightarrow{q'} x$. Then for any $n \in \mathbb{N}$ we have $\mathbb{U}_p^n(\mathbb{U}_p(\mathbb{F})) = \mathbb{U}_p^{n+1}(\mathbb{F}) \xrightarrow{q} x$, which means $\mathbb{U}_p(\mathbb{F}) \xrightarrow{q'} x$. Thus (X, q') is p-topological.

(4) Let (X, r) be p-topological and $q \leq r$. Then $q' \leq r$. Indeed, let $\mathbb{F} \xrightarrow{r} x$ then for any $n \in \mathbb{N}$, by Proposition 3(1) we have $\mathbb{U}_p^n(\mathbb{F}) \xrightarrow{r} x$ and so $\mathbb{U}_p^n(\mathbb{F}) \xrightarrow{q} x$ by $q \leq r$. That means $\mathbb{F} \xrightarrow{q'} x$.

(1)–(4) show that q' is the coarsest p-topological $\top$-convergence structure on X which is finer than q. Thus $\tau^p q = q'$. $\square$

Theorem 11. *Let $f : (X, q) \longrightarrow (X', q')$ be a continuous function, and $f : (X, p) \longrightarrow (X', p')$ be an interior function between $\top$-convergence spaces. If $\tau^p q$ and $\tau^{p'} q'$ exist then $f : (X, \tau^p q) \longrightarrow (X', \tau^{p'} q')$ is also continuous.*

Proof. Let $\mathbb{F} \xrightarrow{\tau^p q} x$. Then $\forall n \in \mathbb{N}, \mathbb{U}_p^n(\mathbb{F}) \xrightarrow{q} x$. Because $f : (X, q) \longrightarrow (X', q')$ is a continuous function and $f : (X, p) \longrightarrow (X', p')$ is an interior function we have

$$\forall n \in \mathbb{N}, \mathbb{U}_{p'}^n(f^\Rightarrow(\mathbb{F})) \supseteq f^\Rightarrow(\mathbb{U}_p^n(\mathbb{F})) \xrightarrow{q'} f(x),$$

which means $f^\Rightarrow(\mathbb{F}) \xrightarrow{\tau^{p'} q'} f(x)$ as desired. $\square$

The next theorem shows that the upper p-topological modification exhibits comparable behavior relative to initial structures.

Theorem 12. *Let $\{(X_i, q_i, p_i)\}_{i\in I}$ be pairs of spaces in $\top$-**CS** and q be the initial structure w.r.t. The source $(X \xrightarrow{f_i} (X_i, q_i))_{i\in I}$. Let (X, p) be in $\top$-**CS** such that each $f_i : (X, p) \longrightarrow (X_i, p_i)$ is continuous interior function. If $\tau^{p_i} q_i$ exists for all $i \in I$, then $\tau^p q$ exists and is the initial structure w.r.t. The source $(X \xrightarrow{f_i} (X_i, \tau^{p_i} q_i))_{i\in I}$.*

Proof. To prove $\tau^p q$ exists, it suffices, by Theorem 10, to show that $\mathbb{U}^n_p([x]_\top) \xrightarrow{q} x$ for any $x \in X, n \in \mathbb{N}$. Indeed, by the existence of $\tau^{p_i} q_i$ we have $\mathbb{U}^n_{p_i}([f_i(x)]) \xrightarrow{q_i} f_i(x)$ for any $i \in I, x \in X, n \in \mathbb{N}$. It follows by that each $f_i : (X, p) \longrightarrow (X_i, p_i)$ being a continuous interior function we get

$$f_i^{\Rightarrow}(\mathbb{U}^n_p([x]_\top) = \mathbb{U}^n_{p_i}(f_i^{\Rightarrow}([x]_\top)) = \mathbb{U}^n_{p_i}([f_i(x)]_\top) \xrightarrow{q_i} f_i(x),$$

which means $\mathbb{U}^n_p([x]_\top) \xrightarrow{q} x$ for any $x \in X, n \in \mathbb{N}$, i.e., $\tau^p q$ exists.

Let s denote the initial structure on X relative the source $(X \xrightarrow{f_i} (X_i, \tau^{p_i} q_i))_{i\in I}$. Then.

$$\begin{aligned}
\mathbb{F} \xrightarrow{s} x \quad &\Longleftrightarrow \quad \forall i \in I, f_i^{\Rightarrow}(\mathbb{F}) \xrightarrow{\tau^{p_i} q_i} f_i(x) \overset{\text{Theorem 10}}{\Longleftrightarrow} \forall i \in I, \forall n \in \mathbb{N}, \mathbb{U}^n_{p_i}(f_i^{\Rightarrow}(\mathbb{F})) \xrightarrow{q_i} f_i(x) \\
&\overset{\text{Proposition 3}}{\Longleftrightarrow} \quad \forall i \in I, \forall n \in \mathbb{N}, f_i^{\Rightarrow}(\mathbb{U}^n_p(\mathbb{F})) \xrightarrow{q_i} f_i(x) \\
&\Longleftrightarrow \quad \forall n \in \mathbb{N}, \mathbb{U}^n_p(\mathbb{F}) \xrightarrow{q} x \overset{\text{Theorem 10}}{\Longleftrightarrow} \mathbb{F} \xrightarrow{\tau^p q} x. \quad \square
\end{aligned}$$

The following corollary shows that upper p-topological modification exhibits comparable behavior relative to supremum in the lattice $\top(X)$.

Corollary 5. *Let $\{q_i | i \in I\} \subseteq \top(X)$, $p \in \top(X)$ and $q = \sup\{q_i | i \in I\}$. If $\tau^p q_i$ exists for all $i \in I$, then $\tau^p q$ exists and $\tau^p q = \sup\{\tau^p q_i | i \in I\}$.*

5. Conclusions

In this paper, we discussed the p-topologicalness in $\top$-convergence spaces by a Fischer $\top$-diagonal condition and a Gähler $\top$-neighborhood condition, respectively. We proved that the p-topologicalness was preserved under the initial and final structures in the category $\top$-**CS**. As a straightforward conclusion, we further obtained that p-topologicalness was naturally preserved under the infimum and supremum in the lattice $\top(X)$. We also established the relationship between p-topologicalness in $\top$-convergence spaces and p-topologicalness in stratified L-generalized convergence spaces. Furthermore, we defined and studied the lower and upper p-topological modifications in $\top$-convergence spaces. In particular, we proved that the lower (resp., upper) p-topological modification exhibited comparable behavior relative to final (resp., initial) structures.

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