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Green's Relations on a Semigroup of Transformations with Restricted Range that Preserves an Equivalence Relation and a Cross-Section

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Abstract: Let $T(X,Y)$ be the semigroup consisting of all total transformations from X into a fixed nonempty subset Y of X. For an equivalence relation ρ on X, let $\hat{\rho}$ be the restriction of ρ on Y, R a cross-section of $Y/\hat{\rho}$ and define $T(X,Y,\rho,R)$ to be the set of all total transformations α from X into Y such that α preserves both ρ (if $(a,b) \in \rho$, then $(a\alpha, b\alpha) \in \rho$) and R (if $r \in R$, then $r\alpha \in R$). $T(X,Y,\rho,R)$ is then a subsemigroup of $T(X,Y)$. In this paper, we give descriptions of Green's relations on $T(X,Y,\rho,R)$, and these results extend the results on $T(X,Y)$ and $T(X,\rho,R)$ when taking ρ to be the identity relation and $Y = X$, respectively.

Keywords: transformation semigroup; Green's relations; equivalence relation; cross-section

MSC: 20M20

1. Introduction

Let X be a nonempty set and $T(X)$ denote the semigroup containing all full transformations from X into itself with the composition. It is well-known that $T(X)$ is a regular semigroup, as shown in Reference [1]. Various subsemigroups of $T(X)$ have been investigated in different years. One of the subsemigroups of $T(X)$ is related to an equivalence relation ρ on X and a cross-section R of the partition X/ρ (i.e., each ρ-class contains exactly one element of R), namely $T(X,\rho,R)$, which was first considered by Araújo and Konieczny in 2003 [2], and is defined by

$$T(X,\rho,R) = \{\alpha \in T(X) : R\alpha \subseteq R \text{ and } (a,b) \in \rho \Rightarrow (a\alpha, b\alpha) \in \rho\},$$

where $Z\alpha = \{z\alpha : z \in Z\}$. They studied automorphism groups of centralizers of idempotents. Moreover, they also determined Green's relations and described the regular elements of $T(X,\rho,R)$ in 2004 [3].

Let Y be a nonempty subset of the set X. Consider another subsemigroup of $T(X)$, which was first introduced by Symons [4] in 1975, called $T(X,Y)$, defined by

$$T(X,Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\},$$

when $X\alpha$ denotes the image of α. He described all the automorphisms of $T(X,Y)$ and also determined when $T(X_1,Y_1)$ is isomorphic to $T(X_2,Y_2)$. In 2009, Sanwong, Singha and Sullivan [5] described all the maximal and minimal congruences on $T(X,Y)$. Later, in Reference [6], Sanwong and Sommanee studied other algebraic properties of $T(X,Y)$. They gave necessary and sufficient conditions for $T(X,Y)$

to be regular and also determined Green's relations on $T(X,Y)$. Furthermore, they obtained a class of maximal inverse subsemigroups of $T(X,Y)$ and proved that the set

$$F(X,Y) = \{\alpha \in T(X,Y) : X\alpha \subseteq Y\alpha\}$$

contains all regular elements in $T(X,Y)$, and is the largest regular subsemigroup of $T(X,Y)$.

From now on, we study the subsemigroup $T(X,Y,\rho,R)$ of $T(X,Y)$ defined by

$$T(X,Y,\rho,R) = \{\alpha \in T(X,Y) : R\alpha \subseteq R \text{ and } (a,b) \in \rho \Rightarrow (a\alpha, b\alpha) \in \rho\},$$

where ρ is an equivalence relation on X and R is a cross-section of the partition $Y/\hat{\rho}$ in which $\hat{\rho} = \rho \cap (Y \times Y)$. If $Y = X$, then $T(X,Y,\rho,R) = T(X,\rho,R)$; and if ρ is the identity relation, then $T(X,Y,\rho,R) = T(X,Y)$, so we may regard $T(X,Y,\rho,R)$ as a generalization of $T(X,\rho,R)$ and $T(X,Y)$.

Green's relations play a role in semigroup theory, and the aim of this paper is to characterize Green's relations on $T(X,Y,\rho,R)$. As consequences, we obtain Green's relations on $T(X,\rho,R)$ and $T(X,Y)$ as corollaries.

2. Preliminaries and Notations

For any semigroup S, let S^1 be a semigroup obtained from S by adjoining an identity if S has no identity and letting $S^1 = S$ if it already contains an identity. Green's relations of S are equivalence relations on the set S which were first defined by Green. According to such definitions, we define the $\mathcal{L}$-relation as follows. For any $a, b \in S$,

$$a\mathcal{L}b \text{ if and only if } S^1 a = S^1 b,$$

or equivalently, $a\mathcal{L}b$ if and only if $a = xb$ and $b = ya$ for some $x, y \in S^1$.

Furthermore, we dually define the $\mathcal{R}$-relation as follows.

$$a\mathcal{R}b \text{ if and only if } aS^1 = bS^1,$$

or equivalently, $a\mathcal{R}b$ if and only if $a = bx$ and $b = ay$ for some $x, y \in S^1$.

Moreover, we define the $\mathcal{J}$-relation as follows.

$$a\mathcal{J}b \text{ if and only if } S^1 a S^1 = S^1 b S^1,$$

or equivalently, $a\mathcal{J}b$ if and only if $a = xby$ and $b = uav$ for some $x, y, u, v \in S^1$.

Finally, we define $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$, where $\circ$ is the composition of relations. Since the relations $\mathcal{L}$ and $\mathcal{R}$ commute, it follows that $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.

In this paper, we write functions on the right; in particular, this means that for a composition $\alpha\beta$, α is applied first. Furthermore, the cardinality of a set A is denoted by $|A|$.

For each $\alpha \in T(X)$, we denote by $\ker(\alpha)$ the *kernel* of α, the set of ordered pairs in $X \times X$ having the same image under α, that is,

$$\ker(\alpha) = \{(a,b) \in X \times X : a\alpha = b\alpha\}.$$

Moreover, the symbol $\pi(\alpha)$ denotes the partition of X induced by the map α, namely

$$\pi(\alpha) = \{x\alpha^{-1} : x \in X\alpha\}.$$

We observe that $\ker(\alpha)$ is an equivalence relation on X in which the partition $X/\ker(\alpha)$ and $\pi(\alpha)$ coincide. Moreover, for all $\alpha, \beta \in T(X)$, we have $\ker(\alpha) = \ker(\beta)$ if and only if $\pi(\alpha) = \pi(\beta)$.

In addition, if ρ is an equivalence relation on the set X and $a, b \in X$, we sometimes write $a\,\rho\,b$ instead of $(a,b) \in \rho$, and define $a\rho$ to be the equivalence class that contains a, that is, $a\rho = \{b \in X : b\,\rho\,a\}$.

For the subsemigroup $T(X,Y,\rho,R)$ of $T(X)$ where ρ is an equivalence relation on X, Y is a nonempty subset of X and R is a cross-section of $Y/\hat{\rho}$ in which $\hat{\rho} = \rho \cap (Y \times Y)$, we see that if $a \in X$ and $a\rho \cap Y \neq \varnothing$, then there exists a unique $r \in R$ such that $a\,\rho\,r$, and we denote this element by r_a. Furthermore, we observe that $F(X,Y) \cap T(X,Y,\rho,R)$ contains all constant maps whose images belong to R. This implies that $F(X,Y) \cap T(X,Y,\rho,R)$ is a subsemigroup of $T(X,Y,\rho,R)$, which will be denoted by F.

An element a in a semigroup S is said to be *regular* if there exists $x \in S$ such that $a = axa$; and S is a *regular semigroup* if every element of S is regular.

In general, $T(X,Y,\rho,R)$ is not a regular semigroup, so we cannot apply Hall's Theorem to find the $\mathcal{L}$-relation and the $\mathcal{R}$-relation on $T(X,Y,\rho,R)$.

Now, we give an example of a non-regular element in $T(X,Y,\rho,R)$. Let $X = \{1,2,3,4,5\}$, $Y = \{3,4,5\}$, $X/\rho = \{\{1,2\},\{3,4,5\}\}$, $Y/\hat{\rho} = \{\{3,4,5\}\}$ and $R = \{3\}$. Define $\alpha \in T(X,Y,\rho,R)$ by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 3 & 5 & 5 \end{pmatrix}.$$

Suppose that α is regular. Then $\alpha = \alpha\beta\alpha$ for some $\beta \in T(X,Y,\rho,R)$. We see that $4 = 1\alpha = 1(\alpha\beta\alpha) = (4\beta)\alpha$, which implies that $1 = 4\beta \in Y$, a contradiction.

Throughout this paper, the set X we study can be a finite or an infinite set. For convenience, we will denote $T(X,Y,\rho,R)$ by T.

3. Green's Relations on $T(X,Y,\rho,R)$

Unlike $T(X,\rho,R)$, in general T has no identity, as shown in the following example.

Example 1. *Let* $X = \{1,2,3,4,5,6\}, Y = \{1,3\}$, $X/\rho = \{\{1,2\},\{3,4\},\{5,6\}\}$, $Y/\hat{\rho} = \{\{1\},\{3\}\}$ *and* $R = \{1,3\}$. *Suppose that* ε *is an identity element in* T. *Consider* $\alpha \in T$ *defined by*

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 3 & 3 \end{pmatrix}.$$

We see that $(5\varepsilon)\alpha = 5(\varepsilon\alpha) = 5\alpha = 3$, *which implies that* $5\varepsilon \in \{5,6\}$. *This leads to a contradiction, since both* 5 *and* 6 *are not in* Y.

Therefore, we use the semigroup T with identity adjoined, given by T^1, in studying its Green's relations.

From now on, the notation L_α $(R_\alpha, H_\alpha, D_\alpha)$ denote the set of all elements of T which are $\mathcal{L}$-related ($\mathcal{R}$-related, $\mathcal{H}$-related, $\mathcal{D}$-related) to α, where $\alpha \in T$.

Let $\mathscr{A}$ and $\mathscr{B}$ be families of sets. If for each set $A \in \mathscr{A}$ there is a set $B \in \mathscr{B}$ such that $A \subseteq B$, we say that $\mathscr{A}$ refines $\mathscr{B}$, denoted by $\mathscr{A} \hookrightarrow \mathscr{B}$.

In what follows, most of the notation used are taken from Reference [3]. For each $\alpha \in T$, we denote by $\blacktriangledown\alpha$ the family $\{(x\rho)\alpha : x \in X\}$ and $\blacktriangledown^{Y}\alpha$ the family $\{(r\hat{\rho})\alpha : r \in R\}$. Furthermore, we define $\overline{\blacktriangledown}\alpha = \{(x\rho)\alpha^{-1} : x \in X \text{ and } (x\rho)\alpha^{-1} \neq \varnothing\}$. In fact, we see that $\overline{\blacktriangledown}\alpha = \{(r\hat{\rho})\alpha^{-1} : r \in R \text{ and } (r\hat{\rho})\alpha^{-1} \neq \varnothing\}$.

The following example describes the above notation.

Example 2. *Let* $X = \{1,2,3,4,5,6,7,8\}, Y = \{1,2,3,4,5\}, X/\rho = \{\{1,2,3\},\{4,5,6\},\{7,8\}\}$, $Y/\hat{\rho} = \{\{1,2,3\},\{4,5\}\}$ *and* $R = \{1,4\}$. *Define* $\alpha \in T$ *by*

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 5 & 1 & 2 & 3 & 2 & 3 \end{pmatrix}.$$

It follows that

$$\blacktriangledown\alpha = \{(x\rho)\alpha : x \in X\}, \qquad \overset{Y}{\blacktriangledown}\alpha = \{(r\hat{\rho})\alpha : r \in R\},$$
$$= \{\{1,2,3\}\alpha, \{4,5,6\}\alpha, \{7,8\}\alpha\}, \qquad = \{\{1,2,3\}\alpha, \{4,5\}\alpha\},$$
$$= \{\{4,5\}, \{1,2,3\}, \{2,3\}\}, \qquad = \{\{4,5\}, \{1,2\}\},$$

and

$$\begin{aligned}\overline{\blacktriangledown}\alpha &= \{(r\hat{\rho})\alpha^{-1} : r \in R \text{ and } (r\hat{\rho})\alpha^{-1} \neq \varnothing\},\\ &= \{\{1,2,3\}\alpha^{-1}, \{4,5\}\alpha^{-1}\},\\ &= \{\{4,5,6,7,8\}, \{1,2,3\}\}.\end{aligned}$$

3.1. $\mathcal{L}$-Relation and $\mathcal{R}$-Relation

We begin with characterizing the Green's $\mathcal{L}$-relation on T by using the idea of the proof for the $\mathcal{L}$-relation on $T(X,\rho,R)$ (see Reference [3] [Lemma 2.4]) with the idea of restricted range concerned.

The following example shows why the restricted range is involved.

Example 3. *Let* $X = \{1,2,3,4,5,6,7,8,9,10\}$, $Y = \{1,2,3,4,5,6,7\}$, $X/\rho = \{\{1,2,3\},\{4,5\},\{6,7,8\},\{9,10\}\}$, $Y/\hat{\rho} = \{\{1,2,3\},\{4,5\},\{6,7\}\}$ *and* $R = \{1,4,6\}$. *Define* $\alpha, \beta_1, \beta_2, \gamma \in T$ *as follows.*

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 2 & 6 & 7 & 1 & 2 & 2 & 7 & 7 \end{pmatrix}, \beta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 5 & 1 & 3 & 2 & 6 & 7 \end{pmatrix},$$

$$\beta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 2 & 6 & 7 & 4 & 5 & 5 & 3 & 3 \end{pmatrix}, \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 2 & 4 & 5 & 1 & 2 & 2 & 5 & 5 \end{pmatrix}.$$

We see that $\alpha \neq \gamma'\beta_1$ *for all* $\gamma' \in T$, *for if* $\alpha = \gamma'\beta_1$ *for some* $\gamma' \in T$, *then* $7 = 9\alpha = 9\gamma'\beta_1$, *and thus* $10 = 9\gamma' \in Y$, *a contradiction. However,* $\blacktriangledown\alpha = \{\{1,2\},\{6,7\},\{7\}\} \hookrightarrow \{\{1,2,3\},\{4,5\},\{6,7\}\} = \blacktriangledown\beta_1$. *But* $\alpha = \gamma\beta_2$ *since* $\blacktriangledown\alpha \hookrightarrow \overset{Y}{\blacktriangledown}\beta_2 = \{\{1,2\},\{6,7\},\{4,5\}\}$.

Theorem 1. *Let* $\alpha, \beta \in T$. *Then* $\alpha = \gamma\beta$ *for some* $\gamma \in T^1$ *if and only if* $\alpha = \beta$ *or* $\blacktriangledown\alpha \hookrightarrow \overset{Y}{\blacktriangledown}\beta$. *Consequently,* $\alpha\mathcal{L}\beta$ *if and only if* $\alpha = \beta$; *or* $\blacktriangledown\alpha \hookrightarrow \overset{Y}{\blacktriangledown}\beta$ *and* $\blacktriangledown\beta \hookrightarrow \overset{Y}{\blacktriangledown}\alpha$.

Proof. Assume that $\alpha = \gamma\beta$ for some $\gamma \in T^1$. Suppose that $\alpha \neq \beta$. Thus $\gamma \neq 1$. We prove that $\blacktriangledown\alpha \hookrightarrow \overset{Y}{\blacktriangledown}\beta$. Let $A \in \blacktriangledown\alpha$. Then $A = (x\rho)\alpha = ((x\rho)\gamma)\beta$ for some $x \in X$ and $(x\rho)\gamma \subseteq r\hat{\rho}$ for some $r \in R$. Therefore $A \subseteq (r\hat{\rho})\beta \in \overset{Y}{\blacktriangledown}\beta$, and thus, $\blacktriangledown\alpha \hookrightarrow \overset{Y}{\blacktriangledown}\beta$.

Conversely, assume that $\alpha = \beta$ or $\blacktriangledown\alpha \hookrightarrow \overset{Y}{\blacktriangledown}\beta$. If $\alpha = \beta$, then $\alpha = 1\beta$, where $\gamma = 1 \in T^1$. For the case $\blacktriangledown\alpha \hookrightarrow \overset{Y}{\blacktriangledown}\beta$, we define γ on each ρ-class as follows. Let $x\rho \in X/\rho$. Then $(x\rho)\alpha \subseteq (r\hat{\rho})\beta \subseteq s\hat{\rho}$ for some $r, s \in R$ (since $\blacktriangledown\alpha \hookrightarrow \overset{Y}{\blacktriangledown}\beta$). So, for each $a \in x\rho$, we choose $b_a \in r\hat{\rho}$ such that $a\alpha = b_a\beta$ (if $a = t \in R$, we choose $b_a = r$ since $t\alpha = s = r\beta$) and define $a\gamma = b_a$. From $b_a \in r\hat{\rho}$ for all $a \in x\rho$ we obtain $(x\rho)\gamma \subseteq r\hat{\rho}$. By the definition of γ, $t\gamma = b_t = r$. Since $x\rho$ is arbitrary, we conclude that $\gamma \in T$. To see that $\alpha = \gamma\beta$, let $a \in X$. Then $a\gamma\beta = (a\gamma)\beta = b_a\beta = a\alpha$, and so $\alpha = \gamma\beta$. □

If we replace Y with X in Theorem 1, then $T = T(X,\rho,R)$, and $\overset{Y}{\blacktriangledown}\alpha = \blacktriangledown\alpha$ for all $\alpha \in T$. Therefore, we have the $\mathcal{L}$-relation on $T(X,\rho,R)$.

Corollary 1. [3] [Theorem 2.5] *Let* $\alpha, \beta \in T(X,\rho,R)$. *Then* $\alpha\mathcal{L}\beta$ *if and only if* $\blacktriangledown\alpha \hookrightarrow \blacktriangledown\beta$ *and* $\blacktriangledown\beta \hookrightarrow \blacktriangledown\alpha$.

Similarly to $T(X,Y)$, there are two types of $\mathcal{L}$-classes on T. In order to describe these $\mathcal{L}$-classes, the following lemma is needed.

Lemma 1. *Let $\alpha, \beta \in T$ be such that $\blacktriangledown\alpha \hookrightarrow \blacktriangledown^{Y}\beta$ and $\blacktriangledown\beta \hookrightarrow \blacktriangledown^{Y}\alpha$. Then $\alpha, \beta \in F$ and $X\alpha = X\beta$.*

Proof. For each $a\alpha \in X\alpha$, we have $a\alpha \in (a\rho)\alpha \subseteq (r\hat{\rho})\beta$ for some $r \in R$, since $\blacktriangledown\alpha \hookrightarrow \blacktriangledown^{Y}\beta$. Thus, $X\alpha \subseteq Y\beta$. Similarly, $X\beta \subseteq Y\alpha$, since $\blacktriangledown\beta \hookrightarrow \blacktriangledown^{Y}\alpha$. It follows that $X\alpha \subseteq Y\beta \subseteq X\beta \subseteq Y\alpha \subseteq X\alpha$, and thus $\alpha, \beta \in F$ and $X\alpha = X\beta$. $\square$

Corollary 2. *For $\alpha \in T$, the following statements hold.*

(i) *If $\alpha \in F$, then $L_\alpha = \{\alpha\} \cup \{\beta \in F : \blacktriangledown\alpha \hookrightarrow \blacktriangledown^{Y}\beta$ and $\blacktriangledown\beta \hookrightarrow \blacktriangledown^{Y}\alpha\}$.*

(ii) *If $\alpha \in T \backslash F$, then $L_\alpha = \{\alpha\}$.*

Proof. Let α be any element in T and let $\beta \in L_\alpha$. Then $\alpha \mathcal{L} \beta$, which implies that $\alpha = \beta$; or $\blacktriangledown\alpha \hookrightarrow \blacktriangledown^{Y}\beta$ and $\blacktriangledown\beta \hookrightarrow \blacktriangledown^{Y}\alpha$ by Theorem 1.

(i) Assume that $\alpha \in F$. It is clear by Theorem 1 that $\{\alpha\} \cup \{\beta \in F : \blacktriangledown\alpha \hookrightarrow \blacktriangledown^{Y}\beta$ and $\blacktriangledown\beta \hookrightarrow \blacktriangledown^{Y}\alpha\} \subseteq L_\alpha$. To prove the other containment, we consider when $\beta \neq \alpha$. Since $\beta \mathcal{L} \alpha$, we obtain $\blacktriangledown\alpha \hookrightarrow \blacktriangledown^{Y}\beta$ and $\blacktriangledown\beta \hookrightarrow \blacktriangledown^{Y}\alpha$. By Lemma 1, we obtain $\beta \in F$. Thus, we have $L_\alpha \subseteq \{\alpha\} \cup \{\beta \in F : \blacktriangledown\alpha \hookrightarrow \blacktriangledown^{Y}\beta$ and $\blacktriangledown\beta \hookrightarrow \blacktriangledown^{Y}\alpha\}$.

(ii) Assume that $L_\alpha \neq \{\alpha\}$. Then there is $\gamma \neq \alpha$ such that $\gamma \mathcal{L} \alpha$, so $\blacktriangledown\alpha \hookrightarrow \blacktriangledown^{Y}\gamma$ and $\blacktriangledown\gamma \hookrightarrow \blacktriangledown^{Y}\alpha$. By Lemma 1, we get $\alpha \in F$. $\square$

As a direct consequence of Corollary 2, we obtain the $\mathcal{L}$-relation on $T(X, Y)$ as follows.

Corollary 3. [6] [Theorem 3.2] *For $\alpha \in T(X, Y)$, the following statements hold.*

(i) *If $\alpha \in F(X, Y)$, then $L_\alpha = \{\beta \in F(X, Y) : X\alpha = X\beta\}$.*

(ii) *If $\alpha \in T(X, Y) \backslash F(X, Y)$, then $L_\alpha = \{\alpha\}$.*

Proof. If we replace ρ with the identity relation in Corollary 2, then $T = T(X, Y)$, $F = F(X, Y) \cap T(X, Y) = F(X, Y)$ and $Y = R$. Therefore, (ii) holds. To see that (i) holds, it suffices to prove that for $\alpha, \beta \in F$,

$$\blacktriangledown\alpha \hookrightarrow \blacktriangledown^{Y}\beta \text{ and } \blacktriangledown\beta \hookrightarrow \blacktriangledown^{Y}\alpha \text{ if and only if } X\alpha = X\beta.$$

By Lemma 1, we have the "only if" part of the above statement. Now, if $\alpha, \beta \in F$ and $X\alpha = X\beta$, then for each $x\alpha \in X\alpha$ there exist $y \in X$ and $r \in Y$ such that $x\alpha = y\beta = r\beta$, since $X\alpha \subseteq X\beta$ and $\beta \in F$. Hence, $\blacktriangledown\alpha = \{\{x\alpha\} : x \in X\} \hookrightarrow \{\{r\beta\} : r \in Y\} = \blacktriangledown^{Y}\beta$. Similarly, by using $X\beta \subseteq X\alpha$ and $\alpha \in F$, we obtain $\blacktriangledown\beta \hookrightarrow \blacktriangledown^{Y}\alpha$. $\square$

As we know, $\alpha \mathcal{R} \beta$ on $T(X, Y)$ (or $T(X, \rho, R)$) if and only if $\ker(\alpha) = \ker(\beta)$. However, for the semigroup T, this is true only on F (see Corollary 5). For α, β outside F, there are more terminologies involved.

The following example shows that there are $\alpha, \beta \in T$ with $\ker(\beta) \subseteq \ker(\alpha)$ but $\alpha \neq \beta\gamma$ for all $\gamma \in T$.

Example 4. *Considering α, β_1 and β_2 defined in Example 3, we see that $\ker(\beta_1) \subseteq \ker(\alpha)$ but $\alpha \neq \beta_1\gamma$ for all $\gamma \in T$, for if $\alpha = \beta_1\gamma$ for some $\gamma \in T$, then $7 = 9\alpha = 9\beta_1\gamma = 6\gamma \in R$, a contradiction. Moreover, we have $\overline{\blacktriangledown}\beta_1 = \{\{1,2,3,6,7,8\}, \{4,5\}, \{9,10\}\} \hookrightarrow \{\{1,2,3,6,7,8\}, \{4,5,9,10\}\} = \overline{\blacktriangledown}\alpha$ but $(R\beta_1^{-1})\alpha = \{1,4,6,9\}\alpha \nsubseteq R$. In the same way, $\ker(\beta_2) \subseteq \ker(\alpha)$ but $\alpha \neq \beta_2\gamma'$ for all $\gamma' \in T$, for if $\alpha = \beta_2\gamma'$ for some $\gamma' \in T$, then $(1\gamma', 3\gamma') = (1\beta_2\gamma', 9\beta_2\gamma') = (1\alpha, 9\alpha) = (1,7) \notin \rho$, which is a contradiction. Furthermore, $(R\beta_2^{-1})\alpha \subseteq R$ but $\overline{\blacktriangledown}\beta_2 = \{\{1,2,3,9,10\}, \{4,5\}, \{6,7,8\}\} \not\hookrightarrow \overline{\blacktriangledown}\alpha$.*

The proof below is completely different from those for $T(X, Y)$ and $T(X, \rho, R)$, especially when proving the existence of such $\gamma \in T$.

Theorem 2. *Let $\alpha, \beta \in T$. Then $\alpha = \beta\gamma$ for some $\gamma \in T^1$ if and only if $\ker(\beta) \subseteq \ker(\alpha)$, $\overline{\blacktriangledown}\beta \hookrightarrow \overline{\blacktriangledown}\alpha$ and $(R\beta^{-1})\alpha \subseteq R$. Consequently, $\alpha \mathcal{R} \beta$ if and only if $\ker(\alpha) = \ker(\beta)$, $\overline{\blacktriangledown}\alpha = \overline{\blacktriangledown}\beta$ and $(R\beta^{-1})\alpha, (R\alpha^{-1})\beta \subseteq R$.*

Proof. Assume that $\alpha = \beta\gamma$ for some $\gamma \in T^1$. If $\gamma = 1$, then $\alpha = \beta$ and the theorem holds. Now, we prove for $\gamma \in T$. Let $a, b \in X$ be such that $a\beta = b\beta$. Then $a\alpha = a\beta\gamma = b\beta\gamma = b\alpha$. Thus, $\ker(\beta) \subseteq \ker(\alpha)$. For each $U \in \overline{\blacktriangledown}\beta$, $U\beta \subseteq r\hat{\rho}$ for some $r \in R$ and so $U\alpha = U\beta\gamma \subseteq (r\hat{\rho})\gamma \subseteq s\hat{\rho}$ for some $s \in R$. Thus, $U \subseteq (s\hat{\rho})\alpha^{-1} \in \overline{\blacktriangledown}\alpha$, which implies that $\overline{\blacktriangledown}\beta \hookrightarrow \overline{\blacktriangledown}\alpha$. Now, let $c \in (R\beta^{-1})\alpha$. Then $c = d\alpha$ and $d\beta = t$ for some $d \in X$ and $t \in R$. Hence, $c = d\alpha = d\beta\gamma = t\gamma \in R$, that is, $(R\beta^{-1})\alpha \subseteq R$.

Conversely, assume that $\ker(\beta) \subseteq \ker(\alpha), \overline{\blacktriangledown}\beta \hookrightarrow \overline{\blacktriangledown}\alpha$ and $(R\beta^{-1})\alpha \subseteq R$. Let $r_0 \in R$ be fixed, and define $\gamma \in T$ on each ρ-class as follows. Let $x\rho \in X/\rho$.

If $x\rho \cap X\beta = \varnothing$, then define $a\gamma = r_0$ for all $a \in x\rho$. Therefore $(x\rho)\gamma = \{r_0\} \subseteq r_0\hat{\rho}$.

If $x\rho \cap X\beta \neq \varnothing$, then $x\rho \cap Y \neq \varnothing$. Let $x\rho \cap Y = r\hat{\rho}$ for some $r \in R$. We obtain $(r\hat{\rho})\beta^{-1} \neq \varnothing$. Since $\overline{\blacktriangledown}\beta \hookrightarrow \overline{\blacktriangledown}\alpha$, it follows that $(r\hat{\rho})\beta^{-1} \subseteq (s\hat{\rho})\alpha^{-1}$ for some $s \in R$. Now, let $a \in x\rho$ and consider two cases. If $a \notin X\beta$, then define $a\gamma = s \in s\hat{\rho}$. If $a \in X\beta$, then $a \in x\rho \cap Y = r\hat{\rho}$ and $a = b\beta$ for some $b \in X$. Thus, $b \in a\beta^{-1} \subseteq (r\hat{\rho})\beta^{-1} \subseteq (s\hat{\rho})\alpha^{-1}$ which implies $b\alpha \in s\hat{\rho}$. Now, we define $a\gamma = b\alpha \in s\hat{\rho}$ (this is well-defined since $\ker(\beta) \subseteq \ker(\alpha)$). We observe that $(x\rho)\gamma \subseteq s\hat{\rho}$. To see that $r\gamma = s$, we get by the definition of γ that $r\gamma = s$ if $r \notin X\beta$. For $r \in X\beta$, we have $r = c\beta$ for some $c \in X$ and then $c \in r\beta^{-1} \subseteq (r\hat{\rho})\beta^{-1} \subseteq (s\hat{\rho})\alpha^{-1}$, hence $c\alpha \in s\hat{\rho} \cap (R\beta^{-1})\alpha \subseteq R$ since $(R\beta^{-1})\alpha \subseteq R$. Thus, $r\gamma = c\alpha = s$.

To prove that $\alpha = \beta\gamma$, let $b \in X$. Then $b\beta \in X\beta$, and so $(b\beta)\rho \cap X\beta \neq \varnothing$. By the definition of γ, we obtain $b\beta\gamma = (b\beta)\gamma = b\alpha$. Therefore, $\alpha = \beta\gamma$. $\square$

The $\mathcal{R}$-relation on $T(X, Y)$ is as follows.

Corollary 4. [6] [Theorem 3.3] *Let $\alpha, \beta \in T(X, Y)$. Then $\alpha\mathcal{R}\beta$ if and only if $\pi(\alpha) = \pi(\beta)$.*

Proof. If ρ is the identity relation, then $T = T(X, Y)$, $F = F(X, Y)$ and $R = Y$. Moreover, $\overline{\blacktriangledown}\alpha = \{(r\hat{\rho})\alpha^{-1} : r \in Y \text{ and } (r\hat{\rho})\alpha^{-1} \neq \varnothing\} = \{r\alpha^{-1} : r \in X\alpha\} = \pi(\alpha)$ for all $\alpha \in T(X, Y)$. In addition, $(R\beta^{-1})\alpha \subseteq R$ always holds for all $\alpha, \beta \in T(X, Y)$ since $(R\beta^{-1})\alpha = (Y\beta^{-1})\alpha = X\alpha \subseteq Y = R$. Since we have $\ker(\alpha) = \ker(\beta)$ if and only if $\pi(\alpha) = \pi(\beta)$, it follows from Theorem 2 that $\alpha\mathcal{R}\beta$ if and only if $\pi(\alpha) = \pi(\beta)$. $\square$

As one might expect, there are two types of the $\mathcal{R}$-classes on T, the one that lies inside F and the other outside F. To see this, we need the two lemmas below.

Lemma 2. *Let $\alpha, \beta \in F$ and $\ker(\alpha) = \ker(\beta)$. Then the following statements hold.*

(i) $\overline{\blacktriangledown}\alpha = \overline{\blacktriangledown}\beta$.
(ii) $(R\beta^{-1})\alpha, (R\alpha^{-1})\beta \subseteq R$.

Proof. (i) Let $(r\hat{\rho})\alpha^{-1} \in \overline{\blacktriangledown}\alpha$. Then $(r\hat{\rho})\alpha^{-1} = \bigcup_{i \in I} A_i$, where A_i is a ρ-class such that $A_i\alpha \subseteq r\hat{\rho}$ for all $i \in I$. From $\alpha \in F$, there exists A_{i_0} such that $A_{i_0} \cap Y \neq \varnothing$. Therefore there is $s \in A_{i_0} \cap R$ and $s\alpha = r$. Thus, $s\beta = t$ for some $t \in R$, which implies that $A_{i_0}\beta \subseteq t\hat{\rho}$. We prove that $A_i\beta \subseteq t\hat{\rho}$ for all $i \in I$. Let $i \in I$.

If $A_i \cap Y \neq \varnothing$, then there exists $u \in A_i \cap R$ and $u\alpha = r$. It follows that $u\alpha = s\alpha$, and so $u\beta = s\beta = t$ since $\ker(\alpha) = \ker(\beta)$. Thus, $A_i\beta \subseteq t\hat{\rho}$.

If $A_i \cap Y = \varnothing$, then since $A_i\alpha \subseteq r\hat{\rho}$, there exists $a \in A_i$ such that $a\alpha \in r\hat{\rho}$. From $\alpha \in F$, we obtain that $b\alpha = a\alpha \in r\hat{\rho}$ for some $b \in Y$. Hence, $b \in (r\hat{\rho})\alpha^{-1} = \bigcup_{i \in I} A_i$, that is, $b \in A_j \cap Y \neq \varnothing$ for some $j \in I$, which implies $A_j\beta \subseteq t\hat{\rho}$. Since $\ker(\alpha) = \ker(\beta)$, we obtain that $a\beta = b\beta \in A_j\beta \subseteq t\hat{\rho}$, where $a \in A_i$, and it follows that $A_i\beta \subseteq t\hat{\rho}$.

Therefore, $A_i\beta \subseteq t\hat{\rho}$ for all $i \in I$, that is, $(r\hat{\rho})\alpha^{-1} = \bigcup_{i \in I} A_i \subseteq (t\hat{\rho})\beta^{-1}$ and $\overline{\blacktriangledown}\alpha \hookrightarrow \overline{\blacktriangledown}\beta$ as required. Similarly, since $\beta \in F$, we obtain $\overline{\blacktriangledown}\beta \hookrightarrow \overline{\blacktriangledown}\alpha$. Thus, $\overline{\blacktriangledown}\alpha = \overline{\blacktriangledown}\beta$.

(ii) Let $a \in (R\beta^{-1})\alpha$. Then $a = b\alpha$ and $b\beta = r$ for some $b \in X$ and $r \in R$. From $\beta \in F$, we get $b\beta = r = s\beta$ for some $s \in R$. Since $\ker(\alpha) = \ker(\beta)$, we obtain that $a = b\alpha = s\alpha \in R$, and thus $(R\beta^{-1})\alpha \subseteq R$. Similarly, $(R\alpha^{-1})\beta \subseteq R$. $\square$

Lemma 3. *Let $\alpha, \beta \in T$. If $\ker(\alpha) = \ker(\beta)$, then either both α and β are in F, or neither is in F.*

Proof. Assume that $\ker(\alpha) = \ker(\beta)$. Suppose that one of α and β is not in F. Without loss of generality, assume that $\alpha \notin F$. Then there exists $x \in X \backslash Y$ such that $x\alpha \neq y\alpha$ for all $y \in Y$. Thus, $(x, y) \notin \ker(\alpha)$, which implies that $x\beta \neq y\beta$ for all $y \in Y$. Hence, $Y\beta \subsetneq X\beta$, which leads to $\beta \notin F$. □

Using Theorem 2, Lemmas 2 and 3, we have the following corollary.

Corollary 5. *For $\alpha \in T$, the following statements hold.*

(i) *If $\alpha \in F$, then $R_\alpha = \{\beta \in F : \ker(\beta) = \ker(\alpha)\}$.*
(ii) *If $\alpha \in T \backslash F$, then $R_\alpha = \{\beta \in T \backslash F : \ker(\beta) = \ker(\alpha), \bar{\blacktriangledown}\beta = \bar{\blacktriangledown}\alpha$ and $(R\beta^{-1})\alpha, (R\alpha^{-1})\beta \subseteq R\}$.*

If $Y = X$, then $F = F(X, Y) \cap T = T(X) \cap T(X, \rho, R) = T(X, \rho, R)$, and so $T \backslash F = \varnothing$. Thus, Corollary 5 gives us a description of the $\mathcal{R}$-relation on $T(X, \rho, R)$.

Corollary 6. [3] [Theorem 2.3] *Let $\alpha, \beta \in T(X, \rho, R)$. Then $\alpha \mathcal{R} \beta$ if and only if $\ker(\alpha) = \ker(\beta)$.*

As direct consequences of Corollaries 2 and 5, we have the $\mathcal{H}$-relation on T as follows.

Corollary 7. *For $\alpha \in T$, the following statements hold.*

(i) *If $\alpha \in F$, then $H_\alpha = \{\alpha\} \cup \{\beta \in F : \blacktriangledown\alpha \hookrightarrow \blacktriangledown^{Y}\beta, \blacktriangledown\beta \hookrightarrow \blacktriangledown^{Y}\alpha$ and $\ker(\alpha) = \ker(\beta)\}$.*
(ii) *If $\alpha \in T \backslash F$, then $H_\alpha = \{\alpha\}$.*

3.2. $\mathcal{D}$-relation and $\mathcal{J}$-relation

Let $\phi : B \to C$ be a function from a set B to a set C. For a family $\mathscr{A}$ of subsets of B, $(\mathscr{A})\phi$ denotes the family $\{(A)\phi : A \in \mathscr{A}\}$ of subsets of C.

The main results used for characterizing the Green's $\mathcal{D}$-relation on T below are Corollaries 5 and 2. Moreover, the technique for defining such a function ϕ in (i) is taken from Reference [3] [Theorem 2.6].

Theorem 3. *For $\alpha \in T$, the following statements hold.*

(i) *If $\alpha \in F$, then $D_\alpha = \{\beta \in F : \ker(\beta) = \ker(\alpha)$; or there exists a bijection $\phi : X\alpha \to X\beta$ such that $(R \cap X\alpha)\phi \subseteq R$, $(\blacktriangledown\alpha)\phi \hookrightarrow \blacktriangledown^{Y}\beta$ and $\blacktriangledown\beta \hookrightarrow (\blacktriangledown^{Y}\alpha)\phi\}$.*
(ii) *If $\alpha \in T \backslash F$, then $D_\alpha = \{\beta \in T \backslash F : \ker(\beta) = \ker(\alpha), \bar{\blacktriangledown}\beta = \bar{\blacktriangledown}\alpha$ and $(R\beta^{-1})\alpha, (R\alpha^{-1})\beta \subseteq R\}$.*

Proof. Let α be any element in T and let $\beta \in D_\alpha$. Then $\alpha \mathcal{R} \gamma$ and $\gamma \mathcal{L} \beta$ for some $\gamma \in T$.

(i) Assume that $\alpha \in F$. By Corollary 5, $\gamma \in F$ and $\ker(\gamma) = \ker(\alpha)$. By Corollary 2, $\beta \in F$; also, $\beta = \gamma$ or $\blacktriangledown\beta \hookrightarrow \blacktriangledown^{Y}\gamma$, $\blacktriangledown\gamma \hookrightarrow \blacktriangledown^{Y}\beta$. If $\beta = \gamma$, then $\ker(\beta) = \ker(\alpha)$. Now, assume that $\blacktriangledown\beta \hookrightarrow \blacktriangledown^{Y}\gamma$ and $\blacktriangledown\gamma \hookrightarrow \blacktriangledown^{Y}\beta$. Define $\phi : X\alpha \to X\beta$ by $(a\alpha)\phi = a\gamma$. We have $a\gamma \in (a\rho)\gamma \subseteq (r\hat{\rho})\beta \subseteq X\beta$ for some $r \in R$, since $\blacktriangledown\gamma \hookrightarrow \blacktriangledown^{Y}\beta$, so $a\gamma \in X\beta$. Since $\ker(\gamma) = \ker(\alpha)$, we obtain that ϕ is well-defined and injective. To see that ϕ is surjective, let $b\beta \in X\beta$. Then $b\beta \in (b\rho)\beta \subseteq (s\hat{\rho})\gamma$ for some $s \in R$, since $\blacktriangledown\beta \hookrightarrow \blacktriangledown^{Y}\gamma$. It follows that $b\beta = c\gamma$ for some $c \in s\hat{\rho}$, and so $(c\alpha)\phi = c\gamma = b\beta$, hence ϕ is surjective. To show that $(R \cap X\alpha)\phi \subseteq R$, let $t \in R \cap X\alpha$. Since $t \in X\alpha$ and $\alpha \in F$, there exists $p \in R$ such that $t \in (p\hat{\rho})\alpha$, thus $t = p\alpha$ and $t\phi = (p\alpha)\phi = p\gamma \in R$. Moreover, by the definition of ϕ, $(\blacktriangledown\alpha)\phi = \blacktriangledown\gamma$ and $(\blacktriangledown^{Y}\alpha)\phi = \blacktriangledown^{Y}\gamma$. Hence, $(\blacktriangledown\alpha)\phi \hookrightarrow \blacktriangledown^{Y}\beta$ and $\blacktriangledown\beta \hookrightarrow (\blacktriangledown^{Y}\alpha)\phi$.

Conversely, assume that $\lambda \in F$. If $\ker(\lambda) = \ker(\alpha)$, then $\lambda \mathcal{R} \alpha$, and it follows that $\lambda \in D_\alpha$. If there exists a bijection $\phi : X\alpha \to X\lambda$ such that $(R \cap X\alpha)\phi \subseteq R$, $(\blacktriangledown\alpha)\phi \hookrightarrow \blacktriangledown^{Y}\lambda$ and $\blacktriangledown\lambda \hookrightarrow (\blacktriangledown^{Y}\alpha)\phi$, then we define $\gamma : X \to X$ by $a\gamma = (a\alpha)\phi$ for all $a \in X$. From $(R \cap X\alpha)\phi \subseteq R$, we obtain that $r\gamma = (r\alpha)\phi \in R$ for all $r \in R$. Thus, $R\gamma \subseteq R$. To show that $(x\rho)\gamma \subseteq s\hat{\rho}$ for some $s \in R$, considering $(x\rho)\gamma = ((x\rho)\alpha)\phi \in (\blacktriangledown\alpha)\phi$ and $(\blacktriangledown\alpha)\phi \hookrightarrow \blacktriangledown^{Y}\lambda$, we get $(x\rho)\gamma \subseteq (t\hat{\rho})\lambda \subseteq s\hat{\rho}$ for some $s, t \in R$. Thus, $(x\rho)\gamma \subseteq s\hat{\rho}$, that is, $\gamma \in T$. To see that $\gamma \in F$, let $a \in X$. Then $a\gamma = (a\alpha)\phi = (b\alpha)\phi = b\gamma$ for some $b \in Y$, since $\alpha \in F$. It follows that $\gamma \in F(X, Y) \cap T = F$. Since ϕ is an injective map, we obtain that $\ker(\gamma) = \ker(\alpha)$. By the definition of

γ, $\blacktriangledown\gamma = (\blacktriangledown\alpha)\phi$ and $\blacktriangledown^{Y}\gamma = (\blacktriangledown^{Y}\alpha)\phi$. Hence, $\blacktriangledown\gamma \hookrightarrow \blacktriangledown^{Y}\lambda$ and $\blacktriangledown\lambda \hookrightarrow \blacktriangledown^{Y}\gamma$. By Corollaries 5 and 2, $\alpha\mathcal{R}\gamma$ and $\gamma\mathcal{L}\lambda$. Therefore, $\lambda \in D_\alpha$.

(ii) Assume that $\alpha \in T\backslash F$. Corollaries 5 and 2 imply that $\beta = \gamma \in T\backslash F$. Thus, $\alpha\mathcal{R}\beta$. Again by Corollary 5, we have $\ker(\beta) = \ker(\alpha)$, $\overline{\blacktriangledown}\beta = \overline{\blacktriangledown}\alpha$ and $(R\beta^{-1})\alpha, (R\alpha^{-1})\beta \subseteq R$. Therefore, $D_\alpha \subseteq \{\beta \in T\backslash F : \ker(\beta) = \ker(\alpha), \overline{\blacktriangledown}\beta = \overline{\blacktriangledown}\alpha$ and $(R\beta^{-1})\alpha,\ (R\alpha^{-1})\beta \subseteq R\}$. The other containment is clear since $\mathcal{R} \subseteq \mathcal{D}$. $\square$

The two corollaries below are the $\mathcal{D}$-relations on $T(X,\rho,R)$ and $T(X,Y)$, respectively.

Corollary 8. [3] [Theorem 2.6] *Let $\alpha,\beta \in T(X,\rho,R)$. Then $\alpha\mathcal{D}\beta$ if and only if there is a bijection $\phi : X\alpha \to X\beta$ such that $(R \cap X\alpha)\phi \subseteq R$, $(\blacktriangledown\alpha)\phi \hookrightarrow \blacktriangledown\beta$ and $\blacktriangledown\beta \hookrightarrow (\blacktriangledown\alpha)\phi$.*

Proof. If we replace Y with X in Theorem 3, then $T\backslash F = \varnothing$. Therefore we have that: For $\alpha \in T(X,\rho,R)$, $D_\alpha = \{\beta \in T(X,\rho,R) : \ker(\beta) = \ker(\alpha)$; or there is a bijection $\phi : X\alpha \to X\beta$ such that $(R \cap X\alpha)\phi \subseteq R$, $(\blacktriangledown\alpha)\phi \hookrightarrow \blacktriangledown\beta$ and $\blacktriangledown\beta \hookrightarrow (\blacktriangledown\alpha)\phi\}$.

Now, we assert that $\ker(\beta) = \ker(\alpha)$ implies that there is a bijection $\phi : X\alpha \to X\beta$ such that $(R \cap X\alpha)\phi \subseteq R$, $(\blacktriangledown\alpha)\phi \hookrightarrow \blacktriangledown\beta$ and $\blacktriangledown\beta \hookrightarrow (\blacktriangledown\alpha)\phi$. Assume that $\ker(\beta) = \ker(\alpha)$ and define $\phi : X\alpha \to X\beta$ by $(a\alpha)\phi = a\beta$ for all $a \in X$. Then ϕ is a well-defined injective map, since $\ker(\beta) = \ker(\alpha)$. It is obvious that ϕ is surjective. By the definition of ϕ, $[(a\rho)\alpha]\phi = (a\rho)\beta$. Thus, $(\blacktriangledown\alpha)\phi \hookrightarrow \blacktriangledown\beta$ and $\blacktriangledown\beta \hookrightarrow (\blacktriangledown\alpha)\phi$. Finally, $(R \cap X\alpha)\phi \subseteq R\beta \subseteq R$. Hence, we have our assertion, and, therefore, $D_\alpha = \{\beta \in T(X,\rho,R)$: There is a bijection $\phi : X\alpha \to X\beta$ such that $(R \cap X\alpha)\phi \subseteq R$, $(\blacktriangledown\alpha)\phi \hookrightarrow \blacktriangledown\beta$ and $\blacktriangledown\beta \hookrightarrow (\blacktriangledown\alpha)\phi\}$, as required. $\square$

Corollary 9. [6] [Theorem 3.7] *For $\alpha \in T(X,Y)$, the following statements hold.*

(i) *If $\alpha \in F(X,Y)$, then $D_\alpha = \{\beta \in F(X,Y) : |X\beta| = |X\alpha|\}$.*
(ii) *If $\alpha \in T(X,Y)\backslash F(X,Y)$, then $D_\alpha = \{\beta \in T(X,Y)\backslash F(X,Y) : \pi(\beta) = \pi(\alpha)\}$.*

Proof. As in the proof of Corollary 4, if we replace ρ with the identity relation, then (ii) of Theorem 3 is as follows. If $\alpha \in T(X,Y)\backslash F(X,Y)$, then $D_\alpha = \{\beta \in T(X,Y)\backslash F(X,Y) : \pi(\beta) = \pi(\alpha)\}$.

Now, we claim that the conditions $\ker(\beta) = \ker(\alpha)$; or there is a bijection $\phi : X\alpha \to X\beta$ such that $(R \cap X\alpha)\phi \subseteq R$, $(\blacktriangledown\alpha)\phi \hookrightarrow \blacktriangledown^{Y}\beta$ and $\blacktriangledown\beta \hookrightarrow (\blacktriangledown^{Y}\alpha)\phi$ in (i) of Theorem 3 is equivalent to $|X\alpha| = |X\beta|$ for all $\alpha,\beta \in F(X,Y)$. It is clear that the above conditions imply $|X\alpha| = |X\beta|$. Now, let $\alpha,\beta \in F(X,Y)$ and $|X\alpha| = |X\beta|$. Then there is a bijection $\phi : X\alpha \to X\beta$ and $(R \cap X\alpha)\phi \subseteq X\beta \subseteq Y = R$. To see that the remaining conditions hold, we observe that $(Y\alpha)\phi = (X\alpha)\phi = X\beta = Y\beta$, since $\alpha,\beta \in F(X,Y)$. From $(X\alpha)\phi = Y\beta$ and ρ as the identity relation, we obtain $(\blacktriangledown\alpha)\phi = \{\{(x\alpha)\phi\} : x \in X\} = \{\{r\beta\} : r \in Y\} = \blacktriangledown^{Y}\beta$, hence, $(\blacktriangledown\alpha)\phi \hookrightarrow \blacktriangledown^{Y}\beta$. Similarly, from $(Y\alpha)\phi = X\beta$, we obtain $\blacktriangledown\beta \hookrightarrow (\blacktriangledown^{Y}\alpha)\phi$. Therefore, we have our claim. $\square$

To characterize the $\mathcal{J}$-relation on T, we need the terminology below. For each $\alpha \in T$, we define

$$R(\alpha) = \{r \in R : r\hat{\rho} \cap X\alpha \neq \varnothing\}.$$

The following example shows that $R(\alpha)$ is necessary.

Example 5. *Let $X = \{1,2,3,4,5,6,7,8\}$ and $Y = \{3,4,5,6,7,8\}$. Let*

$$X/\rho = \{\{1,2\},\{3,4\},\{5,6,7\},\{8\}\}, Y/\hat{\rho} = \{\{3,4\},\{5,6,7\},\{8\}\} \text{ and } R = \{3,5,8\}.$$

Define $\alpha,\beta \in T$ as follows:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 4 & 5 & 6 & 5 & 6 & 6 & 5 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 7 & 5 & 6 & 3 & 4 & 4 & 8 \end{pmatrix}.$$

Then $R(\alpha) = \{3, 5\} \nsubseteq X\alpha$ and $R(\beta) = \{3, 5, 8\} \subseteq X\beta$. Moreover,

$$\blacktriangledown\alpha = \{\{4\}, \{5, 6\}, \{5\}\} \text{ and } \blacktriangledown^{Y}\beta = \{\{3, 4\}, \{5, 6\}, \{8\}\}.$$

We show further that $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in T$, but there is no function $\phi : Y\beta \to X\alpha$ such that $(R \cap Y\beta)\phi \subseteq R$ and $\blacktriangledown\alpha \hookrightarrow (\blacktriangledown^{Y}\beta)\phi \hookrightarrow Y/\hat{\rho}$. Define $\lambda, \mu \in T$ by

$$\lambda = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 7 & 3 & 4 & 3 & 4 & 4 & 3 \end{pmatrix} \text{ and } \mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 3 & 3 & 4 & 5 & 6 & 5 & 3 \end{pmatrix}.$$

We see that $\alpha = \lambda\beta\mu$. Suppose that there is such a function ϕ, which would imply $\{4\} = \{1, 2\}\alpha \subseteq A\phi \subseteq \{3, 4\}$ for some $A \in \blacktriangledown^{Y}\beta$, and $\varnothing \neq (R \cap A)\phi \subseteq A\phi \cap R = \{3\}$, since $\varnothing \neq R \cap A \subseteq Y\beta$. Thus, $3 \in (R \cap A)\phi \subseteq X\alpha$, which is a contradiction. However, since $3 \in R(\alpha) \backslash X\alpha$, we can define $\phi : Y\beta \to X\alpha \cup R(\alpha)$ satisfying the conditions $(R \cap Y\beta)\phi \subseteq R$ and $\blacktriangledown\alpha \hookrightarrow (\blacktriangledown^{Y}\beta)\phi \hookrightarrow Y/\hat{\rho}$ as follows:

$$\phi = \begin{pmatrix} 3 & 4 & 5 & 6 & 8 \\ 3 & 4 & 5 & 6 & 5 \end{pmatrix}.$$

Note that if $Y = X$ or ρ is the identity relation, then $R(\alpha) \subseteq X\alpha$ for all $\alpha \in T$.

The following result is the key lemma in characterizing the $\mathcal{J}$-relation on T. The outline of the proof is the same as Theorem 2.7 in [3], but there are differences in detail, for example, to prove the "only if" part, the function ϕ has to be defined from $Y\beta$ into $X\alpha \cup R(\alpha)$ in order to make $r_a\mu$ in (2) well-defined. In addition, because of the restricted range of T, the function μ defined in (3) of the "if" part is greatly different from that defined in Theorem 2.7 [3]. Moreover, each step of the proof, the restricted range is involved.

Lemma 4. *Let $\alpha, \beta \in T$. Then $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in T$ if and only if there exists $\phi : Y\beta \to X\alpha \cup R(\alpha)$ such that $(R \cap Y\beta)\phi \subseteq R$ and $\blacktriangledown\alpha \hookrightarrow (\blacktriangledown^{Y}\beta)\phi \hookrightarrow Y/\hat{\rho}$.*

Proof. Assume that $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in T$. We define $\phi : Y\beta \to X\alpha \cup R(\alpha)$ such that $(R \cap Y\beta)\phi \subseteq R$ and $\blacktriangledown\alpha \hookrightarrow (\blacktriangledown^{Y}\beta)\phi \hookrightarrow Y/\hat{\rho}$ as follows.

Fix $r_0 \in X\alpha \cap R$. Let $a \in Y\beta$, and define ϕ in three steps.

(1) If $a \in X(\lambda\beta) \subseteq Y\beta$, we define $a\phi = a\mu \in X\alpha$.
(2) If $a \in Y\beta \backslash X(\lambda\beta)$ and $a\hat{\rho} \cap X(\lambda\beta) \neq \varnothing$, then define $a\phi = r_a\mu$.
(3) If $a \in Y\beta \backslash X(\lambda\beta)$ and $a\hat{\rho} \cap X(\lambda\beta) = \varnothing$, then define $a\phi = r_0 \in X\alpha$.

We observe that $r_a\mu$ in (2) belongs to $R(\alpha)$, since $r_a\mu \in R$ and $r_a\mu\,\hat{\rho}\,b\mu \in X\alpha$ for some $b \in a\hat{\rho} \cap X(\lambda\beta)$.

By the definition of ϕ and the fact that $R\mu \subseteq R$, we obtain that $(R \cap Y\beta)\phi \subseteq R$. To see that $\blacktriangledown\alpha \hookrightarrow (\blacktriangledown^{Y}\beta)\phi$, let $(x\rho)\alpha \in \blacktriangledown\alpha$. Then there exists $r \in R$ such that $(x\rho)\lambda \subseteq r\hat{\rho}$. We have $(x\rho)\alpha = (x\rho)\lambda\beta\mu \subseteq [(r\hat{\rho})\beta \cap X(\lambda\beta)]\mu \subseteq [(r\hat{\rho})\beta]\phi \in (\blacktriangledown^{Y}\beta)\phi$. Thus, $\blacktriangledown\alpha \hookrightarrow (\blacktriangledown^{Y}\beta)\phi$. To show that $(\blacktriangledown^{Y}\beta)\phi \hookrightarrow Y/\hat{\rho}$, let $(r\hat{\rho})\beta \in \blacktriangledown^{Y}\beta$. By the definition of ϕ, either $[(r\hat{\rho})\beta]\phi \subseteq ((r\hat{\rho})\beta)\mu \subseteq s\hat{\rho}$ for some $s \in R$ (if $(r\beta)\hat{\rho} \cap X(\lambda\beta) \neq \varnothing$) or $[(r\hat{\rho})\beta]\phi = \{r_0\} \subseteq r_0\hat{\rho}$ (if $(r\beta)\hat{\rho} \cap X(\lambda\beta) = \varnothing$). Therefore, $(\blacktriangledown^{Y}\beta)\phi \hookrightarrow Y/\hat{\rho}$.

Conversely, assume that there exists $\phi : Y\beta \to X\alpha \cup R(\alpha)$ such that $(R \cap Y\beta)\phi \subseteq R$ and $\blacktriangledown\alpha \hookrightarrow (\blacktriangledown^{Y}\beta)\phi \hookrightarrow Y/\hat{\rho}$. Let $A \in \blacktriangledown\alpha$. Then there is a unique $r_A \in R$ such that $A \subseteq r_A\hat{\rho}$ (note that r_A may not belong to $X\alpha$ but $r_A \in R(\alpha)$). Since $\blacktriangledown\alpha \hookrightarrow (\blacktriangledown^{Y}\beta)\phi \hookrightarrow Y/\hat{\rho}$, there exists $C_A \in \blacktriangledown^{Y}\beta$ such that $A \subseteq C_A\phi \subseteq r_A\hat{\rho}$. From $C_A \in \blacktriangledown^{Y}\beta$, there is $s_A \in R$ such that $C_A = (s_A\hat{\rho})\beta$. Let $t_A = s_A\beta$. Then $C_A = (s_A\hat{\rho})\beta \subseteq t_A\hat{\rho}$. For every $a \in A$, we choose $u_a^A \in C_A$ such that $a = u_a^A\phi$ (since $t_A\phi = r_A$, we may assume that $u_a^A = t_A$ if $a = r_A$). Let $C_A' = \{u_a^A : a \in A\}$. Then $C_A' \subseteq C_A$. For every $b \in C_A'$, we choose $v_b^A \in s_A\hat{\rho}$ such that $b = v_b^A\beta$ (since $s_A\beta = t_A$, we may assume that $v_b^A = s_A$ if $b = t_A$).

We aim to define $\lambda, \mu \in T$ such that $\alpha = \lambda\beta\mu$. We first define λ. Let $x \in X$, $A = (x\rho)\alpha$ and $a = x\alpha$. Then $x\alpha = a \in A$, so there exists $b = u_a^A$ such that $b = v_b^A\beta$, thus we define $x\lambda = v_b^A$. By the definition of λ, $(x\rho)\lambda \subseteq s_A\hat{\rho}$. If $p \in x\rho \cap R$, then $a' = p\alpha = r_A \in R \cap A$, and so $b' = u^A_{a'} = t_A$. Hence $v^A_{b'} = s_A$, which implies that $p\lambda = v^A_{b'} = s_A \in R$. Thus, $\lambda \in T$.

To define μ, fix $r_0 \in R$ and let $x \in X$.

(1) If $x \in C'_A$ for some $A \in \blacktriangledown\alpha$, define $x\mu = x\phi$.
(2) If $x \notin C'_B$ for all $B \in \blacktriangledown\alpha$ and $C_A \subseteq x\rho$ for some $A \in \blacktriangledown\alpha$, define $x\mu = r_A$.
(3) If $x\rho \cap C_A = \varnothing$ for all $A \in \blacktriangledown\alpha$, define $x\mu = r_0$.

To see that the definition of μ in (2) does not depend on the choice of A, we suppose that there are $A, B \in \blacktriangledown\alpha$ such that $C_A, C_B \subseteq x\rho$. Since $C_A = (s_A\hat{\rho})\beta \subseteq t_A\hat{\rho} \subseteq x\rho$ and $C_B = (s_B\hat{\rho})\beta \subseteq t_B\hat{\rho} \subseteq x\rho$, we obtain $t_A = t_B$, and thus $r_A = t_A\phi = t_B\phi = r_B$. Next, we prove that $\mu \in T$. By the definition of μ, we see that $R\mu \subseteq R$. Now, if $x\rho \cap C_A = \varnothing$ for all $A \in \blacktriangledown\alpha$, then $(x\rho)\mu = \{r_0\} \subseteq r_0\hat{\rho}$. For $C_A \subseteq x\rho$ for some $A \in \blacktriangledown\alpha$, we have $x\mu = x\phi \in r_B\hat{\rho} = r_A\hat{\rho}$ if $x \in C'_B$ for some $B \in \blacktriangledown\alpha$ and $x\mu = r_A \in r_A\hat{\rho}$ if $x \notin C'_B$ for all $B \in \blacktriangledown\alpha$. Thus, $(x\rho)\mu \subseteq r_A\hat{\rho}$.

To prove that $\alpha = \lambda\beta\mu$, let $x \in X$, $A = (x\rho)\alpha$ and $a = x\alpha$. Let $b = u^A_a$ (note that $u^A_a \in C'_A$ was selected such that $u^A_a\phi = a$). By the definitions of λ and μ, we have $x\lambda = v^A_b$ (recall that v^A_b was chosen so that $v^A_b\beta = b$) and $b\mu = u^A_a\mu = u^A_a\phi = a = x\alpha$. Thus, $x\lambda\beta\mu = v^A_b\beta\mu = b\mu = x\alpha$, as required. $\square$

If we take $Y = X$ in Lemma 4, then $T = T(X, \rho, R)$, which contains an identity element, the identity map. Thus, we obtain the $\mathcal{J}$-relation on $T(X, \rho, R)$.

Corollary 10. [3] [Theorem 2.8] *Let $\alpha, \beta \in T(X, \rho, R)$. Then $\alpha\mathcal{J}\beta$ if and only if there exist $\phi : X\beta \to X\alpha$ and $\varphi : X\alpha \to X\beta$ such that $(R \cap X\beta)\phi \subseteq R$, $\blacktriangledown\alpha \hookrightarrow (\blacktriangledown\beta)\phi \hookrightarrow X/\rho$; also $(R \cap X\alpha)\varphi \subseteq R$, $\blacktriangledown\beta \hookrightarrow (\blacktriangledown\alpha)\varphi \hookrightarrow X/\rho$.*

Now, we are ready to prove the $\mathcal{J}$-relation on T.

Theorem 4. *Let $\alpha, \beta \in T$. Then $\alpha\mathcal{J}\beta$ if and only if one of the following conditions holds:*

(i) $\ker(\alpha) = \ker(\beta)$, $\overline{\blacktriangledown}\alpha = \overline{\blacktriangledown}\beta$ *and* $(R\beta^{-1})\alpha, (R\alpha^{-1})\beta \subseteq R$;
(ii) *there exist* $\phi : Y\beta \to X\alpha \cup R(\alpha)$ *such that* $(R \cap Y\beta)\phi \subseteq R$, $\blacktriangledown\alpha \hookrightarrow (\overset{Y}{\blacktriangledown}\beta)\phi \hookrightarrow Y/\hat{\rho}$ *and* $\varphi : Y\alpha \to X\beta \cup R(\beta)$ *such that* $(R \cap Y\alpha)\varphi \subseteq R$, $\blacktriangledown\beta \hookrightarrow (\overset{Y}{\blacktriangledown}\alpha)\varphi \hookrightarrow Y/\hat{\rho}$.

Proof. Assume that $\alpha\mathcal{J}\beta$. Then $\alpha = \sigma\beta\delta$ and $\beta = \sigma'\alpha\delta'$ for some $\sigma, \sigma', \delta, \delta' \in T^1$. If $\sigma = 1 = \sigma'$, then $\alpha = \beta\delta$ and $\beta = \alpha\delta'$, which implies that $\alpha\mathcal{R}\beta$, and so $\ker(\alpha) = \ker(\beta)$, $\overline{\blacktriangledown}\alpha = \overline{\blacktriangledown}\beta$ and $(R\beta^{-1})\alpha, (R\alpha^{-1})\beta \subseteq R$. If $\delta = 1 = \delta'$, then $\alpha = \sigma\beta$ and $\beta = \sigma'\alpha$, which implies that $\alpha\mathcal{L}\beta$, and so $\alpha = \beta$; or $\blacktriangledown\alpha \hookrightarrow \overset{Y}{\blacktriangledown}\beta$ and $\blacktriangledown\beta \hookrightarrow \overset{Y}{\blacktriangledown}\alpha$. If $\alpha = \beta$, then (i) holds. If $\blacktriangledown\alpha \hookrightarrow \overset{Y}{\blacktriangledown}\beta$ and $\blacktriangledown\beta \hookrightarrow \overset{Y}{\blacktriangledown}\alpha$, then we define ϕ and φ to be the identity maps on $Y\beta$ and $Y\alpha$, respectively. It follows that $(R \cap Y\beta)\phi \subseteq R$, $\blacktriangledown\alpha \hookrightarrow (\overset{Y}{\blacktriangledown}\beta)\phi \hookrightarrow Y/\hat{\rho}$, $(R \cap Y\alpha)\varphi \subseteq R$ and $\blacktriangledown\beta \hookrightarrow (\overset{Y}{\blacktriangledown}\alpha)\varphi \hookrightarrow Y/\hat{\rho}$. That is, (ii) holds. For the other cases, we can conclude that $\alpha = \lambda\beta\mu$ and $\beta = \lambda'\alpha\mu'$ for some $\lambda, \lambda', \mu, \mu' \in T$ (for example, if $\sigma = 1$ and $\sigma' \in T$, then $\alpha = \beta\delta$ and $\beta = \sigma'\alpha\delta'$ imply $\alpha = \beta\delta = (\sigma'\alpha\delta')\delta = \sigma'\alpha(\delta'\delta) = \sigma'(\beta\delta)\delta'\delta = \sigma'\beta(\delta\delta'\delta)$). Thus, Lemma 4 gives that (ii) holds in all the remaining cases.

Conversely, assume that the statement holds. If $\ker(\alpha) = \ker(\beta)$, $\overline{\blacktriangledown}\alpha = \overline{\blacktriangledown}\beta$ and $(R\beta^{-1})\alpha, (R\alpha^{-1})\beta \subseteq R$, then $\alpha\mathcal{R}\beta$, and so $\alpha\mathcal{J}\beta$. If there exist $\phi : Y\beta \to X\alpha \cup R(\alpha)$ such that $(R \cap Y\beta)\phi \subseteq R$, $\blacktriangledown\alpha \hookrightarrow (\overset{Y}{\blacktriangledown}\beta)\phi \hookrightarrow Y/\hat{\rho}$ and $\varphi : Y\alpha \to X\beta \cup R(\beta)$ such that $(R \cap Y\alpha)\varphi \subseteq R$, $\blacktriangledown\beta \hookrightarrow (\overset{Y}{\blacktriangledown}\alpha)\varphi \hookrightarrow Y/\hat{\rho}$, then $\alpha = \lambda\beta\mu$ and $\beta = \lambda'\alpha\mu'$ for some $\lambda, \lambda', \mu, \mu' \in T$ by Lemma 4. Therefore, $\alpha\mathcal{J}\beta$, as required. $\square$

By setting ρ to be the identity relation in Theorem 4, we obtain the $\mathcal{J}$-relation on $T(X, Y)$ as follows.

Corollary 11. [6] [Theorem 3.9] *Let $\alpha, \beta \in T(X, Y)$. Then $\alpha\mathcal{J}\beta$ if and only if $\pi(\alpha) = \pi(\beta)$ or $|X\alpha| = |Y\alpha| = |Y\beta| = |X\beta|$.*

Proof. If ρ in Theorem 4 is the identity relation, then $T = T(X, Y)$. By the same proof as given for Corollary 4, we have that (i) of Theorem 4 is equivalent to $\pi(\alpha) = \pi(\beta)$. Now, we claim that

(ii) is equivalent to $|X\alpha| = |Y\alpha| = |Y\beta| = |X\beta|$. If (ii) holds, then $\phi : Y\beta \to X\alpha$ is onto, since $\blacktriangledown\alpha = \{\{x\alpha\} : x \in X\} \hookrightarrow \{\{(r\beta)\phi\} : r \in Y\} = (\overset{Y}{\blacktriangledown}\beta)\phi$ implies that for each $x\alpha \in X\alpha$, $x\alpha = (r\beta)\phi$ for some $r \in Y$. Similarly, from $\varphi : Y\alpha \to X\beta$ with $\blacktriangledown\beta \hookrightarrow (\overset{Y}{\blacktriangledown}\alpha)\varphi$, we obtain that φ is onto. Thus, $|X\alpha| \leqslant |Y\beta|$ and $|X\beta| \leqslant |Y\alpha|$, and so $|Y\alpha| \leqslant |X\alpha| \leqslant |Y\beta| \leqslant |X\beta| \leqslant |Y\alpha|$. Therefore, $|X\alpha| = |Y\alpha| = |Y\beta| = |X\beta|$. Conversely, if $|X\alpha| = |Y\alpha| = |Y\beta| = |X\beta|$, then there exist bijections $\phi : Y\beta \to X\alpha$ and $\varphi : Y\alpha \to X\beta$. Since ϕ is onto, it follows that $\blacktriangledown\alpha = \{\{x\alpha\} : x \in X\} \hookrightarrow \{\{(r\beta)\phi\} : r \in Y\} = (\overset{Y}{\blacktriangledown}\beta)\phi \hookrightarrow \{\{y\} : y \in Y\} = Y/\hat{\rho}$. Similarly, as φ is onto, we have $\blacktriangledown\beta \hookrightarrow (\overset{Y}{\blacktriangledown}\alpha)\varphi \hookrightarrow Y/\hat{\rho}$. Moreover, $(R \cap Y\beta)\phi \subseteq (Y\beta)\phi = X\alpha \subseteq Y = R$, and in the same way, $(R \cap Y\alpha)\varphi \subseteq R$. Therefore, we have our claim. $\square$

Recall that $\mathcal{D} \subseteq \mathcal{J}$ on any semigroup and $\mathcal{D} = \mathcal{J}$ on $T(X)$, but in T it is not always true, so we end this section with an example showing that $\mathcal{D} \neq \mathcal{J}$ on T.

Example 6. *Let X be the set of all positive integers and $Y = X \setminus \{1,2\}$. Let*

$$\begin{aligned} X/\rho &= \{\{1,2\},\{3,4,5\}\} \cup \{\{2n+4, 2n+5\} : n \in X\}, \\ Y/\hat{\rho} &= \{\{3,4,5\}\} \cup \{\{2n+4, 2n+5\} : n \in X\} \text{ and} \\ R &= \{3,6,8,10,\ldots\}. \end{aligned}$$

Then we define

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & \ldots \\ 4 & 4 & 8 & 9 & 9 & 12 & 13 & 16 & 17 & 20 & 21 & 24 & 25 & \ldots \end{pmatrix},$$

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & \ldots \\ 9 & 9 & 6 & 7 & 7 & 3 & 4 & 3 & 5 & 6 & 7 & 10 & 11 & \ldots \end{pmatrix}.$$

Thus, $\alpha, \beta \in T \setminus F$ and

$$\begin{aligned} \blacktriangledown\alpha &= \{\{4\},\{8,9\},\{12,13\},\{16,17\},\{20,21\},\{24,25\},\ldots\}, \\ \overset{Y}{\blacktriangledown}\alpha &= \{\{8,9\},\{12,13\},\{16,17\},\{20,21\},\{24,25\},\ldots\}, \\ \blacktriangledown\beta &= \{\{9\},\{6,7\},\{3,4\},\{3,5\},\{10,11\},\{14,15\},\{18,19\},\ldots\}, \\ \overset{Y}{\blacktriangledown}\beta &= \{\{6,7\},\{3,4\},\{3,5\},\{10,11\},\{14,15\},\{18,19\},\ldots\}. \end{aligned}$$

It is clear that $\ker(\alpha) \neq \ker(\beta)$. *Therefore, α and β are not $\mathcal{D}$-related by Theorem 3. However, we can define $\phi : Y\beta \to X\alpha \cup R(\alpha)$ and $\varphi : Y\alpha \to X\beta \cup R(\beta)$ as follows:*

$$\begin{array}{lll} (\{3,4\})\phi = \{3,4\}, & (\{3,5\})\phi = \{3,4\}, & (\{6,7\})\phi = \{8,9\}, \\ (\{10,11\})\phi = \{12,13\}, & (\{14,15\})\phi = \{16,17\}, & (\{18,19\})\phi = \{20,21\},\ldots \\ (\{8,9\})\varphi = \{8,9\}, & (\{12,13\})\varphi = \{6,7\}, & (\{16,17\})\varphi = \{3,4\}, \\ (\{20,21\})\varphi = \{3,5\}, & (\{24,25\})\varphi = \{10,11\}, & (\{28,29\})\varphi = \{14,15\},\ldots. \end{array}$$

Both ϕ and φ satisfy the required properties of Theorem 4. Therefore, α and β are $\mathcal{J}$-related.

However, if X is a finite set, then T is a finite semigroup and it is periodic. Hence $\mathcal{D} = \mathcal{J}$ in this case.

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References

1. Howie, J.M. *Fundamentals of Semigroup Theory*; Oxford University Press: New York, NY, USA, 1995.
2. Araújo, J.; Konieczny, J. Automorphism groups of centralizers of idempotents. *J. Algebra* **2003**, *269*, 227–239. [CrossRef]
3. Araújo, J.; Konieczny, J. Semigroups of transformations preserving an equivalence relation and a cross-section. *Commun. Algebra* **2004**, *32*, 1917–1935. [CrossRef]
4. Symons, J.S.V. Some results concerning a transformation semigroup. *J. Aust. Math. Soc.* **1975**, *19*, 413–425. [CrossRef]
5. Sanwong, J.; Singha, B.; Sullivan, R.P. Maximal and minimal congruences on some semigroups. *Acta Math. Sin. Engl. Ser.* **2009**, *25*, 455–466. [CrossRef]
6. Sanwong, J.; Sommanee, W. Regularity and Green's relations on a semigroup of transformations with restricted range. *Int. J. Math. Math. Sci.* **2008**. [CrossRef]