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A Characterization of Projective Special Unitary Group PSU(3,3) and Projective Special Linear Group PSL(3,3) by NSE

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Abstract: Let G be a finite group and $\omega(G)$ be the set of element orders of G. Let $k \in \omega(G)$ and m_k be the number of elements of order k in G. Let $nse(G) = \{m_k | k \in \omega(G)\}$. In this paper, we prove that if G is a finite group such that $nse(G) = nse(H)$, where $H = PSU(3,3)$ or $PSL(3,3)$, then $G \cong H$.

Keywords: element order; number of elements of the same order; projective special linear group; projective special unitary group; simple K_n-group

1. Introduction

We devote this section to relevant definitions, basic facts about nse, and a brief history of this problem. Throughout this paper, G is a finite group. We express by $\pi(G)$ the set of prime divisors of $|G|$, and by $\omega(G)$, we introduce the set of order of elements from G. Set $m_k = m_k(G) = |\{g \in G | the\ order\ of\ g\ is\ k\}|$ and $nse(G)=\{m_k | k \in \omega(G)\}$. In fact, m_k is the number of elements of order k in G and nse(G) is the set of sizes of elements with the same order in G.

One of the important problems in group theory is characterization of a group by a given property, that is, to prove there exist only one group with a given property (up to isomorphism). A finite nonabelian simple group H is called characterizable by nse if every finite group G with $nse(G) = nse(H)$ implies that $G \cong H$.

After the monumental attempt to classify the finite simple groups, a huge amount of information about these groups has been collected. It has been noticed that some of the known simple groups are characterizable by some of their properties. Until now, different characterization are considered for some simple groups.

The twentieth century mathematician J.G. Thompson posed very interesting problem [1] .

Thompson Problem. Let $T(G)=\{(k, m_k) | k \in \omega(G), m_k \in nse(G)\}$ where m_k is the number of elements with order k. Suppose that $T(G) = T(H)$. If G is a finite solvable group, is it true that H is also necessary solvable?

Characterization of a group G by nse(G) and $|G|$, for short, deals with the number of elements of order k in the group G and $|G|$, where one must answer the question "is a finite group G, can be characterized by the set nse(G) and $|G|$?" While mathematicians might undoubtedly give many answers to such a question, the answer in Shao et al. [2,3] would probably rank near the top of most responses. They proved that if G is a simple k_i $(i = 3, 4)$ group, then G is characterizable by nse(G) and $|G|$. Several groups were characterized by nse and order. For example, in [4,5], it is proved that the Suzuki group, and sporadic groups are characterizable by nse and order. We remark here that not all groups can be characterized by their group orders and the set nse. For example, let $H_1 = C_4 \times C_4$ and $H_2 = C_2 \times Q_8$, where C_2 and C_4 are cyclic groups of order 2 and 4, respectively, and Q_8 is a quaternion group of order 8. It is easy to see that $nse(H_1) = nse(H_2) = \{1, 3, 12\}$ and $|H_1| = |H_2| = 16$ but $H_1 \ncong H_2$.

We know that the set of sizes of conjugacy classes has an essential role in determining the structure of a finite group. Hence, one might ask whether the set of sizes of elements with the same order has an essential role in determining the structure of a finite group. It is claimed that some simple groups could be characterized by exactly the set nse, without considering the order of group. In [6–12], it is proved that the alternating groups A_n, where $n \in \{7, 8\}$, the symmetric groups S_n where $n \in \{3, 4, 5, 6, 7\}$, M_{12}, $L_2(27)$, $L_2(q)$ where $q \in \{16, 17, 19, 23\}$, $L_2(q)$ where $q \in \{7, 8, 11, 13\}$, $L_2(q)$ where $q \in \{17, 27, 29\}$, are uniquely determined by $\mathrm{nse}(G)$. Besides, in [13–16], it is proved that $U_3(4)$, $L_3(4)$, $U_3(5)$, and $L_3(5)$ are uniquely determined by $\mathrm{nse}(G)$. Recently, in [17–19], it is proved that the simple groups $G_2(4)$, $L_2(3^n)$, where $|\pi(L_2(3^n))| = 4$, and $L_2(2^m)$, where $|\pi(L_2(2^m))| = 4$, are uniquely determined by $\mathrm{nse}(G)$. Therefore, it is natural to ask what happens with other kinds of simple groups.

The purpose of this paper is to continue this work by considering the following theorems:

Theorem 1. *Let G be a group such that $nse(G) = nse(PSU(3, 3))$. Then G is isomorphic to $PSU(3, 3)$.*

Theorem 2. *Let G be a group such that $nse(G) = nse(PSL(3, 3))$. Then G is isomorphic to $PSL(3, 3)$.*

2. Notation and Preliminaries

Before we get started, let us fix some notations that will be used throughout the paper. For a natural number n, by $\pi(n)$, we mean the set of all prime divisors of n, so it is obvious that if G is a finite group, then $\pi(G) = \pi(|G|)$. A Sylow r-subgroup of G is denoted by P_r and by $n_r(G)$, we mean the number of Sylow r- subgroup of G. Also the largest element order of P_r is signified by $exp(P_r)$. In addition, G is called a simple K_n group if G is a simple group with $|\pi(G)| = n$. Moreover, we denote by ϕ, the Euler function. In the following, we bring some useful lemmas which be used in the proof of main results.

Remark 1. *If G is a simple K_1- group, then G is a cyclic of prime order.*

Remark 2. *If $|G| = p^a q^b$, with p and q distinct primes, and a, b non-negative integers, then by Burnside's pq-theorem, G is solvable. In particular, there is no simple K_2-groups [20].*

Lemma 1. *Let G be a group containing more than two elements. If the maximal number s of elements of the same order in G is finite, then G is finite and $|G| \leq s(s^2 - 1)$ [21].*

Lemma 2. *Let G be a group. If $1 \neq n \in nse(G)$ and $2 \not| n$, then the following statements hold [12]:*

(1) $2||G|$;

(2) $m_2 = n$;

(3) *for any* $2 < t \in \omega(G)$, $m_t \neq n$.

Lemma 3. *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m||L_m(G)|$ [22].*

Lemma 4. *Let G be a group and P be a cyclic Sylow p-group of G of order p^α. If there is a prime r such that $p^\alpha r \in \omega(G)$, then $m_{p^\alpha r} = m_r(C_G(P))m_{p^\alpha}$. In particular, $\phi(r)m_{p^\alpha}|m_{p^\alpha r}$, where $\phi(r)$ is the Euler function of r [23].*

Lemma 5. *Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$, where $(p, m) = 1$. If P is not cyclic group and $s > 1$, then the number of elements of order n is always a multiple of p^s [24].*

Lemma 6. *Let G be a finite group, $P \in Syl_p(G)$, where $p \in \pi(G)$. Let G have a normal series $1 \trianglelefteq K \trianglelefteq L \trianglelefteq G$. If $P \leq L$ and $p \not| |K|$, then the following hold [3]:*

(1) $N_{\frac{G}{K}}(\frac{PK}{K}) = \frac{N_G(P)K}{K}$;

(2) $|G : N_G(P)| = |L : N_L(P)|$, *that is,* $n_p(G) = n_p(L)$;

(3) $|\frac{L}{K} : N_{\frac{L}{K}}(\frac{PK}{K})|t = |G : N_G(P)| = |L : N_L(P)|$, *that is,* $n_p(\frac{L}{K})t = n_p(G) = n_p(L)$ *for some positive integer* t, *and* $|N_K(P)|t = |K|$.

Lemma 7. *Let G be a finite solvable group and $|G| = mn$, where $m = p_1^{\alpha 1} \cdots p_r^{\alpha r}$, $(m, n) = 1$. Let $\pi = \{p_1, \cdots, p_r\}$ and let h_m be the number of π-Hall subgroups of G. Then $h_m = q_1^{\beta 1} \cdots q_s^{\beta s}$ satisfies the following conditions for all $i \in \{1, 2, \cdots, s\}$ [25]:*

(1) $q_i^{\beta i} = 1\ (mod p_j)$ *for some* p_j;

(2) *The order of some chief factor of G is divisible by $q_i^{\beta i}$.*

Lemma 8. *Let the finite group G act on the finite set X. If the action is semi regular, then $|G| \mid |X|$ [26].*

Let us mention the structure of simple K_3-groups, which will be needed in Section 3.

Lemma 9. *If G is a simple K_3-group, then G is isomorphic to one of the following groups [27]: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, $U_4(2)$.*

3. Main Results

Suppose G is a group such that $nse(G) = nse(H)$, where $H = PSU(3, 3)$, or $PSL(3, 3)$. By Lemma 1, we can assume that G is finite. Let m_n be the number of elements of order n. We notice that $m_n = k\phi(n)$, where k is the number of cyclic subgroups of order n in G. In addition, we notice that if $n > 2$, then $\phi(n)$ is even. If $n \in \omega(G)$, then by Lemma 3 and the above discussion, we have

$$\begin{cases} \phi(n) | m_n \\ n | \sum_{d|n} m_d \end{cases} \tag{1}$$

In the proof of Theorem 1 and Theorem 2, we often apply formula (1) and the above comments.

Proof of Theorem 1. Let G be a group with

$$nse(G) = nse(PSU(3, 3)) = \{1, 63, 504, 728, 1008, 1512, 1728\},$$

where $PSU(3, 3)$ is the projective special unitary group of degree 3 over field of order 3. The proof will be divided into a sequence of lemmas.

Lemma 10. $\pi(G) \subseteq \{2, 3, 7\}$.

Proof. First, since $63 \in nse(G)$, by Lemma 2, $2 \in \pi(G)$ and $m_2 = 63$. Let $2 \neq p \in \pi(G)$, by formula (1), $p|(1 + m_p)$ and $(p - 1)|m_p$, which implies that $p \in \{3, 5, 7, 13, 19, 1009\}$. Now, we prove that $13 \notin \pi(G)$. Conversely, suppose that $13 \in \pi(G)$. Then formula (1), implies $m_{13} = 1728$. On the other hand, by formula (1), we conclude that if $2.13 \in \omega(G)$, then $m_{2.13} \in \{504, 1008, 1512, 1728\}$ and $2.13 | 1 + m_2 + m_{13} + m_{2.13} (= 2296, 2800, 3304, 3520)$. Hence, $(2.13|2296)$, $(2.13|2800)$, $(2.13|3304)$, or $(2.13|3520)$, which is a contradiction, and hence $2.13 \notin \omega(G)$. Since $2.13 \notin \omega(G)$, the group P_{13} acts fixed point freely on the set of elements of order 2, and so, by Lemma 8, $|P_{13}||m_2$, which is a contradiction. Hence $13 \notin \pi(G)$. Similarly, we can prove that the prime numbers 19 and 1009 do not belong to $\pi(G)$. Now, we prove $5 \notin \pi(G)$. Conversely, suppose that $5 \in \pi(G)$. Then formula (1), implies $m_5 = 504$. From the formula (1), we conclude that if $3.5 \in \omega(G)$, then $m_{15} = 1512$. On the other hand, if $3.5 \in \omega(G)$, then by Lemma 4, $m_{3.5} = m_5.\phi(3).t$ for some integer t. Hence $1512 = (504)(2)t$, which is a contradiction and hence $3.5 \notin \omega(G)$. Since $3.5 \notin \omega(G)$, the group P_5 acts fixed point freely on the set of elements of order 3, and so $|P_5||m_3$, which is a contradiction. From what has already been proved, we conclude that $\pi(G) \subseteq \{2,3,7\}$. □

Remark 3. *If* $3,7 \in \pi(G)$*, then, by formula (1),* $m_3 = 728$ *and* $m_7 = 1728$*. If* $7^a \in \omega(G)$*, since* $m_{7^2} \notin nse(G)$*, then* $a = 1$*. By Lemma 3,* $|P_7||(1 + m_7)$ *and so* $|P_7||7$*. Suppose* $7 \in \pi(G)$*. Then since* $|P_7| = 7$*,* $n_7 = \frac{m_7}{\phi(7)} = 3^2.2^5||G|$*. Therefore, if* $7 \in \pi(G)$*, then* $3,2 \in \pi(G)$*. Hence, we only have to consider two proper sets* $\{2\}$*,* $\{2,3\}$*, and finally the whole set* $\{2,3,7\}$*.*

Now, we will show that $\pi(G)$ is not equal $\{2\}$ and $\{2,3\}$. For this purpose at first, we need obtain some information about elements of $\omega(G)$.

If $2^a \in \omega(G)$, then $\phi(2^a) = 2^{a-1}|m_{2^a}$ and so $0 \le a \le 7$.

By Lemma 3, $|P_2||(1 + m_2 + m_{2^2} + \cdots + m_{2^7})$ and so $|P_2||2^{10}$.

If $3^a \in \omega(G)$, then $1 \le a \le 4$.

Lemma 11. $\pi(G) \neq \{2\}$ *and* $\pi(G) \neq \{2,3\}$*.*

Proof. We claim that $\pi(G) \neq \{2\}$. Assume the contrary, that is, let $\pi(G) = \{2\}$. Since $2^8 \notin \omega(G)$, we have $\omega(G) \subseteq \{1,2,2^2,2^3,2^4,2^5,2^6,2^7\}$. Hence $|G| = 2^m = 5544 + 504k_1 + 728k_2 + 1008k_3 + 1512k_4 + 1728k_5$, where k_1, k_2, k_3, k_4, k_5 and m are non-negative integers and $0 \le k_1 + k_2 + k_3 + k_4 + k_5 \le 1$. Since $5544 \le |G| = 2^m \le 5544 + (k_1 + k_2 + k_3 + k_4 + k_5)1728$, we have $5544 \le |G| = 2^m \le 5544 + 1728$. Now, it is easy to check that the equation has no solution, which is a contradiction. Hence $\pi(G) \neq \{2\}$. Our next claim is that $\pi(G) \neq \{2,3\}$. Suppose, contrary to our claim, that $\pi(G) = \{2,3\}$. Since $3^5 \notin \omega(G)$, $exp(P_3) = 3, 3^2, 3^3, 3^4$.

- Let $exp(P_3) = 3$. Then by Lemma 3, $|P_3||(1 + m_3)$ and so $|P_3||3^6$. We will consider six cases for $|P_3|$.

Case 1. If $|P_3| = 3$, then since $n_3 = \frac{m_3}{\phi(3)} = 2^2.7.13||G|$, $13 \in \pi(G)$, which is a contradiction.

Case 2. If $|P_3| = 3^2$, then since $exp(P_3) = 3$ and $2^7.3 \notin \omega(G)$, we have $\omega(G) \subseteq \{1,2,2^2,2^3,2^4,2^5,2^6,2^7\} \cup \{3,3.2,3.2^2,3.2^3,3.2^4,3.2^5,3.2^6\}$, and $|\omega(G)| \le 15$. Therefore, $5544 + 504k_1 + 728k_2 + 1008k_3 + 1512k_4 + 1728k_5 = |G| = 2^a.9$, where k_1, k_2, k_3, k_4, k_5, and a are non-negative integers and $0 \le k_1 + k_2 + k_3 + k_4 + k_5 \le 8$. Since $5544 \le 2^a.9 \le 5544 + 8.1728$, we have $a = 10$ or $a = 11$.
If $a = 11$, then since $|P_2||2^{10}$, we have a contradiction.
If $a = 10$, then $3672 = 504k_1 + 728k_2 + 1008k_3 + 1512k_4 + 1728k_5$ where $0 \le k_1 + k_2 + k_3 + k_4 + k_5 \le 8$. By a computer calculation it is easily seen that the equation has no solution.

Case 3. If $|P_3| = 3^3$, then $5544 + 504k_1 + 728k_2 + 1008k_3 + 1512k_4 + 1728k_5 = |G| = 2^a.27$, where k_1, k_2, k_3, k_4, k_5, and a are non-negative integers and $0 \le k_1 + k_2 + k_3 + k_4 + k_5 \le 8$. Since $5544 \le 2^a.27 \le 5544 + 8.1728$, we have $a = 8$ or $a = 9$.
If $a = 8$, then $1368 = 504k_1 + 728k_2 + 1008k_3 + 1512k_4 + 1728k_5$ where $0 \le k_1 + k_2 + k_3 + k_4 + k_5 \le 8$. By a computer calculation, it is easily seen that the equation has no solution.
If $a = 9$, then $8280 = 504k_1 + 728k_2 + 1008k_3 + 1512k_4 + 1728k_5$ where $0 \le k_1 + k_2 + k_3 + k_4 + k_5 \le 8$. In this case, the equation has nine solutions. For example, $(k_1, k_2, k_3, k_4, k_5) = (1,0,3,2,1)$ is one of the solutions. We show this is impossible. Since $k_2 = 0$ and $m_3 = 728$, it follows that $m_{2^i} \neq 728$ for $1 \le i \le 7$. On the other hand, since $2^8 \notin \omega(G)$, $exp(P_2) = 2,2^2,2^3,2^4,2^5,2^6,2^7$. Hence, if $exp(P_2) = 2^i$ where $1 \le i \le 7$, then $|P_2||(1 + m_2 + m_{2^2} + \cdots + m_{2^i})$ by Lemma 3. Since $m_{2^i} \neq 728$, for $1 \le i \le 7$ by a computer calculation, we have $|P_2||2^7$, which is a contradiction. The same conclusion can be drawn for other solutions.

Case 4. If $|P_3| = 3^4$, then $5544 + 504k_1 + 728k_2 + 1008k_3 + 1512k_4 + 1728k_5 = |G| = 2^a.81$, where k_1, k_2, k_3, k_4, k_5, and a are non-negative integers and $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 8$. Since $5544 \leq 2^a.81 \leq 5544 + 8.1728$, we have $a = 7$. If $a = 7$, then $4824 = 504k_1 + 728k_2 + 1008k_3 + 1512k_4 + 1728k_5$ where $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 8$. One sees immediately that the equation has no solution.

Case 5. If $|P_3| = 3^5$, then $5544 + 504k_1 + 728k_2 + 1008k_3 + 1512k_4 + 1728k_5 = |G| = 2^a.243$ where k_1, k_2, k_3, k_4, k_5, and a are non-negative integers and $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 8$. Since $5544 \leq 2^a.243 \leq 5544 + 8.1728$, we have $a = 5$ or $a = 6$.

If $a = 5$, then $2232 = 504k_1 + 728k_2 + 1008k_3 + 1512k_4 + 1728k_5$. By a computer calculation $(1, 0, 0, 0, 1)$ is the only solution of this equation. Then $|\omega(G)| = 9$, it is clear that $exp(P_2) = 2^4$ or $exp(P_2) = 2^5$. Also since $k_2 = 0$ and $m_3 = 728$, $m_{2^i} \neq 728$ for $1 \leq i \leq 7$.

If $exp(P_2) = 2^5$, then since $|G| = 2^5.3^5$, the number of Sylow 2-subgroups of G is $1, 3, 9, 27, 81, 243$ and so the number of elements of order 2 is $1, 3, 9, 27, 81,\ 243$ but none of which belong to nse(G).

If $exp(P_2) = 2^4$, then $\omega(G) = \{1, 2, 2^2, 2^3, 2^4\} \cup \{3, 3.2, 3.2^2, 3.2^3\}$. Since $3.2^4 \notin \omega(G)$, it follows that the group P_3 acts fixed point freely on the set of elements of order 2^4. Hence, $|P_3|\,|\,m_{2^4}$, which is a contradiction ($m_{2^4} \in \{504, 1008, 1512, 1728\}$).

If $a = 6$, then $10,008 = 504k_1 + 728k_2 + 1008k_3 + 1512k_4 + 1728k_5$. By a computer calculation, $(0, 0, 2, 3, 2)$, and $(1, 0, 0, 4, 2)$ are solutions of this equation. Since $|\omega(G)| = 14$, we have $\omega(G) = \{1, 2, 2^2, 2^3, 2^4, 2^5, 2^6\} \cup \{3, 3.2, 3.2^2, 3.2^3, 3.2^4, 3.2^5, 3.2^6\}$. We know $|G| = 2^6.3^5$. It follows that, the number of Sylow 2-subgroups of G is $1, 3, 9, 27, 81, 243$ and so the number of elements of order 2 is $1, 3, 9, 27, 81, 243$ but none of which belong to nse(G).

Case 6. Similarly, we can rule out $|P_3| = 3^6$.

- Let $exp(P_3) = 3^2$. Then by Lemma 3, $|P_3|\,|\,(1 + m_3 + m_{3^2})$ and so $|P_3|\,|\,3^3$ (for example when $m_9 = 1512$). We will consider two cases for $|P_3|$.

Case 1. If $|P_3| = 3^2$, then $n_3 = \frac{m_9}{\phi(9)}$, since $m_9 \in \{504, 1008, 1512, 1728\}$, $n_3 = 2^2.3.7$ or $n_3 = 2^2.7.3^2$ or $n_3 = 2^3.3.7$, and so $7 \in \pi(G)$, which is a contradiction, and if $n_3 = 2^5.3^2$, since a cyclic group of order 9 has two elements of order 3, $m_3 \leq 2^5.3^2.2 = 576$, which is a contradiction.

Case 2. If $|P_3| = 3^3$, then since $2^7.3 \notin \omega(G)$ and $2^7.3^2 \notin \omega(G)$, $|\omega(G)| \leq 22$. Therefore, $5544 + 504k_1 + 728k_2 + 1008k_3 + 1512k_4 + 1728k_5 = |G| = 2^a.27$, where k_1, k_2, k_3, k_4, k_5, and a are non-negative integers and $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 15$. Since $5544 \leq 2^a.27 \leq 5544 + 15.1728$, we have $a = 8$, $a = 9$, or $a = 10$.

If $a = 8$, then $1368 = 504k_1 + 728k_2 + 1008k_3 + 1512k_4 + 1728k_5$ where $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 15$. By a computer calculation, it is easily seen that the equation has no solution.

If $a = 9$, then $8280 = 504k_1 + 728k_2 + 1008k_3 + 1512k_4 + 1728k_5$ where $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 15$. By a computer calculation, the equation has 22 solutions. For example, $(k_1, k_2, k_3, k_4, k_5) = (1, 0, 0, 4, 1)$. We show this solution is impossible. Since $k_2 = 0$ and $m_3 = 728$, it follows that $m_{2^i} \neq 728$, for $1 \leq i \leq 7$. On the other hand, if $2^a \in \omega(G)$, then $0 \leq a \leq 7$. By Lemma 3, we have $|P_2|\,|\,(1 + m_2 + m_{2^2} + \cdots + m_{2^7})$, since $m_{2^i} \neq 728$ for $1 \leq i \leq 7$, by a computer calculation we have $|P_2|\,|\,2^7$, which is a contradiction. Arguing as above, for other solutions, we have a contradiction.

Similarly, $a = 10$ can be ruled out as the above method.

- Let $exp(P_3) = 3^3$. Then by Lemma 3, $|P_3|\,|\,(1 + m_3 + m_{3^2} + m_{3^3})$ and so $|P_3|\,|\,3^4$ (for example when ($m_9 = 1512$ and $m_{27} = 1728$)). We will consider two cases for $|P_3|$.

Case 1. If $|P_3| = 3^3$, then $n_3 = \frac{m_{27}}{\phi(27)}$, since $m_{27} \in \{504, 1008, 1512, 1728\}$, $n_3 = 2^3.7$ or $n_3 = 2^2.7$ or $n_3 = 2^2.3.7$, and so $7 \in \pi(G)$, which is a contradiction, and if $n_3 = 2^5.3$, since a cyclic group of order 27 has two elements of order 3, $m_3 \leq 2^5.3.2 = 192$, which is a contradiction.

Case 2. If $|P_3| = 3^4$, and P_3 is not cyclic subgroup, then by Lemma 5, $27|m_{27}$. Since $(27 \not| 504)$ and $(27 \not| 1008)$, it is understood that $m_{27} \in \{1512, 1728\}$. Since $2^7.3 \notin \omega(G)$, $2^7.3^2 \notin \omega(G)$, and $2^7.3^3 \notin \omega(G)$, $|\omega(G)| \leq 29$. Therefore $5544 + 504k_1 + 728k_2 + 1008k_3 + 1512k_4 + 1728k_5 = |G| = 2^a.81$, where k_1, k_2, k_3, k_4, k_5, and a are non-negative integers and $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 22$. Since $5544 \leq 2^a.81 \leq 5544 + 22.1728$, we have $a = 7$, $a = 8$, or $a = 9$.

If $a = 7$, then $4824 = 504k_1 + 728k_2 + 1008k_3 + 1512k_4 + 1728k_5$ where $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 22$. By a computer calculation, it is easily seen that the equation has no solution.

If $a = 8$, then $15192 = 504k_1 + 728k_2 + 1008k_3 + 1512k_4 + 1728k_5$ where $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 22$. By a computer calculation, the equation has 22 solutions. For example, $(k_1, k_2, k_3, k_4, k_5) = (0, 0, 2, 3, 5)$. We show this solution is impossible. Since $k_2 = 0$ and $m_3 = 728$, it follows that $m_{2^i} \neq 728$, for $1 \leq i \leq 7$. On the other hand, by Lemma 3, we have $|P_2||(1 + m_2 + m_{2^2} + \cdots + m_{2^7})$, since $m_{2^i} \neq 728$ for $1 \leq i \leq 7$, by a computer calculation we have $|P_2||2^7$, which is a contradiction. Assume $(k_1, k_2, k_3, k_4, k_5) = (0, 9, 0, 0, 5)$ is a solution. Since $|P_2||(1 + m_2 + m_{2^2} + \cdots + m_{2^7})$ by Lemma 3. Indeed, $|P_2||(1 + 63 + 504t_1 + 728t_2 + 1008t_3 + 1512t_4 + 1728t_5)$ where t_1, t_2, t_3, t_4, t_5, are non-negative integers and $0 \leq t_1 + t_2 + t_3 + t_4 + t_5 \leq 6$. Since $k_1 = 0$, $k_2 = 9$, and $k_3 = 0$, $0 \leq t_1 \leq 1$, $0 \leq t_2 \leq 10$, and $0 \leq t_3 \leq 1$. Since $k_4 = 0$ and $m_{27} = 1512$ or 1728, $t_4 = 0$. Also $k_5 = 5$, and thus $0 \leq t_5 \leq 6$. By an easy calculation, this is impossible. Arguing as above, for other solutions, we have a contradiction.

If $a = 9$, then $35928 = 504k_1 + 728k_2 + 1008k_3 + 1512k_4 + 1728k_5$ where $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 22$. By a computer calculation, it is easily seen that the equation has no solution.

- Let $exp(P_3) = 3^4$. Then by Lemma 3, $|P_3||(1 + m_3 + m_{3^2} + m_{3^3} + m_{3^4})$ and so $|P_3||3^4$ (for example when ($m_9 = 504$, $m_{27} = 1008$, and $m_{81} = 1728$)).

  If $|P_3| = 3^4$, then $n_3 = \frac{m_{81}}{\phi(81)}$, since $m_{81} \in \{1512, 1728\}$, $n_3 = 3.7$ or $n_3 = 2^5$. If $n_3 = 3.7$, then $7 \in \pi(G)$ which is a contradiction. If $n_3 = 2^5$, since a cyclic group of order 81 has two elements of order 3, then $m_3 \leq 2^5.2$, which is a contradiction.

□

Remark 4. *According to Lemmas 10 and 11, Remark 3 we have* $\pi(G) = \{2, 3, 7\}$.

Lemma 12. $G \cong PSU(3, 3)$.

Proof. First, we show that $|G| = |PSU(3, 3|$. From the above arguments, we have $|P_7| = 7$. Since $3.7 \notin \omega(G)$, the group P_3 acts fixed point freely on the set of elements of order 7, and so $|P_3||m_7$. Hence $|P_3||3^3$. Likewise, $2.7 \notin \omega(G)$, and so $|P_2||2^6$. Hence, we have $|G| = 2^m.3^n.7$. Since $5544 = 2^3.3^2.7.11 \leq 2^m.3^n.7$, we conclude that $|G| = 2^6.3^3.7$ or $|G| = 2^5.3^3.7$. The proof is completed by showing that there is no group such that $|G| = 2^6.3^3.7$ and $nse(G) = nse(PSU(3, 3))$. First, we claim that G is a non-solvable group. Suppose that G is solvable, since $n_7 = \frac{m_7}{\phi(7)} = 2^5.3^2$, by Lemma 7, $2^5 \equiv 1\ (mod 7)$, which is a contradiction. Therefore, G is a non-solvable group and $7^2 \not| |G|$. Hence, G has a normal series $1 \trianglelefteq N \trianglelefteq H \trianglelefteq G$, such that N is a maximal solvable normal subgroup of G and $\frac{H}{N}$ is a non-solvable minimal normal subgroup of $\frac{G}{N}$. Indeed, $\frac{H}{N}$ is a non-abelian simple K_3-group, and so by Lemma 9, $\frac{H}{N}$ is isomorphic to $L_2(7)$ or $L_2(8)$. Suppose that $\frac{H}{N} \cong L_2(7)$. We know $n_7(L_2(7)) = 8$. From Lemma 6, we have $n_7(\frac{H}{N})t = n_7(G)$, and so, $n_7(G) = 8t$ for some integer t. On the other hand, since $n_7(G)|2^6.3^3$ and $n_7(G) = 1 + 7k$, we have $n_7(G) = 1$, $n_7(G) = 8$, $n_7(G) = 36$, $n_7(G) = 64$, or

$n_7(G) = 288$. If $n_7(G) = 36$, then since $36 = 8t$ has no integer solution, we have a contradiction. Similarly, if $\frac{H}{N} \cong L_2(8)$, we have a contradiction. As a result, $|G| = 2^5.3^3.7 = |PSU(3,3)|$. Hence $|G| = |PSU(3,3)|$, and by assumption, $nse(G) = nse(PSU(3,3))$, so by [2], $G \cong PSU(3,3)$ and the proof is completed. □

The remainder of this section will be devoted to the proof of Theorem 2.

Proof of Theorem 2. Let G be a group with

$$nse(G) = nse(PSL(3,3)) = \{1, 117, 702, 728, 936, 1404, 1728\},$$

where $PSL(3,3)$ is the projective special linear group of degree 3 over field of order 3. The proof will be divided into a sequence of lemmas.

Lemma 13. $\pi(G) \subseteq \{2,3,13\}$.

Proof. First, since $117 \in nse(G)$, by Lemma 2, $2 \in \pi(G)$ and $m_2 = 117$. Applying formula (1), we obtain $\pi(G) \subseteq \{3,5,7,13,19,937\}$. Now, we prove that $7 \notin \pi(G)$. Conversely, suppose that $7 \in \pi(G)$. Then formula (1), implies $m_7 = 1728$. From the formula (1), we conclude that if $2.7 \in \omega(G)$, then $m_{14} = 702$. On the other hand, if $2.7 \in \omega(G)$, then by Lemma 4, $m_{2.7} = m_7.\phi(2).t$ for some integer t. Hence $702 = 1728t$, which is a contradiction and hence $2.7 \notin \omega(G)$. Since $2.7 \notin \omega(G)$, the group P_7 acts fixed point freely on the set of elements of order 2 of G. Hence, by Lemma 8, $|P_7||m_2$, which is a contradiction. In the same manner, we can see that $5 \notin \pi(G)$. Now, we prove $19 \notin \pi(G)$. Conversely, suppose that $19 \in \pi(G)$. Then formula (1), implies $m_{19} \in \{702, 1728\}$. On the other hand, by formula (1), we conclude that if $2.19 \in \omega(G)$, then $m_{2.19} \in \{702, 936, 1404, 1728\}$. Now, if $m_{19} = 702$, then $2.19|1 + m_2 + m_{19} + m_{2.19}(= 1522, 1756, 2224, 2548)$, which is a contradiction, and if $m_{19} = 1728$, $2.19|1 + m_2 + m_{19} + m_{2.19}(= 2548, 2782, 3250, 3574)$ which is a contradiction. Hence $2.19 \notin \omega(G)$. Since $2.19 \notin \omega(G)$, the group P_{19} acts fixed point freely on the set of elements of order 2 of G, and so $|P_{19}||m_2$., which is a contradiction. Similarly, we can prove that $937 \notin \pi(G)$. From what has already been proved, we conclude that $\pi(G) \subseteq \{2,3,13\}$. □

Remark 5. *If* 3, $13 \in \pi(G)$, *then* $m_3 = 728$ *and* $m_{13} = 1728$. *If* $(13)^a \in \omega(G)$, *since* $m_{(13)^2} \notin nse(G)$, *then* $a = 1$. *By Lemma 3*, $|P_{13}||1 + m_{13}$ *and so* $|P_{13}||13$. *Suppose* $13 \in \pi(G)$. *Then since* $|P_{13}| = 13$, $n_{13} = \frac{m_{13}}{\phi(13)} = 3^2.2^4||G|$.*Therefore, if* $13 \in \pi(G)$, *then* $3, 2 \in \pi(G)$. *Hence, we only have to consider two proper sets* $\{2\}$, $\{2,3\}$, *and finally the whole set* $\{2,3,13\}$.

Now, we will show that $\pi(G)$ is not equal $\{2\}$ and $\{2,3\}$. For this purpose at first, we need obtain some information about elements of $\omega(G)$.

If $2^a \in \omega(G)$, then, by formula (1), we have $0 \leq a \leq 4$.

By Lemma 3, $|P_2||(1 + m_2 + m_{2^2} + \cdots + m_{2^4})$ and so $|P_2||2^4$.

If $3^a \in \omega(G)$, then $1 \leq a \leq 4$.

Lemma 14. $\pi(G) \neq \{2\}$ *and* $\pi(G) \neq \{2,3\}$.

Proof. We claim that $\pi(G) \neq \{2\}$. Assume the contrary, that is, let $\pi(G) = \{2\}$. Then $|\omega(G)| \leq 5$. Since, nse(G) has seven elements and $|\omega(G)| \leq 5$, we have a contradiction. Hence $\pi(G) \neq \{2\}$.
Our next claim is that $\pi(G) \neq \{2,3\}$. Suppose, contrary to our claim, that $\pi(G) = \{2,3\}$. Since $3^5 \notin \omega(G)$, $exp(P_3) = 3, 3^2, 3^3, 3^4$.

- Let $exp(P_3) = 3$. Then by Lemma 3, $|P_3||(1 + m_3)$ and so $|P_3||3^6$. We will consider six cases for $|P_3|$.

Case 1. If $|P_3| = 3$, then since $n_3 = \frac{m_3}{\phi(3)} = 2.7.13||G|$, $7 \in \pi(G)$, which is a contradiction.

Case 2. If $|P_3| = 3^2$, then since $exp(P_3) = 3$ and $3.2^5 \notin \omega(G)$, $|\omega(G)| \leq 10$. Therefore $5616 + 702k_1 + 728k_2 + 936k_3 + 1404k_4 + 1728k_5 = |G| = 2^a.9$ where k_1, k_2, k_3, k_4, k_5, and a are non-negative integers and $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 3$. Since $5616 \leq 2^a.9 \leq 5616 + 3.1728$, we have $a = 10$.

If $a = 10$, then since $|P_2||2^4$, we have a contradiction. Similarly, we can rule out other cases.

- Let $exp(P_3) = 3^2$. Then by Lemma 3, $|P_3||(1 + m_3 + m_{3^2})$ and $|P_3||3^3$ (for example when $m_9 = 702$). We will consider two cases for $|P_3|$.
  Case1. If $|P_3| = 3^2$, then $n_3 = \frac{m_9}{\phi(9)}||G|$, since $m_9 \in \{702, 936, 1404, 1728\}$, $n_3 = 3^2.13$, $n_3 = 2^2.13.3$, or $n_3 = 2.3^2.13$, and so $13 \in \pi(G)$, which is a contradiction, and if $n_3 = 2^5.3^2$, since a cyclic group of order 9 has two elements of order 3, $m_3 \leq 2^5.3^2.2 = 576$, which is a contradiction.
  Case 2. If $|P_3| = 3^3$, then since $exp(P_3) = 3^2$, $3.2^5 \notin \omega(G)$, and $3^2.2^5 \notin \omega(G)$, $|\omega(G)| \leq 15$. Therefore $5616 + 702k_1 + 728k_2 + 936k_3 + 1404k_4 + 1728k_5 = |G| = 2^a.27$ where k_1, k_2, k_3, k_4, k_5, and a are non-negative integers and $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 8$. Since $5616 \leq 2^a.27 \leq 5616 + 8.1728$, we have $a = 8$ or $a = 9$, which is a contradiction.
- Let $exp(P_3) = 3^3$. Then by Lemma 3, $|P_3||(1 + m_3 + m_{3^2} + m_{3^3})$ and $|P_3||3^5$ (for example when $m_9 = 702$ and $m_{27} = 1728$). We will consider tree cases for $|P_3|$.
  Case 1. If $|P_3| = 3^3$, then $n_3 = \frac{m_{27}}{\phi(27)}$, since $m_{27} \in \{702, 1404, 1728\}$, $n_3 = 3.13$, or $n_3 = 2.3.13$, and so $13 \in \pi(G)$, which is a contradiction, and if $n_3 = 2^5.3$, since a cyclic group of order 27 has two elements of order 3, $m_3 \leq 2^5.3.2 = 192$, which is a contradiction.
  Case 2. If $|P_3| = 3^4$, then since $exp(P_3) = 3^3$, $3.2^5 \notin \omega(G)$, $3^2.2^5 \notin \omega(G)$, and $3^3.2^5 \notin \omega(G)$, $|\omega(G)| \leq 20$. Therefore $5616 + 702k_1 + 728k_2 + 936k_3 + 1404k_4 + 1728k_5 = |G| = 2^a.81$ where k_1, k_2, k_3, k_4, k_5, and a are non-negative integers and $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 13$. Since $5616 \leq 2^a.81 \leq 5616 + 13.1728$, we have $a = 7$ or $a = 8$, which is a contradiction. In the same way, we can rule out the case $|P_3| = 3^5$
- Let $exp(P_3) = 3^4$. Then by Lemma 3, $|P_3||(1 + m_3 + m_{3^2} + m_{3^3} + m_{3^4})$ and $|P_3||3^5$ (for example when $m_9 = 1404, m_{27} = m_{81} = 1728$).We will consider two cases for $|P_3|$.
  Case 1. If $|P_3| = 3^4$, then $n_3 = \frac{m_{81}}{\phi(81)}$, since $m_{81} \in \{702, 1404, 1728\}$, $n_3 = 13$ or $n_3 = 13.2$ and so $13 \in \pi(G)$, which is a contradiction. If $n_3 = 2^5$, since a cyclic group of order 81 has two elements of order 3, then $m_3 \leq 2^5.2$ which is a contradiction.
  Case 2. If $|P_3| = 3^5$, since $exp(P_3) = 3^4$, $3.2^5 \notin \omega(G)$, $3^2.2^5 \notin \omega(G)$, $3^3.2^5 \notin \omega(G)$, and $3^4.2^5 \notin \omega(G)$, $|\omega(G)| \leq 25$. Therefore, $5616 + 702k_1 + 728k_2 + 936k_3 + 1404k_4 + 1728k_5 = |G| = 2^a.243$ where k_1, k_2, k_3, k_4, k_5, and a are non-negative integers and $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 18$. Since $5616 \leq 2^a.243 \leq 5616 + 18.1728$, we have $a = 5$ or $a = 6$ or $a = 7$, which is a contradiction.

□

Remark 6. *According to Lemmas 13 and 14, and Remark 5, we have* $\pi(G) = \{2, 3, 13\}$.

Lemma 15. $G \cong PSL(3, 3)$.

Proof. We show that $|G| = |PSL(3, 3|$. From the above arguments, we have $|P_{13}| = 13$. Since $2.13 \notin \omega(G)$, it follows that, the group P_2 acts fixed point freely on the set of elements of order 13, and so $|P_2||m_{13}$. Hence, $|P_2||2^6$. Likewise, $3.13 \notin \omega(G)$, and so $|P_3||3^3$. and so $|P_2||m_{13}$. Hence, $|P_2||2^6$. Likewise, $3.13 \notin \omega(G)$, and so $|P_3||3^3$. Hence we have $|G| = 2^m.3^n.13$.

Since $5616 = 2^4.3^3.13 \leq 2^m.3^n.13$, we conclude that $|G| = 2^6.3^3.13$, $|G| = 2^6.3^2.13$, $|G| = 2^5.3^3.13$, or $|G| = 2^4.3^3.13$. The proof is completed by showing that there is no group such that $|G| = 2^6.3^3.13$, $|G| = 2^6.3^2.13$, or $|G| = 2^5.3^3.13$, and $nse(G) = nse(PSL(3, 3))$. First, we show that there is no group such that $|G| = 2^6.3^3.13$ and $nse(G) = nse(PSL(3, 3))$. We claim that G is a non-solvable group. Suppose that G is a solvable group, since $n_{13} = \frac{m_{13}}{\phi(13)} = 2^4.3^2$, by Lemma 7, $2^4 \equiv 1\ (mod 13)$, which is a contradiction. Therefore G is a non-solvable group and $(13)^2 \not| |G|$. Hence, G has a normal series

$1 \trianglelefteq N \trianglelefteq H \trianglelefteq G$, such that N is a maximal solvable normal subgroup of G and $\frac{H}{N}$ is a non-solvable minimal normal subgroup of $\frac{G}{N}$. Indeed, $\frac{H}{N}$ is a non-abelian simple K_3-group, and so by Lemma 9 $\frac{H}{N}$ is isomorphic to one of the simple K_3 groups. In fact, $\frac{H}{N} \cong L_3(3)$. We know $n_{13}(L_3(3)) = 144$. From Lemma 6, we have $n_{13}(\frac{H}{N})t = n_{13}(G)$, and so $n_{13}(G) = 144t$ for some integer t. On the other hand, since $n_{13}(G)|2^6.3^3$ and $n_{13}(G) = 1 + 13k$, we have $n_{13}(G) = 1$, $n_{13}(G) = 27$, or $n_{13}(G) = 144$. If $n_{13}(G) = 27$, then since $27 = 144t$ has no integer solution, we have a contradiction. Similarly, we can rule out the case $|G| = 2^5.3^3.13$ and $nse(G) = nse(PSL(3,3))$. Finally, we have to show that there is no group such that $|G| = 2^6.3^2.13$ and $nse(G) = nse(PSL(3,3))$. By Lemma 7, it is easy to check that G is a non-solvable group, and $(13)^2 \not| |G|$. Hence, G has a normal series $1 \trianglelefteq N \trianglelefteq H \trianglelefteq G$, such that N is a maximal solvable normal subgroup of G and $\frac{H}{N}$ is a non-solvable minimal normal subgroup of $\frac{G}{N}$. Indeed, $\frac{H}{N}$ is a non-abelian simple K_3-group, and so by Lemma 9 $\frac{H}{N}$ is isomorphic to $L_3(3)$. Therefore $|H| = |N|2^4.3^3.13$, which is a contradiction. As a result, $|G| = 2^4.3^3.13 = |PSL(3,3)|$. Hence $|G| = |PSL(3,3)|$ and by assumption, $nse(G) = nse(PSL(3,3))$, so by [2], $G \cong PSL(3,3)$ and the proof is completed. □

4. Conclusions

In this paper, we showed that the groups $PSU(3,3)$ and $PSL(3,3)$ are characterized by *nse*. Further investigations are needed to answer "is a group G isomorphic to $PSU(3,q)$ ($q > 8$ is a prime power) if and only if $nse(G) = nse(PSU(3,q))$?" and "is a group G isomorphic to $PSL(3,q)$ ($q > 8$ is a prime power) if and only if $nse(G) = nse(PSL(3,q))$?". In future work, these questions will be considered.

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References

1. Shi, W.. A new characterization of sporadic simple groups. In *Group Theory, Proceedings of the 1987 Singapore Conference on Group Theory, Singapore, 8–9 June 1987*; Walter de Gruyter: Berlin, Germany, 1989; pp. 531–540.
2. Shoa, C.; Shi, W.; Jiang, Q. A characterization of simple K_3-groups. *Adv. Math.* **2009**, *38*, 327–330.
3. Shoa, C.; Shi,W.; Jiang,Q. Characterization of simple K_4-groups. *Front. Math. China* **2008**, *3*, 355–370. [CrossRef]
4. Iranmanesh, A.; Parvizi Mosaed, H.; Tehranian, A. Characterization of Suzuki group by nse and order of group. *Bull. Korean Math. Soc.* **2016**, *53*, 651–656. [CrossRef]
5. Khalili Asboei, A.; Salehi Amiri, S.S.; Iranmanesh, A.; Tehranian, A. A characterization of sporadic simple groups by nse and order. *J. Algebra Appl.* **2013**, *12*. [CrossRef]
6. Khalili Asboei, A.; Salehi Amiri, S.S.; Iranmanesh, A.; Tehranian, A. A new characterization of A_7, A_8. *An. St. Univ. Ovidius Constanta* **2013**, *21*, 43–50. [CrossRef]
7. Khalili Asboei, A.; Salehi Amiri, S.S.; Iranmanesh, A. A new characterization of Symmetric groups for some n. *Hacet. J. Math. Stat.* **2013**, *43*, 715–723.
8. Khalili Asboei, A.; Salehi Amiri, S.S.; Iranmanesh, A. A new note on characterization of a Mathieu group of degree 12. *Southeast Asian Bull. Math.* **2014**, *38*, 383–388.
9. Khalili Asboei, A. A new characterization of $PSL(2,27)$. *Bol. Soc. Paran. Mat.* **2014**, *32*, 43–50. [CrossRef]
10. Khalili Asboei, A.; Salehi Amiri, S.S.; Iranmanesh, A. A new characterization of PSL(2,q) for some q. *Ukr. Math. J.* **2016**, *67*, 1297–1305. [CrossRef]

11. Khatami, M.; Khosravi, B.; Akhlaghi, Z. A new characterization for some linear groups. *Monatsh. Math.* **2011**, *163*, 39–50. [CrossRef]
12. Shoa, C.; Jiang, Q. Characterization of groups $L_2(q)$ by nse where $q \in \{17, 27, 29\}$. *Chin. Ann. Math.* **2016**, *37B*, 103–110. [CrossRef]
13. Chen, D. A characterization of PSU(3,4) by nse. *Int. J. Algebra Stat.* **2013**, *2*, 51–56. [CrossRef]
14. Liu, S. A characterization of $L_3(4)$. *Sci. Asia* **2013**, *39*, 436–439. [CrossRef]
15. Liu, S. A characterization of projective special unitary group $U_3(5)$ by nse. *Arab J. Math. Sci.* **2014**, *20*, 133–140. [CrossRef]
16. Liu, S. A characterization of projective special linear group $L_3(5)$ by nse. *Ital. J. Pure Appl. Math.* **2014**, *32*, 203–212.
17. Jahandideh Khangheshlaghi, M.; Darafsheh, M.R. Nse characterization of the Chevalley group $G_2(4)$. *Arabian J. Math.* **2018**, *7*, 21–26. [CrossRef]
18. Parvizi Mosaed, H.; Iranmanesh, A.; Tehranian, A. Nse characterization of simple group $L_2(3^n)$. *Publ. Instit. Math. Nouv. Ser.* **2016**, *99*, 193–201. [CrossRef]
19. Parvizi Mosaed, H.; Iranmanesh, A.; Foroudi Ghasemabadi, M.; Tehranian, A. A new characterization of simple group $L_2(2^m)$. *Hacet. J. Math. Stat.* **2016**, *44*, 875–886.
20. Kurzweil, H.; Stellmacher, B. *The Theory of Finite Groups an Introduction*; Springer: New York, NY, USA, 2004 .
21. Shen, R.; Shoa, C.; Q. Jiang, Q.; Shi., W.; Mazurov, V. A new characterization of A_5. *Monatsh. Math.* **2010**, *160*, 337–341. [CrossRef]
22. Frobenius, G. Verallgemeinerung der Sylowschen Satze. *Berl. Ber.* **1895**, *2*, 981–993.
23. Shoa, C.; Jiang, Q. A new characterization of some linear groups by nse. *J. Algebra Its Appl.* **2014**, *13*. [CrossRef]
24. Miller, G.A. Addition to a theorem due to Frobenius. *Bull. Am. Math. Soc.* **1904**, *11*, 6–7. [CrossRef]
25. Hall, M. *The Theory of Groups*; Macmillan: New York, NY, USA, 1959.
26. Passman, D. *Permutation Groups*; W. A. Benjamin: New York, NY, USA, **1968.**
27. Herzog, M. On finite simple groups of order divisible by three primes only. *J. Algebra* **1968**, *10*, 383–388. [CrossRef]