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Resistance Distance in the Double Corona Based on R-Graph

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Abstract: Let G_0 be a connected graph on n vertices and m edges. The R-graph $R(G_0)$ of G_0 is a graph obtained from G_0 by adding a new vertex corresponding to each edge of G_0 and by joining each new vertex to the end points of the edge corresponding to it. Let G_1 and G_2 be graphs on n_1 and n_2 vertices, respectively. The R-graph double corona $G_0^{(R)} \circ \{G_1, G_2\}$ of G_0, G_1 and G_2, is the graph obtained by taking one copy of $R(G_0)$, n copies of G_1 and m copies of G_2 and then by joining the i-th old-vertex of $R(G_0)$ to every vertex of the i-th copy of G_1 and the j-th new vertex of $R(G_0)$ to every vertex of the j-th copy of G_2. In this paper, we consider resistance distance in $G_0^{(R)} \circ \{G_1, G_2\}$. Moreover, we give an example to illustrate the correction and efficiency of the proposed method.

Keywords: graph; double corona; resistance distance; inverse

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1. Introduction

All graphs considered in this paper are simple and undirected. A graph G whose vertex set is $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set is $E(G) = \{e_1, e_2, \ldots, e_m\}$, is denoted by $(V(G), E(G))$. As we know, the conventional distance d_{ij} is the length of a shortest path between vertices v_i and v_j. Connected with practical applications, such as electrical network, Klein and Randić introduced resistance distance [1], which is the effective electrical resistance between two vertices if every edge is replaced by a unit resistor, and is denoted by r_{ij} for resistance distance between v_i and v_j. Some results on resistance distance can be found in [2–4].

One of the main topics about resistance distance is to determine it in various graphs. Now one can easily obtain resistance distance in wheels and fans [5], in subdivision-vertex join and subdivision-edge join of graphs [6], in corona and the neighborhood corona graphs of two disjoint graphs [7], in H-Join of Graphs $G_1, G_2, \ldots, G_k$ [8]. Please turn to [9–15] for more detail.

Motivated by the above works, we consider resistance distance in double corona based on R-graph. The R-graph of G, which is denoted by $R(G)$, appeared in [16]. Moreover, $R(G)$ is defined as the graph obtained from G by adding a new vertex corresponding to each edge of G and by joining each new vertex to the end points of the edge corresponding to it. Recently, in 2017, Barik and Sahoo introduced the R-graph double corona of G_0, G_1 and G_2 [17].

Definition 1 ([17]). *Let G_0 be a connected graph on n vertices and m edges. Let G_1 and G_2 be graphs on n_1 and n_2 vertices, respectively. The R-graph double corona of G_0, G_1 and G_2, denoted by $G_0^{(R)} \circ \{G_1, G_2\}$, is the graph obtained by taking one copy of $R(G_0)$, n copies of G_1 and m copies of G_2 and then by joining the i-th* *old-vertex of $R(G_0)$ to every vertex of the i-th copy of G_1 and the j-th new vertex of $R(G_0)$ to every vertex of the j-th copy of G_2.*

Example 1. *Please refer to Example 3 of [17] for $C_4^{(R)} \circ \{P_3, P_2\}$, where P_n and C_n are a path and a cycle with n vertices. Moreover, one can refer to Example 2 for $P_2^{(R)} \circ \{P_2, P_2\}$.*

This paper will compute resistance distance in $G_0^{(R)} \circ \{G_1, G_2\}$. However, first of all, we turn the readers' attention to some matrices associated with a graph G. The adjacency matrix A_G, which is a $n \times n$-matrix with entry $a_{ij} = 1$ if v_i and v_j are adjacent in G and $a_{ij} = 0$ otherwise, the diagonal matrix D_G with diagonal entries $d_G(v_1), d_G(v_2), \ldots, d_G(v_n)$ and the incidence matrix M_G which is a $n \times m$-matrix with $m_{ij} = 1$ (or 0) if vertex v_i is (not) incident with e_j. Moreover, the Laplacian matrix L_G of G is $D_G - A_G$. For more detail, please refer to [18,19].

Here we list some symbols. Let I_n denote the unit matrix of order n, $\mathbf{1}_n$ be the all-one column vector of dimension n and $J_{n\times m}$ be the all-one $n \times m$-matrix. Recall that the Kronecker product $A \otimes B$ [20] of two matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{p\times q}$ is an $mp \times nq$-matrix obtained from A by replacing every element a_{ij} by $a_{ij}B$. Moreover, $(A \otimes B)(C \otimes D) = AC \otimes BD$, whenever the products AC and BD exist, which implies that $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

2. Preliminaries

Let M be a matrix. If X is a matrix such that $MXM = M$, then X is a $\{1\}$-inverse of M, and X is always denoted by $M^{\{1\}}$. Further assume that M is a square matrix. If X is the matrix satisfying $(1) MXM = M$; $(2) XMX = X$; $(3) MX = XM$, then $X = M^{\#}$ is the group inverse of M. It is well-known that $M^{\#}$ exists if and only if rank(M)=rank(M^2), and $M^{\#}$ is unique.

Let A be a real symmetric matrix. Obviously, $A^{\#}$ exists and it is a $\{1\}$-inverse of A. In fact, assume that U is an orthogonal matrix (i.e., $UU^T = U^TU = I$) such that $A = Udiag\{\lambda_1, \lambda_2, \cdots, \lambda_n\}U^T$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of A. Then $A^{\#} = Udiag\{f(\lambda_1), f(\lambda_2), \cdots, f(\lambda_n)\}U^T$, where $f(\lambda_i) = \begin{cases} 1/\lambda_i, & \text{if } \lambda_i \neq 0, \\ 0, & \text{if } \lambda_i = 0. \end{cases}$ Moreover, in [21], the existence and the representation of the group inverse for the block matrices with an invertible subblock were given.

Lemma 1 ([21])**.** *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a $m \times m$ matrix, where A is an invertible $n \times n$ matrix, $S = D - CA^{-1}B$. If $S^{\#}$ exists, then*

(1) *$M^{\#}$ exists if and only if R is invertible, where $R = A^2 + BS^{\pi}C$ and $S^{\pi} = I_{m-n} - SS^{\#}$;*

(2) *if $M^{\#}$ exists, then $M^{\#} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, where*

$$\begin{aligned} X =& AR^{-1}(A + BS^{\#}C)R^{-1}A, \\ Y =& AR^{-1}(A + BS^{\#}C)R^{-1}BS^{\pi} - AR^{-1}BS^{\#}, \\ Z =& S^{\pi}CR^{-1}(A + BS^{\#}C)R^{-1}A - S^{\#}CR^{-1}A, \\ W =& S^{\pi}CR^{-1}(A + BS^{\#}C)R^{-1}BS^{\pi} - S^{\#}CR^{-1}BS^{\pi} - S^{\pi}CR^{-1}BS^{\#} + S^{\#}. \end{aligned}$$

Please note that, the Laplacian matrix $L(G)$ of a graph G is real symmetric. So $L(G)^{\#}$ exists and consequently, $L(G)^{\{1\}}$ exists. The representations of $L(G)^{\{1\}}$ were investigated in [6,11,22] under different conditions. We list two in the next lemma.

Lemma 2 ([6,11,22])**.** *Let $L = \begin{pmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{pmatrix}$ be the Laplacian matrix of a connected graph. Assume that L_1 is nonsingular. Denote $S = L_3 - L_2^T L_1^{-1} L_2$. Then*

(1) $\begin{pmatrix} L_1^{-1}+L_1^{-1}L_2S^{\#}L_2^TL_1^{-1} & -L_1^{-1}L_2S^{\#} \\ -S^{\#}L_2^TL_1^{-1} & S^{\#} \end{pmatrix}$ *is a symmetric* $\{1\}$*-inverse of* L;

(2) *If each column vector of* L_2 *is* $\mathbf{1}$ *or a zero vector, then* $\begin{pmatrix} L_1^{-1} & 0 \\ 0 & S^{\#} \end{pmatrix}$ *is a symmetric* $\{1\}$*-inverse of* L.

To compute the inverse of a matrix, the next lemma is useful.

Lemma 3 ([6])**.** *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be a nonsingular matrix. If* A *and* D *are nonsingular, then*

$$M^{-1} = \begin{pmatrix} A^{-1}+A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix},$$

where $S = D - CA^{-1}B$ *is the Schur complement of* A *in* M.

This paper is devoted to the compute of resistance distance in $G_0^{(R)} \circ \{G_1, G_2\}$. In [6], authors obtained the formulae for resistance distance by elements of group inverse $L(G)^{\#}$ or $\{1\}$-inverse $L(G)^{\{1\}}$ of $L(G)$, where $G = G_0^{(R)} \circ \{G_1, G_2\}$.

Lemma 4 ([6])**.** *Let* G *be a connected graph and* $(A)_{ij}$ *be the* (i,j)*-entry of a matrix* A*. Then*

$$\begin{aligned} r_{ij}(G) &= (L(G)^{\{1\}})_{ii} + (L(G)^{\{1\}})_{jj} - (L(G)^{\{1\}})_{ij} - (L(G)^{\{1\}})_{ji} \\ &= (L(G)^{\#})_{ii} + (L(G)^{\#})_{jj} - 2(L(G)^{\#})_{ij}. \end{aligned}$$

Keep Lemma 4 in mind, we only need to compute $L(G)^{\#}$ or $L(G)^{\{1\}}$. Before calculating, we list more preliminaries below.

Lemma 5 ([11])**.** *For any graph* G, $L(G)^{\#}\mathbf{1} = 0$.

Lemma 6 ([23])**.** *Let* G *be a simple connected graph. Then its adjacency matrix* $A(G)$*, diagonal matrix* $D(G)$ *and incidence matrix* $M(G)$ *satisfy* $M(G)M(G)^T = A(G) + D(G)$.

Lemma 7. *Assume that* A *is symmetric and* 0 *is a simple eigenvalue. Let* u *be the unitary* 0*-eigenvector of* A*. Then the group inverse* $A^{\#}$ *is characterized as the unique singular matrix satisfying*

$$AA^{\#} = A^{\#}A = I - uu^T.$$

Proof. Assume that λ_i is a non-zero eigenvalue of A and u_i is the unitary λ_i-eigenvector of A, for $i = 1, 2, \cdots, n-1$. Let $U = (u_1\ u_2\ \cdots u_{n-1}\ u)$. Then

$$A = U diag\{\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, 0\}U^T, A^{\#} = U diag\{\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots, \frac{1}{\lambda_{n-1}}, 0\}U^T.$$

Clearly, $I - AA^{\#} = I - A^{\#}A$ and

$$I - A^{\#}A = U \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{pmatrix} U^T = U \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix} U^T = uu^T.$$

□

Lemma 8. *Let M be a matrix and X be a $\{1\}$-inverse of M. If X_0 is a matrix satisfying $MX_0M = 0$, then $X - X_0$ is also a $\{1\}$-inverse of M.*

Proof. Please note that $M(X - X_0)M = MXM - MX_0M = M - 0 = M$. Thus, $X - X_0$ is a $\{1\}$-inverse of M. $\square$

3. Main Results

In this section, $G = G_0^{(R)} \circ \{G_1, G_2\}$, where G_0 is a connected r-regular graph on n vertices $V(G_0) = \{v_1, v_2, \ldots, v_n\}$ and m edges $E(G_0) = \{e_1, e_2, \ldots, e_m\}$, and G_1, G_2 are graphs on n_1 vertices $V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\}$ and n_2 vertices $V(G_2) = \{w_1, w_2, \ldots, w_{n_2}\}$. We give a $\{1\}$-inverse of $L(G)$ in Theorem 1. However, before Theorem 1, we show the labeling rule of vertices of G.

(1) For $i = 1, 2, \ldots, m$, label the n_2 vertices of the i-th copy of G_2 with

$$V(G_2)_i = \{w_1^{(i-1)n_2+1}, w_2^{(i-1)n_2+2}, \ldots, w_{n_2}^{in_2}\}.$$

(2) For $j = 1, 2, \ldots, n$, label the n_1 vertices of the j-th copy of G_1 with

$$V(G_1)_j = \{u_1^{mn_2+(j-1)n_1+1}, u_2^{mn_2+(j-1)n_1+2}, \ldots, u_{n_1}^{mn_2+jn_1}\}.$$

(3) Label the m new-vertices of $R(G_0)$ corresponding to edges of $E(G_0)$ with

$$\{e_1^{mn_2+nn_1+1}, e_2^{mn_2+nn_1+2}, \ldots, e_m^{m(n_2+1)+nn_1}\}.$$

(4) Label the n old-vertices $V(G_0) = \{v_1, v_2, \ldots, v_n\}$ of $R(G_0)$ with

$$\{v_1^{m(n_2+1)+nn_1+1}, v_2^{m(n_2+1)+nn_1+2}, \ldots, v_n^{m(n_2+1)+n(n_1+1)}\}.$$

Thus,

$$\begin{aligned} V(G) =& V(G_2)_1 \cup \cdots \cup V(G_2)_m \cup V(G_1)_1 \cup \cdots \cup V(G_1)_n \\ & \cup \{e_1^{mn_2+nn_1+1}, \ldots, e_m^{m(n_2+1)+nn_1}\} \cup \{v_1^{m(n_2+1)+nn_1+1}, \ldots, v_n^{m(n_2+1)+n(n_1+1)}\}. \end{aligned}$$

Theorem 1. *The following matrix is a $\{1\}$-inverse of $L(G)$,*

$$\begin{pmatrix} I_m \otimes \left((L_{G_2} + I_{n_2})^{-1} + \frac{1}{2}J_{n_2 \times n_2}\right) & 0 & \frac{1}{2}I_m \otimes \mathbf{1}_{n_2} & 0 \\ 0 & I_n \otimes (L_{G_1} + I_{n_1})^{-1} & 0 & 0 \\ \frac{1}{2}I_m \otimes \mathbf{1}_{n_2}^T & 0 & \frac{1}{2}I_m & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} \frac{1}{2}M_{G_0}^T \otimes \mathbf{1}_{n_2} \\ I_n \otimes \mathbf{1}_{n_1} \\ \frac{1}{2}M_{G_0}^T \\ I_n \end{pmatrix} \frac{2}{3}L_{G_0}^{\#} \left(\frac{1}{2}M_{G_0} \otimes \mathbf{1}_{n_2}^T \quad I_n \otimes \mathbf{1}_{n_1}^T \quad \frac{1}{2}M_{G_0} \quad I_n\right).$$

Moreover, denote $\begin{pmatrix} \frac{1}{2}M_{G_0}^T \\ I_n \end{pmatrix} \frac{2}{3}L_{G_0}^{\#} \begin{pmatrix} \frac{1}{2}M_{G_0} & I_n \end{pmatrix} = \begin{pmatrix} \frac{1}{6}M_{G_0}^T L_{G_0}^{\#} M_{G_0} & \frac{1}{3}M_{G_0}^T L_{G_0}^{\#} \\ \frac{1}{3}L_{G_0}^{\#} M_{G_0} & \frac{2}{3}L_{G_0}^{\#} \end{pmatrix}$ *by* $\begin{pmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{pmatrix}$. *Then the second term in the above matrix is*

$$\begin{pmatrix} X_{11} \otimes J_{n_2 \times n_2} & X_{12} \otimes J_{n_2 \times n_1} & X_{11} \otimes \mathbf{1}_{n_2} & X_{12} \otimes \mathbf{1}_{n_2} \\ X_{12}^T \otimes J_{n_1 \times n_2} & X_{22} \otimes J_{n_1 \times n_1} & X_{12}^T \otimes \mathbf{1}_{n_1} & X_{22} \otimes \mathbf{1}_{n_1} \\ X_{11} \otimes \mathbf{1}_{n_2}^T & X_{12} \otimes \mathbf{1}_{n_1}^T & X_{11} & X_{12} \\ X_{12}^T \otimes \mathbf{1}_{n_2}^T & X_{22} \otimes \mathbf{1}_{n_1}^T & X_{12}^T & X_{22} \end{pmatrix}.$$

Proof. By the definition, it is easy to show that $G = G_0^{(R)} \circ \{G_1, G_2\}$ is connected. Furthermore, from the vertex-labeling rule of G, we know that all the diagonal matrix D_G, the adjacency matrix A_G and the Laplacian matrix L_G are partitioned $(m+n+2) \times (m+n+2)$-matrices. Particularly, the Laplacian matrix of G is

$$L_G = \begin{pmatrix} I_m \otimes (L_{G_2} + I_{n_2}) & 0 & -I_m \otimes \mathbf{1}_{n_2} & 0 \\ 0 & I_n \otimes (L_{G_1} + I_{n_1}) & 0 & -I_n \otimes \mathbf{1}_{n_1} \\ -I_m \otimes \mathbf{1}_{n_2}^T & 0 & (2+n_2)I_m & -M_{G_0}^T \\ 0 & -I_n \otimes \mathbf{1}_{n_1}^T & -M_{G_0} & 2D_{G_0} - A_{G_0} + n_1 I_{n_1} \end{pmatrix}.$$

We proceed via the following steps.

Step 1. To use Lemma 2, we further divide L_G into blocks $L_G = \begin{pmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{pmatrix}$, where

$$L_1 = \begin{pmatrix} I_m \otimes (L_{G_2} + I_{n_2}) & 0 \\ 0 & I_n \otimes (L_{G_1} + I_{n_1}) \end{pmatrix};$$
$$L_2 = \begin{pmatrix} -I_m \otimes \mathbf{1}_{n_2} & 0 \\ 0 & -I_n \otimes \mathbf{1}_{n_1} \end{pmatrix}; \; L_2^T = \begin{pmatrix} -I_m \otimes \mathbf{1}_{n_2}^T & 0 \\ 0 & -I_n \otimes \mathbf{1}_{n_1}^T \end{pmatrix};$$
$$L_3 = \begin{pmatrix} (2+n_2)I_m & -M_{G_0}^T \\ -M_{G_0} & 2D_{G_0} - A_{G_0} + n_1 I_{n_1} \end{pmatrix}.$$

Clearly, $L_1^{-1} = \begin{pmatrix} I_m \otimes (L_{G_2} + I_{n_2})^{-1} & 0 \\ 0 & I_n \otimes (L_{G_1} + I_{n_1})^{-1} \end{pmatrix}$.

Step 2. Please note that $L_{G_1}\mathbf{1}_{n_1} = 0$. So $(L_{G_1} + I_{n_1})\mathbf{1}_{n_1} = \mathbf{1}_{n_1}$, which shows that

$$(L_{G_1} + I_{n_1})^{-1}\mathbf{1}_{n_1} = \mathbf{1}_{n_1}.$$

Furthermore,

$$\mathbf{1}_{n_1}^T (L_{G_1} + I_{n_1})^{-1}\mathbf{1}_{n_1} = n_1.$$

Similarly, we have $\mathbf{1}_{n_2}^T (L_{G_2} + I_{n_2})^{-1}\mathbf{1}_{n_2} = n_2$. Keep these in mind and recall the Kronecker product. Then

$$\begin{aligned} L_2^T L_1^{-1} L_2 &= \begin{pmatrix} I_m \otimes \mathbf{1}_{n_2}^T (L_{G_2} + I_{n_2})^{-1}\mathbf{1}_{n_2} & 0 \\ 0 & I_n \otimes \mathbf{1}_{n_1}^T (L_{G_1} + I_{n_1})^{-1}\mathbf{1}_{n_1} \end{pmatrix} \\ &= \begin{pmatrix} n_2 I_m & 0 \\ 0 & n_1 I_n \end{pmatrix}. \end{aligned}$$

From this result, it follows that

$$S = L_3 - L_2^T L_1^{-1} L_2 = \begin{pmatrix} (2+n_2)I_m & -M_{G_0}^T \\ -M_{G_0} & 2D_{G_0} - A_{G_0} + n_1 I_{n_1} \end{pmatrix} - \begin{pmatrix} n_2 I_m & 0 \\ 0 & n_1 I_n \end{pmatrix}$$
$$= \begin{pmatrix} 2I_m & -M_{G_0}^T \\ -M_{G_0} & 2D_{G_0} - A_{G_0} \end{pmatrix}.$$

Since S is real symmetric, $S^{\#}$ exists.

Step 3. Here we computer $S^{\#}$ by Lemma 1. Please note that $M_{G_0} M_{G_0}^T = D_{G_0} + A_{G_0}$. We denote

$$S_0 = 2D_{G_0} - A_{G_0} - M_{G_0} \left(\frac{1}{2} I_m \right) M_{G_0}^T.$$

Then $S_0 = \frac{3}{2} L_{G_0}$. Consequently, $S_0^{\#}$ exists and in fact, $S_0^{\#} = \frac{2}{3} L_{G_0}^{\#}$. Denote $I_n - S_0 S_0^{\#}$ by S_0^{π}. Combing with Lemma 7, we have

$$S_0^{\pi} = I_n - L_{G_0} L_{G_0}^{\#} = \frac{1}{n} J_{n \times n}.$$

Please note that $\begin{pmatrix} 1 \cdots 1 \end{pmatrix} M_{G_0} = 2 \begin{pmatrix} 1 \cdots 1 \end{pmatrix}$ and $M_{G_0}^T \mathbf{1}_n = 2\mathbf{1}_n$. So, we further have

$$R = 4I_m + M_{G_0}^T \left(\frac{1}{n} J_{n \times n} \right) M_{G_0} = 4 \left(I_m + \frac{1}{n} J_{m \times m} \right).$$

Therefore, it is easy to obtain that

$$R^{-1} = \frac{1}{4} \left(I_m - \frac{1}{n+m} J_{m \times m} \right).$$

Please note that G is r-regular. So $M_{G_0} \mathbf{1}_m = r \mathbf{1}_m$. Moreover, $L_{G_0}^{\#} \mathbf{1}_n = 0$ according to Lemma 5. In general, by Lemma 1, we finally have $S^{\#} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, where

$$X = 2I_m \frac{1}{4} \left(I_m - \frac{1}{n+m} J_{m \times m} \right) 2 \left(I_m + \frac{1}{3} M_{G_0}^T L_{G_0}^{\#} M_{G_0} \right) \frac{1}{4} \left(I_m - \frac{1}{n+m} J_{m \times m} \right) 2I_m$$
$$= \frac{1}{2} \left(I_m - \frac{1}{n+m} J_{m \times m} \right) \left(I_m + \frac{1}{3} M_{G_0}^T L_{G_0}^{\#} M_{G_0} \right) \left(I_m - \frac{1}{n+m} J_{m \times m} \right)$$
$$= \frac{1}{2} \left(I_m + \frac{1}{3} M_{G_0}^T L_{G_0}^{\#} M_{G_0} - \frac{2n+m}{(n+m)^2} J_{m \times m} \right),$$

$$
\begin{aligned}
Y =& 2I_m\frac{1}{4}\left(I_m-\frac{1}{n+m}J_{m\times m}\right)2\left(I_m+\frac{1}{3}M_{G_0}^TL_{G_0}^{\#}M_{G_0}\right)\frac{1}{4}\left(I_m-\frac{1}{n+m}J_{m\times m}\right)(-M_{G_0}^T)\frac{1}{n}J_{n\times n}\\
&-2I_m\frac{1}{4}\left(I_m-\frac{1}{n+m}J_{m\times m}\right)(-M_{G_0}^T)\frac{2}{3}L_{G_0}^{\#}\\
=&\frac{-1}{4n}\left(I_m-\frac{1}{n+m}J_{m\times m}\right)\left(I_m+\frac{1}{3}M_{G_0}^TL_{G_0}^{\#}M_{G_0}\right)\left(I_m-\frac{1}{n+m}J_{m\times m}\right)M_{G_0}^TJ_{n\times n}\\
&+\frac{1}{3}\left(I_m-\frac{1}{n+m}J_{m\times m}\right)M_{G_0}^TL_{G_0}^{\#}\\
=&\frac{1}{3}M_{G_0}^TL_{G_0}^{\#}-\frac{1}{4n}\left(I_m+\frac{1}{3}M_{G_0}^TL_{G_0}^{\#}M_{G_0}-\frac{2n+m}{(n+m)^2}J_{m\times m}\right)M_{G_0}^TJ_{n\times n}\\
=&\frac{1}{3}M_{G_0}^TL_{G_0}^{\#}-\frac{n}{2(n+m)^2}J_{m\times n},
\end{aligned}
$$

$$
\begin{aligned}
Z =& \frac{1}{n}J_{n\times n}(-M_{G_0})\frac{1}{4}\left(I_m-\frac{1}{n+m}J_{m\times m}\right)2\left(I_m+\frac{1}{3}M_{G_0}^TL_{G_0}^{\#}M_{G_0}\right)\frac{1}{4}\left(I_m-\frac{1}{n+m}J_{m\times m}\right)2I_m\\
&-\frac{2}{3}L_{G_0}^{\#}(-M_{G_0})\frac{1}{4}\left(I_m-\frac{1}{n+m}J_{m\times m}\right)2I_m\\
=&\frac{1}{3}L(G)^{\#}M_{G_0}-\frac{1}{4n}J_{n\times n}M_{G_0}\left(I_m+\frac{1}{3}M_{G_0}^TL_{G_0}^{\#}M_{G_0}-\frac{2n+m}{(n+m)^2}J_{m\times m}\right)\\
=&\frac{1}{3}L_{G_0}^{\#}M_{G_0}-\frac{n}{2(n+m)^2}J_{n\times m},
\end{aligned}
$$

and

$$
\begin{aligned}
W =& \frac{1}{n}J_{n\times n}(-M_{G_0})\frac{1}{4}\left(I_m-\frac{1}{n+m}J_{m\times m}\right)2\left(I_m+\frac{1}{3}M_{G_0}^TL_{G_0}^{\#}M_{G_0}\right)\frac{1}{4}\left(I_m-\frac{1}{n+m}J_{m\times m}\right)(-M_{G_0}^T)\frac{1}{n}J_{n\times n}\\
&-\frac{2}{3}L_{G_0}^{\#}(-M_{G_0})\frac{1}{4}\left(I_m-\frac{1}{n+m}J_{m\times m}\right)(-M_{G_0}^T)\frac{1}{n}J_{n\times n}\\
&-\frac{1}{n}J_{n\times n}(-M_{G_0})\frac{1}{4}\left(I_m-\frac{1}{n+m}J_{m\times m}\right)(-M_{G_0}^T)\frac{2}{3}L_{G_0}^{\#}+\frac{2}{3}L_{G_0}^{\#}\\
=&\frac{1}{2n^2}J_{n\times m}\left(I_m-\frac{1}{n+m}J_{m\times m}\right)\left(I_m+\frac{1}{3}M_{G_0}^TL_{G_0}^{\#}M_{G_0}\right)\left(I_m-\frac{1}{n+m}J_{m\times m}\right)J_{m\times n}+\frac{2}{3}L_{G_0}^{\#}\\
=&\frac{2}{3}L_{G_0}^{\#}+\frac{m}{2(n+m)^2}J_{n\times n}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
S^{\#} =& \frac{1}{2}\begin{pmatrix} I_m & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}\frac{1}{2}M_{G_0}^T\\ I_n\end{pmatrix}\frac{2}{3}L_{G_0}^{\#}\left(\frac{1}{2}M_{G_0}\ \ I_n\right)\\
&+\frac{1}{2(n+m)}\begin{pmatrix}-J_{m\times m} & 0\\ 0 & J_{n\times n}\end{pmatrix}-\frac{n}{2(n+m)^2}J_{(m+n)\times(m+n)}.
\end{aligned}
$$

Step 4. In this step, we compute $-L_1^{-1}L_2S^{\#}$, $-S^{\#}L_2^TL_1^{-1}$ and $L_1^{-1}L_2S^{\#}L_2^TL_1^{-1}$.

$$\begin{aligned}-L_1^{-1}L_2S^{\#} =& \begin{pmatrix} I_m \otimes (L_{G_2}+I_{n_2})^{-1}\mathbf{1}_{n_2} & 0 \\ 0 & I_n \otimes (L_{G_1}+I_{n_1})^{-1}\mathbf{1}_{n_1} \end{pmatrix} S^{\#} \\ =& \begin{pmatrix} I_m \otimes \mathbf{1}_{n_2} & 0 \\ 0 & I_n \otimes \mathbf{1}_{n_1} \end{pmatrix} S^{\#} \\ =& \frac{1}{2}\begin{pmatrix} I_m \otimes \mathbf{1}_{n_2} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}M_{G_0}^T \otimes \mathbf{1}_{n_2} \\ I_n \otimes \mathbf{1}_{n_1} \end{pmatrix} \frac{2}{3}L_{G_0}^{\#}\begin{pmatrix} \frac{1}{2}M_{G_0} & I_n \end{pmatrix} \\ &+ \frac{1}{2(n+m)}\begin{pmatrix} -J_{mn_2\times m} & 0 \\ 0 & J_{nn_1\times n} \end{pmatrix} - \frac{n}{2(n+m)^2}J_{(mn_2+nn_1)\times(m+n)},\end{aligned}$$

and similarly, we would have that

$$\begin{aligned}-S^{\#}L_2^TL_1^{-1} =& S^{\#}\begin{pmatrix} I_m \otimes \mathbf{1}_{n_2}^T(L_{G_2}+I_{n_2})^{-1} & 0 \\ 0 & I_n \otimes \mathbf{1}_{n_1}^T(L_{G_1}+I_{n_1})^{-1} \end{pmatrix} \\ =& S^{\#}\begin{pmatrix} I_m \otimes \mathbf{1}_{n_2}^T & 0 \\ 0 & I_n \otimes \mathbf{1}_{n_1}^T \end{pmatrix} \\ =& \frac{1}{2}\begin{pmatrix} I_m \otimes \mathbf{1}_{n_2}^T & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}M_{G_0}^T \\ I_n \end{pmatrix} \frac{2}{3}L_{G_0}^{\#}\begin{pmatrix} \frac{1}{2}M_{G_0}\otimes \mathbf{1}_{n_2}^T & I_n \otimes \mathbf{1}_{n_1}^T \end{pmatrix} \\ &+ \frac{1}{2(n+m)}\begin{pmatrix} -J_{m\times mn_2} & 0 \\ 0 & J_{n\times nn_1} \end{pmatrix} - \frac{n}{2(n+m)^2}J_{(m+n)\times(mn_2+nn_1)}.\end{aligned}$$

Furthermore,

$$\begin{aligned}L_1^{-1}L_2S^{\#}L_2^TL_1^{-1} =& \frac{1}{2}\begin{pmatrix} I_m \otimes J_{n_2\times n_2} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}M_{G_0}^T \otimes \mathbf{1}_{n_2} \\ I_n \otimes \mathbf{1}_{n_1} \end{pmatrix} \frac{2}{3}L_{G_0}^{\#}\begin{pmatrix} \frac{1}{2}M_{G_0}\otimes \mathbf{1}_{n_2}^T & I_n \otimes \mathbf{1}_{n_1}^T \end{pmatrix} \\ &+ \frac{1}{2(n+m)}\begin{pmatrix} -J_{mn_2\times mn_2} & 0 \\ 0 & J_{nn_1\times nn_1} \end{pmatrix} - \frac{n}{2(n+m)^2}J_{(mn_2+nn_1)\times(mn_2+nn_1)}.\end{aligned}$$

Step 5. Since L_G is the Laplacian matrix of $G = G_0^{(R)} \circ \{G_1, G_2\}$, we have

$$L_G\mathbf{1}_{m(n_2+1)+n(n_1+1)} = 0,\ \ \mathbf{1}_{m(n_2+1)+n(n_1+1)}^T L_G = 0,$$

which shows that $L_G J_{m(n_2+1)+n(n_1+1)} L_G = 0$. Moreover,

$$\begin{aligned}& L_G \begin{pmatrix} -J_{mn_2\times mn_2} & 0 & -J_{mn_2\times m} & 0 \\ 0 & J_{nn_1\times nn_1} & 0 & J_{nn_1\times n} \\ -J_{m\times mn_2} & 0 & -J_{m\times m} & 0 \\ 0 & J_{n\times nn_1} & 0 & J_{n\times n} \end{pmatrix} L_G \\ =& \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2J_{m\times mn_2} & -2J_{m\times nn_1} & -2J_{m\times m} & -2J_{m\times n} \\ rJ_{n\times mn_2} & rJ_{n\times nn_1} & rJ_{n\times m} & rJ_{n\times n} \end{pmatrix} L_G \\ =& 0.\end{aligned}$$

So from Lemmas 2 and 8, we know that the following matrix is also a symmetric $\{1\}$-inverse of L_G,

$$\begin{pmatrix} I_m \otimes (L_{G_2}+I_{n_2})^{-1} & 0 & 0 & 0 \\ 0 & I_n \otimes (L_{G_1}+I_{n_1})^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} I_m \otimes J_{n_2\times n_2} & 0 & I_m \otimes \mathbf{1}_{n_2} & 0 \\ 0 & 0 & 0 & 0 \\ I_m \otimes \mathbf{1}_{n_2}^T & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$+\begin{pmatrix} \frac{1}{2}M_{G_0}^T \otimes \mathbf{1}_{n_2} \\ I_n \otimes \mathbf{1}_{n_1} \\ \frac{1}{2}M_{G_0}^T \\ I_n \end{pmatrix} \frac{2}{3}L_{G_0}^{\#} \begin{pmatrix} \frac{1}{2}M_{G_0} \otimes \mathbf{1}_{n_2}^T & I_n \otimes \mathbf{1}_{n_1}^T & \frac{1}{2}M_{G_0} & I_n \end{pmatrix}.$$

Denote $\begin{pmatrix} \frac{1}{2}M_{G_0}^T \\ I_n \end{pmatrix} \frac{2}{3}L_{G_0}^{\#} \begin{pmatrix} \frac{1}{2}M_{G_0} & I_n \end{pmatrix} = \begin{pmatrix} \frac{1}{6}M_{G_0}^T L_{G_0}^{\#} M_{G_0} & \frac{1}{3}M_{G_0}^T L_{G_0}^{\#} \\ \frac{1}{3}L_{G_0}^{\#} M_{G_0} & \frac{2}{3}L_{G_0}^{\#} \end{pmatrix}$ by $\begin{pmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{pmatrix}$. Then

$$\begin{pmatrix} \frac{1}{2}M_{G_0}^T \otimes \mathbf{1}_{n_2} \\ I_n \otimes \mathbf{1}_{n_1} \\ \frac{1}{2}M_{G_0}^T \\ I_n \end{pmatrix} \frac{2}{3}L_{G_0}^{\#} \begin{pmatrix} \frac{1}{2}M_{G_0} \otimes \mathbf{1}_{n_2}^T & I_n \otimes \mathbf{1}_{n_1}^T & \frac{1}{2}M_{G_0} & I_n \end{pmatrix}$$
$$= \begin{pmatrix} X_{11} \otimes J_{n_2\times n_2} & X_{12} \otimes J_{n_2\times n_1} & X_{11} \otimes \mathbf{1}_{n_2} & X_{12} \otimes \mathbf{1}_{n_2} \\ X_{12}^T \otimes J_{n_1\times n_2} & X_{22} \otimes J_{n_1\times n_1} & X_{12}^T \otimes \mathbf{1}_{n_1} & X_{22} \otimes \mathbf{1}_{n_1} \\ X_{11} \otimes \mathbf{1}_{n_2}^T & X_{12} \otimes \mathbf{1}_{n_1}^T & X_{11} & X_{12} \\ X_{12}^T \otimes \mathbf{1}_{n_2}^T & X_{22} \otimes \mathbf{1}_{n_1}^T & X_{12}^T & X_{22} \end{pmatrix}.$$

Therefore, we complete the proof of Theorem 1. □

Please note that $\begin{pmatrix} 2I_m & -M_{G_0}^T \\ -M_{G_0} & 2D_{G_0} - A_{G_0} \end{pmatrix}$ is just the Laplacian matrix of $G_0^{(R)}$. So, from Step 3, we obtain the group inverse of $L(G_0^{(R)})$.

Corollary 1. *Let G_0 be a connected r-regular graph on n vertices and m edges, whose incidence matrix is M_{G_0}. Then*

$$L(G_0^{(R)})^{\#} = \frac{1}{2}\begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}M_{G_0}^T \\ I_n \end{pmatrix} \frac{2}{3}L_{G_0}^{\#} \begin{pmatrix} \frac{1}{2}M_{G_0} & I_n \end{pmatrix}$$
$$+ \frac{1}{2(n+m)}\begin{pmatrix} -J_{m\times m} & 0 \\ 0 & J_{n\times n} \end{pmatrix} - \frac{n}{2(n+m)^2}J_{(m+n)\times(m+n)}.$$

Next we consider two special situations of $G_0^{(R)} \circ \{G_1, G_2\}$. By choosing G_2 as a null-graph, we would reduce $G = G_0^{(R)} \circ \{G_1, G_2\}$ to R-vertex corona $G_0 \odot G_1$ [24]. If G_1 is a null-graph, then $G = G_0^{(R)} \circ \{G_1, G_2\}$ reduces to R-edge corona $G_0 \ominus G_2$ [24]. Thus, from Theorem 1, we obtain a $\{1\}$-inverse of $L(G_0 \odot G_1)$ and $L(G_0 \ominus G_2)$.

Corollary 2. *Denote* $\begin{pmatrix} \frac{1}{6}M_{G_0}^T L_{G_0}^{\#} M_{G_0} & \frac{1}{3}M_{G_0}^T L_{G_0}^{\#} \\ \frac{1}{3}L_{G_0}^{\#} M_{G_0} & \frac{2}{3}L_{G_0}^{\#} \end{pmatrix}$ *by* $\begin{pmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{pmatrix}$.

(1) *The following matrix is a* $\{1\}$*-inverse of* $L(G_0 \odot G_1)$,

$$\begin{pmatrix} I_n \otimes (L_{G_1}+I_{n_1})^{-1} & 0 & 0 \\ 0 & \frac{1}{2}I_m & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} X_{22}\otimes J_{n_1\times n_1} & X_{12}^T \otimes \mathbf{1}_{n_1} & X_{22}\otimes \mathbf{1}_{n_1} \\ X_{12}\otimes \mathbf{1}_{n_1}^T & X_{11} & X_{12} \\ X_{22}\otimes \mathbf{1}_{n_1}^T & X_{12}^T & X_{22} \end{pmatrix}.$$

(2) *The following matrix is a* $\{1\}$*-inverse of* $L(G_0^{(R)} \ominus G_2)$,

$$\begin{pmatrix} I_m \otimes \left((L_{G_2}+I_{n_2})^{-1} + \frac{1}{2}J_{n_2\times n_2}\right) & \frac{1}{2}I_m \otimes \mathbf{1}_{n_2} & 0 \\ \frac{1}{2}I_m \otimes \mathbf{1}_{n_2}^T & \frac{1}{2}I_m & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} X_{11}\otimes J_{n_2\times n_2} & X_{11} \otimes \mathbf{1}_{n_2} & X_{12}\otimes \mathbf{1}_{n_2} \\ X_{11}\otimes \mathbf{1}_{n_2}^T & X_{11} & X_{12} \\ X_{12}^T\otimes \mathbf{1}_{n_2}^T & X_{12}^T & X_{22} \end{pmatrix}.$$

Example 2. *Compute resistance distance in* $G = P_2^{(R)} \circ \{P_2, P_2\}$*, see the following Figure 1.*

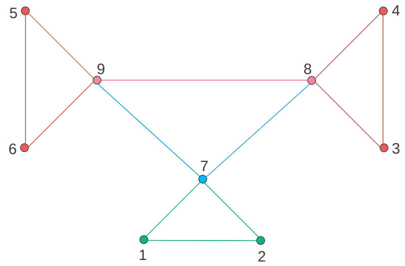

Figure 1. $P_2^{(R)} \circ \{P_2, P_2\}$.

Step 1. $L_{G_2} + I_{n_2} = L_{G_1} + I_{n_1} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. So

$$(L_{G_2}+I_{n_2})^{-1} = (L_{G_1}+I_{n_1})^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

and $\begin{pmatrix} I_m \otimes (L_{G_2}+I_{n_2})^{-1} & 0 & 0 & 0 \\ 0 & I_n \otimes (L_{G_1}+I_{n_1})^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I_3 \otimes \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$

Step 2. $\frac{1}{2}\begin{pmatrix} I_m \otimes J_{n_2\times n_2} & 0 & I_m \otimes \mathbf{1}_{n_2} & 0 \\ 0 & 0 & 0 & 0 \\ I_m \otimes \mathbf{1}_{n_2}^T & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} J_{2\times 2} & 0 & \mathbf{1}_2 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbf{1}_2^T & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

Step 3. $M_{G_0} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $L_{G_0} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Hence

$$L_{G_0}^{\#} = \frac{1}{4}L_{G_0} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad L_{G_0}^{\#}M_{G_0} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Moreover,

$$M_{G_0}^T L_{G_0}^{\#} = (0\,0), \quad M_{G_0}^T L_{G_0}^{\#} M_{G_0} = 0.$$

Therefore,

$$\begin{pmatrix} \frac{1}{6}M_{G_0}^T L_{G_0}^{\#} M_{G_0} & \frac{1}{3}M_{G_0}^T L_{G_0}^{\#} \\ \frac{1}{3}L_{G_0}^{\#} M_{G_0} & \frac{2}{3}L_{G_0}^{\#} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{6}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{pmatrix},$$

and

$$\begin{pmatrix} \frac{1}{2}M_{G_0}^T \otimes \mathbf{1}_{n_2} \\ I_n \otimes \mathbf{1}_{n_1} \\ \frac{1}{2}M_{G_0}^T \\ I_n \end{pmatrix} \frac{2}{3}L_{G_0}^{\#} \left(\frac{1}{2}M_{G_0} \otimes \mathbf{1}_{n_2}^T \;\; I_n \otimes \mathbf{1}_{n_1}^T \;\; \frac{1}{2}M_{G_0} \;\; I_n \right)$$
$$= \begin{pmatrix} 0_{2\times 2} & 0_{2\times 4} & 0_{2\times 1} & 0_{2\times 2} \\ 0_{4\times 2} & \frac{1}{6}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix} & 0_{4\times 1} & \frac{1}{6}\begin{pmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \\ -1 & 1 \end{pmatrix} \\ 0_{1\times 2} & 0_{1\times 4} & 0 & 0_{1\times 2} \\ 0_{2\times 2} & \frac{1}{6}\begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix} & 0_{2\times 1} & \frac{1}{6}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{pmatrix}.$$

Step 4. From the above and Theorem 1, we know that

$$\begin{pmatrix} \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + \frac{1}{2}J_{2\times 2} & 0_{2\times 2} & 0_{2\times 2} & \frac{1}{2}\mathbf{1}_{2\times 1} & 0_{2\times 2} \\ 0_{2\times 2} & \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + \frac{1}{6}J_{2\times 2} & \frac{-1}{6}J_{2\times 2} & 0_{2\times 1} & \frac{1}{6}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \\ 0_{2\times 2} & \frac{-1}{6}J_{2\times 2} & \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + \frac{1}{6}J_{2\times 2} & 0_{2\times 1} & \frac{1}{6}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \\ \frac{1}{2}\mathbf{1}_{2\times 1}^T & 0_{1\times 2} & 0_{1\times 2} & \frac{1}{2} & 0_{1\times 2} \\ 0_{2\times 2} & \frac{1}{6}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} & \frac{1}{6}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} & 0_{2\times 1} & \frac{1}{6}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{pmatrix},$$

is a $\{1\}$-inverse of L_G. From it, we have the matrix whose ij-entry is resistance distance between vertices v^i and v^j, for detail,

$$\begin{pmatrix} 0 & \frac{2}{3} & 2 & 2 & 2 & 2 & \frac{2}{3} & \frac{4}{3} & \frac{4}{3} \\ \frac{2}{3} & 0 & 2 & 2 & 2 & 2 & \frac{2}{3} & \frac{4}{3} & \frac{4}{3} \\ 2 & 2 & 0 & \frac{2}{3} & 2 & 2 & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} \\ 2 & 2 & \frac{2}{3} & 0 & 2 & 2 & \frac{4}{3} & \frac{2}{3} & \frac{4}{3} \\ 2 & 2 & 2 & 2 & 0 & \frac{2}{3} & \frac{4}{3} & \frac{4}{3} & \frac{2}{3} \\ 2 & 2 & 2 & 2 & \frac{2}{3} & 0 & \frac{4}{3} & \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & \frac{4}{3} & \frac{4}{3} & \frac{4}{3} & 0 & \frac{2}{3} & \frac{2}{3} \\ \frac{4}{3} & \frac{4}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & \frac{4}{3} & \frac{2}{3} & 0 & \frac{2}{3} \\ \frac{4}{3} & \frac{4}{3} & \frac{4}{3} & \frac{4}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 \end{pmatrix}.$$

4. Conclusions

The resistance distance which is the effective electrical resistance, has wide application. For this reason, it was widely explored by so many authors. Among topics on resistance distance, its calculation plays an important role. As a continuation of this topic, in this paper, we compute resistance distance in $G = G_0^{(R)} \circ \{G_1, G_2\}$. As we known, there exists relationship between resistance distance and group inverse $L(G)^{\#}$ or $\{1\}$-inverse $L(G)^{\{1\}}$ of $L(G)$. Therefore, we aim to find a $\{1\}$-inverse $L(G)^{\{1\}}$ of $L(G)$, and finally give one in Theorem 1. At the end of this paper, we give an example to illustrate the correction and efficiency of the proposed method.

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