mathematics

Article

Quaternionic Blaschke Group

Margit Pap [1,*] and Ferenc Schipp [2]

1 Faculty of Sciences, University of Pécs, Ifjúság út 6, 7634 Pécs, Hungary
2 Faculty of Informatics, Loránd University, Pázmány Péter sétány 1/C, 1117 Budapest, Hungary; schipp@inf.elte.hu
* Correspondence: papm@gamma.ttk.pte.hu

Received: 28 November 2018; Accepted: 28 December 2018; Published: 31 December 2018

Abstract: In the complex case, the Blaschke group was introduced and studied. It turned out that in the complex case this group plays important role in the construction of analytic wavelets and multiresolution analysis in different analytic function spaces. The extension of the wavelet theory to quaternion variable function spaces would be very beneficial in the solution of many problems in physics. A first step in this direction is to give the quaternionic analogue of the Blaschke group. In this paper we introduce the quaternionic Blaschke group and we study the properties of this group and its subgroups.

Keywords: quaternions; Blaschke functions; Blascke group

1. Introduction

We write $\mathbb{D}$ for the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ for the unit circle. The element $(\mathfrak{e} := (0,1) \in \mathbb{B})$ will play a special role. The *complex Blaschke-functions* are then defined by

$$B_{\mathfrak{a}}(z) := \epsilon \frac{z-a}{1-\bar{a}z} \qquad (\mathfrak{a} := (a,\epsilon) \in \mathbb{B} := \mathbb{D} \times \mathbb{T}, |z| \leq 1). \tag{1}$$

The restrictions of the Blaschke functions on the set $\mathbb{D}$ or on $\mathbb{T}$ with the composition operation $(B_{\mathfrak{a}_1} \circ B_{\mathfrak{a}_2})(z) := B_{\mathfrak{a}_1}(B_{\mathfrak{a}_2}(z))$ $(z \in \overline{\mathbb{D}})$ form transformation groups. $B_{\mathfrak{a}}$ is a one-to-one map on $\mathbb{D}$ as well as on $\mathbb{T}$. The function $B_{\mathfrak{e}}(z) = z$ is the identity map of $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$ and $B_{\mathfrak{a}^{-1}}$ $(\mathfrak{a}^{-1} := (-a\epsilon, \bar{\epsilon}) \in \mathbb{B})$ is the inverse of the function $B_{\mathfrak{a}}$.

In the parameter set $\mathbb{B} := \mathbb{D} \times \mathbb{T}$ let us define the operation induced by the function composition in the following way: $B_{\mathfrak{a}_1 \circ \mathfrak{a}_2} := B_{\mathfrak{a}_1} \circ B_{\mathfrak{a}_2}$. The set of the parameters $\mathbb{B}$ with the induced operation is called the complex *Blaschke group* on $\mathbb{B}$. The components of $\mathfrak{a} = (a,\epsilon) = \mathfrak{a}_1 \circ \mathfrak{a}_2$ are given by the following formulas:

$$a = B_{\mathfrak{a}_2^{-1}}(a_1), \quad \epsilon = \epsilon_1 B_{-a_1\bar{a}_2}(\epsilon_2). \tag{2}$$

The complex Blaschke functions play important role in the theory of Hardy spaces and in the control theory. Using the Blaschke functions one of the basic results of the theory of Hardy spaces, the factorization theorem, can be formulated in a natural way (see for ex. [1]).

The Blaschke group is related to well known matrix groups. The special linear group $SL(2,\mathbb{R})$ is the group of 2×2 real matrices with determinant one. $SL(2,\mathbb{R})$ is isomorphic to the group of all linear transformations of $\mathbb{R}^2$ that preserve oriented area, and is isomorphic to the generalized special unitary group $SU(1,1)$. $SL(2,\mathbb{R})$ acts on the complex upper half-plane by fractional linear transformations.

The group action factors through the quotient $PSL(2,\mathbb{R})$ (the 2×2 projective special linear group over $\mathbb{R}$). More specifically, $PSL(2,R) = SL(2,R)/\{I,-I\}$, where I denotes the 2×2 identity matrix.

The quotient $PSL(2,\mathbb{R})$ has several interesting descriptions. $PSL(2,R)$ is the group of conformal automorphisms of the upper half-plane, which is isomorphic with the group of conformal automorphisms of the unit disc, i.e., with the Blaschke group .

The topological group $SU(1,1)$ is homeomorphic to the space $\mathbb{B} = \mathbb{D} \times \mathbb{T}$. With the Blaschke group we can realize another parametrization of the $SU(1,1)$, on which wavelet transforms were introduced earlier. For the descriptions of the mentioned matrix groups and the related transforms see for example [2–5].

Using the parametrization of the Blaschke group reflects better in the same time the properties of the covering group and the action of the representations on different analytic function spaces, see [6], where it is explained in detail the relation between $SU(1,1)$ and the Blaschke group, and why we consider the Blaschke group useful in order to develop wavelet analysis on this group. One reason is that the techniques of the complex analysis can be applied more directly in the study of the properties of the voice transforms (so called hyperbolic wavelet transforms) generated by representations of the Blaschke group on different analytic function spaces (see [7–11]). The discretization of these special wavelet transforms leads to the construction of analytic rational orthogonal wavelets, and multiresolution analysis (MRA) in the Hardy space of the unit disc, upper half plane, and in weighted Bergman spaces (see [11–14]). The Blaschke functions are closely related to the generator functions of the Zernike functions often used in optical tests. They can be expressed as matrixelements of the representation of the Blasche group on the Hardy space of the unit circle. An important consequence of this relation is the addition formula for these functions (see [7,8,11]). In the same time using the parametrization of the Blaschke group it was easier to apply the coorbit theory (see [15]) in order to obtain atomic decompositions in weighted Bergman spaces (see [6,10]). In this way as a special case we get back well known atomic decompositions in the weighted Bergman spaces obtained by complex techniques, but in addition some new atomic decompositions can be presented. This is the reason why we consider that Blaschke group is very interesting and the wavelet transforms on Blaschke group are worth to be studied.

In this paper we introduce the quaternionic analogue of the Blaschke group, and we will study the properties of this group.

2. The Blaschke Group over the Set of Quaternions.

Quaternions play important role in modeling the time and space dependent problems in physics and engineering. For example in engineering applications unit quaternions are used to describe three dimensional rotations. In the last years quaternions have gained a new life due to their applicability in signal processing, for example by the use of quaternion-valued functions for the coding of color-coded images as well as the link to new concepts of higher-dimensional phases, like the hypercomplex signal of Bülow or the monogenic signal by Larkin and Felsberg. Quaternions are also of interest in connection with quantum theory. Thus there is a strong motivation to extend key results of modern harmonic analysis, like the wavelet theory, to spaces of functions with quaternion variables. As a first step in this direction we propose the foundations of a quaternionic analogue of the Blaschke group. The main obstacle in the study of quaternion-valued matrices and functions, as expected, comes from the non-commutative nature of quaternionic multiplication.

Our work was inspired by [16], where monogenic wavelet transform for quaternion valued functions on the three dimensional unit ball in $\mathbb{R}^3$ was introduced. The construction is based on representations of the group of Möbius transformations which maps the three dimensional unit ball onto itself.

Quaternions are extensions of complex numbers. There is an useful representation of the quaternions: The matrix representation. The matrix representation makes possible to use the properties of the matrices at different computations.

Let us denote by

$$E := E_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; E_1 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; E_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; E_3 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{3}$$

the quaternion units, where $i \in \mathbb{C}$ is the complex imaginary unit. Analogous with the property of the complex unit $i^2 = -1$, the quaternion units satisfy the following equations $E_j^2 = -E \ (j = 1, 2, 3)$. Since $E_1E_2 = -E_2E_1 = E_3$, $E_2E_3 = -E_3E_2 = E_1$,$E_3E_1 = -E_1E_3 = E_2$, the set $\{\pm E_j : j = 0, 1, 2, 3\}$ is closed with respect to matrix multiplication. Let us denote by

$$\mathbf{Q} := \Big\{ Z := \sum_{j=0}^{3} z_j E_j : z = (z_0, z_1, z_2, z_3) \in \mathbb{R}^4 \Big\}, \tag{4}$$

the set of quaternions, which is a non-commutative field with the unit element E and null element the null-matrix $\Theta \in \mathbb{C}^{2\times 2}$. Let us denote by

$$\overline{Z} := z_0 E_0 - \sum_{j=1}^{3} z_j E_j = Z^*, \ |Z| := \Big(\sum_{j=0}^{3} z_j^2 \Big)^{1/2}, \ ZZ^* = |Z|^2 E,$$

the analogue of the conjugate which in matrix representation is Z^*, the adjoint matrix of $Z \in \mathbb{C}^{2\times 2}$, and the absolute value of the $Z = \sum_{j=0}^{3} z_j E_j \in \mathbf{Q}$. The map $Z \to |Z|$ defines a multiplicative norm:

$$|Z_1 + Z_2| \leq |Z_1| + |Z_2|, \ \ |Z_1 \cdot Z_2| = |Z_1|\,|Z_2| \ \ (Z_1, Z_2 \in \mathbf{Q}).$$

The multiplicative inverse of a nonzero quaternion $Z \in \mathbf{Q}, Z \neq \Theta$ in matrix representation is $Z^{-1} = Z^*/|Z|^2$. The analogue of the complex torus and unit disc in the set of the quaternions are defined by $\mathbf{T} := \{Z \in \mathbf{Q} : |Z| = 1\}$, and $\mathbf{D} := \{Z \in \mathbf{Q} : |Z| < 1\}$ respectively. From the property of the norm it follows that $\mathbf{T}$ is a multiplicative subgroup of the multiplicative group of $\mathbf{Q}$, which can be identified by the matrix group $\mathbb{SU}_2$.

The set $\mathbf{Q}$ with matrix addition and multiplication is a skew-field denoted by $(\mathbf{Q}, +, \cdot)$. Taking into account that $\mathbb{R}E$ and $\mathbb{R}$ are isomorphic ($\mathbb{R}E \approxeq \mathbb{R}$) and $\mathbb{C}E \approxeq \mathbb{C}$ the field $(\mathbf{Q}, +, \cdot)$ can be considered as an extension of $\mathbb{R}$ and $\mathbb{C}$, respectively. The purely imaginary quaternion $I_{\mathbf{c}} := \sum_{j=1}^{3} c_j E_j$ ($\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{R}^3$) satisfy the equation $I_{\mathbf{c}}^2 = -|\mathbf{c}|^2 E$. The map $\mathbf{c} \to I_{\mathbf{c}}$ is a linear isomorphism between $\mathbb{R}^3$ and the set of purely imaginary quaternion $\mathbf{J} := \{Z = I_{\mathbf{c}} : \mathbf{c} \in \mathbb{R}^3\} = \{Z \in \mathbf{Q} : \mathrm{spur}(Z) = 0\}$, consequently $\mathbb{R}^3$ and $\mathbf{J}$ can be identified.

The two dimensional subspace

$$\mathbf{Q}_{\mathbf{c}} := \{Q_{\mathbf{c}}(z) := xE + yI_{\mathbf{c}} : z = x + iy \in \mathbb{C}\} \subset \mathbf{Q} \ \ (\mathbf{c} \in \mathbb{R}^3, |\mathbf{c}| = 1) \tag{5}$$

of $\mathbf{Q}$ is called *the slice of* $\mathbf{Q}$ *in the direction of the vector* $\mathbf{c}$. The map $Q_{\mathbf{c}} : \mathbb{C} \to \mathbf{Q}_{\mathbf{c}}$ is a linear isomorphism. From $I_{\mathbf{c}}^2 = -E \ (|\mathbf{c}| = 1)$ follows that

$$Q_{\mathbf{c}}(z_1 + z_2) = Q_{\mathbf{c}}(z_1) + Q_{\mathbf{c}}(z_2), \ Q_{\mathbf{c}}(z_1 z_2) = Q_{\mathbf{c}}(z_1)\, Q_{\mathbf{c}}(z_2) \ (z_1, z_2 \in \mathbb{C}), \tag{6}$$

and obviously $Q_{\mathbf{c}}(\overline{z}) = Q_{\mathbf{c}}^*(z)\ (z \in \mathbb{C})$. This implies that the map $Q_{\mathbf{c}}$ is an isometric isomorphism between the fields $\mathbb{C}$ and $\mathbf{Q_c}$.

The complex numbers and their extensions, the quaternions are very useful in the description of many problems in geometry and physics. For example the rotations in the Euclidian plane $\mathbb{C}$ can be described using the map $z \to \epsilon z$ where $\epsilon, z \in \mathbb{C}$ are complex numbers and $\epsilon = e^{i\alpha} \in \mathbb{T}\ (\alpha \in \mathbb{R})$. In this case α is the angle of the rotation.

If instead of complex numbers we use quaternions, we can describe the rotations in $\mathbb{R}^3$ with a relatively simple map. In order to illustrate this, we use the analogue of the Euler formula $e^{it} = \cos t + i \sin t\ (t \in \mathbb{R})$:

$$e^{tI_c} = E \cos t + I_{\mathbf{c}} \sin t \quad (t \in \mathbb{R}, \mathbf{c} \in \mathbb{R}^3, |\mathbf{c}| = 1). \tag{7}$$

From this it follows that, analogue to unit complex numbers, every unit quaternion $S = z_0 E + I_{\mathbf{z}}$ $(\mathbf{z} = (z_1, z_2, z_3) \in \mathbb{R}^3), |S| = 1$ can be represented as $S = e^{tI_{\mathbf{c}}}$, where $\cos t = z_0$, $\mathbf{c} = \mathbf{z}/|\mathbf{z}|$.

The relation $\mathrm{spur}(SZS^*) = \mathrm{spur}(Z)\ (S \in \mathbf{T}, Z \in \mathbf{Q})$ implies that the map $Z \to SZS^*$ takes the subspace $\mathbf{J}$, which is isomorphic with $\mathbb{R}^3$, in itself and can be interpreted as a rotation around the axis $\mathbf{c}$ of the space $\mathbb{R}^3$ with angle $2t$. The image of the slice $\mathbf{Q_c}$ trough this rotation will be the slice $\mathbf{Q_b}$ for which $I_{\mathbf{b}} = SI_{\mathbf{c}}S^*\ (S \in \mathbf{T})$, i.e., $\mathbf{Q_b} = S\mathbf{Q_c}S^*$. The polar representation of the quaternion $Z \in \mathbf{Q}$ can be written as

$$Z = \rho e^{tI_{\mathbf{c}}} \quad (\rho = |Z|, t \in \mathbb{R}, I_{\mathbf{c}} \in \mathbf{J}). \tag{8}$$

3. The Quaternion Blaschke Group

The Blaschke functions can be defined also among quaternion. The formulas are very similar to the complex case:

$$B_A(Z) := (Z - A)(E - A^*Z)^{-1} \quad (A \in \mathbf{D}, Z \in \overline{\mathbf{D}} := \{Z \in \mathbf{Q} : |Z| \le 1\}). \tag{9}$$

It can be proved that these quaternion Blaschke functions have many analogue properties of the complex Blaschke functions (see [17]). One of this is:

$$1 - |B_A(Z)|^2 = \frac{(1 - |A|^2)(1 - |Z|^2)}{|E - A^*Z|^2} \quad (A \in \mathbf{D}, Z \in \overline{\mathbf{D}}). \tag{10}$$

From this follows that, similar to the complex case, for any $A \in \mathbf{D}$ the function B_A takes the quaternion unit disc $\mathbf{D}$ into $\mathbf{D}$, and the quaternion unit torus $\mathbf{T}$ into $\mathbf{T}$.

Because of the non commutativity of the product operation in $\mathbf{Q}$, in order to generate the quaternion analogue of the complex Blaschke group, we have to introduce a right and left unit quaternion factor from $\mathbf{T}$ in (9) instead of the multiplication by complex $\epsilon \in \mathbb{T}$. We consider in $\mathbf{Q}$ the following function:

$$C_A(Z) := (E - ZA^*)_0 := \frac{E - ZA^*}{|E - ZA^*|} \quad (A \in \mathbf{D}, Z \in \overline{\mathbf{D}}). \tag{11}$$

It is obvious that C_A takes $\overline{\mathbf{D}}$ into $\mathbf{T}$, and $C_Z(A) = C_A^*(Z)\ \ (A, Z \in \mathbf{D})$.

First we show that for the extended quaternion Blaschke functions, given by (9), an analogue rule of composition hold.

Theorem 1. *For every $A_1, A_2 \in \mathbf{D}$ and $Z \in \overline{\mathbf{D}}$ we have*

$$B_{A_1}(B_{A_2}(Z)) = UB_A(Z)V^*,$$

where

$$A = B_{-A_2}(A_1),\;\; U = C_{-A_2}(A_1),\; V = C_{-A_2^*}(A_1^*). \tag{12}$$

We observe that for the complex unit parameter ϵ (see formula (1)), in the quaternion case corresponds a right and left unit quaternion. In the complex case we can interchange the order of the terms in the product and obtain analogue of ϵ factor.

To get a collection of functions closed with respect to the composition operation $\circ$ it is convenient to introduce the parameter set $\mathbf{B} := \mathbf{T} \times \mathbf{D} \times \mathbf{T}$ and the function set

$$\mathfrak{B} := \{\mathcal{B}_{\mathfrak{a}} := UB_AV^* : \mathfrak{a} = (U, A, V) \in \mathbf{B}\}. \tag{13}$$

For the extended quaternion Blaschke functions we have

$$|\mathcal{B}_{\mathfrak{a}}(Z)| = |B_A(Z)| \leq \frac{|A| + |Z|}{1 + |A||Z|} \;\; (A \in \mathbf{D}, Z \in \overline{\mathbf{D}}) \tag{14}$$

and $\mathcal{B}_{\mathfrak{a}}$ takes $\overline{\mathbf{D}}$ into $\overline{\mathbf{D}}$. Applying formula (12) for $A_1 = A,\; A_2 = -A$ we get $U = V = E$ and

$$B_A(B_{-A}(Z)) = B_{-A}(B_A(Z)) = Z \;\; (Z \in \overline{\mathbf{D}}, A \in \mathbf{T}).$$

This implies that $B_A : \mathbf{D} \to \mathbf{D}, B_A : \mathbf{T} \to \mathbf{T}$ is bijective and $B_A^{-1} = B_{-A}$.

The set of functions $\mathfrak{B}$ is closed with respect to the inverse operation. In order to prove this we will use the following formula

$$U^* B_A(UZV^*)V = B_{U^*AV}(Z) \;\; (A \in \mathbf{D}, U, V \in \mathbf{T}). \tag{15}$$

Let us introduce the map $\mathfrak{a} = (U, A, V) \to \widehat{\mathfrak{a}} := UAV^*$ from $\mathbf{B}$ to $\mathbf{D}$. Based on the previous relation it follows that any function of the form

$$\mathcal{B}_{\mathfrak{a}} = UB_AV^* \;\; (\mathfrak{a} = (U, A, V) \in \mathbf{B})$$

has an inverse and

$$\mathcal{B}_{\mathfrak{a}}^{-1}(Z) = U^* B_{-UAV^*}(Z)V = U^* B_{-\widehat{\mathfrak{a}}}(Z)V. \tag{16}$$

Indeed $\mathcal{B}_{\mathfrak{a}}(X) = UB_A(X)V^* = Z$ is equivalent to, $\mathcal{B}_{\mathfrak{a}}^{-1}(Z) = X = B_{-A}(U^*ZV)$. From this we get

$$\mathcal{B}_{\mathfrak{a}}^{-1}(Z) = U^* B_{-UAV^*}(Z)V = U^* B_{-\widehat{\mathfrak{a}}}(Z)V.$$

It can be proved that the set of functions $\mathfrak{B}$ is closed with respect to function composition, consequently $(\mathfrak{B}, \circ)$ is a transformation group on $\mathbf{D}$ and $\mathbf{T}$ respectively, called *quaternion Blascke transformation group*.

Theorem 2. *For any two functions $\mathcal{B}_{\mathfrak{a}_1}, \mathcal{B}_{\mathfrak{a}_2} \in \mathfrak{B}$ $(\mathfrak{a}_j = (U_j, A_j, V_j) \in \mathbf{B}, j = 1, 2)$, we have*

$$\mathcal{B}_{\mathfrak{a}_1} \circ \mathcal{B}_{\mathfrak{a}_2} = \mathcal{B}_{\mathfrak{a}} \quad (\mathfrak{a} = (U, A, V) \in \mathbf{B}),$$

where

$$A = \mathcal{B}_{\mathfrak{a}_2}^{-1}(A_1),\;\; U = U_1 C_{-\widehat{\mathfrak{a}}_2}(A_1) U_2,\;\; V = V_1 C_{-(\widehat{\mathfrak{a}}_2)^*}(A_1^*) V_2. \tag{17}$$

The unit element of this group is $\mathcal{B}_{\mathfrak{e}}$, where $\mathfrak{e} = (E, \Theta, E)$.

The bijection $\mathbf{B} \ni \mathfrak{a} \to \mathcal{B}_{\mathfrak{a}} \in \mathfrak{B}$ induces in the set of the parameters $\mathbf{B}$ an operation, $\mathfrak{a}_1 \odot \mathfrak{a}_2 = \mathfrak{a}$ for which $\mathcal{B}_{\mathfrak{a}_1} \circ \mathcal{B}_{\mathfrak{a}_2} = \mathcal{B}_{\mathfrak{a}}$. The set of the parameters with the induced operation $(\mathbf{B}, \odot)$ is a group. In the set of the parameters the inverse $\mathfrak{a}^-$ of an element $\mathfrak{a} = (U, A, V)$ is the element for which $\mathcal{B}_{\mathfrak{a}^-} = \mathcal{B}_{\mathfrak{a}}^{-1}$ where $\mathfrak{a}^- = (U^*, -\widehat{\mathfrak{a}}, V^*)$.

If instead of $\mathfrak{a}_1$ we set $\mathfrak{z}$ and instead of $\mathfrak{a}_2$ $\mathfrak{a}^-$, then $\widehat{\mathfrak{a}_2^-} = -U_2^* U_2 A V_2^* V_2 = -A$, and in the set of the parameters the right translations $\mathfrak{z} \to \mathfrak{z} \odot \mathfrak{a}^-$ can be described as follows:

$$\mathfrak{z} \odot \mathfrak{a}^- = (U_1 C_A(Z) U_2^*, U_2 B_A(Z) V_2^*,\ V_1 C_{A^*}(Z^*) V_2^*). \tag{18}$$

In the papers [18–22]the operations $C_{-a}(z), B_a(z)$ were studied, also for higher dimensions, which we describe now as follows:

$$a \oplus z = B_{-a}(z),\ gyr[a, z] = C_{-a}(z). \tag{19}$$

They have been also used (19) to describe the *gyro group*. Our description makes possible to avoid the complicated gyro group description. It is also more useful from the point of view of the extensions for higher dimension.

4. Subgroups of $\mathfrak{B}$

The set $\{B_{\rho E} : \rho \in \mathbb{I} := (-1, 1)\}$ is subgroup of $\mathfrak{B}$, satisfying $B_{\rho_1 E} \circ B_{\rho_2 E} = B_{\rho_1 \circ \rho_2 E}$, where

$$\rho_1 \circ \rho_2 = \frac{\rho_1 + \rho_2}{1 + \rho_1 \rho_2} \ \ (\rho_1, \rho_2 \in \mathbb{I}) \tag{20}$$

is the real Blaschke group operation on $\mathbb{I}$.

Another subgroup can be generated if we choose the parameters and variable Z on the same slice. First let us observe that if $A_j = Q_{\mathbf{c}}(a_j)$ $(j = 1, 2)$ and $Z = Q_{\mathbf{c}}(z)$ belong to the same slice, then

$$B_{A_j}(Z) = Q_{\mathbf{c}}(B_{a_j}(z)),\ B_{A_1}(B_{A_2}(Z)) = Q_{\mathbf{c}}(B_{a_1}(B_{a_2}(z)),\ B_{A_1}^{-1}(Z) = Q_{\mathbf{c}}(B_{a_1}^{-1}(z)). \tag{21}$$

This implies

$$\begin{aligned} &B_{A_1}(B_{A_2}(Z)) = Q_{\mathbf{c}}(B_{a_1}(B_{a_2}(z))),\ B_A^{-1}(Z) = Q_{\mathbf{c}}(B_a^{-1}(z)), \\ &A = Q_{\mathbf{c}}(a), A_j = Q_{\mathbf{c}}(a_j),\ Z = Q_{\mathbf{c}}(z)\ (a, a_j \in \mathbb{D}, z \in \overline{\mathbb{D}}, j = 1, 2). \end{aligned}$$

Set $\mathbf{D_c} = \mathbf{D} \cap \mathbf{Q_c}$, $\mathbf{T_c} = \mathbf{T} \cap \mathbf{Q_c}$. Then it follows that the collection

$$\mathfrak{B}_{\mathbf{c}} := \{U B_A V^* : A \in \mathbf{D_c}, U, V \in \mathbf{T_c}\}$$

is a transformation group on $\mathbf{D_c}$ and $\mathbf{T_c}$ respectively, isomorph to the complex Blaschke transformation group.

Another interesting subgroup of the quaternion Blaschke group is induced by the following subset:

Theorem 3. *Let $\epsilon(A) := (E - A^*)/|E - A|$ $(A \in \mathbf{D})$. Then the subset*

$$\mathfrak{A} := \{\mathcal{A}_A = \epsilon(A) B_A \epsilon(A) :\ A \in \mathbf{D}\} \subset \mathfrak{B} \tag{22}$$

is a one parameter subgroup of $\mathfrak{B}$. *Moreover*

$$\begin{aligned}
&(i)\ \ \mathcal{A}_A(E) = E\ \ (A \in \mathbf{D}),\\
&(ii)\ \ \mathcal{A}_A^{-1} = \mathcal{A}_{A^-},\ A^- = -\epsilon(A)A\epsilon(A),\\
&(iii)\ \ \mathcal{A}_{A_1} \circ \mathcal{A}_{A_2} = \mathcal{A}_A,\ A = \mathcal{A}_{A_2^-}(A_1).
\end{aligned} \tag{23}$$

From

$$1 - |\mathcal{A}_A(Z)|^2 = \frac{(1-|A|^2)(1-|Z|^2)}{|1-A^*Z|^2},\ |\mathcal{A}_A(B)| \le \frac{|A|+|B|}{1+|A||B|} \le |A|+|B| \tag{24}$$

it follows that the function $\mathcal{A}_A$ $(A \in \mathbf{D})$ are bijections on $\mathbf{D}$ and on $\mathbf{T}$ respectively, consequently $\mathfrak{A}$ is a transformation subgroup on $\mathbf{D}$. The bijection $A \to \mathcal{A}_A$ between the sets $\mathbf{D}$ and $\mathfrak{A}$ induces a group structure $(\mathbf{D}, \bullet)$, where

$$A_1 \bullet A_2^- = \mathcal{A}_{A_2}(A_1)\ \ (A_1, A_2 \in \mathbf{D}). \tag{25}$$

The unit element of this subgroup is the nullmatrix $O \in \mathbf{Q}$ and the inverse element of $A \in \mathbf{D}$ is given by $A^- = -\epsilon(A)A\epsilon(A)$.

The map $A \to |A|$ defines a norm on the group $(\mathbf{D}, \bullet)$. Denote $\rho(A_1, A_2) := |A_1 \bullet A_2^-|$ the metric induced by this norm. It can be proved that the group operation $(A_1, A_2) \to A_1 \bullet A_2^-$ is continuous with respect to this metric.

5. Proofs

First we prove relation (10), i.e.,:

$$1 - |B_A(Z)|^2 = \frac{(1-|A|^2)(1-|Z|^2)}{|E-A^*Z|^2}.$$

During the proofs we will use in several places the following identity:

$$AA^* = A^*A = |A|^2E\ \ (A \in \mathbf{Q}).$$

We start from the left hand side of the equality (10), which is equal to:

$$\begin{aligned}
E - B_A(Z)B_A^*(Z) &= E\left(1 - \frac{|Z-A|^2}{|E-A^*Z|^2}\right) = E\frac{|E-A^*Z|^2 - |Z-A|^2}{|E-A^*Z|^2}\\
&= E\frac{(1+|A|^2|Z|^2 - A^*Z - AZ^*) - (|Z|^2 + |A|^2 - A^*Z - AZ^*)}{|E-A^*Z|^2}\\
&= E\frac{(1-|A|^2)(1-|Z|^2)}{|E-A^*Z|^2}.
\end{aligned}$$

Another frequently used relation is the following:

$$|E + A_1A_2^*| = |E + A_1^*A_2|\ \ (A_1, A_2 \in \mathbf{D}).$$

This is equivalent to

$$(E + A_1A_2^*)(E + A_2A_1^*) = (E + A_1^*A_2)(E + A_2^*A_1),$$

and

$$A_1A_2^* + A_2A_1^* = A_1^*A_2 + A_2^*A_1.$$

This last equality follows from the following identity:

$$(A_1 + A_2)(A_1^* + A_2^*) = (A_1^* + A_2^*)(A_1 + A_2).$$

We will use also the following property:

$$B_A^*(Z) = B_{A^*}(Z^*),$$

which is equivalent to the following relations

$$(E - Z^*A)^{-1}(Z^* - A^*) = (Z^* - A^*)(E - AZ^*)^{-1},$$

and

$$(Z^* - A^*)(E - AZ^*) = (E - Z^*A)(Z^* - A^*).$$

This last one is true, because

$$Z^* - A^* - Z^*AZ^* + |A|^2Z^* = Z^* - A^* - Z^*AZ^* + Z^*|A|^2.$$

Proof of Theorem 1. As in the complex case this identity can be proved directly:

$$\begin{aligned}
&B_{A_1}(B_{A_2}(Z)) = \\
&= \left[(Z - A_2)(E - A_2^*Z)^{-1} - A_1\right]\left[E - A_1^*(Z - A_2)(E - A_2^*Z)^{-1}\right]^{-1} \\
&= \left[\Big((Z - A_2) - A_1(E - A_2^*Z)\Big)(E - A_2^*Z)^{-1}\right]\cdot \\
&\cdot \left[\Big((E - A_2^*Z) - A_1^*(Z - A_2)\Big)(E - A_2^*Z)^{-1}\right]^{-1} \\
&= [(E + A_1A_2^*)Z - (A_1 + A_2)][(E + A_1^*A_2) - (A_1^* + A_2^*)Z]^{-1}.
\end{aligned}$$

From this relation it follows that

$$\begin{aligned}
&B_{A_1}(B_{A_2}(Z)) = \\
&= \left[(E + A_1A_2^*)\Big(Z - (E + A_1A_2^*)^{-1}(A_1 + A_2)\Big)\right]\cdot \\
&\cdot \left[(E + A_1^*A_2)\Big(E - (E + A_1^*A_2)^{-1}(A_1^* + A_2^*)Z\Big)\right]^{-1} \\
&= (E + A_1A_2^*)\Big((Z - P)(E - QZ)^{-1}\Big)(E + A_1^*A_2)^{-1},
\end{aligned}$$

where

$$\begin{aligned}
P^* &= (A_1^* + A_2^*)(E + A_2A_1^*)^{-1} = B_{-A_2^*}(A_1^*) \\
Q^* &= (A_1 + A_2)(E + A_2^*A_1)^{-1} = B_{-A_2}(A_1).
\end{aligned}$$

But we have $P = B^*_{-A_2^*}(A_1^*) = B_{-A_2}(A_1) = Q^*$, and let us denote $A := P = Q^*$. Using this notation we get that

$$B_{A_1}(B_{A_2}(Z))(E+A_1A_2^*)\Big((Z-A)(E-A^*Z)^{-1}\Big)(E+A_1^*A_2)^{-1}$$
$$=\frac{E+A_1A_2^*}{|E+A_1A_2^*|}B_A(Z)\left(\frac{E+A_1^*A_2}{|E+A_1^*A_2|}\right)^{-1},$$

and Theorem 1 is proved. □

Relation (14) says:

$$|B_A(Z)|\leq\frac{|A|+|Z|}{1+|A||Z|}\ \ (A\in\mathbf{D},Z\in\overline{\mathbf{D}}).$$

This follows from:

$$1-|B_{\mathcal{A}}(Z)|^2=\frac{(1-|A|^2)(1-|Z|^2)}{|E-A^*Z|^2}\geq$$
$$\geq\frac{(1-|A|^2)(1-|Z|^2)}{(1+|A|\,|Z|)^2},$$

which implies (14).

Relation (15) says:

$$U^*B_A(UZV^*)V=B_{U^*AV}(Z)\ \ (A\in\mathbf{D},U,V\in\mathbf{T}).$$

Proof of relation (15):

$$U^*B_A(UZV^*)V=U^*(UZV^*-A)(E-A^*UZV^*)^{-1}V=$$
$$=(Z-U^*AV)V^*(E-A^*UZV^*)^{-1}V=(Z-U^*AV)(V^*(E-A^*UZV^*)V)^{-1}=$$
$$=(Z-U^*AV)(E-V^*A^*UZ)^{-1}=B_{U^*AV}(Z).\ \square$$

Proof of Theorem 2. We use that $B_{A_3}(U_3ZV_3^*)=U_3B_{U_3^*A_3V_3}(Z)V_3^*$ with the following parameters $U_3=U_2,V_3=V_2,U_3^*A_3V_3=A_2$. Then $A_3=U_3A_2V_3^*=U_2A_2V_2^*$, and the following relation is true:

$$\mathcal{B}_{\mathfrak{a}_2}(Z)=U_2B_{A_2}(Z)V_2^*=B_{U_2A_2V_2^*}(U_2ZV_2^*)=B_{\widehat{\mathfrak{a}}_2}(\widehat{\mathfrak{z}}),$$

where $\widehat{\mathfrak{a}}_2=U_2A_2V_2^*,\widehat{\mathfrak{z}}=U_2ZV_2^*$. Using the previous relation and Theorem 1. we get

$$\mathcal{B}_{\mathfrak{a}_1}(\mathcal{B}_{\mathfrak{a}_2}(Z))=U_1B_{A_1}(B_{\widehat{\mathfrak{a}}_2}(\widehat{\mathfrak{z}}))V_1^*.$$

Applying again Theorem 1. for the parameters $A_1,\widehat{\mathfrak{a}}_2,\widehat{\mathfrak{z}}$:

$$B_{A_1}(B_{\widehat{\mathfrak{a}}_2}(\widehat{\mathfrak{z}}))=U'B_{A'}(U_2ZV_2^*)V'^*=U'U_2B_{U_2^*A'V_2}(Z)V_2^*V'^*,$$

where

$$A'=B_{-\widehat{\mathfrak{a}}_2}(A_1),U'=C_{-\widehat{\mathfrak{a}}_2}(A_1),V'=C_{-\widehat{\mathfrak{a}}_2^*}(A_1^*).$$

From here we get the formula

$$\mathcal{B}_{\mathfrak{a}_1}\circ\mathcal{B}_{\mathfrak{a}_2}=\mathcal{B}_{\mathfrak{a}}=UB_AV^*,$$
$$A=U_2^*B_{-\widehat{\mathfrak{a}}_2}(A_1)V_2,\ \ U=U_1C_{-\widehat{\mathfrak{a}}_2}(A_1)U_2,\ \ V=V_1C_{-\widehat{\mathfrak{a}}_2^*}(A_1^*)V_2.$$

□

Proof of Theorem 3. (i) From

$$\epsilon(A) = (E - A^*)/|E - A| = (E - A)^{-1}|E - A|$$

it follows that $\mathcal{A}_A(E) = E$ $(A \in \mathbf{D})$.

(ii) Let $A^- := -\epsilon(A)A\epsilon(A)$. First we prove

$$\epsilon(A^-) = \epsilon^*(A) \ \ (A \in \mathbf{D}). \tag{26}$$

Applying $\epsilon(A)^{-1} = \epsilon^*(A), |\epsilon(A)| = 1$ we get

$$\begin{aligned} &E - A^- = E + \epsilon(A)A\epsilon(A) = \epsilon(A)(\epsilon^*(A) + A\epsilon(A)) \\ &= \epsilon(A)|E - A|^{-1}((E - A) + A(E - A^*)) = \epsilon(A)|E - A|^{-1}(1 - |A|^2). \end{aligned}$$

Hence we get

$$\epsilon(A^-) = (E - A^-)^{-1}|E - A^-| = [\epsilon(A)|E - A|^{-1}(1 - |A|^2)]^{-1}|E - A|^{-1}(1 - |A|^2) = \epsilon^*(A).$$

Using (26) we have that

$$\mathcal{A}_A^{-1} = \epsilon^*(A)B_{-\epsilon(A)A\epsilon(A)}\epsilon^*(A) = \mathcal{A}_{A^-}.$$

(iii) To prove (iii) we use the equation

$$B_{A_1}(\epsilon(A_2)Z\epsilon(A_2)) = \epsilon(A_2)B_{A_3}(Z)\epsilon(A_2), \ A_3 := \epsilon^*(A_2)A_1\epsilon^*(A_2) \tag{27}$$

and Theorem 1 in the following form:

$$B_{A_3} \circ B_{A_2} = \epsilon(-A_2A_3^*)B_A\epsilon^*(-A_2^*A_3), \ A = B_{-A_2}(A_3),$$

where by (27) $A = \mathcal{A}_{A_2^-}(A_1)$.

Then we get

$$\begin{aligned} &\mathcal{A}_{A_1}(\mathcal{A}_{A_2}(Z)) = \epsilon(A_1)B_{A_1}(\epsilon(A_2)B_{A_2}(Z)\epsilon(A_2))\epsilon(A_1) \\ &= \epsilon(A_1)\epsilon(A_2)B_{A_3}(B_{A_2}(Z))\epsilon(A_2)\epsilon(A_1) \\ &= \epsilon(A_1)\epsilon(A_2)\epsilon(-A_2A_3^*)B_A(Z)\epsilon^*(-A_2^*A_3)\epsilon(A_2)\epsilon(A_1) \\ &= X\mathcal{A}_A(Z)Y, \end{aligned}$$

where

$$X = \epsilon(A_1)\epsilon(A_2)\epsilon(-A_2A_3^*)\epsilon^*(A), \ Y = \epsilon^*(A)\epsilon^*(-A_2^*A_3)\epsilon(A_2)\epsilon(A_1).$$

We show that $X = Y = E$. Since by i) $XY = E$ it is enough to see that $X = E$, or which is the same

$$\epsilon(A_1)\epsilon(A_2)\epsilon(-A_2A_3^*) = \epsilon(A). \tag{28}$$

Indeed

$$\begin{aligned}
E - A &= E - B_{-A_2}(A_3) = E - (E + A_3 A_2^*)^{-1}(A_3 + A_2) \\
&= (E + A_3 A_2^*)^{-1}((E - A_2) - A_3(E - A_2^*)) \\
&= (E + A_3 A_2^*)^{-1}|E - A_2|(\epsilon^*(A_2) - \epsilon^*(A_2)A_1\epsilon^*(A_2)\epsilon(A_2)) \\
&= (E + A_3 A_2^*)^{-1}|E - A_2|\epsilon^*(A_2)(E - A_1) \\
&= (E + A_3 A_2^*)^{-1}|E - A_2||E - A_1|\epsilon^*(A_2)\epsilon^*(A_1),
\end{aligned}$$

consequently

$$\epsilon(A) = |E - A|(E - A)^{-1} = \epsilon(A_1)\epsilon(A_2)(E + A_3 A_2^*)/|E + A_3 A_2^*| = \epsilon(A_1)\epsilon(A_2)\epsilon(-A_2 A_3^*)$$

and (iii) is proved. □

Author Contributions: conceptualization, M.P. and F.S.; methodology, M.P. and F.S.; validation, M.P. and F.S.; formal analysis, M.P. and F.S.; investigation, M.P. and F.S.; resources, M.P. and F.S.; writing—original draft preparation, M.P. and F.S.; writing—review and editing, M.P.; supervision, M.P.; project administration, M.P.; funding acquisition, M.P.

Funding: The first author of the present scientific contribution has been supported by the European Union, co-financed by the European Social Fund EFOP-3.6.1.-16-2016-00004.

Conflicts of Interest: The authors declare no conflict of interest.

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