

On the Most Extended Modal Operator of First Type over Interval-Valued Intuitionistic Fuzzy Sets

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Abstract: The definition of the most extended modal operator of first type over interval-valued intuitionistic fuzzy sets is given, and some of its basic properties are studied.

Keywords: interval-valued intuitionistic fuzzy set; intuitionistic fuzzy set; modal operator

1. Introduction

Intuitionistic fuzzy sets (IFSs; see [1–5]) were introduced in 1983 as an extension of the fuzzy sets defined by Lotfi Zadeh (4.2.1921–6.9.2017) in [6]. In recent years, the IFSs have also been extended: intuitionistic L -fuzzy sets [7], IFSs of second [8] and n th [9–12] types, temporal IFSs [4,5,13], multidimensional IFSs [5,14], and others. Interval-valued intuitionistic fuzzy sets (IVIFSs) are the most detailed described extension of IFSs. They appeared in 1988, when Georgi Gargov (7.4.1947–9.11.1996) and the author read Gorzalczany's paper [15] on the interval-valued fuzzy set (IVFS). The idea of IVIFS was announced in [16,17] and extended in [4,18], where the proof that IFSs and IVIFSs are equipollent generalizations of the notion of the fuzzy set is given.

Over IVIFS, many (more than the ones over IFSs) relations, operations, and operators are defined. Here, similar to the IFS case, the standard modal operators \square and \diamond have analogues, but their extensions—the intuitionistic fuzzy extended modal operators of the first type—already have two different forms. In the IFS case, there is an operator that includes as a partial case all other extended modal operators. In the present paper, we construct a similar operator for the case of IVIFSs and study its properties.

2. Preliminaries

Let us have a fixed universe E and its subset A . The set

$$A = \{\langle x, M_A(x), N_A(x) \rangle \mid x \in E\},$$

where $M_A(x) \subset [0, 1]$ and $N_A(x) \subset [0, 1]$ are closed intervals and for all $x \in E$:

$$\sup M_A(x) + \sup N_A(x) \leq 1 \quad (1)$$

is called IVIFS, and functions $M_A : E \rightarrow \mathcal{P}([0, 1])$ and $N_A : E \rightarrow \mathcal{P}([0, 1])$ represent the *set of degrees of membership (validity, etc.)* and *the set of degrees of non-membership (non-validity, etc.)* of element $x \in E$ to a fixed set $A \subseteq E$, where $\mathcal{P}(Z) = \{Y \mid Y \subseteq Z\}$ for an arbitrary set Z .

Obviously, both intervals have the representation:

$$M_A(x) = [\inf M_A(x), \sup M_A(x)],$$

$$N_A(x) = [\inf N_A(x), \sup N_A(x)].$$

Therefore, when

$$\inf M_A(x) = \sup M_A(x) = \mu_A(x) \quad \text{and} \quad \inf N_A(x) = \sup N_A(x) = \nu_A(x),$$

the IVIFS A is transformed to an IFS.

We must mention that in [19,20] the second geometrical interpretation of the IFSs is given (see Figure 1).

IVIFSs have geometrical interpretations similar to, but more complex than, those of the IFSs. For example, the analogue of the geometrical interpretation from Figure 1 is shown in Figure 2.

Obviously, each IVFS A can be represented by an IVIFS as

$$\begin{aligned} A &= \{\langle x, M_A(x), N_A(x) \rangle \mid x \in E\} \\ &= \{\langle x, M_A(x), [1 - \sup M_A(x), 1 - \inf M_A(x)] \rangle \mid x \in E\}. \end{aligned}$$

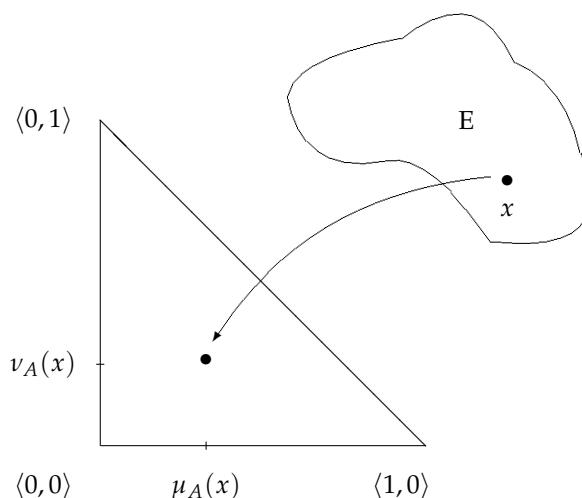


Figure 1. The second geometrical interpretation of an intuitionistic fuzzy set (IFS).

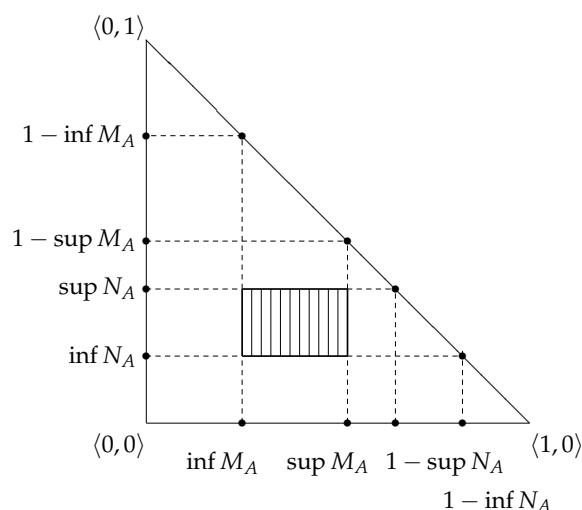


Figure 2. The second geometrical interpretation of an interval-valued intuitionistic fuzzy set (IVIFS).

The geometrical interpretation of the IVFS A is shown in Figure 3. It has the form of a section lying on the triangle's hypotenuse.

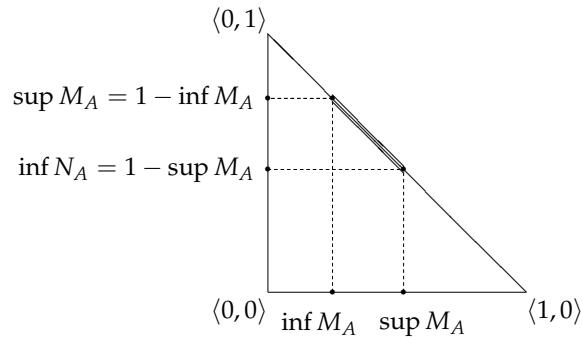


Figure 3. The second geometrical interpretation of an IVFS.

Modal-type operators are defined similarly to those defined for IFSs, but here they have two forms: shorter and longer. The shorter form is:

$$\begin{aligned}\square A &= \{\langle x, M_A(x), [\inf N_A(x), 1 - \sup M_A(x)] \rangle \mid x \in E\}, \\ \diamond A &= \{\langle x, [\inf M_A(x), 1 - \sup N_A(x)], N_A(x) \rangle \mid x \in E\}, \\ D_\alpha(A) &= \{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha(1 - \sup M_A(x) - \sup N_A(x))], \\ &\quad [\inf N_A(x), \sup N_A(x) + (1 - \alpha)(1 - \sup M_A(x) - \sup N_A(x))] \rangle \\ &\mid x \in E\}, \\ F_{\alpha,\beta}(A) &= \{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha(1 - \sup M_A(x) - \sup N_A(x))], \\ &\quad [\inf N_A(x), \sup N_A(x) + \beta(1 - \sup M_A(x) - \sup N_A(x))] \rangle \\ &\mid x \in E\}, \text{ for } \alpha + \beta \leq 1, \\ G_{\alpha,\beta}(A) &= \{\langle x, [\alpha \inf M_A(x), \alpha \sup M_A(x)], [\beta \inf N_A(x), \beta \sup N_A(x)] \rangle \\ &\mid x \in E\}, \\ H_{\alpha,\beta}(A) &= \{\langle x, [\alpha \inf M_A(x), \alpha \sup M_A(x)], [\inf N_A(x), \sup N_A(x) \\ &\quad + \beta(1 - \sup M_A(x) - \sup N_A(x))] \rangle \mid x \in E\}, \\ H_{\alpha,\beta}^*(A) &= \{\langle x, [\alpha \inf M_A(x), \alpha \sup M_A(x)], [\inf N_A(x), \sup N_A(x) \\ &\quad + \beta(1 - \alpha \sup M_A(x) - \sup N_A(x))] \rangle \mid x \in E\}, \\ J_{\alpha,\beta}(A) &= \{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha(1 - \sup M_A(x) \\ &\quad - \sup N_A(x))], [\beta \inf N_A(x), \beta \sup N_A(x)] \rangle \mid x \in E\}, \\ J_{\alpha,\beta}^*(A) &= \{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha(1 - \sup M_A(x) \\ &\quad - \beta \sup N_A(x))], [\beta \inf N_A(x), \beta \sup N_A(x)] \rangle \mid x \in E\},\end{aligned}$$

where $\alpha, \beta \in [0, 1]$.

Obviously, as in the case of IFSs, the operator D_α is an extension of the intuitionistic fuzzy forms of (standard) modal logic operators \square and \diamond , and it is a partial case of $F_{\alpha,\beta}$.

The longer form of these operators (operators \square , \diamond , and D do not have two forms—only the one above) is (see [4]):

$$\begin{aligned}\bar{F}_{\left(\begin{array}{cc} \alpha & \gamma \\ \beta & \delta \end{array}\right)}(A) = & \{\langle x, [\inf M_A(x) + \alpha(1 - \sup M_A(x) - \sup N_A(x)), \\ & \sup M_A(x) + \beta(1 - \sup M_A(x) - \sup N_A(x))], \\ & [\inf N_A(x) + \gamma(1 - \sup M_A(x) - \sup N_A(x)), \\ & \sup N_A(x) + \delta(1 - \sup M_A(x) - \sup N_A(x))] \rangle \mid x \in E\} \\ & \text{where } \beta + \delta \leq 1, \\ \bar{G}_{\left(\begin{array}{cc} \alpha & \gamma \\ \beta & \delta \end{array}\right)}(A) = & \{\langle x, [\alpha \inf M_A(x), \beta \sup M_A(x)], \\ & [\gamma \inf N_A(x), \delta \sup N_A(x)] \rangle \mid x \in E\}, \\ \bar{H}_{\left(\begin{array}{cc} \alpha & \gamma \\ \beta & \delta \end{array}\right)}(A) = & \{\langle x, [\alpha \inf M_A(x), \beta \sup M_A(x)], \\ & [\inf N_A(x) + \gamma(1 - \sup M_A(x) - \sup N_A(x)), \\ & \sup N_A(x) + \delta(1 - \sup M_A(x) - \sup N_A(x))] \rangle \mid x \in E\}, \\ \bar{H}^*_{\left(\begin{array}{cc} \alpha & \gamma \\ \beta & \delta \end{array}\right)}(A) = & \{\langle x, [\alpha \inf M_A(x), \beta \sup M_A(x)], \\ & [\inf N_A(x) + \gamma(1 - \beta \sup M_A(x) - \sup N_A(x)), \\ & \sup N_A(x) + \delta(1 - \beta \sup M_A(x) - \sup N_A(x))] \rangle \mid x \in E\}, \\ \bar{J}_{\left(\begin{array}{cc} \alpha & \gamma \\ \beta & \delta \end{array}\right)} = & \{\langle x, [\inf M_A(x) + \alpha(1 - \sup M_A(x) - \sup N_A(x)), \\ & \sup M_A(x) + \beta(1 - \sup M_A(x) - \sup N_A(x))], \\ & [\gamma \inf N_A(x), \delta \sup N_A(x)] \rangle \mid x \in E\}, \\ \bar{J}^*_{\left(\begin{array}{cc} \alpha & \gamma \\ \beta & \delta \end{array}\right)}(A) = & \{\langle x, [\inf M_A(x) + \alpha(1 - \delta \sup M_A(x) - \sup N_A(x)), \\ & \sup M_A(x) + \beta(1 - \sup M_A(x) - \delta \sup N_A(x))], \\ & [\gamma \inf N_A(x), \delta \sup N_A(x)] \rangle \mid x \in E\},\end{aligned}$$

where $\alpha, \beta, \gamma, \delta \in [0, 1]$ such that $\alpha \leq \beta$ and $\gamma \leq \delta$.

Figure 4 shows to which region of the triangle the element $x \in E$ (represented by the small rectangular region in the triangle) will be transformed by the operators F, G, \dots , irrespective of whether they have two or four indices.

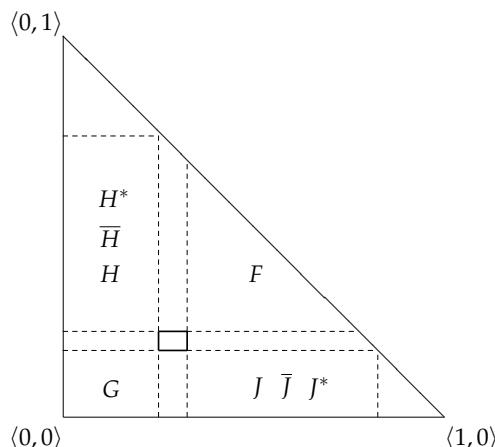


Figure 4. Region of transformation by the application of the operators.

3. Operator X

Now, we introduce the new operator

$$\begin{aligned} & X_{\left(\begin{array}{cccccc} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{array} \right)}(A) \\ &= \{ \langle x, [a_1 \inf M_A(x) + b_1(1 - \inf M_A(x)) - c_1 \inf N_A(x)), \\ & \quad a_2 \sup M_A(x) + b_2(1 - \sup M_A(x)) - c_2 \sup N_A(x)], \\ & \quad [d_1 \inf N_A(x) + e_1(1 - f_1 \inf M_A(x)) - \inf N_A(x)), \\ & \quad d_2 \sup N_A(x) + e_2(1 - f_2 \sup M_A(x)) - \sup N_A(x))] \mid x \in E \}, \end{aligned}$$

where $a_1, b_1, c_1, d_1, e_1, f_1, a_2, b_2, c_2, d_2, e_2, f_2 \in [0, 1]$, the following three conditions are valid for $i = 1, 2$:

$$a_i + e_i - e_i f_i \leq 1, \quad (2)$$

$$b_i + d_i - b_i c_i \leq 1, \quad (3)$$

$$b_i + e_i \leq 1, \quad (4)$$

and

$$a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2, d_1 \leq d_2, e_1 \leq e_2, f_1 \leq f_2. \quad (5)$$

Theorem 1. For every IVIFS A and for every $a_1, b_1, c_1, d_1, e_1, f_1, a_2, b_2, c_2, d_2, e_2, f_2 \in [0, 1]$ that satisfy (2)–(5), $X_{\left(\begin{array}{cccccc} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{array} \right)}(A)$ is an IVIFS.

Proof. Let $a_1, b_1, c_1, d_1, e_1, f_1, a_2, b_2, c_2, d_2, e_2, f_2 \in [0, 1]$ satisfy (2)–(5) and let A be a fixed IVIFS. Then, from (5) it follows that

$$\begin{aligned} & a_1 \inf M_A(x) + b_1(1 - \inf M_A(x)) - c_1 \inf N_A(x)) \\ & \leq a_2 \sup M_A(x) + b_2(1 - \sup M_A(x)) - c_2 \sup N_A(x)) \end{aligned}$$

and

$$\begin{aligned} & d_1 \inf N_A(x) + e_1(1 - f_1 \inf M_A(x)) - \inf N_A(x)) \\ & \leq d_2 \sup N_A(x) + e_2(1 - f_2 \sup M_A(x)) - \sup N_A(x)). \end{aligned}$$

Now, from (5) it is clear that it will be enough to check that

$$\begin{aligned} X &= a_2 \sup M_A(x) + b_2(1 - \sup M_A(x)) - c_2 \sup N_A(x)) \\ &+ d_2 \sup N_A(x) + e_2(1 - f_2 \sup M_A(x)) - \sup N_A(x)) \\ &= (a_2 - b_2 - e_2 f_2) \sup M_A(x) + (d_2 - e_2 - b_2 c_2) \sup N_A(x) + b_2 + e_2 \leq 1. \end{aligned}$$

In fact, from (2),

$$a_2 - b_2 - e_2 f_2 \leq 1 - b_2 - e_2$$

and from (3):

$$d_2 - e_2 - b_2 c_2 \leq 1 - b_2 - e_2.$$

Then, from (1),

$$X \leq (1 - b_2 - e_2)(\sup M_A(x) + \sup N_A(x)) + b_2 + e_2$$

$$\leq 1 - b_2 - e_2 + b_2 + e_2 = 1.$$

Finally, when $\sup M_A(x) = \inf N_A(x) = 0$ and from (4),

$$X = b_2(1 - 0 - 0) + e_2(1 - 0 - 0) = b_2 + e_2 \leq 1.$$

Therefore, the definition of the IVIFS is correct. \square

All of the operators described above can be represented by the operator $X\left(\begin{array}{cccccc} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{array}\right)$ at suitably chosen values of its parameters. These representations are the following:

$$\begin{aligned} \square A &= X\left(\begin{array}{cccccc} 1 & 0 & r_1 & 1 & 0 & s_1 \\ 1 & 0 & r_2 & 1 & 1 & 1 \end{array}\right)(A), \\ \diamond A &= X\left(\begin{array}{cccccc} 1 & 0 & r_1 & 1 & 0 & s_1 \\ 1 & 1 & 1 & 1 & 0 & s_2 \end{array}\right)(A), \\ D_\alpha(A) &= X\left(\begin{array}{cccccc} 1 & 0 & r_1 & 1 & \alpha & 1 \\ 1 & 0 & r_2 & 1 & 1-\alpha & 1 \end{array}\right)(A), \\ F_{\alpha,\beta}(A) &= X\left(\begin{array}{cccccc} 1 & 0 & r_1 & 1 & 0 & s_1 \\ 1 & \alpha & 1 & 1 & \beta & 1 \end{array}\right)(A), \\ G_{\alpha,\beta}(A) &= X\left(\begin{array}{cccccc} \alpha & 0 & r_1 & \beta & 0 & s_1 \\ \alpha & 0 & r_2 & \beta & 0 & s_2 \end{array}\right)(A), \\ H_{\alpha,\beta}(A) &= X\left(\begin{array}{cccccc} \alpha & 0 & r_1 & 1 & 0 & s_1 \\ \alpha & 0 & r_2 & 1 & \beta & 1 \end{array}\right)(A), \\ H_{\alpha,\beta}^*(A) &= X\left(\begin{array}{cccccc} \alpha & 0 & r_1 & \alpha & 0 & s_1 \\ 1 & 0 & r_2 & 1 & \beta & \alpha \end{array}\right)(A), \\ J_{\alpha,\beta}(A) &= X\left(\begin{array}{cccccc} 1 & 0 & r_1 & \beta & 0 & s_1 \\ 1 & \alpha & 1 & \beta & 0 & s_2 \end{array}\right)(A), \\ J_{\alpha,\beta}^*(A) &= X\left(\begin{array}{cccccc} 1 & 0 & r_1 & \beta & 0 & s_1 \\ 1 & \alpha & \beta & \beta & 0 & s_2 \end{array}\right)(A), \\ \bar{F}\left(\begin{array}{cc} \alpha & \gamma \\ \beta & \delta \end{array}\right)(A) &= X\left(\begin{array}{cccccc} 1 & \alpha & 1 & 1 & \gamma & 1 \\ 1 & \beta & 1 & 1 & \delta & s_1 \end{array}\right)(A), \\ \bar{G}\left(\begin{array}{cc} \alpha & \gamma \\ \beta & \delta \end{array}\right)(A) &= X\left(\begin{array}{cccccc} \alpha & 0 & r_1 & \beta & 0 & s_1 \\ \gamma & 0 & r_2 & \delta & 0 & s_2 \end{array}\right)(A), \\ \bar{H}\left(\begin{array}{cc} \alpha & \gamma \\ \beta & \delta \end{array}\right)(A) &= X\left(\begin{array}{cccccc} \alpha & 0 & r_1 & 1 & \gamma & 1 \\ \beta & 0 & r_2 & 1 & \delta & 1 \end{array}\right)(A), \\ \bar{H}^*\left(\begin{array}{cc} \alpha & \gamma \\ \beta & \delta \end{array}\right)(A) &= X\left(\begin{array}{cccccc} \alpha & 0 & r_1 & 1 & \gamma & 1 \\ \beta & 0 & r_2 & 1 & \delta & \beta \end{array}\right)(A), \\ \bar{J}\left(\begin{array}{cc} \alpha & \gamma \\ \beta & \delta \end{array}\right)(A) &= X\left(\begin{array}{cccccc} 1 & \alpha & 1 & \gamma & 0 & s_1 \\ 1 & \beta & 1 & \delta & 0 & s_2 \end{array}\right)(A), \\ \bar{J}^*\left(\begin{array}{cc} \alpha & \gamma \\ \beta & \delta \end{array}\right)(A) &= X\left(\begin{array}{cccccc} 1 & \alpha & \delta & \gamma & 0 & s_1 \\ 1 & \beta & \delta & \delta & 0 & s_2 \end{array}\right)(A), \end{aligned}$$

where r_1, r_2, s_1, s_2 are arbitrary real numbers in the interval $[0, 1]$.

Three of the operations, defined over two IVIFSs A and B , are the following:

$$\begin{aligned} \neg A &= \{\langle x, N_A(x), M_A(x) \rangle \mid x \in E\}, \\ A \cap B &= \{\langle x, [\min(\inf M_A(x), \inf M_B(x)), \min(\sup M_A(x), \sup M_B(x))], \\ &\quad [\max(\inf N_A(x), \inf N_B(x)), \max(\sup N_A(x), \sup N_B(x))] \rangle \mid x \in E\}, \\ A \cup B &= \{\langle x, [\max(\inf M_A(x), \inf M_B(x)), \max(\sup M_A(x), \sup M_B(x))], \\ &\quad [\min(\inf N_A(x), \inf N_B(x)), \min(\sup N_A(x), \sup N_B(x))] \rangle \mid x \in E\}. \end{aligned}$$

For any two IVIFSs A and B , the following relations hold:

$$A \subset B \quad \text{iff} \quad \forall x \in E, \inf M_A(x) \leq \inf M_B(x), \inf N_A(x) \geq \inf N_B(x), \\ \sup M_A(x) \leq \sup M_B(x) \text{ and } \sup N_A(x) \geq \sup N_B(x),$$

$$A \supset B \quad \text{iff} \quad B \subset A,$$

$$A = B \quad \text{iff} \quad A \subset B \text{ and } B \subset A.$$

Theorem 2. For every two IVIFSs A and B and for every $a_1, b_1, c_1, d_1, e_1, f_1, a_2, b_2, c_2, d_2, e_2, f_2 \in [0, 1]$ that satisfy (2)–(5),

$$\text{(a)} \quad \neg X_{\left(\begin{array}{cccccc} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{array}\right)}(\neg A) = X_{d,e,f,a,b,c}(A),$$

$$\text{(b)} \quad X_{\left(\begin{array}{cccccc} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{array}\right)}(A \cap B)$$

$$\subset X_{\left(\begin{array}{cccccc} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{array}\right)}(A) \cap X_{\left(\begin{array}{cccccc} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{array}\right)}(B),$$

$$\text{(c)} \quad X_{\left(\begin{array}{cccccc} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{array}\right)}(A \cup B)$$

$$\supset X_{\left(\begin{array}{cccccc} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{array}\right)}(A) \cup X_{\left(\begin{array}{cccccc} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{array}\right)}(B).$$

Proof. (c) Let $a_1, b_1, c_1, d_1, e_1, f_1, a_2, b_2, c_2, d_2, e_2, f_2 \in [0, 1]$ satisfy (2)–(5), and let A and B be fixed IVIFSs. First, we obtain:

$$Y = X_{\left(\begin{array}{cccccc} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{array}\right)}(A \cup B)$$

$$= X_{\left(\begin{array}{cccccc} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{array}\right)}(\{\langle x, [\max(\inf M_A(x), \inf M_B(x)), \\ \max(\sup M_A(x), \sup M_B(x))] \mid x \in E\})$$

$$\begin{aligned} &= \{\langle x, [a_1 \max(\inf M_A(x), \inf M_B(x)) + b_1(1 - \max(\inf M_A(x), \inf M_B(x))) \\ &\quad - c_1 \min(\inf N_A(x), \inf N_B(x))), a_2 \max(\sup M_A(x) \sup M_B(x)) \\ &\quad + b_2(1 - \max(\sup M_A(x) \sup M_B(x))) - c_2 \min(\sup N_A(x), \sup N_B(x))] \mid x \in E\}, \\ &\quad [d_1 \min(\inf N_A(x), \inf N_B(x)) + e_1(1 - f_1 \max(\inf M_A(x), \inf M_B(x))) \\ &\quad - \min(\inf N_A(x), \inf N_B(x))), d_2 \min(\sup N_A(x), \sup N_B(x)) \\ &\quad + e_2(1 - f_2 \max(\sup M_A(x) \sup M_B(x))) - \min(\sup N_A(x), \sup N_B(x))] \mid x \in E\}. \end{aligned}$$

Second, we calculate:

$$Z = X_{\left(\begin{array}{cccccc} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{array}\right)}(A) \cup X_{\left(\begin{array}{cccccc} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{array}\right)}(B)$$

$$\begin{aligned} &= \{\langle x, [a_1 \inf M_A(x) + b_1(1 - \inf M_A(x)) - c_1 \inf N_A(x)), \\ &\quad a_2 \sup M_A(x) + b_2(1 - \sup M_A(x)) - c_2 \sup N_A(x)], \\ &\quad [d_1 \inf N_A(x) + e_1(1 - f_1 \inf M_A(x)) - \inf N_A(x)), \\ &\quad d_2 \sup N_A(x) + e_2(1 - f_2 \sup M_A(x)) - \sup N_A(x)] \mid x \in E\} \end{aligned}$$

$$\begin{aligned}
& \cup \{ \langle x, [a_1 \inf M_B(x) + b_1(1 - \inf M_B(x) - c_1 \inf N_B(x)), \\
& \quad a_2 \sup M_B(x) + b_2(1 - \sup M_B(x) - c_2 \sup N_B(x))], \\
& \quad [d_1 \inf N_B(x) + e_1(1 - f_1 \inf M_B(x) - \inf N_B(x)), \\
& \quad d_2 \sup N_B(x) + e_2(1 - f_2 \sup M_B(x) - \sup N_B(x))] \rangle \mid x \in E \} \\
\\
& = \{ \langle x, [\max(a_1 \inf M_A(x) + b_1(1 - \inf M_A(x) - c_1 \inf N_A(x)), \\
& \quad a_1 \inf M_B(x) + b_1(1 - \inf M_B(x) - c_1 \inf N_B(x))), \\
& \quad \max(a_2 \sup M_A(x) + b_2(1 - \sup M_A(x) - c_2 \sup N_A(x)), \\
& \quad a_2 \sup M_B(x) + b_2(1 - \sup M_B(x) - c_2 \sup N_B(x))], \\
& \quad [\min(d_1 \inf N_A(x) + e_1(1 - f_1 \inf M_A(x) - \inf N_A(x)), \\
& \quad d_1 \inf N_B(x) + e_1(1 - f_1 \inf M_B(x) - \inf N_B(x))), \\
& \quad \min(d_2 \sup N_A(x) + e_2(1 - f_2 \sup M_A(x) - \sup N_A(x)), \\
& \quad d_2 \sup N_B(x) + e_2(1 - f_2 \sup M_B(x) - \sup N_B(x))] \rangle \mid x \in E \}.
\end{aligned}$$

Let

$$\begin{aligned}
P &= a_1 \max(\inf M_A(x), \inf M_B(x)) + b_1(1 - \max(\inf M_A(x), \inf M_B(x))) \\
&\quad - c_1 \min(\inf N_A(x), \inf N_B(x)) - \max(a_1 \inf M_A(x) + b_1(1 - \inf M_A(x) - c_1 \inf N_A(x)), \\
&\quad a_1 \inf M_B(x) + b_1(1 - \inf M_B(x) - c_1 \inf N_B(x))) \\
\\
&= a_1 \max(\inf M_A(x), \inf M_B(x)) + b_1 - b_1 \max(\inf M_A(x), \inf M_B(x)) \\
&\quad - b_1 c_1 \min(\inf N_A(x), \inf N_B(x)) - \max((a_1 - b_1) \inf M_A(x) + b_1 - b_1 c_1 \inf N_A(x), \\
&\quad (a_1 - b_1) \inf M_B(x) + b_1 - b_1 c_1 \inf N_B(x)) \\
\\
&= a_1 \max(\inf M_A(x), \inf M_B(x)) - b_1 \max(\inf M_A(x), \inf M_B(x)) \\
&\quad - b_1 c_1 \min(\inf N_A(x), \inf N_B(x)) - \max((a_1 - b_1) \inf M_A(x) - b_1 c_1 \inf N_A(x), \\
&\quad (a_1 - b_1) \inf M_B(x) - b_1 c_1 \inf N_B(x)).
\end{aligned}$$

Let $\inf M_A(x) \geq \inf M_B(x)$. Then

$$\begin{aligned}
P &= (a_1 - b_1) \inf M_A(x) - b_1 c_1 \min(\inf N_A(x), \inf N_B(x)) - \max((a_1 - b_1) \inf M_A(x) \\
&\quad - b_1 c_1 \inf N_A(x), (a_1 - b_1) \inf M_B(x) - b_1 c_1 \inf N_B(x)).
\end{aligned}$$

Let $(a_1 - b_1) \inf M_A(x) - b_1 c_1 \inf N_A(x) \geq (a_1 - b_1) \inf M_B(x) - b_1 c_1 \inf N_B(x)$. Then

$$\begin{aligned}
P &= (a_1 - b_1) \inf M_A(x) - b_1 c_1 \min(\inf N_A(x), \inf N_B(x)) - (a_1 - b_1) \inf M_A(x) \\
&\quad + b_1 c_1 \inf N_A(x) \\
&= b_1 c_1 \inf N_A(x) - b_1 c_1 \min(\inf N_A(x), \inf N_B(x)) \geq 0.
\end{aligned}$$

If $(a_1 - b_1) \inf M_A(x) - b_1 c_1 \inf N_A(x) < (a_1 - b_1) \inf M_B(x) - b_1 c_1 \inf N_B(x)$. Then

$$P = (a_1 - b_1) \inf M_A(x) - b_1 c_1 \min(\inf N_A(x), \inf N_B(x)) - (a_1 - b_1) \inf M_B(x)$$

$$\begin{aligned}
& + b_1 c_1 \inf N_B(x)). \\
& = b_1 c_1 \inf N_B(x) - b_1 c_1 \min(\inf N_A(x), \inf N_B(x)) \geq 0.
\end{aligned}$$

Therefore, the $\inf M_A$ -component of IVIFS Y is higher than or equal to the $\inf M_A$ -component of IVIFS Z . In the same manner, it can be checked that the same inequality is valid for the $\sup M_A$ -components of these IVIFSs. On the other hand, we can check that the $\inf N_A$ - and $\sup N_A$ -components of IVIFS Y are, respectively, lower than or equal to the $\inf N_A$ and $\sup N_A$ -components of IVIFS Z . Therefore, the inequality (c) is valid. \square

4. Conclusions

In the near future, the author plans to study some other properties of the new operator $X \left(\begin{array}{cccccc} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{array} \right)$.

In [21], it is shown that the IFSs are a suitable tool for the evaluation of data mining processes and objects. In the near future, we plan to discuss the possibilities of using IVIFSs as a similar tool.

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