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A New Extension of the τ-Gauss Hypergeometric Function and Its Associated Properties

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Abstract: In this article, we define an extended version of the Pochhammer symbol and then introduce the corresponding extension of the τ-Gauss hypergeometric function. The basic properties of the extended τ-Gauss hypergeometric function, including integral and derivative formulas involving the Mellin transform and the operators of fractional calculus, are derived. We also consider some new and known results as consequences of our proposed extension of the τ-Gauss hypergeometric function.

Keywords: gamma function and its extension; Pochhammer symbol and its extensions; hypergeometric function and its extensions; τ-Gauss hypergeometric function and its extensions; τ-Kummer hypergeometric function; Fox-Wright function

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1. Introduction

Throughout this article, we denote the sets of positive integers, negative integers, and complex numbers by $\mathbb{N}$, $\mathbb{Z}^-$, and $\mathbb{C}$, respectively. We also set

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} \qquad \text{and} \qquad \mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}.$$

During the past few decades, various extensions and generalizations of well-known special functions have been studied by various researchers (see, for example, [1–6]). For example, a two-parameter extension of the gamma function $\Gamma(\xi)$ with the parameters p and v) was defined in [2] by

$$\Gamma_v(\xi;p) = \begin{cases} \sqrt{\dfrac{2p}{\pi}} \displaystyle\int_0^\infty t^{\xi-\frac{3}{2}}\, e^{-t}\, K_{v+\frac{1}{2}}\left(\frac{p}{t}\right) dt & \big(\min\{\Re(p), \Re(v)\} > 0;\ \xi \in \mathbb{C}\big), \\ \Gamma_p(\xi) & \big(v = 0;\ \Re(\xi) > 0\big), \end{cases} \tag{1}$$

where $K_v(z)$ is the modified Bessel function (or the Macdonald function) of order v and $\Gamma_p(\xi)$ was studied in [2,7]. Indeed, if we set $v = 0$ in (1) and make use of the following relationship:

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z},$$

then this extended gamma function $\Gamma_p(\xi)$ is given by (see [2,7])

$$\Gamma_p(\xi) = \int_0^\infty t^{\xi-1}\, e^{-t-\frac{p}{t}}\, dt \qquad \big(\Re(p) > 0;\ \Re(\xi) > 0\big). \tag{2}$$

In the year 2012, Srivastava et al. [8] (see also [9]) defined the incomplete Pochhammar symbols in terms of the incomplete gamma functions. Another generalization of the Pochhammer symbol was defined in [10] by

$$(\xi; p)_\mu = \begin{cases} \dfrac{\Gamma_p(\xi+\mu)}{\Gamma(\xi)} & \big(\Re(p) > 0;\ \xi, \mu \in \mathbb{C}\big), \\ (\xi)_\mu & \big(p = 0;\ \xi, \mu \in \mathbb{C} \setminus \{0\}\big). \end{cases} \tag{3}$$

Here, in our present investigation, we first introduce a new extension $(\xi; p, v)_\mu$ of the Pochhammer symbol $(\xi; p)_\mu$ in (3), which is defined by

$$(\xi; p, v)_\mu = \begin{cases} \dfrac{\Gamma_v(\xi+\mu; p)}{\Gamma(\xi)} & \big(\min\{\Re(p), \Re(v)\} > 0;\ \xi, \mu \in \mathbb{C}\big), \\ (\xi; p)_\mu & \big(v = 0;\ \xi, \mu \in \mathbb{C} \setminus \{0\}\big), \end{cases} \tag{4}$$

where, as we mentioned above in connection with (3), the generalized Pochhmmer symbol $(\xi; p)_\mu$ was studied by Srivastava et al. [10]. The integral representation of the extended Pochhammer symbol $(\xi; p, v)_\mu$ is given by

$$(\xi; p, v)_\mu = \sqrt{\frac{2p}{\pi}}\, \frac{1}{\Gamma(\xi)} \int_0^\infty t^{\xi+\mu-\frac{3}{2}}\, e^{-t}\, K_{v+\frac{1}{2}}\left(\frac{p}{t}\right)\, dt, \tag{5}$$

which, in the special case when $v = 0$, yields the following result due to Srivastava et al. [10]):

$$(\xi; p, 0)_\mu = (\xi; p)_\mu = \frac{1}{\Gamma(\xi)} \int_0^\infty t^{\xi+\mu-1}\, e^{-t-\frac{p}{t}}\, dt \tag{6}$$

$$\big(\Re(p) > 0;\ \Re(\xi+\mu) > 0 \quad \text{when} \quad p = 0\big).$$

By using the definition (4), we now define an extension of the generalized hypergeometric function ${}_{\mathfrak{p}}F_{\mathfrak{q}}$ (with $\mathfrak{p}$ numerator parameters and $\mathfrak{q}$ denominator parameters) as follows:

$${}_{\mathfrak{p}}F_{\mathfrak{q}}\left[\begin{matrix} (\rho_1; p, v), \rho_2, \cdots, \rho_{\mathfrak{p}}; \\ \\ \sigma_1, \cdots, \sigma_{\mathfrak{q}}; \end{matrix}\ z\right] = \sum_{n=0}^{\infty} \frac{(\rho_1; p, v)_n\, (\rho_2)_n \cdots (\rho_{\mathfrak{p}})_n}{(\sigma_1)_n \cdots (\sigma_{\mathfrak{q}})_n}\, \frac{z^n}{n!}, \tag{7}$$

where

$$\rho_j \in \mathbb{C}\ \ (j = 1, \cdots, \mathfrak{p}) \qquad \text{and} \qquad \sigma_j \in \mathbb{C} \setminus \mathbb{Z}_0^-\ \ (j = 1, \cdots, \mathfrak{q}).$$

Another interesting extension of the Pochhammer symbol and the associated hypergeometric functions was recently given by Srivastava et al. in [11].

We next recall that Virchenko et al. [12] studied the following τ-Gauss hypergeometric function ${}_2R_1^{\tau}(z)$ defined by (see also [13,14])

$$ {}_2R_1^{\tau}(z) = {}_2R_1(\delta_1, \delta_2; \delta_3; \tau; z) = \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)} \sum_{n=0}^{\infty} \frac{(\delta_1)_n \, \Gamma(\delta_2 + n\tau)}{\Gamma(\delta_3 + n\tau)} \, \frac{z^n}{n!} \tag{8} $$

$$ \big(\tau > 0;\ |z| < 1;\ \Re(\delta_3) > \Re(\delta_2) > 0 \ \text{ when } \ |z| = 1\big), $$

for which they derived an integral representation in the form

$$ {}_2R_1(\delta_1, \delta_2; \delta_3; \tau; z) = \frac{1}{B(\delta_2; \delta_3 - \delta_2)} \int_0^{\infty} t^{\delta_2 - 1} \, (1-t)^{\delta_3 - \delta_2 - 1} \, (1 - zt^{\tau})^{-\delta_1} \, dt \tag{9} $$

$$ \big(\tau > 0;\ |\arg(1-z)| < \pi;\ \Re(\delta_3) > \Re(\delta_2) > 0\big) $$

in terms of the classical beta function $B(\alpha, \beta)$ defined by

$$ B(\alpha, \beta) := \begin{cases} \displaystyle\int_0^1 t^{\alpha-1} \, (1-t)^{\beta=a} \, dt & \big(\min\{\Re(\alpha), \Re(\beta)\} > 0\big), \\[2ex] \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & \big(\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-\big). \end{cases} \tag{10} $$

Remark 1. *For $\tau = 1$, (8) and (9) would immediately yield the definition of the Gauss hypergeoemtric function ${}_2F_1(\delta_1, \delta_2; \delta_3; z)$ and its Eulerian integral representation (see, for details, [15]).*

Remark 2. *The so-called τ-Gauss hypergeometric function in (8) is, in fact, a rather specialized case of the widely-studied Fox-Wright extension ${}_{\mathfrak{p}}\Psi_{\mathfrak{q}}$ of the generalized hypergeometric function ${}_{\mathfrak{p}}F_{\mathfrak{q}}$ in (7) involving $\mathfrak{p}$ numerator and $\mathfrak{q}$ denominator parameters (see, for example, [16]).*

2. An Extension of the τ-Gauss Hypergeometric Function

In this section, we first introduce the following extension of the τ-Gauss hypergeometric function ${}_2R_1^{\tau}(z)$ in terms of the Pochhammer symbol $(\xi; p, v)_{\mu}$ defined by (4) for $\delta_1, \delta_2 \in \mathbb{C}$ and $\delta_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-$:

$$ \begin{aligned} &{}_2R_1\big[(\delta_1; p, v), \delta_2; \delta_3; \tau; z\big] \\ &\quad = \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)} \sum_{n=0}^{\infty} \frac{(\delta_1; p, v)_n \, \Gamma(\delta_2 + n\tau)}{\Gamma(\delta_3 + n\tau)} \, \frac{z^n}{n!} \end{aligned} \tag{11} $$

$$ \big(p \geqq 0;\ v > 0;\ \tau > 0;\ |z| < 1;\ \Re(\delta_3) > \Re(\delta_2) > 0 \ \text{ when } \ |z| = 1 \ \text{ and } \ p = 0\big). $$

Remark 3. *The following are some of the special cases of τ-Gauss hypergeometric functions defined by (11).*
(i) When $v = 0$, (11) reduces to the following extended τ-Gauss hypergeometric function (see [17]):

$$ {}_2R_1\big[(\delta_1; p), \delta_2; \delta_3; \tau; z\big] = \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)} \sum_{n=0}^{\infty} \frac{(\delta_1; p)_n \, \Gamma(\delta_2 + n\tau)}{\Gamma(\delta_3 + n\tau)} \, \frac{z^n}{n!} \tag{12} $$

$$ \big(p \geqq 0;\ \tau > 0;\ |z| < 1;\ \Re(\delta_3) > \Re(\delta_2) > 0 \ \text{ when } \ |z| = 1 \ \text{ and } \ p = 0\big). $$

(ii) When $\tau = 1$, (11) will yield the following extended Gauss hypergeometric function:

$$ {}_2F_1\big[(\delta_1; p, v); \delta_2; \delta_3; z\big] = \sum_{n=0}^{\infty} \frac{(\delta_1; p, v)_n \, (\delta_2)_n}{(\delta_3)_n} \, \frac{z^n}{n!}. $$

(iii) When $v = 0$ and $\tau = 1$, (11) will reduce to the following extended Gauss hypergeometric function (see [10]):

$$ {}_2F_1\left[(\delta_1;p);\delta_2;\delta_3;z\right] = \sum_{n=0}^{\infty} \frac{(\delta_1;p)_n\,(\delta_2)_n}{(\delta_3)_n}\,\frac{z^n}{n!}. $$

3. Integral Representations and Derivative Formulas

In this section, we obtain the Eulerian and Laplace-type integral representations and some derivative formulas of the extended τ-Gauss hypergeometric function defined by (11).

Theorem 1. *The following Eulerian representation holds true for* (11):

$$ \begin{aligned} &{}_2R_1\left[(\delta_1;p,v),\delta_2;\delta_3;\tau;z\right] \\ &\quad = \frac{1}{B(\delta_2,\delta_3-\delta_2)} \int_0^1 t^{\delta_2-1}\,(1-t)^{\delta_3-\delta_2-1}\;{}_1F_0[(\delta_1;p,v);\text{—}\,;zt^{\tau}]\,dt \end{aligned} \tag{13} $$

$$ \big(\Re(p) > 0;\ v > 0;\ \tau > 0;\ |z| < 1;\ \Re(\delta_3) > \Re(\delta_2) > 0\big), $$

where $B(\alpha,\beta)$ denotes the classical beta function defined by (10).

Proof. Using the following well-known identity involving the beta function $B(\alpha,\beta)$:

$$ \frac{(\delta_2)_{\tau n}}{(\delta_3)_{\tau n}} = \frac{B(\delta_2+\tau n,\delta_3-\delta_2)}{B(\delta_2;\delta_3-\delta_2)} = \frac{1}{B(\delta_2,\delta_3-\delta_2)} \int_0^1 t^{\delta_2+\tau n-1}\,(1-t)^{\delta_3-\delta_2-1}\,dt $$

$$ \big(\Re(\delta_3) > \Re(\delta_2) > 0\big) $$

in (11) and using the definition (7), we get the desired assertion (13) of Theorem 1. □

Theorem 2. *The following Laplace-type representation holds true for* (11):

$$ {}_2R_1\left[(\delta_1;p,v),\delta_2;\delta_3;\tau;z\right] = \frac{\sqrt{\dfrac{2p}{\pi}}}{\Gamma(\delta_1)} \int_0^{\infty} t^{\delta_1-\frac{3}{2}}\,e^{-t}\,K_{v+\frac{1}{2}}\left(\frac{p}{t}\right)\;{}_1\Phi_1^{\tau}\left[\delta_2;\delta_3;zt\right]dt \tag{14} $$

$$ \big(\Re(p) > 0;\ v > 0;\ \tau > 0;\ \Re(z) < 1;\ \Re(\delta_1) > 0\big), $$

where ${}_1\Phi_1^{\tau}\left[\delta_2;\delta_3;zt\right]$ is the τ-Kummer hypergeometric function defined by

$$ {}_1\Phi_1^{\tau}(z) = {}_1\Phi_1^{\tau}[\delta_2;\delta_3;zt] = \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)} \sum_{n=0}^{\infty} \frac{\Gamma(\delta_2+\tau n)}{\Gamma(\delta_3+\tau n)}\,\frac{z^n}{n!} \tag{15} $$

$$ (\tau > 0;\ \delta_2 \in \mathbb{C};\ \delta_3 \in \mathbb{C}\setminus\mathbb{Z}_0^-). $$

Proof. By first utilizing (5) in (11) and then applying (15), we obtain the assertion (14) of Theorem 2. □

Remark 4. *When $\tau = 1$,* (13) *and* (14) *yield the following special cases:*

$$ \begin{aligned} &{}_2F_1\left[(\delta_1;p,v);\delta_2;\delta_3;z\right] \\ &\quad = \frac{1}{B(\delta_2,\delta_3-\delta_2)} \int_0^1 t^{\delta_2-1}\,(1-t)^{\delta_3-\delta_2-1}\;{}_1F_0\left[(\delta_1;p,v);\text{—}\,;zt\right]dt \end{aligned} \tag{16} $$

and

$$ {}_2F_1\left[(\delta_1;p,v);\delta_2;\delta_3;z\right]=\frac{\sqrt{\dfrac{2p}{\pi}}}{\Gamma(\delta_1)}\int_0^\infty t^{\delta_1-\frac{3}{2}}\,e^{-t}K_{v+\frac{1}{2}}\left(\frac{p}{t}\right)\;{}_1F_1[\delta_2;\delta_3;zt]\,dt, \tag{17} $$

respectively. Similarly, when $v=0$*, our integral representations* (13) *and* (14) *reduce to the following known results (see [17]):*

$$ \begin{aligned} &{}_2R_1\left[(\delta_1;p),\delta_2;\delta_3;\tau;z\right]\\ &\quad=\frac{1}{B(\delta_2,\delta_3-\delta_2)}\int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}\;{}_1F_0\left[(\delta_1;p);\text{—};zt^\tau\right]dt \end{aligned} $$

and

$$ {}_2R_1\left[(\delta_1;p),\delta_2;\delta_3;\tau;z\right]=\frac{1}{\Gamma(\delta_1)}\int_0^\infty t^{\delta_1-1}\,e^{-t-\frac{p}{t}}\;{}_1\Phi_1^\tau[\delta_2;\delta_3;zt]\,dt, $$

respectively. Moreover, when $\tau=1$ *and* $v=0$*,* (13) *and* (14) *yield the following known results (see [10]):*

$$ \begin{aligned} &{}_2F_1\left[(\delta_1;p),\delta_2;\delta_3;z\right]\\ &\quad=\frac{1}{B(\delta_2,\delta_3-\delta_2)}\int_0^1 t^{\delta_2-1}\,(1-t)^{\delta_3-\delta_2-1}\;{}_1F_0[(\delta_1;p);\text{—};zt]\,dt \end{aligned} $$

and

$$ {}_2F_1\left[(\delta_1;p,v),\delta_2;\delta_3;z\right]=\frac{1}{\Gamma(\delta_1)}\int_0^\infty t^{\delta_1-1}\,e^{-t-\frac{p}{t}}\;{}_1F_1\left[\delta_2;\delta_3;zt\right]dt, $$

respectively.

Theorem 3. *Each of the following derivative formulas holds true for the extended* τ*-Gauss hypergeometric function defined by* (11)*:*

$$ \begin{aligned} &\frac{d^n}{dz^n}\left\{{}_2R_1[(\delta_1;p,v),\delta_2;\delta_3;\tau;z]\right\}\\ &\quad=\frac{(\delta_1)_n\,\Gamma(\delta_2+n\tau)\Gamma(\delta_3)}{\Gamma(\delta_3+n\tau)\Gamma(\delta_2)}\;{}_2R_1\left[(\delta_1+n;p,v),\delta_2+n\tau;\delta_3+n\tau;\tau;z\right] \end{aligned} \tag{18} $$

and

$$ \begin{aligned} &\frac{d^n}{dz^n}\left\{z^{\delta_3-1}\;{}_2R_1\left[(\delta_1;p,v),\delta_2;\delta_3;\tau;\omega z^\tau\right]\right\}\\ &\quad=\frac{z^{\delta_3-n-1}\Gamma(\delta_3)}{\Gamma(\delta_3-n)}\;{}_2R_1\left[(\delta_1;p,v),\delta_2;\delta_3-n;\tau;\omega z^\tau\right]. \end{aligned} \tag{19} $$

Proof. Upon differentiating both sides of (11) with respect to z, we get

$$ \frac{d}{dz}\left\{{}_2R_1\left[(\delta_1;p,v),\delta_2;\delta_3;\tau;z\right]\right\}=\frac{\Gamma(\delta_3)}{\Gamma(\delta_2)}\sum_{n=1}^\infty\frac{(\delta_1;p,v)_n\,\Gamma(\delta_2+n\tau)}{\Gamma(\delta_3+n\tau)}\,\frac{z^{n-1}}{(n-1)!}. \tag{20} $$

Replacing n by $n+1$ in (20), we have

$$ \frac{d}{dz}\left\{{}_2R_1\left[(\delta_1;p,v),\delta_2;\delta_3;\tau;z\right]\right\}=\frac{\Gamma(\delta_3)}{\Gamma(\delta_2)}\sum_{n=0}^\infty\frac{(\delta_1;p,v)_{n+1}\,\Gamma\big(\delta_2+(n+1)\tau\big)}{\Gamma\big(\delta_3+(n+1)\tau\big)}\,\frac{z^n}{n!}, $$

which, after simplification, yields

$$\begin{aligned}&\frac{d}{dz}\left\{ {}_2R_1\left[(\delta_1;p,v),\delta_2;\delta_3;\tau;z\right]\right\}\\&\quad=\frac{\delta_1\Gamma(\delta_3)\Gamma(\delta_2+\tau)}{\Gamma(\delta_3+\tau)\Gamma(\delta_2)}\sum_{n=0}^{\infty}\frac{(\delta_1+1;p,v)_n\,\Gamma(\delta_2+\tau+n\tau)}{\Gamma(\delta_3+\tau+n\tau)}\,\frac{z^n}{n!}\\&\quad=\frac{\delta_1\,\Gamma(\delta_3)\Gamma(\delta_2+\tau)}{\Gamma(\delta_3+\tau)\Gamma(\delta_2)}\;{}_2R_1[(\delta_1+1;p,v),\delta_2+\tau;\delta_3+\tau;\tau;z].\end{aligned}$$

By iterating this differentiation process n times, we are led to the desired assertion (18) of Theorem 3.

Similarly, in order to prove the assertion (19) of Theorem 3, we observe that

$$\begin{aligned}&\frac{d^n}{dz^n}\left\{z^{\delta_3-1}\;{}_2R_1\left[(\delta_1;p,v),\delta_2;\delta_3;\tau;\omega z^{\tau}\right]\right\}\\&=\frac{\Gamma(\delta_3)}{\Gamma(\delta_2)}\sum_{m=0}^{\infty}\frac{(\delta_1;p,v)_m\,\Gamma(\delta_2+m\tau)}{\Gamma(\delta_3+m\tau)}\,\frac{\omega^m}{m!}\,\frac{d^n}{dz^n}\left\{z^{\delta_3+\tau m-1}\right\}\\&=\frac{\Gamma(\delta_3)}{\Gamma(\delta_2)}\sum_{m=0}^{\infty}\frac{(\delta_1;p,v)_m\,\Gamma(\delta_2+m\tau)}{\Gamma(\delta_3+m\tau)}\,\frac{\omega^m}{m!}\\&\quad\cdot\Big[(\delta_3+\tau m-1)(\delta_3+\tau m-2)\cdots(\delta_3+\tau m-n-1)\Big]z^{\delta_3+\tau m-n-1}\\&=\frac{z^{\delta_3-n-1}\,\Gamma(\delta_3)}{\Gamma(\delta_2)}\sum_{m=0}^{\infty}\frac{(\delta_1;p,v)_m\,\Gamma(\delta_2+m\tau)}{\Gamma(\delta_3+m\tau)}\,\frac{(\omega z^{\tau})^m}{m!}\,\frac{\Gamma(\delta_3+m\tau)}{\Gamma(\delta_3+\tau m-n)}\\&=\frac{z^{\delta_3-n-1}\Gamma(\delta_3)\Gamma(\delta_3-n)}{\Gamma(\delta_3-n)\Gamma(\delta_2)}\sum_{m=0}^{\infty}\frac{(\delta_1;p,v)_m\,\Gamma(\delta_2+m\tau)}{\Gamma(\delta_3+\tau m-n)}\,\frac{(\omega z^{\tau})^m}{m!},\end{aligned}$$

which, in view of (11), gives the derivative formula (19) asserted by Theorem 3. □

4. Application of the Mellin Transform

The well-known Mellin transform of a given integrable function $f(t)$ is defined by

$$\mathfrak{M}\{f(t):t\to s\}=\int_0^{\infty}t^{s-1}\,f(t)\,dt, \tag{21}$$

provided that the improper integral in (21) exists.

Theorem 4. *The Mellin transform of the extended τ-Gauss hypergeometric function,*

$${}_2R_1\left[(\delta_1;p,v),\delta_2;\delta_3;\tau;z\right],$$

is given by

$$\begin{aligned}&\mathfrak{M}\left\{{}_2R_1\left[(\delta_1;p,v),\delta_2;\delta_3;\tau;z\right]:p\to s\right\}\\&\quad=\frac{2^{s-1}}{\sqrt{\pi}}\,(\delta_1)_s\,\Gamma\left(\frac{s-v}{2}\right)\Gamma\left(\frac{s+v+1}{2}\right)\;{}_2R_1(\delta_1+s;\delta_2,\delta_3;\tau;z)\end{aligned} \tag{22}$$

$$\big(\Re(s-v)>0;\;\Re(\delta_1+s)>-1\big).$$

Proof. Applying the definition (21) of the Mellin transform on both sides of (11), we get

$$\mathfrak{M}\left\{{}_2R_1\left[(\delta_1;p,v),\delta_2;\delta_3;\tau;z\right]:p\to s\right\}$$

$$= \int_0^\infty p^{s-1} \left(\frac{\Gamma(\delta_3)}{\Gamma(\delta_2)} \sum_{n=0}^\infty \frac{(\delta_1; p, v)_n \Gamma(\delta_2 + n\tau)}{\Gamma(\delta_3 + n\tau)} \frac{z^n}{n!} \right) dp$$

$$= \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)} \sum_{n=0}^\infty \frac{\Gamma(\delta_2 + n\tau)}{\Gamma(\delta_3 + n\tau)} \frac{z^n}{n!} \frac{1}{\Gamma(\delta_1)} \int_0^\infty p^{s-1}\, \Gamma_v(\delta_1 + n; p)\, dp. \tag{23}$$

Using the following result given by Chaudhry and Zubair ([2], Eq. 4.105),

$$\int_0^\infty p^{s-1}\, \Gamma_v(\delta_1 + n; p) dp = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s-v}{2}\right) \Gamma\left(\frac{s+v+1}{2}\right) \Gamma(\delta_1 + n + s), \tag{24}$$

in (23), we find that

$$\begin{aligned} &\mathfrak{M}\Big\{ {}_2R_1\big[(\delta_1; p, v), \delta_2; \delta_3; \tau; z\big] : p \to s \Big\} \\ &\quad = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s-v}{2}\right) \Gamma\left(\frac{s+v+1}{2}\right) \frac{\Gamma(\delta_3)\Gamma(\delta_1 + s)}{\Gamma(\delta_1 + s)\Gamma(\delta_1)\Gamma(\delta_2)} \\ &\qquad \cdot \sum_{n=0}^\infty \frac{\Gamma(\delta_1 + n + s)\Gamma(\delta_2 + n\tau)}{\Gamma(\delta_3 + n\tau)} \frac{z^n}{n!} \\ &\quad = \frac{2^{s-1}}{\sqrt{\pi}} (\delta_1)_s\, \Gamma\left(\frac{s-v}{2}\right) \Gamma\left(\frac{s+v+1}{2}\right) \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)} \\ &\qquad \cdot \sum_{n=0}^\infty \frac{(\delta_1 + s)_n\, \Gamma(\delta_2 + n\tau)}{\Gamma(\delta_3 + n\tau)} \frac{z^n}{n!}, \end{aligned} \tag{25}$$

which, in view of (8), yields the Mellin transform formula (22) asserted by Theorem 4. □

5. Use of the Operators of Fractional Calculus

In this section, we recall the operators $\mathfrak{I}_{\rho+}$ and $\mathfrak{D}_{\rho+}$ of the fractional integral and fractional derivatives of order $\mu \in \mathbb{C}$ $\big(\Re(\mu) > 0\big)$, which are defined by (see [18,19])

$$\left(\mathfrak{I}^\mu_{\rho+} f\right)(x) = \frac{1}{\Gamma(\mu)} \int_0^x \frac{f(t)}{(x-t)^{1-\mu}}\, dt \qquad (\mu \in \mathbb{C};\ \Re(\mu) > 0) \tag{26}$$

and

$$\left(\mathfrak{D}^\mu_{\rho+} f\right)(x) = \frac{d^n}{dx^n} \left\{ \left(\mathfrak{I}^{n-\mu}_{\rho+}\, f\right)(x) \right\} \qquad (\mu \in \mathbb{C};\ \Re(\mu) > 0;\ n = [\Re(\mu)] + 1), \tag{27}$$

respectively.

We now prove the following fractional integral and fractional derivative formulas associated with the extended τ-Gauss hypergeometric function:

$${}_2R_1\big[(\delta_1; p, v), \delta_2; \delta_3; \tau; z\big].$$

Theorem 5. *Let* $\rho \in \mathbb{R}_+ = [0, \infty)$, $\delta_1, \delta_2, \delta_3, \omega \in \mathbb{C}$, *and* $\min\{\Re(mu), \Re(\delta_3), \Re(\tau) > 0\}$. *Then the following formulas hold true for* $x > \rho$:

$$\begin{aligned} &\left(\mathfrak{I}^\mu_{\rho+} \left[(t-\rho)^{\delta_3 - 1}\ {}_2R_1\big[(\delta_1; p, v), \delta_2; \delta_3; \tau; \omega(t-\rho)^\tau\big]\right]\right)(x) \\ &\quad = \frac{(x-\rho)^{\delta_3 + \mu - 1}\, \Gamma(\delta_3)}{\Gamma(\delta_3 + \mu)}\ {}_2R_1\big[(\delta_1; p, v), \delta_2; \delta_3 + \mu; \tau; \omega(x-\rho)^\tau\big] \end{aligned} \tag{28}$$

and

$$\left(\mathfrak{D}^{\mu}_{\rho+}\left[(t-\rho)^{\delta_3-1}\ {}_2R_1\left[(\delta_1;p,v),\delta_2;\delta_3;\tau;\omega(t-\rho)^{\tau}\right]\right]\right)(x)$$
$$= \frac{(x-\rho)^{\delta_3-\mu-1}\,\Gamma(\delta_3)}{\Gamma(\delta_3-\mu)}\ {}_2R_1\left[(\delta_1;p,v),\delta_2;\delta_3-\mu;\tau;\omega(x-\rho)^{\tau}\right]. \tag{29}$$

Proof. Using the following well-known relation (see [18,19]),

$$\left(\mathfrak{I}^{\mu}_{\rho+}\ \left[(t-\rho)^{\delta_3-1}\right]\right)(x) = \frac{\Gamma(\delta_3)}{\Gamma(\delta_3+\mu)}(x-\rho)^{\delta_3+\mu-1} \qquad (x>\rho), \tag{30}$$

we have

$$\left(\mathfrak{I}^{\mu}_{\rho+}\left[(t-\rho)^{\delta_3-1}\ {}_2R_1\left[(\delta_1;p,v),\delta_2;\delta_3;\tau;\omega(t-\rho)^{\tau}\right]\right]\right)(x)$$
$$= \left(\mathfrak{I}^{\mu}_{\rho+}\ \left[\frac{\Gamma(\delta_3)}{\Gamma(\delta_2)}\sum_{n=0}^{\infty}\frac{(\delta_1;p,v)_n\ \Gamma(\delta_2+n\tau)}{\Gamma(\delta_3+n\tau)}\ \frac{\omega^n}{n!}(t-\rho)^{\delta_3+\tau n-1}\right]\right)$$
$$= \frac{(x-\rho)^{\delta_3+\mu-1}\Gamma(\delta_3)}{\Gamma(\delta_3+\mu)}\ {}_2R_1\left[(\delta_1;p,v),\delta_2;\delta_3+\mu;\tau;\omega(x-\rho)^{\tau}\right],$$

which proves the assertion (28) of Theorem 5.

Next, in view of (27) and (11), we have

$$\left(\mathfrak{D}^{\mu}_{\rho+}\ \left[(t-\rho)^{\delta_3-1}\ {}_2R_1\left[(\delta_1;p,v),\delta_2;\delta_3;\tau;\omega(t-\rho)^{\tau}\right]\right]\right)(x)$$
$$= \frac{d^n}{dx^n}\left\{\left(\mathfrak{I}^{n-\mu}_{\rho+}\ \left[(t-\rho)^{\delta_3-1}\ {}_2R_1\left[(\delta_1;p,v),\delta_2;\delta_3;\tau;\omega(t-\rho)^{\tau}\right]\right]\right)(x)\right\}$$
$$= \frac{d^n}{dx^n}\left\{\frac{(x-\rho)^{\delta_3+n-\mu-1}\ \Gamma(\delta_3)}{\Gamma(\delta_3-\mu+n)}\ {}_2R_1\left[(\delta_1;p,v),\delta_2;\delta_3+n-\mu;\tau;\omega(x-\rho)^{\tau}\right]\right\}. \tag{31}$$

Finally, by applying (19) to the equation (31), we are led to the assertion (29) of Theorem 5. □

6. Concluding Remarks

In our present investigation, we have first introduced an extension of the τ-Gauss hypergeometric function in terms of a certain extended Pochhammer symbol. We have then derived its various properties, including (for example) integral representations, derivative formulas, Mellin transform formulas, as well as the fractional integral and fractional derivative formulas. We have observed that by letting $v = 0$, the various results derived in this paper will reduce to the corresponding results proved earlier in [17]. Moreover, if we set $\tau = 1$, then we get several interesting new or known formulas for the extended Gauss hypergeometric function. Finally, we have observed that, if $v = 0$ and $\tau = 1$, then we get some new or known results for the extended Gauss hypergeometric function defined and studied by Srivastava et al. [10].

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References

1. Arshad, M.; Mubeen, S.; Nisar, K.S.; Rahman, G. Extended Wright-Bessel function and its properties. *Commun. Korean Math. Soc.* **2018**, *33*, 143–155.
2. Chaudhry, M.A.; Zubair, S.M. *On a Class of Incomplete Gamma Functions with Applications*; Chapman and Hall (CRC Press Company): Boca Raton, NJ, USA; London, UK; New York, NY, USA; Washington, DC, USA, 2001.
3. Chaudhry, M.A.; Qadir, A.; Srivastava, H.M.; Paris, R.B. Extended hypergeometric and confluent hypergeometric functions. *Appl. Math. Comput.* **2004**, *159*, 589–602. [CrossRef]
4. Mubeen, S.; Rahman, G.; Nisar, K.S.; Choi, J.; Arshad, M. An extended beta function and its properties. *Far East J. Math. Sci.* **2017**, *102*, 1545–1557. [CrossRef]
5. Özarslan, M.A.; Yilmaz, B. The extended Mittag-Leffler function and its properties. *J. Inequal. Appl.* **2014**, *2014*, 85. [CrossRef]
6. Özergin, E.; Özarslan, M.A.; Altın, A. Extension of gamma, beta and hypergeometric functions. *J. Comput. Appl. Math.* **2011**, *235*, 4601–4610. [CrossRef]
7. Chaudhry, M.A.; Zubair, S.M. Generalized incomplete gamma functions with applications. *J. Comput. Appl. Math.* **1994**, *55*, 99–124. [CrossRef]
8. Srivastava, H.M.; Chaudhry, M.A.; Agarwal, R.P. The incomplete Pochhammer symbols and their applications to hypergeometric and related functions. *Integral Transforms Spec. Funct.* **2012**, *23*, 659–683. [CrossRef]
9. Srivastava, R. Some classes of generating functions associated with a certain family of extended and generalized hypergeometric functions. *Appl. Math. Comput.* **2014**, *243*, 132–137. [CrossRef]
10. Srivastava, H.M.; Çetinkaya, A.; Kıymaz, İ.O. A certain generalized Pochhammer symbol and its applications to hypergeometric functions. *Appl. Math. Comput.* **2014**, *226*, 484–491. [CrossRef]
11. Srivastava, H.M.; Rahman, G.; Nisar, K.S. Some Extensions of the Pochhammer Symbol and the Associated Hypergeometric Functions. *Iran. J. Sci. Tech. Trans. A Sci.* **2019**, *43*, 2601–2606. [CrossRef]
12. Virchenko, N. On some generalizations of the functions of hypergeometric type. *Fract. Calc. Appl. Anal.* **1999**, *2*, 233–244.
13. Galué, L.; Al-Zamel, A.; Kalla, S.L. Further results on generalized hypergeometric functions. *Appl. Math. Comput.* **2013**, *136*, 17–25. [CrossRef]
14. Virchenko, N.; Kalla, S.L.; Al-Zamel, A. Some results on a generalized hypergeometric function. *Integral Transforms Spec. Funct.* **2001**, *12*, 89–100. [CrossRef]
15. Rainville, E.D. *Special Functions*; Macmillan Company: New York, NY, USA, 1960; Reprinted by Chelsea Publishing Company: New York, NY, USA, 1971.
16. Srivastava, H.M.; Karlsson, P.W. *Multiple Gaussian Hypergeometric Series*; Halsted Press (Ellis Horwood Limited): Chichester, UK; John Wiley and Sons: New York, NY, USA; Chichester, UK; Brisbane, Australia; Toronto, ON, Canada, 1985.
17. Parmar, R.K. Extended τ-hypergeometric functions and associated properties. *C. R. Math. Acad. Sci. Paris Sér. I* **2015**, *353*, 421–426. [CrossRef]
18. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematical Studies; Elsevier (North-Holland) Science Publishers: Amsterdam, The Netherlands; London, UK; New York, NY, USA, 2006; Volume 204.
19. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Translated from the Russian: Integrals and Derivatives of Fractional Order and Some of Their Applications; ("Nauka i Tekhnika", Minsk, 1987); Gordon and Breach Science Publishers: Reading, UK, 1993.