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Some Common Fixed Point Theorems in Ordered Partial Metric Spaces via $\mathcal{F}$-Generalized Contractive Type Mappings

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Abstract: In the present work, the concept of $\mathcal{F}$-generalized contractive type mappings by using C-class functions is introduced, and some common fixed point results for weakly isotone increasing set-valued mappings in the setting of ordered partial metric spaces are studied. These results improve and generalize various results existing in the literature. The effectiveness of the obtained results is verified with the help of some comparative examples.

Keywords: fixed point; ordered partial metric space; $\mathcal{F}$-generalized contractive type mappings

1. Introduction

The study of common fixed points was initiated by Gerald Jungck [1] in 1986, and this concept has attracted many researchers to prove the existence of fixed points by using various metrical contractions. On the other hand, the notion of partial metric spaces was presented by S.G. Matthews [2] and has been considered one of the most interesting, robust, and outstanding generalizations of metric spaces. Many authors have generalized this notion in different ways (see [3–9]). In 2010, Hong [10] defined the concept of approximative values to prove the existence of common fixed points for multivalued operators in the framework of ordered metric spaces. After that, Erduran [11] extended this concept and studied some fixed point results for multivalued mappings in partial metric spaces. In 2014, Arslan Hojat Ansari [12] introduced C-class functions defined on R.

In this paper, the notion of $\mathcal{F}$-generalized contractive type mappings is introduced, and some common fixed point theorems for multivalued mappings in ordered partial metric spaces using C-class functions are obtained.

Definition 1. *([2]) Let U be a nonempty set. A function $p : U \times U \to R^+$ is said to be a partial metric on U if the following postulates hold true for all $u, v, w \in U$:*

- *(p1) $u = v$ if, and only if, $p(u, u) = p(v, v) = p(u, v)$;*
- *(p2) $p(u, u) \leq p(u, v)$ (small self-distance axiom);*
- *(p3) $p(u, v) = p(v, u)$ (symmetry);*
- *(p4) $p(u, w) \leq p(u, v) + p(v, w) - p(v, v)$ (modified triangle inequality).*

The pair (U, p) is then called a partial metric space (in short: PMS). Each partial metric p on U generates a T_0 topology τ_p on U which has a base, the family of open p-balls $\{B_p(u, \epsilon), u \in U, \epsilon > 0\}$, where $B_p(u, \epsilon) = \{v \in U: p(u, v) < p(u, u) + \epsilon\}$ for all $u \in U$ and $\epsilon > 0$.

If p is a partial metric defined on U, then the mapping $d_p : U \times U \to R^+$ given by $d_p(u, v) = 2p(u, v) - p(u, u) - p(v, v)$ is a metric on U.

Definition 2. *([2]) For a partial metric space $(U,\ p)$, a sequence $\{u_n\}$ in U is said to be*

(i) convergent if there exists a point $u \in U$ such that $p(u,\ u) = \lim_{n\to\infty} p(u_n,\ u)$;

(ii) a Cauchy sequence if the limit $\lim_{n,m\to\infty} p(u_n,\ u_m)$ exists (and is finite).

Definition 3. *([2]) A partial metric space $(U,\ p)$ is said to be complete if every Cauchy sequence $\{u_n\}$ in U converges w.r.t. τ_p to a point $u \in U$ such that $p(u,\ u) = \lim_{n,m\to\infty} p(u_n,\ u_m)$.*

Lemma 1. *([2]) Let $(U,\ p)$ be a partial metric space. Then*

(i) $\{u_n\}$ is said to be a Cauchy sequence in $(U,\ p)$ if it is a Cauchy sequence in the metric space $(U,\ d_p)$;

(ii) $(U,\ p)$ is complete if the metric space $(U,\ d_p)$ is complete. Also,

$$\lim_{n\to\infty} d_p(u_n,\ u) = 0 \ if\ p(u,\ u) = \lim_{n\to\infty} p(u_n,\ u) = \lim_{n,m\to\infty} p(u_n,\ u_m).$$

Lemma 2. *([13]) Let $(U,\ p)$ be a partial metric space and let $\{u_n\}$ be a sequence in U such that $\lim_{n\to\infty} p(u_n,\ u_{n+1}) = 0$.*

If the sequence $\{u_{2n}\}$ is not a Cauchy sequence in $(U,\ p)$, then there exist $\epsilon > 0$ and two sequences $\left\{u_{m(k)}\right\}$ and $\left\{u_{n(k)}\right\}$ of positive integers with $n(k) > m(k) > k$ such that the four sequences

$$p\left(u_{2m(k)},\ u_{2n(k)+1}\right),\quad p\left(u_{2m(k)},\ u_{2n(k)}\right),\quad p\left(u_{2m(k)-1},\ u_{2n(k)+1}\right),\quad p\left(u_{2m(k)-1},\ u_{2n(k)}\right)$$

tend to $\epsilon > 0$ when $k \to \infty$.

Lemma 3. *([2]) If the sequence $\{u_n\}$ with $\lim_{n\to\infty} d_p(u_{n+1},\ u_n) = 0$ is not a Cauchy sequence in $(U,\ p)$, then for each $\epsilon > 0$, there exist two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers with $n(k) > m(k) > k$ such that the four sequences*

$$p\left(u_{m(k)},\ u_{n(k)+1}\right),\quad p\left(u_{m(k)},\ u_{n(k)}\right),\quad p\left(u_{m(k)-1},\ u_{n(k)+1}\right),\quad p\left(u_{m(k)-1},\ u_{n(k)}\right)$$

tend to $\epsilon > 0$ when $k \to \infty$.

Let $CB^p(U)$ be a family of all nonempty, closed, and bounded subsets of the partial metric space $(U,\ p)$. Note that the notion of a closed set is obvious as τ_p is the topology induced by p and boundedness in its standard form is given as follows: A_1 is a bounded subset in $(U,\ p)$ if there exist $M \geq 0$ and $u_0 \in U$ such that for each $a_1 \in A_1$, we have $a_1 \in B_p(u_0,\ M)$, i.e., $p(u_0,\ a_1) < p(a_1,\ a_1) + M$.

For all $A_1, A_2 \in CB^p(U)$ and $u \in$ U,

$$p(u,\ A_1) = \inf\{p(u,\ v : v \in A_1\},$$

$$\delta_p(A_1,\ A_2) = \sup\{p(a_1,\ A_2) : a_1 \in A_1\},$$

$$\delta_p(A_2,\ A_1) = \sup\{p(A_1,\ a_2) : a_2 \in A_2\},$$

and

$$P_h(A_1,\ A_2) = \max\{\ \delta_p(A_1,\ A_2),\ \delta_p(A_2,\ A_1)\}.$$

Note that $p(u,\ A_1) = 0$ implies $d_p(u,\ A_1) = 0$ where $d_p(u,\ A_1) = \inf\{d_p(u,\ a_1) \colon a_1 \in A_1\}$.

Corollary 1. *([14]) Let $(U,\ p)$ be a partial metric space and let A_1 be any nonempty set in $(U,\ p)$, then $a_1 \in \overline{A_1}$ if, and only if, $p(a_1,\ A_1) = p(a_1,\ a_1)$, where $\overline{A_1}$ denotes the closure of A_1 w.r.t. the partial metric p. We say that A_1 is closed in $(U,\ p)$ if, and only if, $\overline{A_1} = A_1$.*

Proposition 1. *([9]) Let $(U,\ p)$ be a partial metric space. For all $A_1,\ A_2, A_3 \in CB^p(U)$, we have*

- (h1) $P_h(A_1,\ A_1) \leq P_h(A_1,\ A_2)$,
- (h2) $P_h(A_1,\ A_2) = P_h(A_2,\ A_1)$,
- (h3) $P_h(A_1,\ A_2) \leq\ P_h(A_1,\ A_3) + P_h(A_3,\ A_2) - inf_{a_3 \in A_3} p(a_3,\ a_3)$,
- (h4) $P_h(A_1,\ A_2) = 0 \implies\ A_1 =\ A_2$.

The mapping P_h: $CB^p(U) \times CB^p(U) \to [0, +\infty)$ is called the Partial Hausdorff metric induced by p. Every Hausdorff metric is a Partial Hausdorff metric but the converse need not be true (Example 2.6, [3]).

Definition 4. *([15]) For a nonempty set U, The space $(U,\ p, \preccurlyeq)$ is called an ordered partial metric space if $(U,\ p)$ is a partial metric space and $(U,\ \preccurlyeq)$ is a partially ordered set.*

Let $(U,\ \preccurlyeq)$ be a partially ordered set. Then $u,\ v \in\ U$ are called comparable if $u \preccurlyeq v$ or $v \preccurlyeq u$.

Definition 5. *([10]) Let A_1 and A_2 be any two nonempty subsets of an ordered set $(U,\ \preccurlyeq)$. The relation $\preccurlyeq_2$ between A_1 and A_2 is defined as follows:*

$A_1 \preccurlyeq_2 A_2$ if $a_1 \preccurlyeq a_2$ for each $a_1 \in A_1$ and $a_2 \in A_2$.

Definition 6. *([16]) Let $(U,\ \preccurlyeq)$ be a partially ordered set. Two maps $S,\ T:\ U \to 2^U$ are said to be weakly isotone increasing if for any $u \in\ U$, we have $Su \preccurlyeq_2\ Tv$ for all $v \in\ Su$ and $Tu \preccurlyeq_2\ Sv$ for all $v \in\ Tu$.*

In particular, the mappings $S,\ T:\ U \to U$ are called weakly isotone increasing if $Su\ \preccurlyeq\ TSu$ and $Tu \preccurlyeq STu$ hold for each $u \in\ U$.

Definition 7. *([11]) An ordered partial metric space is said to have a sequential limit comparison property if for every nonincreasing sequence (or nondecreasing sequence) $\{u_n\}$ in U, we have $u_n \to u$ implies $u \leq\ u_n$ (or $u_n \leq u$, respectively).*

Definition 8. *([11]) A subset A of set U is said to be approximative if the set $P_A(u) = \{v \in\ A\ :\ p(u,\ v) =\ p(A,\ u)\}\ \forall\ u \in\ U$ is nonempty. A set-valued mapping T is said to have approximate values in U if Tu is an approximative for each $u \in\ U$.*

Definition 9. *([17]) Denote by Υ the set of all functions $\xi:\ [0, +\infty)^4 \to [0, +\infty)$ with the following properties:*

(1) ξ is nondecreasing in third and fourth variables.
(2) $\xi(s_1,\ s_2,\ s_3,\ s_4) = 0$ if, and only if, $s_1 s_2 s_3 s_4 = 0$.
(3) ξ is continuous.

The following functions belong to Υ:

(1) $\xi(s_1,\ s_2,\ s_3,\ s_4) = L\ min\{s_1,\ s_2,\ s_3,\ s_4\}$ where $L > 0$,
(2) $\xi(s_1,\ s_2,\ s_3,\ s_4) = s_1 s_2 s_3 s_4$,
(3) $\xi(s_1,\ s_2,\ s_3,\ s_4) = ln(1 +\ s_1 s_2 s_3 s_4)$,
(4) $\xi(s_1,\ s_2,\ s_3,\ s_4) = exp(s_1 s_2 s_3 s_4) - 1$.

For two mappings $S,\ \ T:\ U \to 2^U$, we define

$$M(u,\ v) = \max\left\{p(u,\ v),\ \ p(u,\ Tu),\ \ p(v,\ Sv),\ \ \frac{1}{2}[p(v,\ Tu) +\ p(u,\ Sv)]\right\}.$$

In 2014, the concept of C-class functions was introduced by A.H. Ansari [12]. By using this concept, many fixed point theorems in the literature can be generalized.

Definition 10. *([12]) A mapping* $\mathcal{F}: [0,\infty)^2 \to R$ *is called a C-class function if it is continuous and satisfies the following axioms:*

(1) $\mathcal{F}(t_1, t_2) \le t_1$,
(2) $\mathcal{F}(t_1, t_2) = t_1$ *implies that either* $t_1 = 0$ or $t_2 = 0$ for all $t_1, t_2 \in [0,\infty)$.

We denote C-class functions by $\mathcal{C}$.

Example 1. *([12]) Following are some members of class* $\mathcal{C}$ *for all* $t_1, t_2 \in [0,\infty)$.

(1) $\mathcal{F}(t_1, t_2) = t_1 - t_2, \mathcal{F}(t_1, t_2) = t_1 \Longrightarrow t_2 = 0;$
(2) $\mathcal{F}(t_1, t_2) = m\, t_1, 0 < m < 1, \mathcal{F}(t_1, t_2)\ \ t_1 \Longrightarrow t_1 = 0;$
(3) $\mathcal{F}(t_1, t_2) = \frac{t_1}{(1+t_2)^r}, r \in (0, \infty), \mathcal{F}(t_1, t_2) = t_1 \Longrightarrow t_1 = 0 \text{ or } t_2 = 0;$
(4) $\mathcal{F}(t_1, t_2) = \frac{log(t_2 + a^{t_1})}{(1 + t_2)}, a > 1, \mathcal{F}(t_1, t_2)\ \ t_1 \Longrightarrow t_1 = 0 \text{ or } t_2 = 0;$
(5) $\mathcal{F}(t_1, t_2) = \frac{ln(1 + a^{t_1})}{2}, a > e, \mathcal{F}(t_1, t_2) = t_1 \Longrightarrow t_1 = 0;$
(6) $\mathcal{F}(t_1, t_2) = (t_1 + l)^{(1/(1+t_2)^r)} - l, l > 1, r \in (0, \infty), \mathcal{F}(t_1, t_2) = t_1 \Longrightarrow t_2 = 0;$
(7) $\mathcal{F}(t_1, t_2) = t_1 log_{t_2 + a} a, a > 1, \mathcal{F}(t_1, t_2) = t_1 \Longrightarrow t_1 = 0 \text{ or } t_2 = 0;$
(8) $\mathcal{F}(t_1, t_2) = t_1 - \left(\frac{1+t_1}{2+t_1}\right)\left(\frac{t_2}{1+t_2}\right), \mathcal{F}(t_1, t_2) = t_1 \Longrightarrow t_2 = 0;$
(9) $\mathcal{F}(t_1, t_2) = t_1 \beta(t_1)$ *where* β: $[0,\infty) \to [0,1)$ *is continuous*, $\mathcal{F}(t_1, t_2) = t_1 \Longrightarrow t_1 = 0;$
(10) $\mathcal{F}(t_1, t_2) = t_1 - \frac{t_2}{k+t_2}, \mathcal{F}(t_1, t_2) = t_1 \Longrightarrow t_2 = 0;$
(11) $\mathcal{F}(t_1, t_2) = t_1 - \varphi(t_1), \mathcal{F}(t_1, t_2) = t_1 \Longrightarrow t_1 = 0$ *where* $\varphi : [0, \infty) \to [0, \infty)$ *is a continuous function such that* $\varphi(t_2) = 0$ *if, and only if,* $t_2 = 0;$
(12) $\mathcal{F}(t_1, t_2) = t_1 h(t_1, t_2), \mathcal{F}(t_1, t_2) = t_1 \Longrightarrow t_1 = 0$ *where* $h : [0, \infty) \times [0, \infty) \to [0, \infty)$ *is a continuous function such that* $h(t_2, t_1) < 1$ *for all* $t_2, t_1 > 0;$
(13) $\mathcal{F}(t_1, t_2) = t_1 - \left(\frac{2+t_2}{1+t_2}\right) t_2, \mathcal{F}(t_1, t_2)\ \ t_1 \Longrightarrow t_2 = 0;$
(14) $\mathcal{F}(t_1, t_2) = \sqrt[n]{\ln(1 + t_1{}^n)}, \mathcal{F}(t_1, t_2) = t_1 \Longrightarrow t_1 = 0;$
(15) $\mathcal{F}(t_1, t_2) = \phi(t_1), \mathcal{F}(t_1, t_2) = t_1 \Longrightarrow t_1 = 0$ *where* $\phi : [0, \infty) \to [0, \infty)$ *is an upper semicontinuous function such that* $\phi(0) = 0$ *and* $\phi(t_2) < t_2$ *for* $t_2 > 0;$
(16) $\mathcal{F}(t_1, t_2) = \vartheta(t_1)$ *where* $\vartheta : R^+ \times R^+ \to R^+$ *is a generalized Mizoguchi–Takahashi type function,*
$\mathcal{F}(t_1, t_2) = t_1 \Longrightarrow t_1 = 0;$
(17) $\mathcal{F}(t_1, t_2) = \frac{t_1}{\Gamma(\frac{1}{2})} \int_0^\infty \frac{e^{-u}}{\sqrt{u} + t_2}$ *where* Γ *is the Euler Gamma function.*

Let Ψ *be the family of continuous and monotone nondecreasing functions* ψ: $[0, \infty) \to [0, \infty)$ *such that* $\psi(t) = 0$ *if, and only if,* $t = 0$; *let* Φ_1 *be the family of continuous functions* φ: $[0, \infty) \to [0, \infty)$ *such that* $\varphi(t) = 0$ *if, and only if,* $t = 0$; *and let* Φ_u *be the family of continuous functions* φ: $[0, \infty) \to [0, \infty)$ *such that* $\varphi(0) \ge 0$. *Note that* $\Phi_1 \subset \Phi_u$.

2. Main Results

In this section, $\mathcal{F}$-generalized (ψ, φ, ξ)-contractive type mappings are defined and some common fixed point theorems are proved.

Definition 11. *Let U be an ordered partial metric space. Two mappings* $S,\ T: U \to 2^U$ *are said to be* $\mathcal{F}$*-generalized* $(\psi,\ \varphi,\ \xi)$*-contractive type mappings if*

$$\psi\big(H_p(Tu,\ Sv)\big) \le \mathcal{F}(\psi(M(u,\ v)\ ),\ \ \varphi(\psi(M(u,\ v)\ )\ )$$
$$+\ \xi(p(u,\ Tu),\ p(v,\ Sv),\ p(v,\ Tu) - p(v,\ v),\ p(u,\ Sv) -\ p(u,\ u)),$$

for all $u,\ v \in U$ *with u and v comparable and* $\psi \in \Psi,\ \ \varphi \in \Phi_u,\ \ \xi \in \Upsilon,\ \mathcal{F} \in \mathcal{C}$.

Definition 12. *Limit comparison property: A nonempty set U is said to hold the limit comparison property if for a sequence* $\{u_n\} \in U$ *such that* $u_n \to u$ *implies that* u_n *is comparable to u for all* $n\ \in N$.

Theorem 1. *Let* $(U,\ d,\ \preccurlyeq)$ *be a complete ordered partial metric space with the limit comparison property. Assume that* $S,\ T: U \to 2^U$ *are weakly isotone increasing* $\mathcal{F}$*-generalized* $(\psi,\ \varphi,\ \xi)$*-contractive type mappings and satisfy the approximative property. Suppose that there exists* $u_0 \in U$ *such that* $\{u_0\} \preccurlyeq_2 Tu_0$.

Then T, S have a common fixed point $u \in U$ *such that* $p(u, u) = 0$.

Proof. Firstly, it is proved that if u is a fixed point of T such that $p(u, u) = 0$, then it is a common fixed point of T and S.

By using the given contractive condition and property 2) of ξ,

$$\begin{aligned}
\psi(p(u,\ Su)) &\le \psi\big(H_p(Tu,\ Su)\big) \\
&\le \mathcal{F}(\psi(M(u,\ u),\ \varphi(\psi(M(u,\ u)))) \\
&+\ \xi(p(u,\ Tu),\ p(u,\ Su),\ p(u,\ Tu) - p(u,\ u),\ p(u,\ Su) - p(u,\ u)) \qquad (1)\\
&=\ \mathcal{F}(\psi(M(u,\ u),\ \varphi(\psi(M(u,\ u))))\ +\ \xi(0,\ p(u,\ Su),0,\ p(u,\ Su) - 0) \\
&=\ \mathcal{F}(\psi(M(u,\ u),\ \varphi(\psi(M(u,\ u))))
\end{aligned}$$

where

$$\begin{aligned}
M(u,\ u) &=\ Max\left\{p(u,\ u),\ p(u,\ Tu),\ p(u,\ Su),\ \frac{p(u,\ Su) +\ p(u,\ Tu)}{2}\right\} \\
&\le Max\left\{p(u,\ u),\ p(u,\ u),\ p(u,\ Su),\ \frac{p(u,\ Su) +\ p(u,\ u)}{2}\right\} \\
&=\ p(u,\ Su).
\end{aligned}$$

Thus, by (1),

$$\begin{aligned}
\psi(p(u,\ Su)) &\le\ \mathcal{F}(\psi(p(u,\ Su),\ \varphi(\psi(p(Su,\ u)))) \\
&=\ \mathcal{F}(\psi(p(u,\ Su))\ \varphi(\psi(p(u,\ Su)))).
\end{aligned}$$

This implies that $\psi(p(u,\ Su))\ =\ 0$ or $\varphi(\psi(p(u,\ Su)))\ =\ 0$; therefore, $p(Su,\ u)\ =\ 0$. Since Su satisfies the approximative property, there exists $v \in P_{Su}(u)$ such that $p(v,\ u)\ =\ p(u,\ Su)\ =\ 0$, i.e., $v =\ u$. Thus $u \in Su$.

Let $u_0\ \in\ U$; if $u_0\ \in\ Tu_0$, the proof is complete. Otherwise, from the fact that Tu_0 has the approximative property, it follows that there exists $u_1 \in\ Tu_0$ with $u_1 \ne u_0$ such that

$$p(u_0, u_1) = inf_{u\in\ Tu_0} p(u,\ u_0) =\ p(Tu_0,\ u_0).$$

Again, if $u_1\ \in\ Su_1$, the proof is complete. Otherwise, since Su_1 has the approximative property, it follows that there exists $u_2 \in\ Su_1$ with $u_2 \ne u_1$ such that

$$p(u_1, u_2) = inf_{u\in\ Su_1} p(u,\ u_1) =\ p(Su_1,\ u_1).$$

By repeating this process, we can find a sequence $\{u_n\}$ in U such that $u_{2n+1} \in Tu_{2n}$ and $p(u_{2n+1}, u_{2n}) = p(Tu_{2n}, u_{2n})$ and $u_{2n+2} \in Su_{2n+1}$ with $p(u_{2n+2}, u_{2n+1}) = p(Su_{2n+1}, u_{2n+1})$, On the other hand,

$$\begin{aligned} p(Tu_{2n}, u_{2n}) &\leq sup_{u \in Su_{2n-1}} p(Tu_{2n}, u) \\ &\leq Hp(Tu_{2n}, Su_{2n-1}). \end{aligned}$$

Therefore,

$$p(u_{2n+1}, u_{2n}) \leq Hp(Tu_{2n}, Su_{2n-1}) \tag{2}$$

and similarly

$$p(u_{2n+2}, u_{2n+1}) \leq Hp(Su_{2n+1}, Tu_{2n}) \tag{3}$$

since $u_0 \preccurlyeq_2 Tu_0$ and $u_1 \in Tu_0 \Rightarrow u_0 \preccurlyeq_2 u_1$. Also, since T and S are isotone increasing, $Tu_0 \preccurlyeq_2 Sv$ for all $v \in Tu_0$; thus, $Tu_0 \preccurlyeq_2 Su_1$. In particular, $u_1 \preccurlyeq_2 u_2$. Continuing this process, we obtain

$$u_1 \preccurlyeq u_2 \preccurlyeq \ldots \preccurlyeq u_n \preccurlyeq u_{n+1} \preccurlyeq \ldots$$

Now it is required to show that $\lim_{n \to \infty} p(u_{n+1}, u_n) = 0$.

Using (2) and the fact that T and S are $\mathcal{F}$-generalized (ψ, φ, ξ)-contractive mappings, we get

$$\begin{aligned} \psi(p(u_{2n+1}, u_{2n})) &\leq \psi(Hp(Tu_{2n}, Su_{2n-1})) \\ &\leq \mathcal{F}(\psi(M(u_{2n}, u_{2n-1}), \varphi(\psi(M(u_{2n}, u_{2n-1})))) \\ &+ \xi(p(u_{2n}, Tu_{2n}), p(u_{2n-1}, Su_{2n-1}), p(u_{2n}, Su_{2n-1}) - p(u_{2n}, u_{2n}) \\ &, p(u_{2n+1}, Tu_{2n}) - p(u_{2n}, u_{2n})) \\ &= \mathcal{F}(\psi(M((u_{2n}, u_{2n-1}), \varphi(\psi(M(u_{2n}, u_{2n-1})))) \\ &+ \xi(p(u_{2n}, u_{2n+1}), p(u_{2n-1}, Su_{2n-1}), p(u_{2n}, u_{2n}) - p(u_{2n}, u_{2n}) \\ &, p(u_{2n+1}, u_{2n+1}) - p(u_{2n-1}, u_{2n-1})) \\ &\leq \mathcal{F}(\psi(M((u_{2n}, u_{2n-1}), \varphi(\psi(M(u_{2n}, u_{2n-1})))) \end{aligned} \tag{4}$$

where

$$\begin{aligned} M(u_{2n}, u_{2n-1}) &= Max\left\{p(u_{2n}, u_{2n-1}), p(u_{2n}, Tu_{2n}), p(u_{2n-1}, Su_{2n-1}), \frac{p(u_{2n-1}, Tu_{2n}) + p(u_{2n}, Su_{2n-1})}{2}\right\} \\ &\leq Max\left\{p(u_{2n}, u_{2n-1}), p(u_{2n}, u_{2n+1}), p(u_{2n-1}, u_{2n}), \frac{p(u_{2n-1}, u_{2n+1}) + p(u_{2n}, u_{2n})}{2}\right\} \\ &\leq Max\left\{p(u_{2n}, u_{2n-1}), p(u_{2n}, u_{2n+1}), p(u_{2n-1}, u_{2n}), \frac{p(u_{2n-1}, u_{2n-1}) + p(u_{2n}, u_{2n+1})}{2}\right\} \\ &= Max\{p(u_{2n}, u_{2n-1}), p(u_{2n}, u_{2n+1})\}. \end{aligned}$$

If $Max\{p(u_{2n}, u_{2n-1}), p(u_{2n}, u_{2n+1})\} = p(u_{2n}, u_{2n+1})$, then by (4),

$$\psi(p(u_{2n}, u_{2n+1})) \leq \mathcal{F}(\psi(p(u_{2n}, u_{2n-1}), \varphi(\psi(p(u_{2n+1}, u_{2n}))))$$

which implies that $\psi(p(u_{2n}, u_{2n+1})) = 0$ or $\varphi(\psi(p(u_{2n+1}, u_{2n}))) = 0$. Therefore, $p(u_{2n}, u_{2n+1}) = 0$, which is a contradiction. Thus, $p(u_{2n}, u_{2n-1}) \leq M(u_{2n}, u_{2n-1}) \leq p(u_{2n}, u_{2n-1})$ and so $M(u_{2n}, u_{2n-1}) = p(u_{2n}, u_{2n-1})$.

Also, by using (4), we get

$$\psi(p(u_{2n}, u_{2n+1})) \leq \mathcal{F}(\psi(p(u_{2n}, u_{2n-1})), \varphi(\psi(p(u_{2n}, u_{2n-1})))) \leq \psi(p(u_{2n}, u_{2n-1})). \tag{5}$$

Proceeding as above,

$$\psi(p(u_{2n+1}, u_{2n+2})) \leq \mathcal{F}(\psi(p(u_{2n}, u_{2n+1})), \varphi(\psi(p(u_{2n}, u_{2n+1})))) \leq \psi(p(u_{2n}, u_{2n+1})). \tag{6}$$

By (5) and (6),

$$p(u_{n+1},\ u_n) \le\ p(u_n,\ u_{n-1}) \text{ for each } n \in N.$$

Therefore, the sequence $\{p(u_n,\ u_{n+1})\}$ is a nonnegative and nonincreasing sequence; thus, there exists $r > 0$ such that

$$lim_{n\to\infty}\ p(u_n,\ u_{n+1}) =\ r.$$

Now, since φ is lower semicontinuous,

$$\varphi(\psi(r)) \le lim\ _{n\to\infty}\inf\varphi(\psi(p(u_n,\ u_{n-1}))).$$

Therefore, by (5), we obtain

$$\psi(r) \le\ \mathcal{F}(\psi(r),\ \varphi(\psi(r))).$$

This implies that $\psi(r) = 0$ or $\varphi(\psi(r)) = 0$. Hence, $r =\ 0$.

Next it remains to show that $\{u_n\}$ is a Cauchy sequence in U, i.e., to prove that

$$lim_{n,m\to\infty} p(u_n,\ u_m) =\ 0.$$

Assume that the sequence $\{u_{2n}\}$ is not a Cauchy sequence in $(U,\ p)$; then, by Lemma 2, there exist $\epsilon > 0$ and two sequences $\left\{u_{m(k)}\right\}$ and $\left\{u_{n(k)}\right\}$ of $\{u_n\}$ with $n(k) > m(k) >\ k$ such that the sequences

$$p\left(u_{2m(k)},\ u_{2n(k)+1}\right),\quad p\left(u_{2m(k)},\ u_{2n(k)}\right),\quad p\left(u_{2m(k)-1},\ u_{2n(k)+1}\right),\quad p\left(u_{2m(k)-1},\ u_{2n(k)}\right)$$

tend to $\epsilon > 0$ when $k \to \infty$.

Using the given contractive condition,

$$\begin{aligned}\psi\left(p\left(u_{2m(k)},\ u_{2n(k)+1}\right)\right) &\le \psi\left(P_h\left(Tu_{2m(k)-1}, Su_{2n(k)}\right)\right)\\ &\le\ \mathcal{F}\left(\psi\left(M\left(u_{2m(k)-1}, u_{2n(k)}\right)\right),\ \varphi\left(\psi\left(M(u_{2m(k)-1}, Su_{2n(k)})\right)\right)\right)\end{aligned} \tag{7}$$

where

$$\begin{aligned}M(u_{2m(k)-1}, Su_{2n(k)}) &= Max\left\{\begin{array}{c} p\left(u_{2m(k)-1}, u_{2n(k)}\right),\ p\left(u_{2m(k)-1}, Tu_{2m(k)-1}\right),\ p\left(u_{2n(k)}, Su_{2n(k)}\right)\\ ,\ \frac{p\left(u_{2n(k)},\ Tu_{2m(k)-1}\right)+\ p\left(u_{2m(k)-1}, Su_{2n(k)}\right)}{2}\end{array}\right\}\\ &\le Max\left\{\begin{array}{c} p\left(u_{2m(k)-1}, u_{2n(k)}\right),\ p\left(u_{2m(k)-1}, u_{2m(k)}\right),\ p\left(u_{2n(k)}, u_{2n(k)+1}\right)\\ ,\ \frac{p\left(u_{2n(k)},\ u_{2m(k)}\right)+\ p\left(u_{2m(k)-1}, u_{2n(k)+1}\right)}{2}\end{array}\right\}\\ &\to\ Max\{\epsilon,\ 0, 0, \epsilon\}\ =\ \epsilon \text{ as } k \to \infty.\end{aligned}$$

Thus, by (7) and for any $k \to \infty$,

$$\psi(\epsilon) \le\ \mathcal{F}(\psi(\epsilon),\ \varphi(\psi(\epsilon))).$$

This implies that $\psi(\epsilon) = 0$ or $\varphi(\psi(\epsilon)) = 0$ and thus $\epsilon = 0$, which is a contradiction. Therefore, the sequence $\{u_n\}$ is a Cauchy sequence. As $(U,\ p)$ is complete, the space $\left(U,\ d_p\right)$ is complete. Therefore, $lim_{n\to\infty} d_p(u_n,\ u) = 0$ for some $u \in U$. Now, by Lemma 1,

$$p(u,\ u) =\ lim_{n\to\infty} p(u_n,\ u) = lim_{m,n\to\infty} p(u_n,\ u_m)\ =\ 0.$$

Since U has the limit comparison property, for $n \in N,\ \ u_n$ is comparable to u; therefore,

$$p(u_{2n+2}, Tu) \le sup_{u\in Su_{2n+1}} p(u,\ Tu) \le\ Hp(Su_{2n+1},\ Tu).$$

Thus,

$$
\begin{aligned}
\psi(p(u_{2n+2}, Tu)) &\leq \psi(Hp(Su_{2n+1},\ Tu)) \\
&\leq \mathcal{F}(\psi(M(u_{2n+1},\ u))\ \varphi(\psi(M(u_{2n+1},\ u)))) \\
&+ \xi\begin{pmatrix} p(u_{2n+1},\ Su_{2n+1}),\ p(u,\ Tu),\ p(u_{2n+1},\ Tu) - p(u_{2n+1}, u_{2n+1}),\ p(u,\ Su_{2n+1}) \\ -p(u,\ u) \end{pmatrix} \\
&\leq \mathcal{F}(\psi(M(u_{2n+1},\ u)),\ \varphi(\psi(M(u_{2n+1},\ u)))) \\
&+ \xi\begin{pmatrix} p(u_{2n+1},\ u_{2n+2}),\ p(u,\ Tu),\ p(u_{2n+1},\ Tu) - p(u_{2n+1}, u_{2n+1}),\ p(u,\ u_{2n+2}) \\ -p(u,\ u) \end{pmatrix}
\end{aligned} \tag{8}
$$

where

$$
\begin{aligned}
p(u, Tu) &\leq M(u_{2n+1},\ u) \\
&= Max\left\{p(u_{2n+1},\ u),\ p(u_{2n+1},\ Su_{2n+1}),\ p(u, Tu),\ \frac{p(u_{2n+1},\ Tu) +\ p(u, Su_{2n+1})}{2}\right\} \\
&\leq Max\left\{p(u_{2n+1},\ u),\ p(u_{2n+1},\ u_{2n+2}),\ p(u, Tu),\ \frac{p(u_{2n+1},\ Tu) +\ p(u, u_{2n+2})}{2}\right\}.
\end{aligned}
$$

Taking the limit as $n \to \infty$, we get $lim_{n\to\infty} M(u_{2n+1},\ u) = d(u,\ Tu)$. Since φ is lower semicontinuous, taking the limit as $n \to \infty$, in (8) implies

$$\psi(p(u,\ Tu)) \leq \mathcal{F}(\psi(p(u,\ Tu)),\ \varphi(\psi(p(u,\ Tu)))),$$

which further implies that $\psi(p(u,\ Tu)) = 0$ or $\varphi(\psi(p(u,\ Tu))) = 0$.

Thus, $p(u,\ Tu) = 0$. Since Tu has the approximative property, there exists $v \in P_{Tu}$ such that $p(v,\ u) = 0$, i.e., $v = u$; therefore, $u \in Tu$. Thus, u is a fixed point of T. This completes the proof. □

By putting $\mathcal{F}(t_1,\ t_2) = t_1 - t_2$, the following result holds:

Corollary 1. *Let U be a complete ordered partial metric space satisfying the limit comparison property. Let $S,\ T: U \to 2^U$ be two weakly isotone increasing mappings with the approximative property such that*

$$
\begin{aligned}
\psi(H_p(Tu,\ Sv)) &\leq \psi(M(u,\ v)) - \varphi(\psi(M(u,\ v))) \\
&+ \xi(p(u,\ Tu),\ p(v,\ Sv),\ p(v,\ Tu) - p(v,\ v),\ p(u,\ Sv) - p(u,\ u)).
\end{aligned}
$$

Suppose that there exists $u_0 \in U$ such that $\{u_0\} \preccurlyeq_2 Tu_0$. Then T, S have a common fixed point $u \in U$ such that $p(u, u) = 0$.

On putting $\mathcal{F}(t_1,\ t_2) = mt_1$ and $\psi(t) = t$, the following result is obtained:

Corollary 2. *Let U be a complete ordered partial metric space satisfying the limit comparison property. Let $S,\ T: U \to 2^U$ be two weakly isotone increasing mappings with the approximative property, and suppose there exists $k \in [0,\ 1)$ such that*

$$H_p(Tu,\ Sv)\ \leq m\ M(u,\ v) + \xi(p(u,\ Tu),\ p(v,\ Sv),\ p(v,\ Tu) - p(v,\ v),\ p(u,\ Sv) - p(u,\ u))$$

for all $u, v \in U$ with u and v comparable and $\xi \in \Upsilon$. Suppose that there exists $u_0 \in U$ such that $\{u_0\} \preccurlyeq_2 Tu_0$. Then T, S have a common fixed point $u \in U$ such that $p(u, u) = 0$.

By putting $S = T$ in Theorem 1, the following corollary holds:

Corollary 3. *Let $(U, d, \preccurlyeq)$ be a complete ordered partial metric space with the limit comparison property. Assume that $T: U \to 2^U$ is a weakly isotone increasing $\mathcal{F}$-generalized (ψ, φ, ξ)-contractive type mapping and satisfies the approximative property. Suppose that there exists $u_0 \in U$ such that $\{u_0\} \preccurlyeq_2 Tu_0$.*

Then T has a fixed point $u \in U$ such that $p(u, u) = 0$.

Example 2. *Let $U = [0,1]$ be equipped with partial metric p defined by $p(u, v) = max\{u, v\}$ for each $u, v \in U$. Define the partial order on U by*

$$u \preccurlyeq v \iff p(u, u) = p(u, v) \iff u = \max\{u, v\} \iff v \le u.$$

It is easy to check that (U, $\preccurlyeq$) is a totally ordered set and (U, p) is a complete partial metric space. Also, the mappings T and S are defined as

$$Tu = \left\{ \begin{array}{cc} \{0\} & if\ u \in \left\{0, \frac{1}{2}\right\}, \\ \left\{0, \frac{1}{2}\right\} & otherwise. \end{array} \right\} \quad \text{and} \quad Su = \left\{ \begin{array}{cc} \{0\} & if\ u \in \left\{0, \frac{1}{2}\right\}, \\ \left\{\frac{1}{2}\right\} & otherwise. \end{array} \right\}.$$

Note that T and S are weakly isotone increasing as for $z \in Su,\ w \in Tv \Longrightarrow w = 0$. Thus, $w \le z \Longrightarrow z \preccurlyeq w$. Hence, for each $u \in U$; $Su \preccurlyeq_2 Tv$ for each $v \in Su$. Similarly, for each $u \in U$, it can be easily shown that $Tu \preccurlyeq_2 Sv$ for all $v \in Tu$.

$$\text{Let } \psi(t) = 2t \text{ and } \varphi(t) = \frac{t}{2},\ \mathcal{F}(t_1,\ t_2) = \frac{1}{2}t_1\ (s_1,\ s_2,\ s_3,\ s_4) = s_1 s_2 s_3 s_4.$$

Next it is proved that the mappings T and S are $\mathcal{F}$-generalized (ψ, φ, ξ)-contractive type mappings. The following cases arise:

Case I. *If $u, v \in \left\{0, \frac{1}{2}\right\}$, then*

$$\begin{aligned} \psi(Hp(Tu, Sv)) &= \psi(Hp(\{0\}, \{0\})) \\ &= \psi(0) \\ &= 0 \le F(\psi(M(u, v)), \varphi(\psi(M(u, v))) \\ &\quad + \xi(p(u, Tu), p(v, Sv), p(v, Tu) - p(v, v), p(u, Sv) \\ &\quad - p(u, u)) \end{aligned}$$

Case II. *If $u = v = 1$, then*

$$\begin{aligned} \psi(Hp(Tu, Sv)) &= \psi\left(Hp\left(\left\{0, \frac{1}{2}\right\}, \left\{\frac{1}{2}\right\}\right)\right) \\ &= \psi\left(\frac{1}{2}\right) = 1. \end{aligned}$$

Now

$$\begin{aligned} M(u, v) &= Max\left\{p(u, v),\ p(u, Tu),\ p(v, Sv),\ \frac{p(u, Sv) + p(v, Tu)}{2}\right\} \\ &\le Max\left\{p(1,1),\ p\left(1,\ \left\{0, \frac{1}{2}\right\}\right),\ p\left(1,\ \frac{1}{2}\right),\ \frac{p\left(1,\ \left\{0, \frac{1}{2}\right\}\right) + p\left(1,\ \frac{1}{2}\right)}{2}\right\} \\ &= Max\left\{1,\ 1, 1,\ \frac{1}{2}(1+1)\right\} \\ &= 1, \end{aligned}$$

so

$$\begin{aligned} F(\psi(M(u,\, v)), \varphi(\psi(M(u,\, v))) + \xi(p(u,\, Tu),\; p(v,\, Sv),\; p(v,\, Tu) - p(v,\, v),\; p(u,\, Sv) - p(u,\, u)) \\ = \frac{1}{2}\psi(1) + \xi(1, 1, 1-1, 1-1) \\ = 1. \end{aligned}$$

Thus, the contractive condition is proved. Similarly, the remaining cases can be discussed and proved. Hence, all the hypotheses of Theorem 1 are fulfilled. Therefore, T, S have a common fixed point $u = 0$.

3. Discussion

The notion of $\mathcal{F}$-generalized contractive type mappings and some common fixed point theorems for multivalued mappings in ordered partial metric spaces using C-class functions are obtained in this paper. These results improve and extend many relevant results existing in literature. These results can be applied for various types of generalized mappings as well as for various abstract spaces.

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