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Article

Higher Order Hamiltonian Systems with Generalized Legendre Transformation

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Abstract: The aim of this paper is to report some recent results regarding second order Lagrangians corresponding to 2nd and 3rd order Euler–Lagrange forms. The associated 3rd order Hamiltonian systems are found. The generalized Legendre transformation and geometrical correspondence between solutions of the Hamilton equations and the Euler–Lagrange equations are studied. The theory is illustrated on examples of Hamiltonian systems satisfying the following conditions: (a) the Hamiltonian system is strongly regular and the Legendre transformation exists; (b) the Hamiltonian system is strongly regular and the Legendre transformation does not exist; (c) the Legendre transformation exists and the Hamiltonian system is not regular but satisfies a weaker condition.

Keywords: Hamilton equations; Lagrangian; regular and strongly regular systems

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1. Introduction

Hamiltonian theory on manifolds has been intensively studied since the 1970s (see e.g., [1–10]). The aim of this paper is to apply an extension of the classical Hamilton–Cartan variational theory on fibered manifolds, recently proposed by Krupková [11,12], to the case of a class of second order Lagrangians and third order Hamiltonian systems. In the generalized Hamiltonian field theory, one can associate different Hamilton equations corresponding to different Lepagean equivalents of the Euler–Lagrange form with a variational problem represented by a Lagrangian. With the help of Lepagean equivalents of a Lagrangian, one obtains an intrinsic formulation of the Euler–Lagrange and Hamilton equations. The arising Hamilton equations and regularity conditions depend not only on a Lagrangian but also on some "free" functions, which correspond to the choice of a concrete Lapagean equivalent. Consequently, one has many different "Hamilton theories" associated to a given variational problem. A regularization of some interesting singular physical fields, the Dirac field, the electromagnetic field, and the Scalar Curvature Lagrangian by various methods has been studied in [3,6,13–15]. Some second order Lagrangians have also been discussed in [16–18].

The multisymplectic approach was proposed in [2,4,8,10]. This approach is not well adapted to study Lagrangians that are singular in the standard sense. Note that an alternative approach to the study of "degenerated" Lagrangians (singular in the standard sense) is the constraint theory from mechanics (see [19,20]) and in the field theory [21].

In this work, we are interested in second order Lagrangians that give rise to Euler–Lagrange equations of the 3rd order or non-affine 2nd order. All these Lagrangians are singular in the standard Hamilton–De Donder theory and do not have a Legendre transformation. Examples of these Lagrangians are afinne (scalar curvature Lagrangians) and many Lagrangians quadratic in second derivatives. However, in the generalized setting, the question on existence of regular Hamilton equations makes sense. For such a Lagrangian, we find the set of Lepagean equivalents (respectively family of Hamilton equations) that are regular in the generalized sense, as well as a generalized

Legendre transformation. We note that the generalized momenta p_σ^{ij} satisfy $p_\sigma^{ij} \neq p_\sigma^{ji}$. We study the correspondence between solutions of Euler–Lagrange and Hamilton equations. The regularity conditions are found (ensuring that the Hamilton extremals are holonomic up to the second order). These conditions depend on a choice of a Hamiltonian system (i.e., on a choice of "free" functions). We study the correspondence between the regularity conditions and the existence of the Legendre transformation. Contrary to the classical approach, the regularity conditions do not guarantee the existence of a generalized Legendre transformation. On the other hand, the generalized Legendre conditions do not guarantee regularity. The existence of a generalized Legendre transformation guarantees that the Hamilton extremals are holonomic up to the first order. The regularization procedure and properties of the Legendre transformation are illustrated in three examples. We consider three different Hamiltonian systems for a given Lagrangian. The first system is regular and possesses a generalized Legendre transformation. The second Hamiltonian system is regular and a generalized Legendre transformation does not exist. The last one is not regular but a generalized transformation exists.

Throughout the paper, all manifolds and mappings are smooth and the summation convention is used. We consider a fibered manifold (i.e., surjective submersion) $\pi : Y \to X$, $\dim X = n$, $\dim Y = n + m$. Its r-jet prolongation is $\pi_r : J^rY \to X$, $r \geq 1$ and its canonical jet projections are $\pi_{r,k} : J^rY \to J^kY$, $0 \leq k \leq r$ (with the obvious notation $J^0Y = Y$). A fibered chart on Y (respectively associated fibered chart on J^rY) is denoted by (V, ψ), $\psi = (x^i, y^\sigma)$ (respectively (V_r, ψ_r), $\psi_r = (x^i, y^\sigma, y_i^\sigma, \dots, y^\sigma_{i_1 \dots i_r})$).

A vector field ξ on J^rY is called π_r -vertical (respectively $\pi_{r,k}$ -vertical) if it projects onto the zeroth vector field on X (respectively on J^kY).

Recall that every q-form η on J^rY admits a unique (canonical) decomposition into a sum of q-forms on $J^{r+1}Y$ as follows [7]:

$$\pi^*_{r+1,r}\eta = h\eta + \sum_{k=1}^{q} p_k\eta,$$

where $h\eta$ is a horizontal form, called the horizontal part of η, and $p_k\eta$, $1 \leq k \leq q$, is a k-contact part of η.

We use the following notations:

$$\omega_0 = dx^1 \wedge dx^2 \wedge \dots \wedge dx^n, \ \omega_i = i_{\partial/\partial x^i}\omega_0, \ \omega_{ij} = i_{\partial/\partial x^j}\omega_i,$$

and

$$\omega^\sigma = dy^\sigma - y_j^\sigma dx^j, \ \dots \ , \ \omega^\sigma_{i_1 i_2 \dots i_k} = dy^\sigma_{i_1 i_2 \dots i_k} - y^\sigma_{i_1 i_2 \dots i_k j} dx^j.$$

For more details on fibered manifolds and the corresponding geometric structures, we refer to sources such as [22].

2. Lepagean Equivalents and Hamiltonian Systems

In this section we briefly recall the basic concepts on Lepagean equivalents of Lagrangians according to Krupka [7,23], and on Lepagean equivalents of Euler–Lagrange forms and generalized Hamiltonian field theory according to Krupková [11,12].

By an *r-th order Lagrangian* we shall mean a horizontal n-form λ on J^rY.

An n-form ρ is called a *Lepagean equivalent of a Lagrangian* λ if (up to a projection) $h\rho = \lambda$ and $p_1 d\rho$ is a $\pi_{r+1,0}$-horizontal form.

For an r-th order Lagrangian we have all its Lepagean equivalents of order $(2r-1)$ characterized by the following formula

$$\rho = \Theta + \mu, \tag{1}$$

where Θ is a (global) Poincaré–Cartan form associated to λ and μ is an arbitrary n-form of order of contactness ≥ 2, i.e., such that $h\mu = p_1\mu = 0$. Recall that for a Lagrangian of order 1, $\Theta = \theta_\lambda$ where θ_λ is the classical Poincaré–Cartan form of λ. If $r \geq 2$, Θ is no longer unique, however there is a non-invariant decomposition

$$\Theta = \theta_\lambda + p_1 d\nu, \tag{2}$$

where

$$\theta_\lambda = L\omega_0 + \sum_{k=0}^{r-1} \left(\sum_{l=0}^{r-k-1} (-1)^l d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial L}{\partial y^\sigma_{j_1 \dots j_k p_1 \dots p_l i}} \right) \omega^\sigma_{j_1 \dots j_k} \wedge \omega_i, \tag{3}$$

and ν is an arbitrary at least 1-contact $(n-1)$-form (see [7,23]).

A closed $(n+1)$-form α is called a *Lepagean equivalent of an Euler–Lagrange form* $E = E_\sigma \omega^\sigma \wedge \omega_0$ if $p_1\alpha = E$.

Recall that the Euler–Lagrange form corresponding to an r-th order $\lambda = L\omega_0$ is the following $(n+1)$-form of order $\leq 2r$:

$$E = E_\sigma\, \omega^\sigma \wedge \omega_0 = \left(\frac{\partial L}{\partial y^\sigma} - \sum_{l=1}^{r} (-1)^l d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial L}{\partial y^\sigma_{p_1 \dots p_l}} \right) \omega^\sigma \wedge \omega_0.$$

By definition of a Lepagean equivalent of E, one can find Poincaré lemma local forms ρ such that $\alpha = d\rho$, where ρ is a Lepagean equivalent of a Lagrangian for E. The family of Lepagean equivalents of E is also called a *Lagrangian system* and denoted by $[\alpha]$. The corresponding Euler–Lagrange equations now take the form

$$J^s\gamma^* i_{J^s\xi}\alpha = 0 \quad \text{for every } \pi\text{−vertical vector field } \xi \text{ on Y}, \tag{4}$$

where α is any representative of order s of the class $[\alpha]$. A (single) Lepagean equivalent α of E on J^sY is also called a *Hamiltonian system of order* s and the equations

$$\delta^* i_\xi \alpha = 0 \quad \text{for every } \pi_s\text{−vertical vector field } \xi \text{ on } \mathrm{J^sY} \tag{5}$$

are called *Hamilton equations*. They represent equations for integral sections δ (called *Hamilton extremals*) of the *Hamilton ideal*, generated by the system $\mathcal{D}^s_\alpha$ of n-forms $i_\xi\alpha$, where ξ runs over π_s-vertical vector fields on J^sY. Also, considering π_{s+1}-vertical vector fields on $J^{s+1}Y$, one has the ideal $\mathcal{D}^{s+1}_{\hat{\alpha}}$ of n-forms $i_\xi\hat{\alpha}$ on $J^{s+1}Y$, where $\hat{\alpha}$ (called *principal part* of α) denotes the at most 2-contact part of α. Its integral sections, which annihilate all at least 2-contact forms, are called *Dedecker–Hamilton extremals*. It holds that if γ is an extremal then its s-prolongation (respectively $(s+1)$-prolongation) is a Hamilton (respectively Dedecker–Hamilton) extremal, and (up to projection) every Dedecker–Hamilton extremal is a Hamilton extremal (see [11,12]).

Denote by r_0 the minimal order of Lagrangians corresponding to E. A Hamiltonian system α on J^sY, $s \geq 1$, associated with E is called *regular* if the system of local generators of $\mathcal{D}^{s+1}_{\hat{\alpha}}$ contains all the n-forms

$$\omega^\sigma \wedge \omega_i,\ \omega^\sigma_{(j_1} \wedge \omega_{i)},\ \dots,\ \omega^\sigma_{(j_1 \dots j_{r_0-1}} \wedge \omega_{i)}, \tag{6}$$

where $(\ldots)$ denotes symmetrization in the indicated indices. If α is regular then every Dedecker–Hamilton extremal is holonomic up to the order r_0, and its projection is an extremal. (In the case of first order Hamiltonian systems, there is a bijection between extremals and Dedecker–Hamilton extremals). α is called *strongly regular* if the above correspondence holds between extremals and Hamilton extremals. It can be proved that every strongly regular Hamiltonian system is regular, and it is clear that if α is regular and such that $\alpha = \hat{\alpha}$ then it is strongly regular. A Lagrangian system is called *regular* (respectively*strongly regular*) if it has a regular (respectively strongly regular) associated Hamiltonian system [11].

3. Regular and Strongly Regular 3rd Order Hamiltonian Systems

In this section we discuss a part of variational theory which is singular in the standard sense. In general, a second order Lagrangian gives rise to an Euler–Lagrange form on J^4Y. We shall consider second order Lagrangians λ that satisfy one of the following conditions:

(1) The corresponding Euler–Lagrange form is of order 3, i.e., the Lagrangians satisfy the conditions

$$\left(\frac{\partial^2 L}{\partial y^\sigma_{ij} \partial y^\nu_{kl}} \right)_{Sym(ijkl)} = 0, \tag{7}$$

where $Sym(ijkl)$ means symmetrization in the indicated indices.

(2) The Euler–Lagrange expressions E_σ (4) of λ are second order and "non-affine" in the second derivatives

$$\frac{\partial^2 E_\sigma}{\partial y^\nu_{kl} \partial y^\kappa_{ij}} \neq 0 \tag{8}$$

for some indices $i, j, k, l, \sigma, \nu, \kappa$.

In what follows, we shall study Hamiltonian systems corresponding to a special choice of a Lepagean equivalent of such Lagrangians, namely α of order 3 and $\alpha = d\rho$, where

$$\begin{aligned} \rho &= L\omega_0 + \left(\frac{\partial L}{\partial y^\sigma_j} - d_k \frac{\partial L}{\partial y^\sigma_{jk}} \right) \omega^\sigma \wedge \omega_j + \frac{\partial L}{\partial y^\sigma_{ij}} \omega^\sigma_i \wedge \omega_j + \bar{\mu} \\ &+ a^{ij}_{\sigma\nu} \omega^\sigma \wedge \omega^\nu \wedge \omega_{ij} + b^{kij}_{\sigma\nu} \omega^\sigma \wedge \omega^\nu_k \wedge \omega_{ij} \\ &+ c^{klij}_{\sigma\nu} \omega^\sigma \wedge \omega^\nu_{kl} \wedge \omega_{ij}, \end{aligned} \tag{9}$$

with an arbitrary at least 3-contact n-form $\bar{\mu}$ and functions $a^{ij}_{\sigma\nu}$, $b^{kij}_{\sigma\nu}$, $c^{klij}_{\sigma\nu}$ dependent on variables x^k, y^κ, y^κ_k, y^κ_{kl} and satisfying the conditions

$$\begin{aligned} &a^{ij}_{\sigma\nu} = -\, a^{ji}_{\sigma\nu},\ a^{ij}_{\sigma\nu} = -\, a^{ij}_{\nu\sigma};\ \ b^{kij}_{\sigma\nu} = -\, b^{kji}_{\sigma\nu}; \\ &c^{klij}_{\sigma\nu} = c^{lkij}_{\sigma\nu},\ c^{klij}_{\sigma\nu} = -\, c^{klji}_{\sigma\nu}. \end{aligned} \tag{10}$$

Theorem 1. *Ref. [18] Let* $\dim X \geq 2$. *Let* $\lambda = L\omega_0$ *be a second order Lagrangian with the Euler–Lagrange form (7) or (8), and* $\alpha = d\rho$ *with* ρ *of the form (9), (10), be its Lepagean equivalent. Assume that the matrix*

$$P^{ijkl}_{\sigma\nu} = \left(\frac{\partial^2 L}{\partial y^\nu_{ij} \partial y^\sigma_{kl}} + 2\, c^{klij}_{\nu\sigma} \right)_{Sym(jkl)}, \tag{11}$$

with mn^3 rows (respectively mn columns) labelled by σjkl (respectively νi) has maximal rank equal to mn and the matrix

$$Q^{ijkl}_{\sigma\nu} = \left(\frac{\partial^2 L}{\partial y^\sigma_{ij} \partial y^\nu_{kl}} - 2c^{klij}_{\sigma\nu} \right), \tag{12}$$

with mn^2 rows (respectively mn^2 columns) labelled by σij (respectively νkl) has maximal rank equal to $mn(n+1)/2$. Then the Hamiltonian system $\alpha = d\rho$ is regular (i.e. every Dedecker–Hamilton extremal is of the form $\pi_{3,2} \circ \delta_D = J^2\gamma$, where γ is an extremal of λ).

Moreover, if $\bar{\mu}$ is closed then the Hamiltonian system $\alpha = d\rho$ is strongly regular (i.e., every Hamilton extremal is of the form $\pi_{3,2} \circ \delta = J^2\gamma$, where γ is an extremal of λ).

Proof. Explicit computation $\alpha = d\rho$ gives:

$$\begin{aligned}
\pi^*_{4,3}\alpha \;=\;& E_\sigma \omega^\sigma \wedge \omega_0 + \left(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu} - \frac{\partial}{\partial y^\nu} d_j \frac{\partial L}{\partial y^\sigma_{ij}} - 2d_j a^{ij}_{\sigma\nu} \right) \omega^\nu \wedge \omega^\sigma \wedge \omega_i \\
&+ \left(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_k} - \frac{\partial^2 L}{\partial y^\sigma \partial y^\nu_{ik}} - \frac{\partial}{\partial y^\nu_k} d_j \frac{\partial L}{\partial y^\sigma_{ij}} + 4a^{ik}_{\nu\sigma} - 2d_j b^{kij}_{\sigma\nu} \right) \omega^\nu_k \wedge \omega^\sigma \wedge \omega_i \\
&+ \left(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_{kl}} - \frac{\partial}{\partial y^\nu_{kl}} d_j \frac{\partial L}{\partial y^\sigma_{ij}} - 2(b^{kil}_{\sigma\nu})_{Sym(kl)} - 2d_j c^{klij}_{\sigma\nu} \right) \omega^\nu_{kl} \wedge \omega^\sigma \wedge \omega_i \\
&- \left(\frac{\partial^2 L}{\partial y^\sigma_{ij} \partial y^\nu_{kl}} + 2c^{klij}_{\sigma\nu} \right)_{Sym(jkl)} \omega^\nu_{jkl} \wedge \omega^\sigma \wedge \omega_i \\
&+ \left(\frac{\partial^2 L}{\partial y^\sigma_{ij} \partial y^\nu_k} - 4(b^{kij}_{\sigma\nu})_{Alt((\sigma j)(\nu k))} \right) \omega^\nu_k \wedge \omega^\sigma_j \wedge \omega_i \\
&+ \left(\frac{\partial^2 L}{\partial y^\sigma_{ij} \partial y^\nu_{kl}} - 2c^{klij}_{\sigma\nu} \right) \omega^\nu_{kl} \wedge \omega^\sigma_j \wedge \omega_i + \left(\frac{\partial a^{ij}_{\sigma\nu}}{\partial y^\kappa} \right)_{Alt(\kappa\sigma\nu)} \\
& \omega^\kappa \wedge \omega^\sigma \wedge \omega^\nu \wedge \omega_{ij} + \left(\frac{\partial a^{ij}_{\sigma\nu}}{\partial y^\kappa_p} + \frac{\partial b^{pij}_{\nu\kappa}}{\partial y^\sigma} \right)_{Alt(\sigma\nu)} \omega^\kappa_p \wedge \omega^\sigma \wedge \omega^\nu \wedge \omega_{ij} \\
&+ \left(\left(\frac{\partial a^{ij}_{\sigma\nu}}{\partial y^\kappa_{pq}} \right)_{Sym(pq)} + \left(\frac{\partial c^{pqij}_{\nu\kappa}}{\partial y^\sigma_{pq}} \right)_{Alt(\sigma\nu)} \right) \omega^\kappa_{pq} \wedge \omega^\sigma \wedge \omega^\nu \wedge \omega_{ij} \\
&+ \left(\frac{\partial b^{qij}_{\sigma\nu}}{\partial y^\kappa_p} \right)_{Alt((\kappa p)(\nu q))} \omega^\sigma \wedge \omega^\nu_q \wedge \omega^\kappa_p \wedge \omega_{ij} + \left(\frac{\partial b^{kij}_{\sigma\nu}}{\partial y^\kappa_{pq}} - \frac{\partial c^{pqij}_{\sigma\kappa}}{\partial y^\nu_k} \right)_{Sym(pq)} \\
& \omega^\sigma \wedge \omega^\nu_k \wedge \omega^\kappa_{pq} \wedge \omega_{ij} - \left(\frac{\partial c^{klij}_{\sigma\nu}}{\partial y^\kappa_{pq}} \right)_{Alt((\kappa pq)(\nu kl))} \omega^\sigma \wedge \omega^\kappa_{pq} \wedge \omega^\nu_{kl} \wedge \omega_{ij} + d\bar{\mu},
\end{aligned} \tag{13}$$

where $Alt((\dots)\ \dots\ (\dots))$ means alternation in the indicated multi-indices and $Sym(\dots)$ means symmetrization in the indicated indices.

In the notation of Equations (11) and (12), the principal part of α (13) takes the form

$$
\begin{aligned}
\hat{\alpha} \;=\;& E_\sigma \omega^\sigma \wedge \omega_0 + \left(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu} - \frac{\partial}{\partial y^\nu} d_j \frac{\partial L}{\partial y^\sigma_{ij}} - 2 d_j a^{ij}_{\sigma\nu} \right) \omega^\nu \wedge \omega^\sigma \wedge \omega_i \\
+& \left(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_k} - \frac{\partial^2 L}{\partial y^\sigma \partial y^\nu_{ik}} - \frac{\partial}{\partial y^\nu_k} d_j \frac{\partial L}{\partial y^\sigma_{ij}} + 4 a^{ik}_{\nu\sigma} - 2 d_j b^{kij}_{\sigma\nu} \right) \omega^\nu_k \wedge \omega^\sigma \wedge \omega_i \\
+& \left(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_{kl}} - \frac{\partial}{\partial y^\nu_{kl}} d_j \frac{\partial L}{\partial y^\sigma_{ij}} - 2 (b^{kil}_{\sigma\nu})_{Sym(kl)} - 2 d_j c^{klij}_{\sigma\nu} \right) \omega^\nu_{kl} \wedge \omega^\sigma \wedge \omega_i \\
+& \left(\frac{\partial^2 L}{\partial y^\sigma_{ij} \partial y^\nu_k} - 4 (b^{kij}_{\sigma\nu})_{Alt((\sigma j)(\nu k))} \right) \omega^\nu_k \wedge \omega^\sigma_j \wedge \omega_i \\
-& P^{ijkl}_{\nu\sigma} \omega^\nu_{jkl} \wedge \omega^\sigma \wedge \omega_i + Q^{ijkl}_{\sigma\nu} \omega^\nu_{kl} \wedge \omega^\sigma_j \wedge \omega_i,
\end{aligned}
\tag{14}
$$

Expressing the generators of the ideal $\mathcal{D}^4_{\hat{\alpha}}$, we obtain

$$
\begin{aligned}
i_{\frac{\partial}{\partial y^\nu}} \hat{\alpha} \;=\;& E_\nu \omega_0 + 2 \left(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu} - \frac{\partial}{\partial y^\nu} d_j \frac{\partial L}{\partial y^\sigma_{ij}} - 2 d_j a^{ij}_{\sigma\nu} \right) \omega^\sigma \wedge \omega_i \\
-& \left(\frac{\partial^2 L}{\partial y^\nu_i \partial y^\sigma_k} - \frac{\partial^2 L}{\partial y^\nu \partial y^\sigma_{ik}} - \frac{\partial}{\partial y^\sigma_k} d_j \frac{\partial L}{\partial y^\nu_{ij}} + 4 a^{ik}_{\sigma\nu} - 2 d_j b^{kij}_{\nu\sigma} \right) \omega^\sigma_k \wedge \omega_i \\
-& \left(\frac{\partial^2 L}{\partial y^\nu_i \partial y^\sigma_{kl}} - \frac{\partial}{\partial y^\sigma_{kl}} d_j \frac{\partial L}{\partial y^\nu_{ij}} - 2 (b^{kil}_{\nu\sigma})_{Sym(kl)} - 2 d_j c^{klij}_{\nu\sigma} \right) \omega^\sigma_{kl} \wedge \omega_i \\
+& P^{ijkl}_{\sigma\nu} \omega^\sigma_{jkl} \wedge \omega_i, \\
i_{\frac{\partial}{\partial y^\nu_k}} \hat{\alpha} \;=\;& \left(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_k} - \frac{\partial^2 L}{\partial y^\sigma \partial y^\nu_{ik}} - \frac{\partial}{\partial y^\nu_k} d_j \frac{\partial L}{\partial y^\sigma_{ij}} + 4 a^{ik}_{\nu\sigma} - 2 d_j b^{kij}_{\sigma\nu} \right) \omega^\sigma \wedge \omega_i \\
+& 2 \left(\frac{\partial^2 L}{\partial y^\sigma_{ij} \partial y^\nu_k} - 4 (b^{kij}_{\sigma\nu})_{Alt((\sigma j)(\nu k))} \right) \omega^\sigma_j \wedge \omega_i + Q^{ikjl}_{\nu\sigma} \omega^\sigma_{jl} \wedge \omega_i, \\
i_{\frac{\partial}{\partial y^\nu_{kl}}} \hat{\alpha} \;=\;& \left(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_{kl}} - \frac{\partial}{\partial y^\nu_{kl}} d_j \frac{\partial L}{\partial y^\sigma_{ij}} - 2 (b^{kil}_{\sigma\nu})_{Sym(kl)} - 2 d_j c^{klij}_{\sigma\nu} \right) \omega^\sigma \wedge \omega_i \\
+& Q^{ijkl}_{\sigma\nu} \omega^\sigma_j \wedge \omega_i, \\
i_{\frac{\partial}{\partial y^\nu_{jkl}}} \hat{\alpha} \;=\;& -P^{ijkl}_{\sigma\nu} \omega^\nu \wedge \omega_i
\end{aligned}
\tag{15}
$$

Since the ranks of the matrices $P^{ijkl}_{\nu\sigma}$, $Q^{ijkl}_{\sigma\nu}$ are maximal then the $\omega^\sigma \wedge \omega_i$ and $\omega^\sigma_{(j} \wedge \omega_{i)}$ are generators of the ideal $\mathcal{D}^4_{\hat{\alpha}}$. For Dedecker–Hamilton extremals, we obtain $\delta_D\ \pi_{3,2} \circ \delta_D = J^2\gamma$, where γ is a section of π. Substituting this into Equation (5), we get

$$\delta_D{}^* i_{\frac{\partial}{\partial y^\sigma}} \hat{\alpha} = E_\sigma \circ J^3\gamma$$

for the 3rd order Euler–Lagrange form (7) and

$$\delta_D{}^* i_{\frac{\partial}{\partial y^\sigma}} \hat{\alpha} = E_\sigma \circ J^2\gamma$$

for the 2nd order Euler–Lagrange form (8) and γ is an extremal of λ.

Let us prove strong regularity. We have to show that under our assumptions, for every section δ satisfying the Hamilton equations, $\pi_{3,2} \circ \delta = J^2\gamma$, where γ is a solution of the Euler–Lagrange equations of the Lagrangian λ. Assuming $d\bar{\mu} = 0$, we obtain $\delta^*(i_{\partial/\partial y^\sigma_{jkl}}\alpha) = \delta^*(P^{ijkl}_{\sigma\nu}\omega^\nu \wedge \omega_i) = 0$, i.e., $\delta^*\omega^\nu = 0$ by the rank condition on $P^{ijkl}_{\sigma\nu}$, i.e., $\partial(y^\sigma \circ \delta)/\partial x^i = y^\sigma_i \circ \delta$. Hence, $\delta^*(i_{\partial/\partial y^\nu_{kl}}\alpha) = \delta^*\left(Q^{ijkl}_{\sigma\nu}\omega^\sigma_j \wedge \omega_i\right) = 0$.

Note that the matrix $Q^{ijkl}_{\sigma\nu}$ is symmetric in indices kl and its maximal rank is $mn(n+1)/2$. Due to the rank condition on $Q^{ijkl}_{\sigma\nu}$, $\delta^*\omega^\sigma_j = 0$, i.e., $\left(\partial(y^\sigma_j \circ \delta)/\partial x^i\right)_{Sym(ij)} = y^\sigma_{ij} \circ \delta$. The conditions for δ obtained above mean that every solution of Hamilton equations is holonomic up to the second order, i.e., we can write $\pi_{3,2} \circ \delta = J^2\gamma$, where γ is a section of π. Now, the equations $J^3(\pi_{3,0} \circ \delta)^*(i_{\partial/\partial y^\sigma_k}\alpha) = 0$ are satisfied identically and the last set of Hamilton equations—$J^3(\pi_{3,0} \circ \delta)^*(i_{\partial/\partial y^\sigma}\alpha) = 0$—take the form $E_\sigma \circ J^3\gamma = 0$ (7) or $E_\sigma \circ J^2\gamma = 0$ (8), proving that γ is an extremal of λ. □

In the next propositon we study a weaker condition which the Hamilton extremals satisfy.

Theorem 2. *Let* $\dim X \geq 2$. *Let* $\lambda = L\omega_0$ *be a second order Lagrangian with the Euler–Lagrange form (7) or (8), and* $\alpha = d\rho$ *with* ρ *of the form (9) and (10) be its Lepagean equivalent. Assume that* $\bar{\mu}$ *is closed and the matrix*

$$P^{ijkl}_{\sigma\nu} = \left(\frac{\partial^2 L}{\partial y^\nu_{ij}\partial y^\sigma_{kl}} + 2\,c^{klij}_{\nu\sigma}\right)_{Sym(jkl)}, \tag{16}$$

with mn^3 *rows (respectively* mn *columns) labelled by* σ, j, k, l *(respectively* νi*) has rank* mn *.*

Then every Hamilton extremal $\delta : \pi(U) \subset V \to J^2Y$ *of the Hamiltonian system* $\alpha = d\rho$ *is of the form* $\pi_{3,1} \circ \delta = J^1\gamma$ *(i.e.,* $\frac{\partial y^\sigma}{\partial x^i} = y^\sigma_i$*), where* γ *is an extremal of* λ*.*

Proof. The assertion of Theorem 2 follows from the proof of Theorem 1. □

4. Legendre Transformation

In this section the Hamiltonian systems admitting Legendre transformation are studied. By the Legendre transformation we understand the coordinate transformation onto J^3Y.

Writing the Lepagean equivalent ρ (9), (10) in the form of a noninvariant decomposition, we get

$$\begin{aligned} \rho &= -H\omega_0 + p^j_\sigma dy^\sigma \wedge \omega_j + p^{ij}_\sigma dy^\sigma_i \wedge \omega_j + 2c^{klij}_{\sigma\nu} y^\sigma_j dy^\nu_{kl} \wedge \omega_i \\ &+ a^{ij}_{\sigma\nu} dy^\sigma \wedge dy^\nu \wedge \omega_{ij} + b^{kij}_{\sigma\nu} dy^\sigma \wedge dy^\nu_k \wedge \omega_{ij} \\ &+ c^{klij}_{\sigma\nu} dy^\sigma \wedge dy^\nu_{kl} \wedge \omega_{ij} + \bar{\mu}, \end{aligned} \tag{17}$$

where

$$\begin{aligned} H &= -L + \left(\frac{\partial L}{\partial y^\sigma_i} - d_j \frac{\partial L}{\partial y^\sigma_{ij}}\right) y^\sigma_i + \frac{\partial L}{\partial y^\sigma_{ij}} y^\sigma_{ij} - 2a^{ij}_{\sigma\nu} y^\sigma_i y^\nu_j \\ &- 2(b^{kij}_{\sigma\nu})_{Sym(ki)} y^\sigma_i y^\nu_{kj} - 2(c^{klij}_{\sigma\nu})_{Sym(klj)} y^\sigma_i y^\nu_{klj}, \\ p^j_\sigma &= \frac{\partial L}{\partial y^\sigma_j} - d_i \frac{\partial L}{\partial y^\sigma_{ij}} + 4a^{ij}_{\sigma\nu} y^\nu_i + 2(b^{kij}_{\sigma\nu})_{Sym(ki)} y^\nu_{ki} + 2(c^{klij}_{\sigma\nu})_{Sym(kli)} y^\nu_{kli}, \\ p^{ij}_\sigma &= \frac{\partial L}{\partial y^\sigma_{ij}} + 2b^{ijk}_{\nu\sigma} y^\nu_k. \end{aligned} \tag{18}$$

Moreover, if the matrix

$$\begin{pmatrix} \frac{\partial p^i_\sigma}{\partial y^\nu_{kl}} & \frac{\partial p^i_\sigma}{\partial y^\nu_{klm}} \\ \frac{\partial p^{ij}_\sigma}{\partial y^\nu_{kl}} & \frac{\partial p^{ij}_\sigma}{\partial y^\nu_{klm}} \end{pmatrix} \tag{19}$$

has maximal rank, then

$$(x^i,\, y^\sigma,\, y^\sigma_i,\, p^i_\sigma, p^{ij}_\sigma)$$

is part of coordinate system.

We note that the functions p^{ij}_σ do not depend on the variables y^ν_{klm}. Then the submatrix of the Jacobi matrix of the transformation takes the form

$$\begin{pmatrix} \frac{\partial p^i_\sigma}{\partial y^\nu_{kl}} & \frac{\partial p^i_\sigma}{\partial y^\nu_{klm}} \\ \frac{\partial p^{ij}_\sigma}{\partial y^\nu_{kl}} & 0 \end{pmatrix}. \tag{20}$$

The above matrix has maximal rank if and only if the matrices $(\partial p^i_\sigma/\partial y^\nu_{klm})$ and $\left(\partial p^{ij}_\sigma/\partial y^\nu_{kl}\right)$ have maximal ranks. Explicit computations lead to

$$\begin{aligned} \left(\frac{\partial p^i_\sigma}{\partial y^\nu_{klm}}\right) &= \left(\frac{\partial^2 L}{\partial y^\nu_{im}\partial y^\sigma_{kl}} + 2\, c^{klim}_{\nu\sigma}\right)_{Sym(klm)}, \\ \left(\frac{\partial p^{ij}_\sigma}{\partial y^\nu_{kl}}\right) &= \frac{\partial^2 L}{\partial y^\sigma_{ij}\partial y^\nu_{kl}} + 2\frac{\partial b^{ijq}_{\kappa\sigma}}{\partial y^\nu_{kl}} y^\kappa_q. \end{aligned} \tag{21}$$

Note that in the notation of Equation (11), $\left(P^{ijkl}_{\sigma\nu}\right)^T = \left(\partial p^i_\sigma/\partial y^\nu_{jkl}\right)$ and the maximal rank is equal to mn. The matrix $\left(\partial p^{ij}_\sigma/\partial y^\nu_{kl}\right)$ is symmetric in the indices kl and therefore the maximal rank of the matrix is equal to $mn\,(n+1)\,/2$, i.e., the number of independent p^{ij}_σ is $mn\,(n+1)\,/2$. Contrary to the situation in Hamilton–De Donder theory, the functions p^{ij}_σ are not symmetric in the indices ij.

If we suppose that the matrix (19) has maximal rank, then

$$\psi_3 = (x^k,\, y^\nu,\, y^\nu_k,\, y^\nu_{kl}, y^\nu_{klm}) \to (x^i,\, y^\sigma,\, y^\sigma_i,\, p^i_\sigma,\, p^{ij}_\sigma, z^B) = \chi \tag{22}$$

is a coordinate transformation over an open set $U \subset V_2$, where z^B, $1 \leq B \leq mn(n^2+3n-1)/6$ are arbitrary coordinate functions. We call it a *generalized Legendre transformation* and χ (22) the *generalized Legendre coordinates*. Accordingly, H, p^i_σ, p^{ij}_σ are called *generalized Hamiltonian* and *generalized momenta*, respectively.

Writing the Lepagean equivalent ρ (9) and (10) in the generalized Legendre transformation, we get

$$\begin{aligned} \rho &= -H\omega_0 + p^j_\sigma dy^\sigma \wedge \omega_j + p^{ij}_\sigma dy^\sigma_i \wedge \omega_j \\ &+ 2c^{klij}_{\sigma\nu} y^\sigma_j \left(\frac{\partial y^\nu_{kl}}{\partial p^q_\beta} dp^q_\beta + \frac{\partial y^\nu_{kl}}{\partial p^{qr}_\beta} dp^{qr}_\beta + \frac{\partial y^\nu_{kl}}{\partial z^B} dz^B\right) \wedge \omega_i \\ &+ a^{ij}_{\sigma\nu} dy^\sigma \wedge dy^\nu \wedge \omega_{ij} + b^{kij}_{\sigma\nu} dy^\sigma \wedge dy^\nu_k \wedge \omega_{ij} \\ &+ c^{klij}_{\sigma\nu} dy^\sigma \wedge \left(\frac{\partial y^\nu_{kl}}{\partial p^q_\beta} dp^q_\beta + \frac{\partial y^\nu_{kl}}{\partial p^{qr}_\beta} dp^{qr}_\beta + \frac{\partial y^\nu_{kl}}{\partial z^B} dz^B\right) \wedge \omega_{ij} + \bar{\mu}, \end{aligned} \tag{23}$$

where y^ν_{kl} are functions of variables p^i_σ, p^{ij}_σ, z^B.

The Hamilton Equation (5) in these generalized Legendre coordinates take a rather complicated form, see Appendix A.

An interesting case. However, if $d\eta = 0$, where

$$\begin{aligned}\eta &= 2c^{klij}_{\sigma\nu} y^\sigma_j dy^\nu_{kl} \wedge \omega_i + a^{ij}_{\sigma\nu} dy^\sigma \wedge dy^\nu \wedge \omega_{ij} + b^{kij}_{\sigma\nu} dy^\sigma \wedge dy^\nu_k \wedge \omega_{ij} \\ &+ c^{klij}_{\sigma\nu} dy^\sigma \wedge dy^\nu_{kl} \wedge \omega_{ij} + d^{klij}_{\sigma\nu} dy^\sigma_k \wedge dy^\nu_l \wedge \omega_{ij}\end{aligned} \tag{24}$$

then the Hamilton Equation (5) have the following form

$$\frac{\partial H}{\partial y^\kappa} = -\frac{\partial p^j_\kappa}{\partial x^j}, \; \frac{\partial H}{\partial y^\kappa_q} = -\frac{\partial p^{qj}_\kappa}{\partial x^j}, \; \frac{\partial H}{\partial p^q_\kappa} = \frac{\partial y^\kappa}{\partial x^q}, \; \frac{\partial H}{\partial p^{qr}_\kappa} = \frac{\partial y^\kappa_q}{\partial x^r}, \; \frac{\partial H}{\partial z^M} = 0.$$

Contrary to the Hamilton–De Donder theory, the regularity conditions of the Lepagean form (9), (10) and regularity of the generalized Legendre transformation (21) do not coincide. The regularity conditions do not guarantee the existence of the Legendre transformation. On the other hand, the existence of the Legendre transformation does not guarantee the regularity. But we can see that the existence of a Legendre transformation (22) guarantees a weaker relation: $\pi_{3,1} \circ \delta = J^1\gamma$, where γ is an extremal of λ.

Theorem 3. *Let* $\dim X \geq 2$. *Let* $\lambda = L\omega_0$ *be a second order Lagrangian with the Euler–Lagrange form (7) or (8), and* $\alpha = d\rho$ *with* ρ *of the form (9), and Equation (10) be the expression of its Lepagean equivalent in a fiber chart* (V, ψ), $\psi = (x^i, y^\sigma)$.

Suppose that $\bar{\mu}$ *is closed and* ρ *admits Legendre transformation (22) defined by Equation (18).*

Then $\pi_{3,1} \circ \delta = J^1\gamma$, *where* γ *is an extremal of* λ.

Proof. The form ρ admits Legendre transformation, so the matrix

$$\left(\frac{\partial p^i_\sigma}{\partial y^\nu_{jkl}}\right) = \left(\frac{\partial^2 L}{\partial y^\nu_{ij} \partial y^\sigma_{kl}} + 2\, c^{klij}_{\nu\sigma}\right)_{Sym(jkl)}$$

has maximal rank equal to mn. In the notation of (11), $\left(P^{ijkl}_{\sigma\nu}\right)^T = \left(\partial p^i_\sigma / \partial y^\nu_{jkl}\right)$. Acordingly, from Proposition 2, we obtain $\pi_{3,1} \circ \delta = J^1\gamma$, where γ is an extremal of λ. □

5. Examples

The above results (the regularity conditions and the Legendre transformation) can be directly applied to concrete Lagrangians. Let us consider the following examples as an illustration. For a given Lagrangian, we find three different Hamiltonian systems satisfying:

(a) The Hamiltonian system is strongly regular and the Legendre transformation exists. (See examples of strongly regular systems in [17]).
(b) The Hamiltonian system is strongly regular and the Legendre transformation does not exist.
(c) The Legendre transformation exists and the Hamiltonian system is not regular but satisfies a weaker condition.

Let $X = R^2$, $Y = R^2 \times R^2$ (i.e., $n = 2$, $m = 2$). Denote (V, ψ), $\psi = (x^i, y^\sigma)$ a fibered chart on $R^2 \times R^2$. Let us consider the following Lagrangian

$$\lambda = L\omega_0, \; L = y^1_{11} y^2_{22} - y^1_{22} y^2_{11} \tag{25}$$

which satisfies (7).

5.1. Example (a)

View of the above considerations, we take a Lepagean equivalent ρ (of the Euler–Lagrange form E of Lagrangian (25)) in the form $\alpha = d\rho$, where ρ is (9), (10).

We consider functions $a^{ij}_{\sigma\nu}$, $b^{kij}_{\sigma\nu}$, $c^{ijkl}_{\sigma\nu}$ (see Equation (10)) on an open set $U \subset J^3R^2$ with the conditions $y^1_1 \neq 0$, $y^1_2 \neq 0$, $y^1_{12} \neq 0$ and $y^2_{12} \neq 0$.

The functions $a^{ij}_{\sigma\nu}$ are arbitrary. The functions $b^{ijp}_{\kappa\sigma}$ are linear in variables y^ν_{kl}. We denote $d^{ijpkl}_{\kappa\sigma\nu} = \partial b^{ijp}_{\kappa\sigma}/\partial y^\nu_{kl}$. Suppose that $d^{ijpkl}_{\kappa\sigma\nu}$ are constant functions, then we have only eight non-zero constants and we put $d^{12112}_{112} = d^{12121}_{112} = -d^{11212}_{112} = -d^{11221}_{112} = 1$ and $d^{22112}_{121} = d^{22121}_{121} = -d^{21212}_{121} = -d^{21221}_{121} = 1$. Similarly, we assume that $c^{ijkl}_{\sigma\nu}$ are constant functions. We have again only eight non-zero constants, and we choose $c^{1212}_{11} = c^{2112}_{11} = -c^{2121}_{11} = -c^{1221}_{11} = 1$ and $c^{1212}_{22} = c^{2112}_{22} = -c^{2121}_{22} = -c^{1221}_{22} = 1$. Then the Lepagean equivalent takes the form

$$\begin{aligned}\rho_1 = \; & \theta_\lambda + a^{ij}_{\sigma\nu}\omega^\sigma \wedge \omega^\nu \wedge \omega_{ij} - 4y^1_{12}\omega^2 \wedge \omega^1_2 \wedge \omega_{12} - 4y^2_{12}\omega^1 \wedge \omega^2_1 \wedge \omega_{12} \\ + \; & 4\,\omega^1 \wedge \omega^1_{12} \wedge \omega_{12} + 4\,\omega^2 \wedge \omega^2_{12} \wedge \omega_{12} + \bar{\mu},\end{aligned}$$

where $\bar{\mu}$ is an arbitrary closed n-form.

The matrices (11), (12), and (21) take the following form

$$(P^{ijkl}_{\sigma\nu})^T = \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 4 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & -4 & -4 & -4 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -4 & -4 & -4 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$Q^{ijkl}_{\sigma\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\left(\frac{\partial p^{ij}_\sigma}{\partial y^\nu_{kl}}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -y^1_2 & -y^1_2 & 1 \\ 0 & 0 & 0 & 0 & 0 & y^1_1 & y^1_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -y^1_2 & -y^1_2 & 0 & 0 & 0 & 0 & 0 \\ 1 & y^1_1 & y^1_1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

We can easily see that $\mathrm{rank}(P^{ijkl}_{\sigma\nu}) = 4$ and $\mathrm{rank}(Q^{ijkl}_{\sigma\nu}) = 6$. Since $y^1_1 \neq 0$ and $y^1_2 \neq 0$ $\mathrm{rank}\left(\partial p^{ij}_\sigma/\partial y^\nu_{kl}\right) = 6$. The form $\alpha = d\rho$ is strongly regular and a generalized Legendre transformation exists.

The generalized Hamiltonian and momenta (18) take the form

$$
\begin{aligned}
H &= -y^1_{12}y^2_{22} + y^1_{22}y^2_{11} - y^1_1(8y^1_{122} + y^2_{122}) + y^1_2(8y^1_{112} + y^2_{122}) \\
&- y^2_1(8y^2_{122} + y^1_{122}) + y^2_2(8y^2_{112} + y^1_{122}) - 4a^{12}_{12}(y^1_1y^2_2 - y^1_2y^2_1),
\end{aligned}
$$

$$
\begin{aligned}
p^1_1 &= y^2_{122} - 8y^1_{122} + 4a^{12}_{12}y^2_2, \;\; p^1_2 = -y^1_{122} + 8y^2_{122} - 4a^{12}_{12}y^1_2, \\
p^2_1 &= -y^2_{112} - 8y^1_{112} - 4a^{12}_{12}y^2_1, \;\; p^2_2 = y^1_{112} + 8y^2_{112} + 4a^{12}_{12}y^1_1,
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
p^{11}_1 &= y^2_{22} - 4y^1_2y^2_{12}, \;\; p^{12}_1 = 4y^1_1y^2_{12}, \;\; p^{22}_1 = -y^2_{11}, \\
p^{22}_2 &= y^1_{11} + 4y^1_1y^1_{12}, \;\; p^{21}_2 = -4y^1_2y^1_{12}, \;\; p^{22}_1 = -y^1_{22}.
\end{aligned}
$$

We have only six independent generalized momenta p^{ij}_σ. We note that $p^{21}_1 = p^{12}_2 = 0$.

5.2. Example (b)

For the given Lagrangian (25), we consider another Hamiltonian system on an open set $U \subset J^3R^2$

$$
\begin{aligned}
\rho_2 = &\quad \theta_\lambda + a^{ij}_{\sigma\nu}\omega^\sigma \wedge \omega^\nu \wedge \omega_{ij} + b^{kij}_{\sigma\nu}\omega^\sigma \wedge \omega^\nu_k \wedge \omega_{ij} \\
+ &\quad 4\,\omega^1 \wedge \omega^1_{12} \wedge \omega_{12} + 4\,\omega^2 \wedge \omega^2_{12} \wedge \omega_{12} + \bar{\mu},
\end{aligned}
$$

where $a^{ij}_{\sigma\nu}$, $b^{kij}_{\sigma\nu}$ are arbitrary constant functions satisfying Equation (10) and $\bar{\mu}$ is an arbitrary closed n-form.

We can easily see that matrices (11) and (12) have the same form as in Example (a), i.e., the Hamiltonian system is strongly regular. The matrix (21) takes the form

$$
\left(\frac{\partial p^{ij}_\sigma}{\partial y^\nu_{kl}}\right) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

and rank $\left(\partial p^{ij}_\sigma / \partial y^\nu_{kl}\right) = 4$. Therefore the generalized Legendre transformation does not exist.

5.3. Example (c)

On an open set $U \subset J^3R^2$ where $y^1_1 \neq 0$, $y^1_2 \neq 0$, $y^1_{12} \neq 0$ and $y^2_{12} \neq 0$, the Lepagean equivalent takes the form

$$
\begin{aligned}
\rho_3 = &\quad \theta_\lambda + a^{ij}_{\sigma\nu}\omega^\sigma \wedge \omega^\nu \wedge \omega_{ij} - 4y^1_{12}\omega^2 \wedge \omega^1_2 \wedge \omega_{12} - 4y^2_{12}\omega^1 \wedge \omega^2_1 \wedge \omega_{12} \\
+ &\quad 4\,\omega^1 \wedge \omega^1_{12} \wedge \omega_{12} + \bar{\mu},
\end{aligned}
$$

where $\bar{\mu}$ is an arbitrary closed n-form and $a^{ij}_{\sigma\nu}$ are arbitrary functions satisfying Equation (10).

It is easy to see that rank $\left(\partial p^{ij}_\sigma / \partial y^\nu_{kl}\right) = 6$ and the matrix has the same form as in Example (a).

The matrices (11) and (12) take take the form

$$(P^{ijkl}_{\sigma\nu})^T = \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 4 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & -4 & -4 & -4 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Q^{ijkl}_{\sigma\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $\operatorname{rank}(P^{ijkl}_{\sigma\nu}) = 4$ and $\operatorname{rank}(Q^{ijkl}_{\sigma\nu}) = 5$. The Hamiltonian system is not regular but it is holonomic up to first order and the generalized Legendre transformation exists (see Theorem 3).

6. Conclusions

This paper presents a generalization of classical Hamiltonian field theory on a fibered manifold. The regularization procedure of the first order Lagrangians proposed by Krupkova and Smetanová is applied to the case of a third order Hamiltonian system satisfying the conditions (7) or (8). Hamilton equations are created from the Lepagean equivalent whose order of contactness is more than 2-contact (contrary to the Hamilton p2-equations in [16]). The generalized Legendre transformation was studied and the generalized momenta $p^{ij}_\sigma \neq p^{ji}_\sigma$ were found. The theory was illustrated using examples of Hamilton systems satisfying:

(a) The Hamiltonian system is strongly regular and the Legendre transformation exists.
(b) The Hamiltonian system is strongly regular and the Legendre transformation does not exist.
(c) The Legendre transformation exists and the Hamiltonian system is not regular but satisfies a weaker condition.

Contrary to the standard approach, where all afinne and many quadratic Lagrangians are singular, we show that these Lagrangians are regularizable, admit Legendre transformation, and provide Hamilton equations that are equivalent to the Euler–Lagrange equations (i.e., they do not contain constraints). Within this setting, a proper choice of a Lepagean equivalent can lead to a "regularization" of a Lagrangian. The method proposed in this article is appropriate for the regularization of 2nd order Lagrangians (e.g., scalar curvature Lagrangians). The proposed procedure is different from [6,13,15] since it does not change order of the Lepagean equivalent .

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Appendix A

Hamilton Equations (5) with $d\bar{\mu} = 0$ (9) in Legendre coordinates take the following explicit form:

$$
\begin{aligned}
\frac{\partial H}{\partial y^\kappa} \quad = \quad & -\frac{\partial p^j_\kappa}{\partial x^j} + 2\frac{\partial c^{klij}_{\sigma\nu}}{\partial y^\kappa} y^\sigma_j \left(\frac{\partial y^\nu_{kl}}{\partial p^q_\beta}\frac{\partial p^q_\beta}{\partial x^i} + \frac{\partial y^\nu_{kl}}{\partial p^{qr}_\beta}\frac{\partial p^{qr}_\beta}{\partial x^i} + \frac{\partial y^\nu_{kl}}{\partial z^B}\frac{\partial z^B}{\partial x^i} \right) \\
& +4\frac{\partial a^{ij}_{\kappa\nu}}{\partial x^j}\frac{\partial y^\nu}{\partial x^i} + 6\left(\frac{\partial a^{ij}_{\sigma\nu}}{\partial y^\kappa}\right)_{Alt(\kappa\nu\sigma)} \frac{\partial y^\sigma}{\partial x^i}\frac{\partial y^\nu}{\partial x^j} + 4\frac{\partial a^{ij}_{\kappa\nu}}{\partial y^\sigma_q}\frac{\partial y^\sigma_q}{\partial x^i}\frac{\partial y^\nu}{\partial x^j} \\
& +4\frac{\partial a^{ij}_{\kappa\nu}}{\partial p^q_\sigma}\frac{\partial p^q_\sigma}{\partial x^i}\frac{\partial y^\nu}{\partial x^j} + 4\frac{\partial a^{ij}_{\kappa\nu}}{\partial p^{qr}_\sigma}\frac{\partial p^{qr}_\sigma}{\partial x^i}\frac{\partial y^\nu}{\partial x^j} + 4\frac{\partial a^{ij}_{\kappa\nu}}{\partial z^M}\frac{\partial z^M}{\partial x^i}\frac{\partial y^\nu}{\partial x^j} \\
& +2\frac{\partial b^{kij}_{\kappa\nu}}{\partial x^j}\frac{\partial y^\nu_k}{\partial x^i} + 4\left(\frac{\partial b^{ij}_{\sigma\nu}}{\partial y^\kappa}\right)_{Alt(\kappa\sigma)} \frac{\partial y^\sigma}{\partial x^i}\frac{\partial y^\nu_k}{\partial x^j} \\
& +2\left(\frac{\partial b^{kij}_{\kappa\nu}}{\partial y^\sigma_q}\right)_{Alt((\nu k)(\sigma q))} \frac{\partial y^\nu_k}{\partial x^i}\frac{\partial y^\sigma_q}{\partial x^j} + 2\frac{\partial b^{kij}_{\kappa\nu}}{\partial p^q_\sigma}\frac{\partial y^\nu_k}{\partial x^i}\frac{\partial p^q_\sigma}{\partial x^j} \\
& +2\frac{\partial b^{kij}_{\kappa\nu}}{\partial p^{qr}_\sigma}\frac{\partial y^\nu_k}{\partial x^i}\frac{\partial p^{qr}_\sigma}{\partial x^j} + 2\frac{\partial b^{kij}_{\kappa\nu}}{\partial z^M}\frac{\partial y^\nu}{\partial x^i}\frac{\partial z^M}{\partial x^j} \\
& +2\frac{\partial c^{klij}_{\kappa\nu}}{\partial x^j} \left(\frac{\partial y^\nu_{kl}}{\partial p^q_\beta}\frac{\partial p^q_\beta}{\partial x^i} + \frac{\partial y^\nu_{kl}}{\partial p^{qr}_\beta}\frac{\partial p^{qr}_\beta}{\partial x^i} + \frac{\partial y^\nu_{kl}}{\partial z^B}\frac{\partial z^B}{\partial x^i} \right) \\
& +4\left(\frac{\partial c^{klij}_{\sigma\nu}}{\partial y^\kappa}\right)_{Alt(\kappa\sigma)} \frac{\partial y^\sigma}{\partial x^i} \left(\frac{\partial y^\nu_{kl}}{\partial p^q_\beta}\frac{\partial p^q_\beta}{\partial x^j} + \frac{\partial y^\nu_{kl}}{\partial p^{qr}_\beta}\frac{\partial p^{qr}_\beta}{\partial x^j} + \frac{\partial y^\nu_{kl}}{\partial z^B}\frac{\partial z^B}{\partial x^j} \right) \\
& +2\frac{\partial c^{klji}_{\kappa\nu}}{\partial y^\sigma_q}\frac{\partial y^\sigma_q}{\partial x^i} \left(\frac{\partial y^\nu_{kl}}{\partial p^q_\beta}\frac{\partial p^q_\beta}{\partial x^j} + \frac{\partial y^\nu_{kl}}{\partial p^{qr}_\beta}\frac{\partial p^{qr}_\beta}{\partial x^j} + \frac{\partial y^\nu_{kl}}{\partial z^B}\frac{\partial z^B}{\partial x^j} \right) \\
& +2\frac{\partial c^{klji}_{\kappa\nu}}{\partial p^q_\sigma}\frac{\partial p^q_\sigma}{\partial x^i} \left(\frac{\partial y^\nu_{kl}}{\partial p^q_\beta}\frac{\partial p^q_\beta}{\partial x^j} + \frac{\partial y^\nu_{kl}}{\partial p^{qr}_\beta}\frac{\partial p^{qr}_\beta}{\partial x^j} + \frac{\partial y^\nu_{kl}}{\partial z^B}\frac{\partial z^B}{\partial x^j} \right) \\
& +2\frac{\partial c^{klji}_{\kappa\nu}}{\partial p^q_\sigma}\frac{\partial p^q_\sigma}{\partial x^i} \left(\frac{\partial y^\nu_{kl}}{\partial p^q_\beta}\frac{\partial p^q_\beta}{\partial x^j} + \frac{\partial y^\nu_{kl}}{\partial p^{qr}_\beta}\frac{\partial p^{qr}_\beta}{\partial x^j} + \frac{\partial y^\nu_{kl}}{\partial z^B}\frac{\partial z^B}{\partial x^j} \right) \\
& +2\frac{\partial c^{klji}_{\kappa\nu}}{\partial p^{qr}_\sigma}\frac{\partial p^{qr}_\sigma}{\partial x^i} \left(\frac{\partial y^\nu_{kl}}{\partial p^q_\beta}\frac{\partial p^q_\beta}{\partial x^j} + \frac{\partial y^\nu_{kl}}{\partial p^{qr}_\beta}\frac{\partial p^{qr}_\beta}{\partial x^j} + \frac{\partial y^\nu_{kl}}{\partial z^B}\frac{\partial z^B}{\partial x^j} \right) \\
& +2\frac{\partial c^{klji}_{\kappa\nu}}{\partial z^M}\frac{\partial z^M}{\partial x^i} \left(\frac{\partial y^\nu_{kl}}{\partial p^q_\beta}\frac{\partial p^q_\beta}{\partial x^j} + \frac{\partial y^\nu_{kl}}{\partial p^{qr}_\beta}\frac{\partial p^{qr}_\beta}{\partial x^j} + \frac{\partial y^\nu_{kl}}{\partial z^B}\frac{\partial z^B}{\partial x^j} \right)
\end{aligned} \tag{A1}
$$

$$
\begin{aligned}
\frac{\partial H}{\partial y^\kappa_q} \quad = \quad & -\frac{\partial p^{qj}_\kappa}{\partial x^j} + 2c^{kliq}_{\kappa\nu} \left(\frac{\partial y^\nu_{kl}}{\partial p^q_\beta}\frac{\partial p^q_\beta}{\partial x^i} + \frac{\partial y^\nu_{kl}}{\partial p^{qr}_\beta}\frac{\partial p^{qr}_\beta}{\partial x^i} + \frac{\partial y^\nu_{kl}}{\partial z^B}\frac{\partial z^B}{\partial x^i} \right) + 2\frac{\partial a^{ij}_{\sigma\nu}}{\partial y^\kappa_q}\frac{\partial y^\sigma}{\partial x^i}\frac{\partial y^\nu}{\partial x^j} \\
& +2\frac{\partial b^{qij}_{\sigma\kappa}}{\partial x^j}\frac{\partial y^\sigma}{\partial x^i} + 2\left(\frac{\partial b^{qij}_{\sigma\kappa}}{\partial y^\nu}\right)_{Alt(\nu\sigma)} \frac{\partial y^\nu}{\partial x^i}\frac{\partial y^\sigma}{\partial x^j} + 4\left(\frac{\partial b^{kij}_{\sigma\nu}}{\partial y^\kappa_q}\right)_{Alt((\kappa q)(\nu k))} \frac{\partial y^\sigma}{\partial x^i}\frac{\partial y^\nu_k}{\partial x^j} \\
& +2\frac{\partial b^{qij}_{\sigma\kappa}}{\partial p^k_\nu}\frac{\partial p^k_\nu}{\partial x^i}\frac{\partial y^\sigma}{\partial x^j} + 2\frac{\partial b^{qij}_{\sigma\kappa}}{\partial p^{kl}_\nu}\frac{\partial p^{kl}_\nu}{\partial x^i}\frac{\partial y^\sigma}{\partial x^j} + 2\frac{\partial b^{qij}_{\sigma\kappa}}{\partial z^M}\frac{\partial z^M}{\partial x^i}\frac{\partial y^\sigma}{\partial x^j} \\
& +2\frac{\partial c^{klji}_{\sigma\nu}}{\partial y^\kappa_q}\frac{\partial y^\sigma}{\partial x^i} \left(\frac{\partial y^\nu_{kl}}{\partial p^q_\beta}\frac{\partial p^q_\beta}{\partial x^j} + \frac{\partial y^\nu_{kl}}{\partial p^{qr}_\beta}\frac{\partial p^{qr}_\beta}{\partial x^j} + 2\frac{\partial y^\nu_{kl}}{\partial z^B}\frac{\partial z^B}{\partial x^j} \right)
\end{aligned} \tag{A2}
$$

$$
\begin{aligned}
\frac{\partial H}{\partial p^q_\kappa} &= \frac{\partial y^\kappa}{\partial x^q} + 2\frac{\partial c^{klji}_{\sigma\nu}}{\partial x^i}\frac{\partial y^\nu_{kl}}{\partial p^q_\kappa} y^\sigma_j + 2\frac{\partial c^{klji}_{\sigma\nu}}{\partial y^\beta}\frac{\partial y^\nu_{kl}}{\partial p^q_\kappa}\frac{\partial y^\beta}{\partial x^i} y^\sigma_j + 2\frac{\partial c^{klji}_{\sigma\nu}}{\partial y^\beta_r}\frac{\partial y^\nu_{kl}}{\partial p^q_\kappa}\frac{\partial y^\beta_r}{\partial x^i} y^\sigma_j \\
&+2\frac{\partial c^{klji}_{\sigma\nu}}{\partial y^\beta_r}\frac{\partial y^\nu_{kl}}{\partial p^q_\kappa}\frac{\partial p^r_\beta}{\partial x^i} y^\sigma_j + +2\frac{\partial c^{klji}_{\sigma\nu}}{\partial p^q_\kappa} y^\sigma_j \left(\frac{\partial y^\nu_{kl}}{\partial p^q_\beta}\frac{\partial p^q_\beta}{\partial x^i} + \frac{\partial y^\nu_{kl}}{\partial p^{qr}_\beta}\frac{\partial p^{qr}_\beta}{\partial x^i} + \frac{\partial y^\nu_{kl}}{\partial z^B}\frac{\partial z^B}{\partial x^i}\right) \\
&+2\frac{\partial c^{klji}_{\sigma\nu}}{\partial p^{rs}_\beta}\frac{\partial y^\nu_{kl}}{\partial p^q_\kappa}\frac{\partial p^{rs}_\beta}{\partial x^i} y^\sigma_j + 2\frac{\partial c^{klji}_{\sigma\nu}}{\partial z^M}\frac{\partial y^\nu_{kl}}{\partial p^q_\kappa}\frac{\partial z^M}{\partial x^i} y^\sigma_j + 2c^{klji}_{\sigma\nu}\frac{\partial y^\nu_{kl}}{\partial p^q_\kappa}\frac{\partial y^\sigma_j}{\partial x^i} \\
&+2\frac{\partial a^{ij}_{\sigma\nu}}{\partial p^q_\kappa}\frac{\partial y^\sigma}{\partial x^i}\frac{\partial y^\nu}{\partial x^j} + 2\frac{\partial b^{kij}_{\sigma\nu}}{\partial p^q_\kappa}\frac{\partial y^\sigma}{\partial x^i}\frac{\partial y^\nu_k}{\partial x^j} + 2\left(\frac{\partial c^{klij}_{\sigma\nu}}{\partial y^\beta}\right)_{Alt(\beta\nu)}\frac{\partial y^\beta}{\partial x^i}\frac{\partial y^\sigma}{\partial x^j}\frac{\partial y^\nu_{kl}}{\partial p^q_\kappa} \\
&+2\frac{\partial c^{klji}_{\sigma\nu}}{\partial x^j}\frac{\partial y^\nu_{kl}}{\partial p^q_\kappa}\frac{\partial y^\sigma}{\partial x^i} + 2\frac{\partial c^{klij}_{\sigma\nu}}{\partial p^q_\kappa}\frac{\partial y^\sigma}{\partial x^i}\left(\frac{\partial y^\nu_{kl}}{\partial p^q_\beta}\frac{\partial p^q_\beta}{\partial x^j} + \frac{\partial y^\nu_{kl}}{\partial p^{qr}_\beta}\frac{\partial p^{qr}_\beta}{\partial x^j} + \frac{\partial y^\nu_{kl}}{\partial z^B}\frac{\partial z^B}{\partial x^j}\right) \\
&+2\frac{\partial c^{klij}_{\sigma\nu}}{\partial y^\beta_r}\frac{\partial y^\nu_{kl}}{\partial p^q_\kappa}\frac{\partial y^\beta_r}{\partial x^i}\frac{\partial y^\sigma}{\partial x^j} + 2\frac{\partial c^{klij}_{\sigma\nu}}{\partial p^r_\beta}\frac{\partial y^\nu_{kl}}{\partial p^q_\kappa}\frac{\partial p^r_\beta}{\partial x^i}\frac{\partial y^\sigma}{\partial x^j} + 2\frac{\partial c^{klij}_{\sigma\nu}}{\partial p^{rs}_\beta}\frac{\partial y^\nu_{kl}}{\partial p^q_\kappa}\frac{\partial p^{rs}_\beta}{\partial x^i}\frac{\partial y^\sigma}{\partial x^j} \\
&+2\frac{\partial c^{klij}_{\sigma\nu}}{\partial z^M}\frac{\partial y^\nu_{kl}}{\partial p^q_\kappa}\frac{\partial z^M}{\partial x^i}\frac{\partial y^\sigma}{\partial x^j}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial H}{\partial p^{qr}_\kappa} &= \frac{\partial y^\kappa_q}{\partial x^r} + 2\frac{\partial c^{klji}_{\sigma\nu}}{\partial x^i}\frac{\partial y^\nu_{kl}}{\partial p^{qr}_\kappa} y^\sigma_j + 2\frac{\partial c^{klji}_{\sigma\nu}}{\partial y^\beta}\frac{\partial y^\nu_{kl}}{\partial p^{qr}_\kappa}\frac{\partial y^\beta}{\partial x^i} y^\sigma_j + 2\frac{\partial c^{klji}_{\sigma\nu}}{\partial y^\beta_s}\frac{\partial y^\nu_{kl}}{\partial p^{qr}_\kappa}\frac{\partial y^\beta}{\partial x^i} y^\sigma_j \\
&+2\frac{\partial c^{klji}_{\sigma\nu}}{\partial p^s_\beta}\frac{\partial y^\nu_{kl}}{\partial p^{qr}_\kappa}\frac{\partial p^s_\beta}{\partial x^i} y^\sigma_j + 2\frac{\partial c^{klji}_{\sigma\nu}}{\partial p^{st}_\beta}\frac{\partial y^\nu_{kl}}{\partial p^{qr}_\kappa}\frac{\partial p^{st}_\beta}{\partial x^i} y^\sigma_j \\
&+2\frac{\partial c^{klij}_{\sigma\nu}}{\partial p^{qr}_\kappa}\frac{\partial y^\sigma}{\partial x^i}\left(\frac{\partial y^\nu_{kl}}{\partial p^s_\beta}\frac{\partial p^s_\beta}{\partial x^i} + \frac{\partial y^\nu_{kl}}{\partial p^{st}_\beta}\frac{\partial p^{st}_\beta}{\partial x^i} + \frac{\partial y^\nu_{kl}}{\partial z^B}\frac{\partial z^B}{\partial x^i}\right) y^\sigma_j \\
&+2\frac{\partial c^{klji}_{\sigma\nu}}{\partial z^M}\frac{\partial y^\nu_{kl}}{\partial p^{qr}_\kappa}\frac{\partial z^M}{\partial x^i} y^\sigma_j + 2c^{klji}_{\sigma\nu}\frac{\partial y^\sigma_j}{\partial x^i}\frac{\partial y^\nu_{kl}}{\partial p^{qr}_\kappa} + 2\frac{\partial a^{ij}_{\sigma\nu}}{\partial p^{qr}_\kappa}\frac{\partial y^\nu}{\partial x^i}\frac{\partial y^\nu_k}{\partial x^j} \\
&+2\frac{\partial b^{kij}_{\sigma\nu}}{\partial p^{qr}_\kappa}\frac{\partial y^\sigma}{\partial x^i}\frac{\partial y^\nu}{\partial x^j} + 2\frac{\partial c^{klji}_{\sigma\nu}}{\partial x^j}\frac{\partial y^\sigma}{\partial x^i}\frac{\partial y^\nu_{kl}}{\partial p^{qr}_\kappa} + 2\left(\frac{\partial c^{klij}_{\sigma\nu}}{\partial y^\beta}\right)_{Alt(\beta\sigma)}\frac{\partial y^\beta}{\partial x^i}\frac{\partial y^\sigma}{\partial x^j}\frac{\partial y^\nu_{kl}}{\partial p^{qr}_\kappa} \\
&+2\frac{\partial c^{klij}_{\sigma\nu}}{\partial y^\beta_s}\frac{\partial y^\nu_{kl}}{\partial p^{qr}_\kappa}\frac{\partial y^\beta_s}{\partial x^i}\frac{\partial y^\sigma}{\partial x^j} + 2\frac{\partial c^{klij}_{\sigma\nu}}{\partial p^s_\beta}\frac{\partial p^s_\beta}{\partial x^i}\frac{\partial y^\sigma}{\partial x^j}\frac{\partial y^\nu_{kl}}{\partial p^{qr}_\kappa} + 2\frac{\partial c^{klij}_{\sigma\nu}}{\partial p^{st}_\beta}\frac{\partial p^{st}_\beta}{\partial x^i}\frac{\partial y^\sigma}{\partial x^j}\frac{\partial y^\nu_{kl}}{\partial p^{qr}_\kappa} \\
&+2\frac{\partial c^{klij}_{\sigma\nu}}{\partial p^{qr}_\kappa}\frac{\partial y^\sigma}{\partial x^i}\left(\frac{\partial y^\nu_{kl}}{\partial p^s_\beta}\frac{\partial p^s_\beta}{\partial x^j} + \frac{\partial y^\nu_{kl}}{\partial p^{st}_\beta}\frac{\partial p^{st}_\beta}{\partial x^j} + \frac{\partial y^\nu_{kl}}{\partial z^B}\frac{\partial z^B}{\partial x^j}\right) \\
&+2\frac{\partial c^{klij}_{\sigma\nu}}{\partial z^M}\frac{\partial y^\nu_{kl}}{\partial p^{qr}_\kappa}\frac{\partial z^M}{\partial x^i}\frac{\partial y^\sigma}{\partial x^j}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial H}{\partial z^M} &= 2\frac{\partial c^{klji}_{\sigma\nu}}{\partial x^i}\frac{\partial y^\nu_{kl}}{\partial z^M} y^\sigma_j + 2\frac{\partial c^{klji}_{\sigma\nu}}{\partial y^\beta}\frac{\partial y^\nu_{kl}}{\partial z^M}\frac{\partial y^\beta}{\partial x^i} y^\sigma_j + 2\frac{\partial c^{klji}_{\sigma\nu}}{\partial y^\beta_s}\frac{\partial y^\nu_{kl}}{\partial z^M}\frac{\partial y^\beta}{\partial x^i} y^\sigma_j \\
&+2\frac{\partial c^{klji}_{\sigma\nu}}{\partial p^s_\beta}\frac{\partial y^\nu_{kl}}{\partial z^M}\frac{\partial p^s_\beta}{\partial x^i} y^\sigma_j + 2\frac{\partial c^{klji}_{\sigma\nu}}{\partial p^{st}_\beta}\frac{\partial y^\nu_{kl}}{\partial z^M}\frac{\partial p^{st}_\beta}{\partial x^i} y^\sigma_j \\
&+2\frac{\partial c^{klij}_{\sigma\nu}}{\partial z^M}\left(\frac{\partial y^\nu_{kl}}{\partial p^s_\beta}\frac{\partial p^s_\beta}{\partial x^i} + \frac{\partial y^\nu_{kl}}{\partial p^{st}_\beta}\frac{\partial p^{st}_\beta}{\partial x^i} + \frac{\partial y^\nu_{kl}}{\partial z^B}\frac{\partial z^B}{\partial x^i}\right) y^\sigma_j
\end{aligned}
$$

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