

Article

Near Fixed Point Theorems in the Space of Fuzzy Numbers

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Abstract: The fuzzy numbers are fuzzy sets owning some elegant mathematical structures. The space consisting of all fuzzy numbers cannot form a vector space because it lacks the concept of the additive inverse element. In other words, the space of fuzzy numbers cannot be a normed space even though the normed structure can be defined on this space. This also says that the fixed point theorems established in the normed space cannot apply directly to the space of fuzzy numbers. The purpose of this paper is to propose the concept of near fixed point in the space of fuzzy numbers and to study its existence. In order to consider the contraction of fuzzy-number-valued function, the concepts of near metric space and near normed space of fuzzy numbers are proposed based on the almost identical concept. The concepts of Cauchy sequences in near metric space and near normed space of fuzzy numbers are also proposed. Under these settings, the existence of near fixed points of fuzzy-number-valued contraction function in complete near metric space and near Banach space of fuzzy numbers are established.

Keywords: Cauchy sequence; near fixed point; near metric space of fuzzy numbers; near normed space of fuzzy numbers; null set

MSC: 03E72; 47H10; 54H25

1. Introduction

The fuzzy numbers can be treated as the imprecise data. For example, in the financial market, the data may not be precisely measured owing to the fluctuation. However, based on the knowledge of experts, it may be said that each numerical data will be around some certain value. In this case, these imprecise data can be described as the fuzzy numbers. In other words, the fuzzy sets theory may provide a useful tool to tackle this kind of imprecision. The basic ideas and applications of fuzzy sets theory can refer to the monographs [1–6].

Let $\mathcal{F}_{cc}(\mathbb{R})$ denote the family of all fuzzy numbers, which will be described in detail below. However, this family $\mathcal{F}_{cc}(\mathbb{R})$ cannot form a vector space. The main reason is that each fuzzy number in $\mathcal{F}_{cc}(\mathbb{R})$ does not have the additive inverse element. Although the space $\mathcal{F}_{cc}(\mathbb{R})$ is not a vector space, the Hahn-Banach extension theorems on $\mathcal{F}_{cc}(\mathbb{R})$ still can be studied by referring to Wu [7]. On the other hand, the fixed point theorems in fuzzy metric space have been studied in [8–19]. However, the fuzzy metric space is completely different from the near metric space of fuzzy numbers that is adopted in this paper. The purpose of this paper is to study the near fixed point theorem in the near metric space $\mathcal{F}_{cc}(\mathbb{R})$.

Some of the conventional fixed point theorems were established in normed space. Since $\mathcal{F}_{cc}(\mathbb{R})$ is not a vector space, it cannot also be a normed space even though we can define a norm structure on $\mathcal{F}_{cc}(\mathbb{R})$. Therefore, the conventional fixed point theorems will not be applicable in $\mathcal{F}_{cc}(\mathbb{R})$. In this paper, based on the norm structure defined on $\mathcal{F}_{cc}(\mathbb{R})$, the concept of Cauchy sequence in $\mathcal{F}_{cc}(\mathbb{R})$ can

be similarly defined. In this case, the Banach space of fuzzy numbers can be defined according to the concept of Cauchy sequence. The main aim of this paper is to study and establish the so-called near fixed point theorems in Banach space of fuzzy numbers.

Let U be a topological space. The *fuzzy subset* \tilde{A} of U is defined by a *membership function* $\zeta_{\tilde{A}} : U \rightarrow [0, 1]$. The α -level set of \tilde{A} , denoted by \tilde{A}_α , is defined by

$$\tilde{A}_\alpha = \{x \in U : \zeta_{\tilde{A}}(x) \geq \alpha\}$$

for all $\alpha \in (0, 1]$. The 0-level set \tilde{A}_0 is defined as the closure of the set $\{x \in U : \zeta_{\tilde{A}}(x) > 0\}$.

Let \odot denote any one of the four basic arithmetic operations $\oplus, \ominus, \otimes, \oslash$ between two fuzzy subsets \tilde{A} and \tilde{B} . The membership function of $\tilde{A} \odot \tilde{B}$ is defined by

$$\zeta_{\tilde{A} \odot \tilde{B}}(z) = \sup_{\{(x,y):z=x \odot y\}} \min \{\zeta_{\tilde{A}}(x), \zeta_{\tilde{B}}(y)\}$$

for all $z \in \mathbb{R}$. More precisely, the membership functions are given by

$$\begin{aligned} \zeta_{\tilde{A} \oplus \tilde{B}}(z) &= \sup_{\{(x,y):z=x+y\}} \min \{\zeta_{\tilde{A}}(x), \zeta_{\tilde{B}}(y)\}; \\ \zeta_{\tilde{A} \ominus \tilde{B}}(z) &= \sup_{\{(x,y):z=x-y\}} \min \{\zeta_{\tilde{A}}(x), \zeta_{\tilde{B}}(y)\}; \\ \zeta_{\tilde{A} \otimes \tilde{B}}(z) &= \sup_{\{(x,y):z=x*y\}} \min \{\zeta_{\tilde{A}}(x), \zeta_{\tilde{B}}(y)\}; \\ \zeta_{\tilde{A} \oslash \tilde{B}}(z) &= \sup_{\{(x,y):z=x/y, y \neq 0\}} \min \{\zeta_{\tilde{A}}(x), \zeta_{\tilde{B}}(y)\}, \end{aligned}$$

where $\tilde{A} \ominus \tilde{B} \equiv \tilde{A} \oplus (-\tilde{B})$.

Let U be a real topological vector space. We denote by $\mathcal{F}_{cc}(U)$ the set of all fuzzy subsets of U such that each $\tilde{a} \in \mathcal{F}_{cc}(U)$ satisfies the the following conditions:

- \tilde{a} is normal, i.e., $\zeta_{\tilde{a}}(x) = 1$ for some $x \in U$;
- \tilde{a} is convex, i.e., the membership function $\zeta_{\tilde{a}}(x)$ is quasi-concave;
- the membership function $\zeta_{\tilde{a}}$ is upper semicontinuous;
- the 0-level set $\tilde{a}^{(0)}$ is a compact subset of U .

In particular, if $U = \mathbb{R}$ then each element of $\mathcal{F}_{cc}(\mathbb{R})$ is called a *fuzzy number*.

For $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$, it is well-known that, for each $\alpha \in [0, 1]$, the α -level set \tilde{a}_α is a bounded closed interval in \mathbb{R} , which is also denoted by $\tilde{a}_\alpha = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U]$.

We say $\tilde{1}_{\{a\}}$ is a *crisp number with value a* if and only if the membership function of $\tilde{1}_{\{a\}}$ is given by

$$\zeta_{\tilde{1}_{\{a\}}}(r) = \begin{cases} 1 & \text{if } r = a \\ 0 & \text{if } r \neq a. \end{cases}$$

It is clear that each α -level set of $\tilde{1}_{\{a\}}$ is a singleton set $\{a\}$ for $\alpha \in [0, 1]$. Therefore, the crisp number $\tilde{1}_{\{a\}}$ can be identified with the real number a . In this case, we have the inclusion $\mathbb{R} \subset \mathcal{F}_{cc}(\mathbb{R})$. For convenience, we also write $\lambda \tilde{a} \equiv \tilde{1}_\lambda \otimes \tilde{a}$.

Let \tilde{a} and \tilde{b} be two fuzzy numbers with $\tilde{a}_\alpha = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U]$ and $\tilde{b}_\alpha = [\tilde{b}_\alpha^L, \tilde{b}_\alpha^U]$ for $\alpha \in [0, 1]$. It is well known that

$$(\tilde{a} \oplus \tilde{b})_\alpha = [\tilde{a}_\alpha^L + \tilde{b}_\alpha^L, \tilde{a}_\alpha^U + \tilde{b}_\alpha^U]$$

and, for $\lambda \in \mathbb{R}$,

$$(\lambda \tilde{a})_\alpha = \begin{cases} [\lambda \tilde{a}_\alpha^L, \lambda \tilde{a}_\alpha^U] & \text{if } \lambda \geq 0 \\ [\lambda \tilde{a}_\alpha^U, \lambda \tilde{a}_\alpha^L] & \text{if } \lambda < 0. \end{cases} \tag{1}$$

For any $\lambda \in \mathbb{R}$ and $\tilde{a}, \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$, it is clear to see that

$$\lambda(\tilde{a} \oplus \tilde{b}) = \lambda\tilde{a} \oplus \lambda\tilde{b}. \tag{2}$$

Suppose that $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$. Then we have

$$(\tilde{a} \ominus \tilde{a})_\alpha = [\tilde{a}_\alpha^L - \tilde{a}_\alpha^U, \tilde{a}_\alpha^U - \tilde{a}_\alpha^L] = \left[-(\tilde{a}_\alpha^U - \tilde{a}_\alpha^L), \tilde{a}_\alpha^U - \tilde{a}_\alpha^L\right],$$

which says that each α -level set $(\tilde{a} \ominus \tilde{a})_\alpha$ is an “approximated real zero number” with symmetric uncertainty $\tilde{a}_\alpha^U - \tilde{a}_\alpha^L$. It is also clear that the real zero number has the highest membership degree 1 given by $\xi_{\tilde{a} \ominus \tilde{a}}(0) = 1$. In this case, we can say that $\tilde{a} \ominus \tilde{a}$ is a fuzzy zero number.

Let

$$\Omega = \{\tilde{a} \ominus \tilde{a} : \tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})\}.$$

Equivalently, $\tilde{\omega} \in \Omega$ if and only if $\tilde{\omega}_\alpha^U \geq 0$ and $\tilde{\omega}_\alpha^L = -\tilde{\omega}_\alpha^U$ for all $\alpha \in [0, 1]$, i.e.,

$$\tilde{\omega}_\alpha = [\tilde{\omega}_\alpha^L, \tilde{\omega}_\alpha^U] = [-\tilde{\omega}_\alpha^U, \tilde{\omega}_\alpha^U],$$

where the bounded closed interval $\tilde{\omega}_\alpha$ is an “approximated real zero number” with symmetric uncertainty $\tilde{\omega}_\alpha^U$. In other words, each $\tilde{\omega} \in \Omega$ is a fuzzy zero number. We also call Ω as the *null set* in $\mathcal{F}_{cc}(\mathbb{R})$. It is also clear that $\tilde{1}_{\{0\}}$ the crisp number with value 0 is in the null set Ω . Since the null set Ω collects all of the fuzzy zero numbers, it can be regarded as a kind of “zero element” of $\mathcal{F}_{cc}(\mathbb{R})$. The true zero element of $\mathcal{F}_{cc}(\mathbb{R})$ is $\tilde{1}_{\{0\}}$, since it is clear that $\tilde{a} \oplus \tilde{1}_{\{0\}} = \tilde{a}$ for any $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$. On the other hand, since $\tilde{a} \ominus \tilde{a}$ is not a zero element of $\mathcal{F}_{cc}(\mathbb{R})$, this says that $\mathcal{F}_{cc}(\mathbb{R})$ cannot form a vector space under the above fuzzy addition and scalar multiplication.

Recall that the (conventional) normed space is based on the vector space. Since $\mathcal{F}_{cc}(\mathbb{R})$ is not a vector space, we cannot consider the (conventional) normed space $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$. Therefore we cannot study the fixed point theorem in $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ using the conventional way. In this paper, although $\mathcal{F}_{cc}(\mathbb{R})$ is not a vector space, we still can endow a norm to $\mathcal{F}_{cc}(\mathbb{R})$ in which the axioms are almost the same as the axioms of conventional norm. The only difference is that the concept of null set is involved in the axioms. Under these settings, we shall study the so-called near fixed point theorem in the near normed space of fuzzy numbers $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$.

Let $\tilde{T} : \mathcal{F}_{cc}(\mathbb{R}) \rightarrow \mathcal{F}_{cc}(\mathbb{R})$ be a function from $\mathcal{F}_{cc}(\mathbb{R})$ into itself. We say that $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ is a fixed point if and only if $\tilde{T}(\tilde{a}) = \tilde{a}$. Since $\mathcal{F}_{cc}(\mathbb{R})$ lacks the vector structure, we cannot expect to obtain the fixed point of the mapping \tilde{T} using the conventional ways. In this paper, we shall try to find a fuzzy number \tilde{a} satisfying $\tilde{T}(\tilde{a}) \oplus \tilde{\omega}^{(1)} = \tilde{a} \oplus \tilde{\omega}^{(2)}$ for some $\tilde{\omega}^{(1)}, \tilde{\omega}^{(2)} \in \Omega$. Since the null set Ω can play the role of “zero element” in $\mathcal{F}_{cc}(\mathbb{R})$, i.e., the elements $\tilde{\omega}^{(1)}$ and $\tilde{\omega}^{(2)}$ can be ignored in some sense, this kind of fuzzy number \tilde{a} is said to be a near fixed point of the mapping \tilde{T} .

In Section 2, the concept of the null set in fuzzy numbers is proposed, where some interesting properties are derived in order to study the near fixed point theorem. In Sections 3 and 4, the concepts of near metric space and near normed space of fuzzy numbers are proposed, where some interesting properties are also derived for further discussion. In Section 5, the concepts of Cauchy sequence in metric space and normed space of fuzzy numbers are similarly defined according to the conventional way. In Section 6, the concept of near fixed point of fuzzy-number-valued function is proposed. Also, three concepts of metric contraction of fuzzy-number-valued functions are proposed. Using the completeness of near metric space of fuzzy numbers, many near fixed point theorems are established. In Section 7, we also propose three concepts of norm contraction of fuzzy-number-valued functions. In this case, many near fixed point theorems in near Banach space of fuzzy numbers are established.

2. Space of Fuzzy Numbers

Under the fuzzy addition and scalar multiplication in $\mathcal{F}_{cc}(\mathbb{R})$, it is clear to see that $\mathcal{F}_{cc}(\mathbb{R})$ cannot form a vector space. One of the reasons is that, given any $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$, the difference $\tilde{a} \ominus \tilde{a}$ is not a zero element of $\mathcal{F}_{cc}(\mathbb{R})$. It is clear to see that $\tilde{1}_{\{0\}}$ is a zero element, since

$$\tilde{a} \oplus \tilde{1}_{\{0\}} = \tilde{1}_{\{0\}} \oplus \tilde{a} = \tilde{a}$$

for any $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$. However, we cannot have $\tilde{a} \ominus \tilde{a} = \tilde{1}_{\{0\}}$ for any $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$. We also recall that the following family

$$\Omega = \{\tilde{a} \ominus \tilde{a} : \tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})\}$$

is called the null set of $\mathcal{F}_{cc}(\mathbb{R})$, which can be regarded as a kind of “zero element” of $\mathcal{F}_{cc}(\mathbb{R})$. In this section, we shall present some properties involving the null set Ω , which will be used for establishing the so-called near fixed point theorems in $\mathcal{F}_{cc}(\mathbb{R})$. For further discussion, we present some useful properties.

Proposition 1. *The following statements hold true.*

- $\lambda(\tilde{a} \oplus \tilde{b}) = \lambda\tilde{a} \oplus \lambda\tilde{b}$ for $\lambda \in \mathbb{R}$ and $\tilde{a}, \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$;
- $\lambda_1(\lambda_2\tilde{a}) = (\lambda_1\lambda_2)\tilde{a}$ for $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$;
- $\tilde{\omega} \in \Omega$ implies $-\tilde{\omega} = \tilde{\omega}$.
- $\lambda\Omega = \Omega$ for $\lambda \in \mathbb{R}$ with $\lambda \neq 0$.
- Ω is closed under the fuzzy addition; that is, $\tilde{\omega}^{(1)} \oplus \tilde{\omega}^{(2)} \in \Omega$ for any $\tilde{\omega}^{(1)}, \tilde{\omega}^{(2)} \in \Omega$.

Since the null set Ω can be regarded as a kind of “zero element”, we can propose the almost identical concept for elements in $\mathcal{F}_{cc}(\mathbb{R})$.

Definition 1. *Given any $\tilde{a}, \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$, we say that \tilde{a} and \tilde{b} are almost identical if and only if there exist $\tilde{\omega}^{(1)}, \tilde{\omega}^{(2)} \in \Omega$ such that $\tilde{a} \oplus \tilde{\omega}^{(1)} = \tilde{b} \oplus \tilde{\omega}^{(2)}$. In this case, we write $\tilde{a} \stackrel{\Omega}{\cong} \tilde{b}$.*

Given any $\tilde{a}, \tilde{b}, \tilde{c} \in \mathcal{F}_{cc}(\mathbb{R})$ with $\tilde{a} \ominus \tilde{b} = \tilde{c}$, we cannot obtain the equality $\tilde{a} = \tilde{b} \oplus \tilde{c}$ as the usual sense. As a matter of fact, we can just have $\tilde{a} \stackrel{\Omega}{\cong} \tilde{b} \oplus \tilde{c}$. Indeed, since $\tilde{a} \ominus \tilde{b} = \tilde{c}$, by adding \tilde{b} on both sides, we obtain $\tilde{a} \oplus \tilde{\omega} = \tilde{b} \oplus \tilde{c}$, where $\tilde{\omega} = \tilde{b} \ominus \tilde{b} \in \Omega$. This says that $\tilde{a} \stackrel{\Omega}{\cong} \tilde{b} \oplus \tilde{c}$.

Proposition 2. *The binary relation $\stackrel{\Omega}{\cong}$ is an equivalence relation.*

Proof. For any $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$, $\tilde{a} = \tilde{a}$ implies $\tilde{a} \stackrel{\Omega}{\cong} \tilde{a}$, which shows the reflexivity. The symmetry is obvious by the definition of the binary relation $\stackrel{\Omega}{\cong}$. Regarding the transitivity, for $\tilde{a} \stackrel{\Omega}{\cong} \tilde{b}$ and $\tilde{b} \stackrel{\Omega}{\cong} \tilde{c}$, we want to claim $\tilde{a} \stackrel{\Omega}{\cong} \tilde{c}$. By definition, we have

$$\tilde{a} \oplus \tilde{\omega}^{(1)} = \tilde{b} \oplus \tilde{\omega}^{(2)} \text{ and } \tilde{b} \oplus \tilde{\omega}^{(3)} = \tilde{c} \oplus \tilde{\omega}^{(4)}$$

for some $\tilde{\omega}_i \in \Omega$ for $i = 1, \dots, 4$. Then

$$\tilde{a} \oplus \tilde{\omega}^{(1)} \oplus \tilde{\omega}^{(3)} = \tilde{b} \oplus \tilde{\omega}^{(2)} \oplus \tilde{\omega}^{(3)} = \tilde{c} \oplus \tilde{\omega}^{(4)} \oplus \tilde{\omega}^{(2)},$$

which shows $\tilde{a} \stackrel{\Omega}{\cong} \tilde{c}$, since Ω is closed under the fuzzy addition as shown in Proposition 1. This completes the proof. \square

According to the equivalence relation $\stackrel{\Omega}{\cong}$, for any $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$, we define the equivalence class

$$[\tilde{a}] = \{\tilde{b} \in \mathcal{F}_{cc}(\mathbb{R}) : \tilde{a} \stackrel{\Omega}{\cong} \tilde{b}\}.$$

The family of all classes $[\tilde{a}]$ for $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ is denoted by $[\mathcal{F}_{cc}(\mathbb{R})]$. In this case, the family $[\mathcal{F}_{cc}(\mathbb{R})]$ is called the *quotient set* of $\mathcal{F}_{cc}(\mathbb{R})$. We also have that $\tilde{b} \in [\tilde{a}]$ implies $[\tilde{a}] = [\tilde{b}]$. In other words, the family of all equivalence classes form a partition of the whole set $\mathcal{F}_{cc}(\mathbb{R})$. We also remark that the quotient set $[\mathcal{F}_{cc}(\mathbb{R})]$ is still not a vector space. The reason is

$$(\alpha + \beta)[\tilde{a}] \neq \alpha[\tilde{a}] \oplus \beta[\tilde{a}]$$

for $\alpha \cdot \beta < 0$, since $(\alpha + \beta)\tilde{a} \neq \alpha\tilde{a} \oplus \beta\tilde{a}$ for $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ with $\alpha \cdot \beta < 0$.

3. Near Metric Space of Fuzzy Numbers

To study the near fixed point in $\mathcal{F}_{cc}(\mathbb{R})$, we are going to consider the metric d defined on $\mathcal{F}_{cc}(\mathbb{R}) \times \mathcal{F}_{cc}(\mathbb{R})$.

Definition 2. For the nonnegative real-valued function $d : \mathcal{F}_{cc}(\mathbb{R}) \times \mathcal{F}_{cc}(\mathbb{R}) \rightarrow \mathbb{R}_+$ defined on the product space $\mathcal{F}_{cc}(\mathbb{R}) \times \mathcal{F}_{cc}(\mathbb{R})$, we consider the following conditions:

- (i) $d(\tilde{a}, \tilde{b}) = 0$ if and only if $\tilde{a} \stackrel{\Omega}{=} \tilde{b}$ for all $\tilde{a}, \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$;
 - (ii) $d(\tilde{a}, \tilde{b}) = d(\tilde{b}, \tilde{a})$ for all $\tilde{a}, \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$;
 - (iii) $d(\tilde{a}, \tilde{b}) \leq d(\tilde{a}, \tilde{c}) + d(\tilde{c}, \tilde{b})$ for all $\tilde{a}, \tilde{b}, \tilde{c} \in \mathcal{F}_{cc}(\mathbb{R})$;
- A pair $(\mathcal{F}_{cc}(\mathbb{R}), d)$ is called a near pseudo-metric space of fuzzy numbers if and only if d satisfies conditions (ii) and (iii).
 - A pair $(\mathcal{F}_{cc}(\mathbb{R}), d)$ is called a near metric space of fuzzy numbers if and only if d satisfies conditions (i), (ii) and (iii).

We say that d satisfies the null equalities if and only if the following condition (iv) is satisfied:

- (iv) for any $\tilde{\omega}^{(1)}, \tilde{\omega}^{(2)} \in \Omega$ and $\tilde{a}, \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$, the following three equalities are satisfied:
 - $d(\tilde{a} \oplus \tilde{\omega}^{(1)}, \tilde{b} \oplus \tilde{\omega}^{(2)}) = d(\tilde{a}, \tilde{b})$;
 - $d(\tilde{a} \oplus \tilde{\omega}^{(1)}, \tilde{b}) = d(\tilde{a}, \tilde{b})$;
 - $d(\tilde{a}, \tilde{b} \oplus \tilde{\omega}^{(2)}) = d(\tilde{a}, \tilde{b})$.

Example 1. Let us define a nonnegative real-valued function $d : \mathcal{F}_{cc}(\mathbb{R}) \times \mathcal{F}_{cc}(\mathbb{R}) \rightarrow \mathbb{R}_+$ by

$$d(\tilde{a}, \tilde{b}) = \sup_{\alpha \in [0,1]} \left| (\tilde{a}_\alpha^L + \tilde{a}_\alpha^U) - (\tilde{b}_\alpha^L + \tilde{b}_\alpha^U) \right|. \tag{3}$$

Then $(\mathcal{F}_{cc}(\mathbb{R}), d)$ is not a (conventional) metric space, since $d(\tilde{a}, \tilde{b}) = 0$ cannot imply $\tilde{a} = \tilde{b}$. However, we are going to claim that $(\mathcal{F}_{cc}(\mathbb{R}), d)$ is a near metric space of fuzzy numbers such that d satisfies the null equality.

- (i) Given any fuzzy numbers \tilde{a} and \tilde{b} , we see that $\tilde{a}_\alpha^L - \tilde{b}_\alpha^L \leq \tilde{a}_\alpha^U - \tilde{b}_\alpha^L$ for all $\alpha \in [0, 1]$. Therefore

$$\text{if } \tilde{a}_\alpha^U - \tilde{b}_\alpha^L < 0 \text{ for some } \alpha \in [0, 1], \text{ then } d(\tilde{a}, \tilde{b}) = \sup_{\alpha \in [0,1]} \left| \tilde{a}_\alpha^L + \tilde{a}_\alpha^U - \tilde{b}_\alpha^L - \tilde{b}_\alpha^U \right| \neq 0.$$

Suppose that

$$0 = d(\tilde{a}, \tilde{b}) = \sup_{\alpha \in [0,1]} \left| \tilde{a}_\alpha^L + \tilde{a}_\alpha^U - \tilde{b}_\alpha^L - \tilde{b}_\alpha^U \right|. \tag{4}$$

We are going to claim $\tilde{a} \stackrel{\Omega}{=} \tilde{b}$. From (4), we must have $\tilde{a}_\alpha^U - \tilde{b}_\alpha^L \geq 0$ for all $\alpha \in [0, 1]$. Now we also have $|\tilde{a}_\alpha^L + \tilde{a}_\alpha^U - \tilde{b}_\alpha^L - \tilde{b}_\alpha^U| = 0$ for all $\alpha \in [0, 1]$, which also says that $\tilde{a}_\alpha^L + \tilde{a}_\alpha^U = \tilde{b}_\alpha^L + \tilde{b}_\alpha^U$ for all $\alpha \in [0, 1]$, i.e., $\tilde{a}_\alpha^L + \tilde{b}_\alpha^L - \tilde{b}_\alpha^U = 2\tilde{b}_\alpha^L - \tilde{a}_\alpha^U$ for all $\alpha \in [0, 1]$. It is easy to see that $\tilde{a}_\alpha^L + \tilde{b}_\alpha^L - \tilde{b}_\alpha^U \leq \tilde{a}_\alpha^U + \tilde{b}_\alpha^U - \tilde{b}_\alpha^L$ and $2\tilde{b}_\alpha^L - \tilde{a}_\alpha^U \leq \tilde{a}_\alpha^U + \tilde{b}_\alpha^U - \tilde{b}_\alpha^L$ by using the facts of $\tilde{a}_\alpha^L \leq \tilde{a}_\alpha^U$, $\tilde{b}_\alpha^L \leq \tilde{b}_\alpha^U$ and $\tilde{a}_\alpha^U \geq \tilde{b}_\alpha^L$. Therefore we can form two identical closed intervals

$$[\tilde{a}_\alpha^L + \tilde{b}_\alpha^L - \tilde{b}_\alpha^U, \tilde{a}_\alpha^U + \tilde{b}_\alpha^U - \tilde{b}_\alpha^L] = [2\tilde{b}_\alpha^L - \tilde{a}_\alpha^U, \tilde{a}_\alpha^U + \tilde{b}_\alpha^U - \tilde{b}_\alpha^L]. \tag{5}$$

Now the closed intervals $[\tilde{a}_\alpha^L + \tilde{b}_\alpha^L - \tilde{b}_\alpha^U, \tilde{a}_\alpha^U + \tilde{b}_\alpha^U - \tilde{b}_\alpha^L]$ and $[2\tilde{b}_\alpha^L - \tilde{a}_\alpha^U, \tilde{a}_\alpha^U + \tilde{b}_\alpha^U - \tilde{b}_\alpha^L]$ can be written as

$$[\tilde{a}_\alpha^L + \tilde{b}_\alpha^L - \tilde{b}_\alpha^U, \tilde{a}_\alpha^U + \tilde{b}_\alpha^U - \tilde{b}_\alpha^L] = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U] + [\tilde{b}_\alpha^L - \tilde{b}_\alpha^U, \tilde{b}_\alpha^U - \tilde{b}_\alpha^L] \tag{6}$$

and

$$[2\tilde{b}_\alpha^L - \tilde{a}_\alpha^U, \tilde{a}_\alpha^U + \tilde{b}_\alpha^U - \tilde{b}_\alpha^L] = [\tilde{b}_\alpha^L, \tilde{b}_\alpha^U] + [\tilde{b}_\alpha^L - \tilde{a}_\alpha^U, \tilde{a}_\alpha^U - \tilde{b}_\alpha^L]. \tag{7}$$

Then we can form two fuzzy numbers $\tilde{\omega}^{(1)}$ and $\tilde{\omega}^{(2)}$ such that

$$\left(\tilde{\omega}^{(1)}\right)_\alpha = [\tilde{b}_\alpha^L - \tilde{b}_\alpha^U, \tilde{b}_\alpha^U - \tilde{b}_\alpha^L] \text{ and } \left(\tilde{\omega}^{(2)}\right)_\alpha = [\tilde{b}_\alpha^L - \tilde{a}_\alpha^U, \tilde{a}_\alpha^U - \tilde{b}_\alpha^L].$$

It is clear to we see that $\tilde{\omega}^{(1)}, \tilde{\omega}^{(2)} \in \Omega$. Therefore, from (5)–(7), we obtain $\tilde{a} \oplus \tilde{\omega}^{(1)} = \tilde{b} \oplus \tilde{\omega}^{(2)}$, which shows $\tilde{a} \stackrel{\Omega}{\cong} \tilde{b}$, since $\tilde{\omega}^{(1)}, \tilde{\omega}^{(2)} \in \Omega$. Conversely, suppose that $\tilde{a} \stackrel{\Omega}{\cong} \tilde{b}$. Then $\tilde{a} \oplus \tilde{\omega}^{(1)} = \tilde{b} \oplus \tilde{\omega}^{(2)}$ for some $\tilde{\omega}^{(1)}, \tilde{\omega}^{(2)} \in \Omega$. By the definition of Ω , we have

$$\left(\tilde{\omega}^{(1)}\right)_\alpha = [-(\tilde{\omega}^{(1)})_\alpha^U, (\tilde{\omega}^{(1)})_\alpha^U] \text{ and } \left(\tilde{\omega}^{(2)}\right)_\alpha = [-(\tilde{\omega}^{(2)})_\alpha^U, (\tilde{\omega}^{(2)})_\alpha^U],$$

where $(\tilde{\omega}^{(1)})_\alpha^U, (\tilde{\omega}^{(2)})_\alpha^U \geq 0$ for all $\alpha \in [0, 1]$. From $\tilde{a} \oplus \tilde{\omega}^{(1)} = \tilde{b} \oplus \tilde{\omega}^{(2)}$, we obtain

$$\left[\tilde{a}_\alpha^L - (\tilde{\omega}^{(1)})_\alpha^U, \tilde{a}_\alpha^U + (\tilde{\omega}^{(1)})_\alpha^U\right] = \left[\tilde{b}_\alpha^L - (\tilde{\omega}^{(2)})_\alpha^U, \tilde{b}_\alpha^U + (\tilde{\omega}^{(2)})_\alpha^U\right],$$

i.e., $\tilde{a}_\alpha^L - (\tilde{\omega}^{(1)})_\alpha^U = \tilde{b}_\alpha^L - (\tilde{\omega}^{(2)})_\alpha^U$ and $\tilde{a}_\alpha^U + (\tilde{\omega}^{(1)})_\alpha^U = \tilde{b}_\alpha^U + (\tilde{\omega}^{(2)})_\alpha^U$ for all $\alpha \in [0, 1]$.

Then we obtain

$$\begin{aligned} d(\tilde{a}, \tilde{b}) &= \sup_{\alpha \in [0,1]} \left| \left(\tilde{a}_\alpha^L - \tilde{b}_\alpha^L\right) + \left(\tilde{a}_\alpha^U - \tilde{b}_\alpha^U\right) \right| \\ &= \sup_{\alpha \in [0,1]} \left| \left((\tilde{\omega}^{(1)})_\alpha^U - (\tilde{\omega}^{(2)})_\alpha^U\right) + \left((\tilde{\omega}^{(2)})_\alpha^U - (\tilde{\omega}^{(1)})_\alpha^U\right) \right| = 0. \end{aligned}$$

(ii) We have

$$d(\tilde{a}, \tilde{b}) = \sup_{\alpha \in [0,1]} \left| \tilde{a}_\alpha^L + \tilde{a}_\alpha^U - \tilde{b}_\alpha^L - \tilde{b}_\alpha^U \right| = \sup_{\alpha \in [0,1]} \left| \tilde{b}_\alpha^L + \tilde{b}_\alpha^U - \tilde{a}_\alpha^L - \tilde{a}_\alpha^U \right| = d(\tilde{b}, \tilde{a}).$$

(iii) Given any $\tilde{a}, \tilde{b}, \tilde{c} \in \mathcal{F}_{cc}(\mathbb{R})$, we have

$$\begin{aligned} d(\tilde{a}, \tilde{b}) &= \sup_{\alpha \in [0,1]} \left| \tilde{a}_\alpha^L + \tilde{a}_\alpha^U - \tilde{b}_\alpha^L - \tilde{b}_\alpha^U \right| = \sup_{\alpha \in [0,1]} \left| \left(\tilde{a}_\alpha^L + \tilde{a}_\alpha^U - \tilde{c}_\alpha^L - \tilde{c}_\alpha^U\right) + \left(\tilde{c}_\alpha^L + \tilde{c}_\alpha^U - \tilde{b}_\alpha^L - \tilde{b}_\alpha^U\right) \right| \\ &\leq \sup_{\alpha \in [0,1]} \left(\left| \tilde{a}_\alpha^L + \tilde{a}_\alpha^U - \tilde{c}_\alpha^L - \tilde{c}_\alpha^U \right| + \left| \tilde{c}_\alpha^L + \tilde{c}_\alpha^U - \tilde{b}_\alpha^L - \tilde{b}_\alpha^U \right| \right) \\ &\leq \sup_{\alpha \in [0,1]} \left| \tilde{a}_\alpha^L + \tilde{a}_\alpha^U - \tilde{c}_\alpha^L - \tilde{c}_\alpha^U \right| + \sup_{\alpha \in [0,1]} \left| \tilde{c}_\alpha^L + \tilde{c}_\alpha^U - \tilde{b}_\alpha^L - \tilde{b}_\alpha^U \right| \\ &= d(\tilde{a}, \tilde{c}) + \tilde{b}_\alpha^U(\tilde{c}, \tilde{b}). \end{aligned}$$

(iv) For any $\tilde{a}, \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\tilde{\omega}^{(1)}, \tilde{\omega}^{(2)} \in \Omega$, i.e., $(\tilde{\omega}^{(1)})_\alpha^U, (\tilde{\omega}^{(2)})_\alpha^U \geq 0$, where $(\tilde{\omega}^{(1)})_\alpha^L = -(\tilde{\omega}^{(1)})_\alpha^U$ and $(\tilde{\omega}^{(2)})_\alpha^L = -(\tilde{\omega}^{(2)})_\alpha^U$, we have

$$\begin{aligned}
 d(\tilde{a} \oplus \tilde{\omega}^{(1)}, \tilde{b} \oplus \tilde{\omega}^{(2)}) &= \sup_{\alpha \in [0,1]} \left| \left(\tilde{a} \oplus \tilde{\omega}^{(1)} \right)_\alpha^L + \left(\tilde{a} \oplus \tilde{\omega}^{(1)} \right)_\alpha^U - \left(\tilde{b} \oplus \tilde{\omega}^{(2)} \right)_\alpha^L - \left(\tilde{b} \oplus \tilde{\omega}^{(2)} \right)_\alpha^U \right| \\
 &= \sup_{\alpha \in [0,1]} \left| \tilde{a}_\alpha^L + (\tilde{\omega}^{(1)})_\alpha^L + \tilde{a}_\alpha^U + (\tilde{\omega}^{(1)})_\alpha^U - \tilde{b}_\alpha^L - (\tilde{\omega}^{(2)})_\alpha^L - \tilde{b}_\alpha^U - (\tilde{\omega}^{(2)})_\alpha^U \right| \\
 &= \sup_{\alpha \in [0,1]} \left| \tilde{a}_\alpha^L - (\tilde{\omega}^{(1)})_\alpha^U + \tilde{a}_\alpha^U + (\tilde{\omega}^{(1)})_\alpha^U - \tilde{b}_\alpha^L + (\tilde{\omega}^{(2)})_\alpha^U - \tilde{b}_\alpha^U - (\tilde{\omega}^{(2)})_\alpha^U \right| \\
 &= \sup_{\alpha \in [0,1]} \left| \left(\tilde{a}_\alpha^L + \tilde{a}_\alpha^U \right) - \left(\tilde{b}_\alpha^L + \tilde{b}_\alpha^U \right) \right| = d(\tilde{a}, \tilde{b})
 \end{aligned}$$

The verification is complete.

4. Near Normed Space of Fuzzy Numbers

Recall that $\mathcal{F}_{cc}(\mathbb{R})$ is not a vector space. Therefore we cannot endow a norm to $\mathcal{F}_{cc}(\mathbb{R})$ in the conventional way to consider the normed space $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$. However, we can propose the so-called near normed space of fuzzy numbers involving the null set Ω as follows.

Definition 3. Given the nonnegative real-valued function $\|\cdot\|: \mathcal{F}_{cc}(\mathbb{R}) \rightarrow \mathbb{R}_+$ defined on $\mathcal{F}_{cc}(\mathbb{R})$, we consider the following conditions:

- (i) $\|\lambda \tilde{a}\| = |\lambda| \cdot \|\tilde{a}\|$ for any $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\lambda \in \mathbb{F}$;
- (i') $\|\lambda \tilde{a}\| = |\lambda| \cdot \|\tilde{a}\|$ for any $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\lambda \in \mathbb{F}$ with $\lambda \neq 0$.
- (ii) $\|\tilde{a} \oplus \tilde{b}\| \leq \|\tilde{a}\| + \|\tilde{b}\|$ for any $\tilde{a}, \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$.
- (iii) $\|\tilde{a}\| = 0$ implies $\tilde{a} \in \Omega$.

We say that $\|\cdot\|$ satisfies the null condition when condition (iii) is replaced by $\|\tilde{a}\| = 0$ if and only if $\tilde{a} \in \Omega$. Different kinds of near normed space of fuzzy numbers are defined below.

- We say that $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ is a near pseudo-seminormed space of fuzzy numbers if and only if conditions (i') and (ii) are satisfied.
- We say that $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ is a near seminormed space of fuzzy numbers if and only if conditions (i) and (ii) are satisfied.
- We say that $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ is a near pseudo-normed space of fuzzy numbers if and only if conditions (i'), (ii) and (iii) are satisfied.
- We say that $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ is a near normed space of fuzzy numbers if and only if conditions (i), (ii) and (iii) are satisfied.

Now we consider the following definitions:

- We say that $\|\cdot\|$ satisfies the null super-inequality if and only if $\|\tilde{a} \oplus \tilde{\omega}\| \geq \|\tilde{a}\|$ for any $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\tilde{\omega} \in \Omega$.
- We say that $\|\cdot\|$ satisfies the null sub-inequality if and only if $\|\tilde{a} \oplus \tilde{\omega}\| \leq \|\tilde{a}\|$ for any $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\tilde{\omega} \in \Omega$.
- We say that $\|\cdot\|$ satisfies the null equality if and only if $\|\tilde{a} \oplus \tilde{\omega}\| = \|\tilde{a}\|$ for any $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\tilde{\omega} \in \Omega$.

For any $\tilde{a}, \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$, since $-(\tilde{b} \ominus \tilde{a}) = \tilde{a} \ominus \tilde{b}$, we have

$$\|\tilde{a} \ominus \tilde{b}\| = \|\tilde{b} \ominus \tilde{a}\|.$$

Example 2. For any $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$, we define

$$\|\tilde{a}\| = \sup_{\alpha \in [0,1]} \left| \tilde{a}_\alpha^L + \tilde{a}_\alpha^U \right|.$$

Then we have the following properties.

- $\|\tilde{a}\| = 0$ if and only if $\tilde{a} \in \Omega$. Indeed, if $\|\tilde{a}\| = 0$, then $|\tilde{a}_\alpha^L + \tilde{a}_\alpha^U| = 0$ for all $\alpha \in [0, 1]$, which also says that $\tilde{a}_\alpha^L = -\tilde{a}_\alpha^U$ for all $\alpha \in [0, 1]$. This shows that $\tilde{a} \in \Omega$. For the converse, if $\tilde{a} \in \Omega$ then $\tilde{a}_\alpha^L = -\tilde{a}_\alpha^U$ for all $\alpha \in [0, 1]$. This shows that $\|\tilde{a}\| = 0$. Therefore $\|\cdot\|$ satisfies the null condition.
- We have

$$\begin{aligned} \|\lambda\tilde{a}\| &= \sup_{\alpha \in [0,1]} |(\lambda\tilde{a})_\alpha^L + (\lambda\tilde{a})_\alpha^U| = \sup_{\alpha \in [0,1]} |\lambda\tilde{a}_\alpha^L + \lambda\tilde{a}_\alpha^U| \text{ (using (1))} \\ &= |\lambda| \cdot \sup_{\alpha \in [0,1]} |\tilde{a}_\alpha^L + \tilde{a}_\alpha^U| = |\lambda| \cdot \|\tilde{a}\|. \end{aligned}$$

- We have

$$\begin{aligned} \|\tilde{a} \oplus \tilde{b}\| &= \sup_{\alpha \in [0,1]} |(\tilde{a} \oplus \tilde{b})_\alpha^L + (\tilde{a} \oplus \tilde{b})_\alpha^U| = \sup_{\alpha \in [0,1]} |\tilde{a}_\alpha^L + \tilde{b}_\alpha^L + \tilde{a}_\alpha^U + \tilde{b}_\alpha^U| \\ &\leq \sup_{\alpha \in [0,1]} (|\tilde{a}_\alpha^L + \tilde{a}_\alpha^U| + |\tilde{b}_\alpha^L + \tilde{b}_\alpha^U|) \leq \sup_{\alpha \in [0,1]} |\tilde{a}_\alpha^L + \tilde{a}_\alpha^U| + \sup_{\alpha \in [0,1]} |\tilde{b}_\alpha^L + \tilde{b}_\alpha^U| \\ &= \|\tilde{a}\| + \|\tilde{b}\|. \end{aligned}$$

- For any $\tilde{\omega} \in \Omega$, i.e., $\tilde{\omega}_\alpha^L = -\tilde{\omega}_\alpha^U$ for all $\alpha \in [0, 1]$, we have

$$\begin{aligned} \|\tilde{a} \oplus \tilde{\omega}\| &= \sup_{\alpha \in [0,1]} |(\tilde{a} \oplus \tilde{\omega})_\alpha^L + (\tilde{a} \oplus \tilde{\omega})_\alpha^U| = \sup_{\alpha \in [0,1]} |\tilde{a}_\alpha^L + \tilde{\omega}_\alpha^L + \tilde{a}_\alpha^U + \tilde{\omega}_\alpha^U| \\ &= \sup_{\alpha \in [0,1]} |\tilde{a}_\alpha^L - \tilde{\omega}_\alpha^U + \tilde{a}_\alpha^U + \tilde{\omega}_\alpha^U| = \sup_{\alpha \in [0,1]} |\tilde{a}_\alpha^L + \tilde{a}_\alpha^U| = \|\tilde{a}\|. \end{aligned}$$

We conclude that $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ is a near normed space of fuzzy numbers such that the null condition and null equality are satisfied.

Proposition 3. Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a near pseudo-seminormed space of fuzzy numbers such that $\|\cdot\|$ satisfies the null super-inequality. For any $\tilde{a}, \tilde{c}, \tilde{b}^{(1)}, \dots, \tilde{b}^{(m)} \in \mathcal{F}_{cc}(\mathbb{R})$, we have

$$\|\tilde{a} \ominus \tilde{c}\| \leq \|\tilde{a} \ominus \tilde{b}^{(1)}\| + \|\tilde{b}^{(1)} \ominus \tilde{b}^{(2)}\| + \dots + \|\tilde{b}^{(j)} \ominus \tilde{b}^{(j+1)}\| + \dots + \|\tilde{b}^{(m)} \ominus \tilde{c}\|.$$

Proof. We have

$$\begin{aligned} \|\tilde{a} \ominus \tilde{c}\| &\leq \|\tilde{a} \oplus (-\tilde{c}) \oplus \tilde{b}^{(1)} \oplus \dots \oplus \tilde{b}^{(m)} \oplus (-\tilde{b}^{(1)}) \oplus \dots \oplus (-\tilde{b}^{(m)})\| \\ &\quad \text{(using the null super-inequality for } m \text{ times)} \\ &= \|\tilde{a} \oplus (-\tilde{b}^{(1)}) \oplus [\tilde{b}^{(1)} \oplus (-\tilde{b}^{(2)})] + \dots + [\tilde{b}^{(j)} \oplus (-\tilde{b}^{(j+1)})] + \dots + [\tilde{b}^{(m)} \oplus (-\tilde{c})\| \\ &\leq \|\tilde{a} \ominus \tilde{b}^{(1)}\| + \|\tilde{b}^{(1)} \ominus \tilde{b}^{(2)}\| + \dots + \|\tilde{b}^{(j)} \ominus \tilde{b}^{(j+1)}\| + \dots + \|\tilde{b}^{(m)} \ominus \tilde{c}\| \\ &\quad \text{(using the triangle inequality)}. \end{aligned}$$

This completes the proof. \square

Proposition 4. According to Definitions 1 and 3, the following statements hold true.

- Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a near pseudo-seminormed space of fuzzy numbers such that $\|\cdot\|$ satisfies the null equality. For any $\tilde{a}, \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$, if $\tilde{a} \stackrel{\Omega}{=} \tilde{b}$, then $\|\tilde{a}\| = \|\tilde{b}\|$.
- Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a near pseudo-normed space of fuzzy numbers. For any $\tilde{a}, \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$, we have that $\|\tilde{a} \ominus \tilde{b}\| = 0$ implies $\tilde{a} \stackrel{\Omega}{=} \tilde{b}$.
- Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a near pseudo-seminormed space of fuzzy numbers such that $\|\cdot\|$ satisfies the null super-inequality and null condition. For any $\tilde{a}, \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$, we have that $\tilde{a} \stackrel{\Omega}{=} \tilde{b}$ implies $\|\tilde{a} \ominus \tilde{b}\| = 0$.

Proof. To prove part (i), we see that $\tilde{a} \stackrel{\Omega}{=} \tilde{b}$ implies $\tilde{a} \oplus \tilde{\omega}^{(1)} = \tilde{b} \oplus \tilde{\omega}^{(2)}$ for some $\tilde{\omega}^{(1)}, \tilde{\omega}^{(2)} \in \Omega$. Therefore, using the null equality, we have

$$\|\tilde{a}\| = \|\tilde{a} \oplus \tilde{\omega}^{(1)}\| = \|\tilde{b} \oplus \tilde{\omega}^{(2)}\| = \|\tilde{b}\|.$$

To prove part (ii), suppose that $\|\tilde{a} \ominus \tilde{b}\| = 0$. Then $\tilde{a} \ominus \tilde{b} \in \Omega$, i.e., $\tilde{a} \ominus \tilde{b} = \tilde{\omega}^{(1)}$ for some $\tilde{\omega}^{(1)} \in \Omega$. By adding \tilde{b} on both sides, we have $\tilde{a} \oplus \tilde{\omega}^{(2)} = \tilde{b} \oplus \tilde{\omega}^{(1)}$ for some $\tilde{\omega}^{(2)} \in \Omega$, which says that $\tilde{a} \stackrel{\Omega}{=} \tilde{b}$.

To prove part (iii), for $\tilde{a} \stackrel{\Omega}{=} \tilde{b}$, we have $\tilde{a} \oplus \tilde{\omega}^{(1)} = \tilde{b} \oplus \tilde{\omega}^{(2)}$ for some $\tilde{\omega}^{(1)}, \tilde{\omega}^{(2)} \in \Omega$. Since Ω is closed under the fuzzy addition, it follows that

$$\tilde{a} \ominus \tilde{b} \oplus \tilde{\omega}^{(1)} = \tilde{a} \oplus \tilde{\omega}^{(1)} \ominus \tilde{b} = \tilde{b} \oplus \tilde{\omega}^{(2)} \ominus \tilde{b} = \tilde{\omega}^{(3)} \tag{8}$$

for some $\tilde{\omega}^{(3)} \in \Omega$. Using the null super-inequality, null condition and (8), we have

$$\|\tilde{a} \ominus \tilde{b}\| \leq \|\tilde{a} \ominus \tilde{b} \oplus \tilde{\omega}^{(1)}\| = \|\tilde{\omega}^{(3)}\| = 0.$$

This completes the proof. \square

5. Cauchy Sequences

In this section, we are going to introduce the concepts of Cauchy sequences and completeness in the near metric space of fuzzy numbers and the near normed space of fuzzy numbers.

5.1. Cauchy Sequences in Near Metric Space of Fuzzy Numbers

We first introduce the concept of limit in the near metric space of fuzzy numbers.

Definition 4. Let $(\mathcal{F}_{cc}(\mathbb{R}), d)$ be a near pseudo-metric space of fuzzy numbers. The sequence $\{\tilde{a}^{(n)}\}_{n=1}^{\infty}$ in $\mathcal{F}_{cc}(\mathbb{R})$ is said to be convergent if and only if

$$\lim_{n \rightarrow \infty} d(\tilde{a}^{(n)}, \tilde{a}) = 0 \text{ for some } \tilde{a} \in \mathcal{F}_{cc}(\mathbb{R}).$$

The element \tilde{a} is called the limit of the sequence $\{\tilde{a}^{(n)}\}_{n=1}^{\infty}$.

Let $\{\tilde{a}^{(n)}\}_{n=1}^{\infty}$ be a sequence in $(\mathcal{F}_{cc}(\mathbb{R}), d)$. If there exist $\tilde{a}, \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$ such that

$$\lim_{n \rightarrow \infty} d(\tilde{a}^{(n)}, \tilde{a}) = 0 = \lim_{n \rightarrow \infty} d(\tilde{a}^{(n)}, \tilde{b}),$$

then, by the triangle inequality (iii) in Definition 2, we have

$$0 \leq d(\tilde{a}, \tilde{b}) \leq d(\tilde{a}, \tilde{a}^{(n)}) + d(\tilde{a}^{(n)}, \tilde{b}) \rightarrow 0 + 0 = 0 \text{ as } n \rightarrow \infty, \tag{9}$$

which says that $d(\tilde{a}, \tilde{b}) = 0$. By condition (i) in Definition 2, we see that $\tilde{a} \stackrel{\Omega}{=} \tilde{b}$, which also says that \tilde{b} is in the equivalence class $[\tilde{a}]$.

Proposition 5. Suppose that d satisfies the null equality (iv) in Definition 2. Let $\{\tilde{a}^{(n)}\}_{n=1}^{\infty}$ be a sequence in $\mathcal{F}_{cc}(\mathbb{R})$ satisfying $d(\tilde{a}^{(n)}, \tilde{a}) \rightarrow 0$ as $n \rightarrow \infty$. Then $d(\tilde{a}^{(n)}, \tilde{b}) \rightarrow 0$ as $n \rightarrow \infty$ for any $\tilde{b} \in [\tilde{a}]$.

Proof. For $\tilde{b} \in [\tilde{a}]$, we have $\tilde{a} \oplus \tilde{\omega}^{(1)} = \tilde{b} \oplus \tilde{\omega}^{(2)}$ for some $\tilde{\omega}^{(1)}, \tilde{\omega}^{(2)} \in \Omega$. Using the null equality, we obtain

$$\begin{aligned} 0 \leq d(\tilde{a}^{(n)}, \tilde{b}) &= d(\tilde{a}^{(n)}, \tilde{\omega}^{(2)} \oplus \tilde{b}) = d(\tilde{a}^{(n)}, \tilde{\omega}^{(1)} \oplus \tilde{a}) \\ &= d(\tilde{a}^{(n)}, \tilde{a}) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

This completes the proof. \square

Inspired by the above result, we propose the following definition.

Definition 5. If $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ is a sequence in $\mathcal{F}_{cc}(\mathbb{R})$ satisfying

$$\lim_{n \rightarrow \infty} d(\tilde{a}^{(n)}, \tilde{a}) = 0$$

for some $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$, then the equivalence class $[\tilde{a}]$ is called the class limit of $\{\tilde{a}^{(n)}\}_{n=1}^\infty$. We also write

$$\lim_{n \rightarrow \infty} \tilde{a}^{(n)} = [\tilde{a}] \text{ or } \tilde{a}^{(n)} \rightarrow [\tilde{a}].$$

Proposition 6. The class limit in the near metric space of fuzzy numbers $(\mathcal{F}_{cc}(\mathbb{R}), d)$ is unique.

Proof. Suppose that the sequence $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ is convergent with the class limits $[\tilde{a}]$ and $[\tilde{b}]$. Then we have

$$\lim_{n \rightarrow \infty} d(\tilde{a}^{(n)}, \tilde{a}) = 0 \text{ and } \lim_{n \rightarrow \infty} d(\tilde{a}^{(n)}, \tilde{b}) = 0,$$

which says that $d(\tilde{a}, \tilde{b}) = 0$ by referring to (9). Therefore we obtain $\tilde{b} \in [\tilde{a}]$, i.e., $[\tilde{a}] = [\tilde{b}]$. This completes the proof. \square

Definition 6. Let $(\mathcal{F}_{cc}(\mathbb{R}), d)$ be a near metric space of fuzzy numbers.

- A sequence $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ in $\mathcal{F}_{cc}(\mathbb{R})$ is called a Cauchy sequence if and only if, given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(\tilde{a}^{(n)}, \tilde{a}^{(m)}) < \epsilon$ for all $n > N$ and $m > N$.
- A subset \mathcal{M} of $\mathcal{F}_{cc}(\mathbb{R})$ is said to be complete if and only if every Cauchy sequence in \mathcal{M} is convergent to some element in \mathcal{M} .

The following result is not hard to prove.

Proposition 7. Every convergent sequence in a near metric space of fuzzy numbers is a Cauchy sequence.

Example 3. Continued from Example 1, we see that d satisfies the null equality, where the metric d is defined in (3). Now we want to show that this space is also complete. Suppose that $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence in the near metric space of fuzzy numbers $(\mathcal{F}_{cc}(\mathbb{R}), d)$. For convenience, the end-points of α -level closed interval of $\tilde{a}^{(n)}$ is written by

$$(\tilde{a}^{(n)})_\alpha^L = \tilde{a}_\alpha^{(n,L)} \text{ and } (\tilde{a}^{(n)})_\alpha^U = \tilde{a}_\alpha^{(n,U)}.$$

Then we have

$$d(\tilde{a}^{(n)}, \tilde{a}^{(m)}) = \sup_{\alpha \in [0,1]} \left| (\tilde{a}_\alpha^{(n,L)} + \tilde{a}_\alpha^{(n,U)}) - (\tilde{a}_\alpha^{(m,L)} + \tilde{a}_\alpha^{(m,U)}) \right| < \epsilon \tag{10}$$

for sufficiently large n and m . For each fixed $\alpha \in [0, 1]$, we define

$$c_\alpha^{(n)} = \tilde{a}_\alpha^{(n,L)} + \tilde{a}_\alpha^{(n,U)} \text{ and } c_\alpha^{(m)} = \tilde{a}_\alpha^{(m,L)} + \tilde{a}_\alpha^{(m,U)}.$$

Let $f_n(\alpha) = c_\alpha^{(n)}$. Then we can consider a sequence of continuous functions $\{f_n(\alpha)\}_{n=1}^\infty$ on $[0, 1]$. Then (10) shows that the sequence of functions $\{f_n(\alpha)\}_{n=1}^\infty$ satisfies the Cauchy condition for uniform convergence by referring to Apostol ([20], Theorem 9.3). This also says that $\{f_n(\alpha)\}_{n=1}^\infty$ converges uniformly to a limit function $f(\alpha) \equiv c_\alpha$ on $[0, 1]$. Therefore, for sufficiently large n , we have

$$\left| c_\alpha^{(n)} - c_\alpha \right| < \frac{\epsilon}{2} \text{ for all } \alpha \in [0, 1]. \tag{11}$$

Since each f_n is continuous on $[0, 1]$, Apostol ([20], Theorem 9.2) also says that the limit function $f(\alpha) \equiv c_\alpha$ is continuous on $[0, 1]$. The continuity of c_α on $[0, 1]$ allows us to find a fuzzy number \tilde{a} such that $\tilde{a}_\alpha^L + \tilde{a}_\alpha^U = c_\alpha$. Therefore, using (16), we have

$$d(\tilde{a}^{(n)}, \tilde{a}) = \sup_{\alpha \in [0,1]} \left| (\tilde{a}_\alpha^{(n,L)} + \tilde{a}_\alpha^{(n,U)}) - (\tilde{a}_\alpha^L + \tilde{a}_\alpha^U) \right| = \sup_{\alpha \in [0,1]} \left| c_\alpha^{(n)} - c_\alpha \right| \leq \frac{\epsilon}{2} < \epsilon$$

for sufficiently large n . This shows that the sequence $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ is convergent, i.e., the space $(\mathcal{F}_{cc}(\mathbb{R}), d)$ is complete.

5.2. Cauchy Sequences in Near Normed Space of Fuzzy Numbers

Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a near pseudo-seminormed space of fuzzy numbers. Given a sequence $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ in $\mathcal{F}_{cc}(\mathbb{R})$, it is clear that $\|\tilde{a}^{(n)} \ominus \tilde{a}\| = \|\tilde{a} \ominus \tilde{a}^{(n)}\|$. The concept of limit is defined below.

Definition 7. Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a near pseudo-seminormed space of fuzzy numbers. A sequence $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ in $\mathcal{F}_{cc}(\mathbb{R})$ is said to converge to $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ if and only if

$$\lim_{n \rightarrow \infty} \|\tilde{a}^{(n)} \ominus \tilde{a}\| = 0.$$

We have the following interesting results.

Proposition 8. Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a near pseudo-normed space of fuzzy numbers with the null set Ω , and let $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ be a sequence in $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$.

- (i) If the sequence $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ converges to \tilde{a} and \tilde{b} simultaneously, then $[\tilde{a}] = [\tilde{b}]$.
- (ii) Suppose that $\|\cdot\|$ satisfies the null equality. If the sequence $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ converges to \tilde{a} , then, give any $\tilde{b} \in [\tilde{a}]$, the sequence $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ converges to \tilde{b} .

Proof. To prove the first case of part (i), we have

$$\lim_{n \rightarrow \infty} \|\tilde{a} \ominus \tilde{a}^{(n)}\| = \lim_{n \rightarrow \infty} \|\tilde{a}^{(n)} \ominus \tilde{b}\| = 0.$$

By Proposition 3, we have

$$0 \leq \|\tilde{a} \ominus \tilde{b}\| \leq \|\tilde{a} \ominus \tilde{a}^{(n)}\| + \|\tilde{a}^{(n)} \ominus \tilde{b}\| \rightarrow 0 + 0 = 0, \tag{12}$$

which says that $\|\tilde{a} \ominus \tilde{b}\| = 0$. By Definition 3, we see that $\tilde{a} \ominus \tilde{b} \in \Omega$, i.e., $\tilde{a} \stackrel{\Omega}{=} \tilde{b}$, which also says that \tilde{b} is in the equivalence class $[\tilde{a}]$.

To prove part (ii), for any $\tilde{b} \in [\tilde{a}]$, i.e., $\tilde{a} \oplus \tilde{\omega}^{(1)} = \tilde{b} \oplus \tilde{\omega}^{(2)}$ for some $\tilde{\omega}^{(1)}, \tilde{\omega}^{(2)} \in \Omega$, using the null equality, we have

$$0 \leq \|\tilde{a}^{(n)} \ominus \tilde{b}\| = \|\tilde{b} \ominus \tilde{a}^{(n)}\| = \|\tilde{\omega}^{(2)} \oplus \tilde{b} \ominus \tilde{a}^{(n)}\| = \|\tilde{\omega}^{(1)} \oplus \tilde{a} \ominus \tilde{a}^{(n)}\| = \|\tilde{a} \ominus \tilde{a}^{(n)}\| \rightarrow 0.$$

This completes the proof. \square

Inspired by part (ii) of Proposition 8, we propose the following concept of limit.

Definition 8. Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a near pseudo-seminormed space of fuzzy numbers. If the sequence $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ in $\mathcal{F}_{cc}(\mathbb{R})$ converges to some $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$, then the equivalence class $[\tilde{a}]$ is called the class limit of $\{\tilde{a}^{(n)}\}_{n=1}^\infty$. We also write

$$\lim_{n \rightarrow \infty} \tilde{a}^{(n)} = [\tilde{a}] \text{ or } \tilde{a}^{(n)} \rightarrow [\tilde{a}].$$

We need to remark that if $[\tilde{a}]$ is a class limit and $\tilde{b} \in [\tilde{a}]$ then it is not necessarily that the sequence $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ converges to \tilde{b} unless $\|\cdot\|$ satisfies the null equality. In other words, for the class limit $[\tilde{a}]$, if $\|\cdot\|$ satisfies the null equality, then part (ii) of Proposition 8 says that sequence $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ converges to \tilde{b} for any $\tilde{b} \in [\tilde{a}]$.

Proposition 9. Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a near pseudo-normed space of fuzzy numbers such that $\|\cdot\|$ satisfies the null super-inequality. Then the class limit is unique.

Proof. Suppose that the sequence $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ is convergent with the class limits $[\tilde{a}]$ and $[\tilde{b}]$. Then, by definition, we have

$$\lim_{n \rightarrow \infty} \|\tilde{a} \ominus \tilde{a}^{(n)}\| = \lim_{n \rightarrow \infty} \|\tilde{a}^{(n)} \ominus \tilde{a}\| = \lim_{n \rightarrow \infty} \|\tilde{b} \ominus \tilde{a}^{(n)}\| = \lim_{n \rightarrow \infty} \|\tilde{a}^{(n)} \ominus \tilde{b}\| = 0,$$

which implies $\|\tilde{a} \ominus \tilde{b}\| = 0$ by referring to (12). By part (ii) of Proposition 4, we have $\tilde{a} \stackrel{\Omega}{=} \tilde{b}$, i.e., $[\tilde{a}] = [\tilde{b}]$. This shows the uniqueness in the sense of class limit. \square

Definition 9. Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a near pseudo-seminormed space of fuzzy numbers.

- A sequence $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ in $\mathcal{F}_{cc}(\mathbb{R})$ is called a Cauchy sequence if and only if, given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|\tilde{a}^{(n)} \ominus \tilde{a}^{(m)}\| = \|\tilde{a}^{(m)} \ominus \tilde{a}^{(n)}\| < \epsilon$$

for $m, n > N$ with $m \neq n$.

- A subset \mathcal{M} of $\mathcal{F}_{cc}(\mathbb{R})$ is said to be complete if and only if every Cauchy sequence in \mathcal{M} is convergent to some element in \mathcal{M} .

Proposition 10. Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a near pseudo-seminormed space of fuzzy numbers such that $\|\cdot\|$ satisfies the null super-inequality. Then every convergent sequence is a Cauchy sequence.

Proof. If the sequence $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ converges to \tilde{a} , then, given any $\epsilon > 0$,

$$\|\tilde{a}^{(n)} \ominus \tilde{a}\| = \|\tilde{a} \ominus \tilde{a}^{(n)}\| < \epsilon/2$$

for sufficiently large n . Therefore, by Proposition 3, we have

$$\|\tilde{a}^{(n)} \ominus \tilde{a}^{(m)}\| = \|\tilde{a}^{(m)} \ominus \tilde{a}^{(n)}\| \leq \|\tilde{a}^{(m)} \ominus \tilde{a}\| + \|\tilde{a} \ominus \tilde{a}^{(n)}\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for sufficiently large n and m , which says that $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence. This completes the proof. \square

Definition 10. Different kinds of near Banach spaces of fuzzy numbers are defined below.

- Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a near pseudo-seminormed space of fuzzy numbers. If $\mathcal{F}_{cc}(\mathbb{R})$ is complete, then it is called a near pseudo-semi-Banach space of fuzzy numbers.
- Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a seminormed space of fuzzy numbers. If $\mathcal{F}_{cc}(\mathbb{R})$ is complete, then it is called a near semi-Banach space of fuzzy numbers.
- Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a near pseudo-normed space of fuzzy numbers. If $\mathcal{F}_{cc}(\mathbb{R})$ is complete, then it is called a near pseudo-Banach space of fuzzy numbers.
- Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a near normed space of fuzzy numbers. If $\mathcal{F}_{cc}(\mathbb{R})$ is complete, then it is called a near Banach space of fuzzy numbers.

Example 4. Continued from Example 2, we want to show that the near normed space of fuzzy numbers $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ is complete. Suppose that $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence in $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$. Then we have $\|\tilde{a}^{(n)} \ominus \tilde{a}^{(m)}\| < \epsilon$ for $m, n > N$ with $m \neq n$, i.e.

$$\begin{aligned} \epsilon > \|\tilde{a}^{(n)} \ominus \tilde{a}^{(m)}\| &= \sup_{\alpha \in [0,1]} \left| \left(\tilde{a}^{(n)} \ominus \tilde{a}^{(m)}\right)_\alpha^L + \left(\tilde{a}^{(n)} \ominus \tilde{a}^{(m)}\right)_\alpha^U \right| \\ &= \sup_{\alpha \in [0,1]} \left| \tilde{a}_\alpha^{(n,L)} - \tilde{a}_\alpha^{(m,U)} + \tilde{a}_\alpha^{(n,U)} - \tilde{a}_\alpha^{(m,L)} \right| \\ &= \sup_{\alpha \in [0,1]} \left| \left(\tilde{a}_\alpha^{(n,L)} + \tilde{a}_\alpha^{(n,U)}\right) - \left(\tilde{a}_\alpha^{(m,L)} + \tilde{a}_\alpha^{(m,U)}\right) \right|. \end{aligned} \tag{13}$$

For each fixed $\alpha \in [0, 1]$, we define

$$c_\alpha^{(n)} = \tilde{a}_\alpha^{(n,L)} + \tilde{a}_\alpha^{(n,U)} \text{ and } c_\alpha^{(m)} = \tilde{a}_\alpha^{(m,L)} + \tilde{a}_\alpha^{(m,U)}.$$

By referring to Example 3, we can find a fuzzy number \tilde{a} such that, for sufficiently large n ,

$$\tilde{a}_\alpha^L + \tilde{a}_\alpha^U = c_\alpha \text{ and } \left| c_\alpha^{(n)} - c_\alpha \right| < \frac{\epsilon}{2} \text{ for all } \alpha \in [0, 1].$$

Therefore, for sufficiently large n , we have

$$\begin{aligned} \|\tilde{a}^{(n)} \ominus \tilde{a}\| &= \sup_{\alpha \in [0,1]} \left| \left(\tilde{a}^{(n)} \ominus \tilde{a}\right)_\alpha^L + \left(\tilde{a}^{(n)} \ominus \tilde{a}\right)_\alpha^U \right| = \sup_{\alpha \in [0,1]} \left| \tilde{a}_\alpha^{(n,L)} - \tilde{a}_\alpha^L + \tilde{a}_\alpha^{(n,U)} - \tilde{a}_\alpha^U \right| \\ &= \sup_{\alpha \in [0,1]} \left| \left(\tilde{a}_\alpha^{(n,L)} + \tilde{a}_\alpha^{(n,U)}\right) - \left(\tilde{a}_\alpha^L + \tilde{a}_\alpha^U\right) \right| = \sup_{\alpha \in [0,1]} \left| c_\alpha^{(n)} - c_\alpha \right| \leq \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

This shows that the sequence $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ is convergent, i.e., $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ is a near Banach space of fuzzy numbers.

6. Near Fixed Point Theorems in Near Metric Space of Fuzzy Numbers

Let $\tilde{T} : \mathcal{F}_{cc}(\mathbb{R}) \rightarrow \mathcal{F}_{cc}(\mathbb{R})$ be a fuzzy-number-valued function from $\mathcal{F}_{cc}(\mathbb{R})$ into itself. We say that $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ is a fixed point of \tilde{T} if and only if $\tilde{T}(\tilde{a}) = \tilde{a}$. The well-known Banach contraction principle presents the fixed point of function \tilde{T} when $(\mathcal{F}_{cc}(\mathbb{R}), d)$ is taken to be a metric space. Since $(\mathcal{F}_{cc}(\mathbb{R}), d)$ presented in Example 1 is not a metric space (it is a near metric space), we cannot study the Banach contraction principle on this space $(\mathcal{F}_{cc}(\mathbb{R}), d)$. In other words, we cannot study the fixed point of contractive mappings defined on $(\mathcal{F}_{cc}(\mathbb{R}), d)$ into itself in the conventional way. However, we can investigate the so-called near fixed point defined below.

Definition 11. Let $\tilde{T} : \mathcal{F}_{cc}(\mathbb{R}) \rightarrow \mathcal{F}_{cc}(\mathbb{R})$ be a fuzzy-number-valued function defined on $\mathcal{F}_{cc}(\mathbb{R})$ into itself. A point $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ is called a near fixed point of \tilde{T} if and only if $\tilde{T}(\tilde{a}) \stackrel{\Omega}{=} \tilde{a}$.

By definition, we see that $\tilde{T}(\tilde{a}) \stackrel{\Omega}{=} \tilde{a}$ if and only if there exist $\tilde{\omega}^{(1)}, \tilde{\omega}^{(2)} \in \Omega$ such that one of the following equalities is satisfied:

- $\tilde{T}(\tilde{a}) \oplus \tilde{\omega}^{(1)} = \tilde{a}$;
- $\tilde{T}(\tilde{a}) = \tilde{a} \oplus \tilde{\omega}^{(1)}$;
- $\tilde{T}(\tilde{a}) \oplus \tilde{\omega}^{(1)} = \tilde{a} \oplus \tilde{\omega}^{(2)}$.

We also see that if $\tilde{T}(\tilde{a}) = \tilde{a}$ then $\tilde{T}(\tilde{a}) \stackrel{\Omega}{=} \tilde{a}$, since the crisp number $\tilde{1}_{\{0\}}$ with value 0 is in Ω and $\tilde{a} \oplus \tilde{1}_{\{0\}} = \tilde{a}$.

Definition 12. A fuzzy-number-valued function $\tilde{T} : (\mathcal{F}_{cc}(\mathbb{R}), d) \rightarrow (\mathcal{F}_{cc}(\mathbb{R}), d)$ is called a metric contraction on $\mathcal{F}_{cc}(\mathbb{R})$ if and only if there is a real number $0 < \lambda < 1$ such that

$$d(\tilde{T}(\tilde{a}), \tilde{T}(\tilde{b})) \leq \lambda \cdot d(\tilde{a}, \tilde{b})$$

for any $\tilde{a}, \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$.

Given any initial element $\tilde{a}^{(0)} \in \mathcal{F}_{cc}(\mathbb{R})$, we define the iterative sequence $\{\tilde{a}^{(n)}\}_{n=1}^{\infty}$ using the fuzzy-number-valued function \tilde{T} as follows:

$$\tilde{a}^{(1)} = \tilde{T}(\tilde{a}^{(0)}), \quad \tilde{a}^{(2)} = \tilde{T}(\tilde{a}^{(1)}) = \tilde{T}^2(\tilde{a}^{(0)}), \dots, \tilde{a}^{(n)} = \tilde{T}^n(\tilde{a}^{(0)}), \dots \tag{14}$$

Under some suitable conditions, we are going to show that the sequence $\{\tilde{a}^{(n)}\}_{n=1}^{\infty}$ can converge to a near fixed point. If the near metric space of fuzzy numbers $(\mathcal{F}_{cc}(\mathbb{R}), d)$ is complete, then it is also called a complete near metric space of fuzzy numbers.

Theorem 1. (Near Fixed Point Theorem) Let $(\mathcal{F}_{cc}(\mathbb{R}), d)$ be a complete near metric space of fuzzy numbers such that d satisfies the null equality. Suppose that the fuzzy-number-valued function $\tilde{T} : (\mathcal{F}_{cc}(\mathbb{R}), d) \rightarrow (\mathcal{F}_{cc}(\mathbb{R}), d)$ is a metric contraction on $\mathcal{F}_{cc}(\mathbb{R})$. Then \tilde{T} has a near fixed point $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ satisfying $\tilde{T}(\tilde{a}) \stackrel{\Omega}{=} \tilde{a}$. Moreover, the near fixed point \tilde{a} is obtained by the limit

$$d(\tilde{a}^{(n)}, \tilde{a}) \rightarrow 0 \text{ as } n \rightarrow \infty$$

in which the sequence $\{\tilde{a}^{(n)}\}_{n=1}^{\infty}$ is generated according to (14). We also have the following properties.

- The uniqueness is in the sense that there is a unique equivalence class $[\tilde{a}]$ such that any $\tilde{a}^\circ \notin [\tilde{a}]$ cannot be a near fixed point.
- Each point $\tilde{a}^\circ \in [\tilde{a}]$ is also a near fixed point of \tilde{T} satisfying $\tilde{T}(\tilde{a}^\circ) \stackrel{\Omega}{=} \tilde{a}^\circ$ and $[\tilde{a}^\circ] = [\tilde{a}]$.
- If \tilde{a}° is a near fixed point of \tilde{T} , then $\tilde{a}^\circ \in [\tilde{a}]$, i.e., $[\tilde{a}^\circ] = [\tilde{a}]$. Equivalently, if \tilde{a} and \tilde{a}° are the near fixed points of \tilde{T} , then $\tilde{a} \stackrel{\Omega}{=} \tilde{a}^\circ$.

Proof. Given any initial element $\tilde{a}^{(0)} \in \mathcal{F}_{cc}(\mathbb{R})$, we have the iterative sequence $\{\tilde{a}^{(n)}\}_{n=1}^{\infty}$ according to (14). We are going to show that $\{\tilde{a}^{(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence. Since \tilde{T} is a metric contraction on $\mathcal{F}_{cc}(\mathbb{R})$, we have

$$\begin{aligned} d(\tilde{a}^{(m+1)}, \tilde{a}^{(m)}) &= d(\tilde{T}(\tilde{a}^{(m)}), \tilde{T}(\tilde{a}^{(m-1)})) \\ &\leq \lambda \cdot d(\tilde{a}^{(m)}, \tilde{a}^{(m-1)}) \\ &= \lambda \cdot d(\tilde{T}(\tilde{a}^{(m-1)}), \tilde{T}(\tilde{a}^{(m-2)})) \\ &\leq \lambda^2 \cdot d(\tilde{a}^{(m-1)}, \tilde{a}^{(m-2)}) \\ &\leq \dots \leq \lambda^m \cdot d(\tilde{a}^{(1)}, \tilde{a}^{(0)}) . \end{aligned}$$

For $n < m$, using the triangle inequality, we obtain

$$\begin{aligned} d(\tilde{a}^{(m)}, \tilde{a}^{(n)}) &\leq d(\tilde{a}^{(m)}, \tilde{a}^{(m-1)}) + d(\tilde{a}^{(m-1)}, \tilde{a}^{(m-2)}) + \dots + d(\tilde{a}^{(n+1)}, \tilde{a}^{(n)}) \\ &\leq (\lambda^{m-1} + \lambda^{m-2} + \dots + \lambda^n) \cdot d(\tilde{a}^{(1)}, \tilde{a}^{(0)}) \\ &= \lambda^n \cdot \frac{1 - \lambda^{m-n}}{1 - \lambda} \cdot d(\tilde{a}^{(1)}, \tilde{a}^{(0)}) . \end{aligned}$$

Since $0 < \lambda < 1$, we have $1 - \lambda^{m-n} < 1$ in the numerator, which says that

$$d(\tilde{a}^{(m)}, \tilde{a}^{(n)}) \leq \frac{\lambda^n}{1-\lambda} \cdot d(\tilde{a}^{(1)}, \tilde{a}^{(0)}) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This proves that $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence. Since the near metric space of fuzzy numbers $\mathcal{F}_{cc}(\mathbb{R})$ is complete, there exists $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ such that $d(\tilde{a}^{(n)}, \tilde{a}) \rightarrow 0$, i.e., $\tilde{a}^{(n)} \rightarrow [\tilde{a}]$ according to Definition 5 and Proposition 6.

We are going to show that any point $\tilde{a}^\circ \in [\tilde{a}]$ is a near fixed point. Now we have

$$\tilde{a}^\circ \oplus \tilde{\omega}^{(1)} = \tilde{a} \oplus \tilde{\omega}^{(2)} \text{ for some } \tilde{\omega}^{(1)}, \tilde{\omega}^{(2)} \in \Omega. \tag{15}$$

Therefore we obtain

$$\begin{aligned} d(\tilde{a}^\circ, \tilde{T}(\tilde{a}^\circ)) &= d(\tilde{a}^\circ \oplus \tilde{\omega}^{(1)}, \tilde{T}(\tilde{a}^\circ)) \text{ (since } d \text{ satisfies the null equality)} \\ &\leq d(\tilde{a}^\circ \oplus \tilde{\omega}^{(1)}, \tilde{a}^{(m)}) + d(\tilde{a}^{(m)}, \tilde{T}(\tilde{a}^\circ)) \text{ (using the triangle inequality)} \\ &= d(\tilde{a}^\circ \oplus \tilde{\omega}^{(1)}, \tilde{a}^{(m)}) + d(\tilde{T}(\tilde{a}^{(m-1)}), \tilde{T}(\tilde{a}^\circ)) \\ &\leq d(\tilde{a}^\circ \oplus \tilde{\omega}^{(1)}, \tilde{a}^{(m)}) + \lambda \cdot d(\tilde{a}^{(m-1)}, \tilde{a}^\circ) \text{ (using the metric contraction)} \\ &= d(\tilde{a}^\circ \oplus \tilde{\omega}^{(1)}, \tilde{a}^{(m)}) + \lambda \cdot d(\tilde{a}^{(m-1)}, \tilde{a}^\circ \oplus \tilde{\omega}^{(1)}) \text{ (since } d \text{ satisfies the null equality)} \\ &= d(\tilde{a} \oplus \tilde{\omega}^{(2)}, \tilde{a}^{(m)}) + \lambda \cdot d(\tilde{a}^{(m-1)}, \tilde{a} \oplus \tilde{\omega}^{(2)}) \text{ (using (15))} \\ &= d(\tilde{a}, \tilde{a}^{(m)}) + \lambda \cdot d(\tilde{a}^{(m-1)}, \tilde{a}) \text{ (since } d \text{ satisfies the null equality),} \end{aligned}$$

which implies $d(\tilde{a}^\circ, \tilde{T}(\tilde{a}^\circ)) = 0$ as $m \rightarrow \infty$, i.e., $\tilde{T}(\tilde{a}^\circ) \stackrel{\Omega}{=} \tilde{a}^\circ$ for any point $\tilde{a}^\circ \in [\tilde{a}]$.

Now we assume that there is another near fixed point \tilde{a}° of \tilde{T} with $\tilde{a}^\circ \notin [\tilde{a}]$, i.e., $\tilde{a}^\circ \stackrel{\Omega}{=} \tilde{T}(\tilde{a}^\circ)$. Then

$$\tilde{a}^\circ \oplus \tilde{\omega}^{(1)} = \tilde{T}(\tilde{a}^\circ) \oplus \tilde{\omega}^{(2)} \text{ and } \tilde{a} \oplus \tilde{\omega}^{(3)} = \tilde{T}(\tilde{a}) \oplus \tilde{\omega}^{(4)}$$

for some $\tilde{\omega}_i \in \Omega, i = 1, \dots, 4$. Since \tilde{T} is a metric contraction on $\mathcal{F}_{cc}(\mathbb{R})$ and d satisfies the null equality, we obtain

$$\begin{aligned} d(\tilde{a}^\circ, \tilde{a}) &= d(\tilde{a}^\circ \oplus \tilde{\omega}^{(1)}, \tilde{a} \oplus \tilde{\omega}^{(3)}) = d(\tilde{T}(\tilde{a}^\circ) \oplus \tilde{\omega}^{(2)}, \tilde{T}(\tilde{a}) \oplus \tilde{\omega}^{(4)}) \\ &= d(\tilde{T}(\tilde{a}^\circ), \tilde{T}(\tilde{a})) \leq \lambda \cdot d(\tilde{a}^\circ, \tilde{a}). \end{aligned}$$

Since $0 < \lambda < 1$, we must have $d(\tilde{a}^\circ, \tilde{a}) = 0$, i.e., $\tilde{a}^\circ \stackrel{\Omega}{=} \tilde{a}$, which contradicts $\tilde{a}^\circ \notin [\tilde{a}]$. Therefore, any $\tilde{a}^\circ \notin [\tilde{a}]$ cannot be a near fixed point. Equivalently, if \tilde{a}° is a near fixed point of \tilde{T} , then $\tilde{a}^\circ \in [\tilde{a}]$. This completes the proof. \square

Example 5. Continued from Example 3, the near metric space of fuzzy numbers $(\mathcal{F}_{cc}(\mathbb{R}), d)$ is complete. Given a real number $0 < \lambda < 1$, we consider the fuzzy-number-valued function

$$\tilde{T}(\tilde{a}) = \tilde{1}_{\{\lambda\}} \otimes \tilde{a}, \text{ where } \tilde{1}_{\{\lambda\}} \text{ is a crisp number with value } \lambda.$$

It is clear to see that

$$(\tilde{T}(\tilde{a}))_\alpha^L = \lambda \cdot \tilde{a}_\alpha^L \text{ and } (\tilde{T}(\tilde{a}))_\alpha^U = \lambda \cdot \tilde{a}_\alpha^U \text{ for } \alpha \in [0, 1]. \tag{16}$$

Then, using (16) and (3), we have

$$\begin{aligned} d(\tilde{T}(\tilde{a}), \tilde{T}(\tilde{b})) &= \sup_{\alpha \in [0,1]} \left| \left((\tilde{T}(\tilde{a}))_{\alpha}^L + (\tilde{T}(\tilde{a}))_{\alpha}^U \right) - \left((\tilde{T}(\tilde{b}))_{\alpha}^L + (\tilde{T}(\tilde{b}))_{\alpha}^U \right) \right| \\ &= \sup_{\alpha \in [0,1]} \left| \lambda \cdot \left(\tilde{a}_{\alpha}^L + \tilde{a}_{\alpha}^U \right) - \lambda \cdot \left(\tilde{b}_{\alpha}^L + \tilde{b}_{\alpha}^U \right) \right| = \lambda \cdot d(\tilde{a}, \tilde{b}), \end{aligned}$$

which says that \tilde{T} is a metric contraction. Theorem 1 says that \tilde{T} has a near fixed point. It is clear to see that the crisp number $\tilde{1}_{\{0\}}$ with value 0 is a near fixed point, since

$$\tilde{T}(\tilde{1}_{\{0\}}) = \tilde{1}_{\{\lambda\}} \otimes \tilde{1}_{\{0\}} = \tilde{1}_{\{0\}}.$$

Now, given any $\tilde{\omega} \in \Omega$, we see that $\tilde{1}_{\{\lambda\}} \otimes \tilde{\omega} \in \Omega$. It is not hard to show that there exists another $\tilde{\omega}^{(1)} \in \Omega$ such that

$$\tilde{1}_{\{\lambda\}} \otimes \tilde{\omega} = \tilde{\omega} \oplus \tilde{\omega}^{(1)}.$$

In this case, we have

$$\tilde{T}(\tilde{\omega}) = \tilde{1}_{\{\lambda\}} \otimes \tilde{\omega} = \tilde{\omega} \oplus \tilde{\omega}^{(1)},$$

which shows that $\tilde{\omega}$ is a near fixed point. Therefore, we obtain the unique equivalence class

$$[\tilde{1}_{\{0\}}] = [\omega] = \Omega$$

for $\omega \in \Omega$, which illustrates the first property of Theorem 1.

Definition 13. A fuzzy-number-valued function $\tilde{T} : (\mathcal{F}_{cc}(\mathbb{R}), d) \rightarrow (\mathcal{F}_{cc}(\mathbb{R}), d)$ is called a weakly strict metric contraction on $\mathcal{F}_{cc}(\mathbb{R})$ if and only if the following conditions are satisfied:

- $\tilde{a} \stackrel{\Omega}{=} \tilde{b}$, i.e., $[\tilde{a}] = [\tilde{b}]$ implies $d(\tilde{T}(\tilde{a}), \tilde{T}(\tilde{b})) = 0$;
- $\tilde{a} \not\stackrel{\Omega}{=} \tilde{b}$, i.e., $[\tilde{a}] \neq [\tilde{b}]$ implies $d(\tilde{T}(\tilde{a}), \tilde{T}(\tilde{b})) < d(\tilde{a}, \tilde{b})$.

It is clear that if \tilde{T} is a metric contraction on $\mathcal{F}_{cc}(\mathbb{R})$, then it is also a weakly strict metric contraction on $\mathcal{F}_{cc}(\mathbb{R})$.

Theorem 2. (Near Fixed Point Theorem) Let $(\mathcal{F}_{cc}(\mathbb{R}), d)$ be a complete near metric space of fuzzy numbers. Suppose that the fuzzy-number-valued function $\tilde{T} : (\mathcal{F}_{cc}(\mathbb{R}), d) \rightarrow (\mathcal{F}_{cc}(\mathbb{R}), d)$ is a weakly strict metric contraction on $\mathcal{F}_{cc}(\mathbb{R})$. If $\{\tilde{T}^n(\tilde{a}^{(0)})\}_{n=1}^{\infty}$ forms a Cauchy sequence for some $\tilde{a}^{(0)} \in \mathcal{F}_{cc}(\mathbb{R})$, then \tilde{T} has a near fixed point $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ satisfying $\tilde{T}(\tilde{a}) \stackrel{\Omega}{=} \tilde{a}$. Moreover, the near fixed point \tilde{a} is obtained by the limit

$$d\left(\tilde{T}^n(\tilde{a}^{(0)}), \tilde{a}\right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Assume further that d satisfies the null equality. Then we also have the following properties.

- The uniqueness is in the sense that there is a unique equivalence class $[\tilde{a}]$ such that any $\tilde{a}^{\circ} \notin [\tilde{a}]$ cannot be a near fixed point.
- Each point $\tilde{a}^{\circ} \in [\tilde{a}]$ is also a near fixed point of \tilde{T} satisfying $\tilde{T}(\tilde{a}^{\circ}) \stackrel{\Omega}{=} \tilde{a}^{\circ}$ and $[\tilde{a}^{\circ}] = [\tilde{a}]$.
- If \tilde{a}° is a near fixed point of \tilde{T} , then $\tilde{a}^{\circ} \in [\tilde{a}]$, i.e., $[\tilde{a}^{\circ}] = [\tilde{a}]$. Equivalently, if \tilde{a} and \tilde{a}° are the near fixed points of \tilde{T} , then $\tilde{a} \stackrel{\Omega}{=} \tilde{a}^{\circ}$.

Proof. Since $\{\tilde{T}^n(\tilde{a}^{(0)})\}_{n=1}^{\infty}$ is a Cauchy sequence, the completeness says that there exists $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ such that $d(\tilde{T}^n(\tilde{a}^{(0)}), \tilde{a}) \rightarrow 0$, i.e., $\tilde{T}^n(\tilde{a}^{(0)}) \rightarrow [\tilde{a}]$ according to Definition 5 and Proposition 6. Therefore, given any $\epsilon > 0$, there exists an integer N such that $d(\tilde{T}^n(\tilde{a}^{(0)}), \tilde{a}) < \epsilon$ for $n \geq N$. Since \tilde{T} is a weakly strict metric contraction on $\mathcal{F}_{cc}(\mathbb{R})$, we consider the following two cases.

- Suppose that $\tilde{T}^n(\tilde{a}^{(0)}) \stackrel{\Omega}{=} \tilde{a}$. Then

$$d(\tilde{T}^{n+1}(\tilde{a}^{(0)}), \tilde{T}(\tilde{a})) = 0 < \epsilon.$$

- Suppose that $\tilde{T}^n(\tilde{a}^{(0)}) \stackrel{\Omega}{\neq} \tilde{a}$. Then

$$d(\tilde{T}^{n+1}(\tilde{a}^{(0)}), \tilde{T}(\tilde{a})) < d(\tilde{T}^n(\tilde{a}^{(0)}), \tilde{a}) < \epsilon \text{ for } n \geq N.$$

The above two cases say that $d(\tilde{T}^{n+1}(\tilde{a}^{(0)}), \tilde{T}(\tilde{a})) \rightarrow 0$. Using the triangle inequality, we obtain

$$d(\tilde{T}(\tilde{a}), \tilde{a}) \leq d(\tilde{T}(\tilde{a}), \tilde{T}^{n+1}(\tilde{a}^{(0)})) + d(\tilde{T}^{n+1}(\tilde{a}^{(0)}), \tilde{a}) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

which says that $d(\tilde{T}(\tilde{a}), \tilde{a}) = 0$, i.e., $\tilde{T}(\tilde{a}) \stackrel{\Omega}{=} \tilde{a}$. This shows that \tilde{a} is a near fixed point.

Assume further that d satisfies the null equality. We are going to claim that each point $\tilde{a}^\circ \in [\tilde{a}]$ is also a near fixed point of \tilde{T} . Since $\tilde{a}^\circ \stackrel{\Omega}{=} \tilde{a}$, we have $\tilde{a}^\circ \oplus \tilde{\omega}^{(1)} = \tilde{a} \oplus \tilde{\omega}^{(2)}$ for some $\tilde{\omega}^{(1)}, \tilde{\omega}^{(2)} \in \Omega$. Then, using the null equality for d , we obtain

$$\begin{aligned} d(\tilde{T}^n(\tilde{a}^{(0)}), \tilde{a}^\circ) &= d(\tilde{T}^n(\tilde{a}^{(0)}), \tilde{a}^\circ \oplus \tilde{\omega}^{(1)}) = d(\tilde{T}^n(\tilde{a}^{(0)}), \tilde{a} \oplus \tilde{\omega}^{(2)}) \\ &= d(\tilde{T}^n(\tilde{a}^{(0)}), \tilde{a}) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

We can similarly obtain $d(\tilde{T}^{n+1}(\tilde{a}^{(0)}), \tilde{T}(\tilde{a}^\circ)) \rightarrow 0$ as $n \rightarrow \infty$. Using the triangle inequality, we have

$$d(\tilde{a}^\circ, \tilde{T}(\tilde{a}^\circ)) \leq d(\tilde{a}^\circ, \tilde{T}^{n+1}(\tilde{a}^{(0)})) + d(\tilde{T}^{n+1}(\tilde{a}^{(0)}), \tilde{T}(\tilde{a}^\circ)) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

which says that $d(\tilde{a}^\circ, \tilde{T}(\tilde{a}^\circ)) = 0$. Therefore we conclude that $\tilde{T}(\tilde{a}^\circ) \stackrel{\Omega}{=} \tilde{a}^\circ$ for any $\tilde{a}^\circ \in [\tilde{a}]$.

Suppose that $\tilde{a}^\circ \notin [\tilde{a}]$ is another near fixed point of \tilde{T} . Then $\tilde{T}(\tilde{a}^\circ) \stackrel{\Omega}{=} \tilde{a}^\circ$ and $[\tilde{a}^\circ] \neq [\tilde{a}]$, i.e., $\tilde{a} \stackrel{\Omega}{\neq} \tilde{a}^\circ$. Then

$$\tilde{T}(\tilde{a}) \oplus \tilde{\omega}^{(1)} = \tilde{a} \oplus \tilde{\omega}^{(2)} \text{ and } \tilde{T}(\tilde{a}^\circ) \oplus \tilde{\omega}^{(3)} = \tilde{a}^\circ \oplus \tilde{\omega}^{(4)}$$

for some $\tilde{\omega}_i \in \Omega$ for $i = 1, 2, 3, 4$. Therefore we obtain

$$\begin{aligned} d(\tilde{a}, \tilde{a}^\circ) &= d(\tilde{a} \oplus \tilde{\omega}^{(2)}, \tilde{a}^\circ \oplus \tilde{\omega}^{(4)}) \text{ (since } d \text{ satisfies the null equality)} \\ &= d(\tilde{T}(\tilde{a}) \oplus \tilde{\omega}^{(1)}, \tilde{T}(\tilde{a}^\circ) \oplus \tilde{\omega}^{(3)}) = d(\tilde{T}(\tilde{a}), \tilde{T}(\tilde{a}^\circ)) \text{ (since } d \text{ satisfies the null equality)} \\ &< d(\tilde{a}, \tilde{a}^\circ) \text{ (since } \tilde{a} \stackrel{\Omega}{\neq} \tilde{a}^\circ \text{ and } \tilde{T} \text{ is a weakly strict metric contraction)}. \end{aligned}$$

This contradiction says that \tilde{a}° cannot be a near fixed point of \tilde{T} . Equivalently, if \tilde{a}° is a near fixed point of \tilde{T} , then $\tilde{a}^\circ \in [\tilde{a}]$. This completes the proof. \square

Now we consider another fixed point theorem based on the weakly uniformly strict metric contraction which was proposed by Meir and Keeler [21]. Under the near metric space of fuzzy numbers $(\mathcal{F}_{cc}(\mathbb{R}), d)$, we have $d(\tilde{a}, \tilde{b}) = 0$ for $\tilde{a} \stackrel{\Omega}{=} \tilde{b}$. Therefore we propose the following different definition.

Definition 14. A fuzzy-number-valued function $\tilde{T} : (\mathcal{F}_{cc}(\mathbb{R}), d) \rightarrow (\mathcal{F}_{cc}(\mathbb{R}), d)$ is called a weakly uniformly strict metric contraction on $\mathcal{F}_{cc}(\mathbb{R})$ if and only if the following conditions are satisfied:

- for $\tilde{a} \stackrel{\Omega}{=} \tilde{b}$, i.e., $[\tilde{a}] = [\tilde{b}]$, $d(\tilde{T}(\tilde{a}), \tilde{T}(\tilde{b})) = 0$;

- for $\tilde{a} \not\stackrel{\Omega}{=} \tilde{b}$, i.e., $[\tilde{a}] \neq [\tilde{b}]$, given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leq d(\tilde{a}, \tilde{b}) < \epsilon + \delta \text{ implies } d(\tilde{T}(\tilde{a}), \tilde{T}(\tilde{b})) < \epsilon.$$

Remark 1. We observe that if \tilde{T} is a weakly uniformly strict metric contraction on $\mathcal{F}_{cc}(\mathbb{R})$, then \tilde{T} is also a weakly strict metric contraction on $\mathcal{F}_{cc}(\mathbb{R})$ by taking $\epsilon = d(\tilde{a}, \tilde{b})$.

Lemma 1. Let $\tilde{T} : (\mathcal{F}_{cc}(\mathbb{R}), d) \rightarrow (\mathcal{F}_{cc}(\mathbb{R}), d)$ be a weakly uniformly strict metric contraction on $\mathcal{F}_{cc}(\mathbb{R})$. Then the sequence

$$\left\{ d(\tilde{T}^n(\tilde{a}), \tilde{T}^{n+1}(\tilde{a})) \right\}_{n=1}^{\infty}$$

is decreasing to zero for any $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$.

Proof. For convenience, we write $\tilde{T}^n(\tilde{a}) = \tilde{a}^{(n)}$ for all n . Let $\eta_n = d(\tilde{a}^{(n)}, \tilde{a}^{(n+1)})$.

- Suppose that $[\tilde{a}^{(n-1)}] \neq [\tilde{a}^{(n)}]$. By Remark 1, since \tilde{T} is also a weakly strict metric contraction on $\mathcal{F}_{cc}(\mathbb{R})$, we have

$$\begin{aligned} \eta_n &= d(\tilde{a}^{(n)}, \tilde{a}^{(n+1)}) = d(\tilde{T}^n(\tilde{a}), \tilde{T}^{n+1}(\tilde{a})) < d(\tilde{T}^{n-1}(\tilde{a}), \tilde{T}^n(\tilde{a})) \\ &= d(\tilde{a}^{(n-1)}, \tilde{a}^{(n)}) = \eta_{n-1}. \end{aligned}$$

- Suppose that $[\tilde{a}^{(n-1)}] = [\tilde{a}^{(n)}]$. Then, by the first condition of Definition 14, we have

$$\eta_n = d(\tilde{T}^n(\tilde{a}), \tilde{T}^{n+1}(\tilde{a})) = d(\tilde{T}(\tilde{a}^{(n-1)}), \tilde{T}(\tilde{a}^{(n)})) = 0 \leq \eta_{n-1}.$$

The above two cases say that the sequence $\{\eta_n\}_{n=1}^{\infty}$ is decreasing. We consider the following cases.

- Let m be the first index in the sequence $\{\tilde{a}^{(n)}\}_{n=1}^{\infty}$ such that $[\tilde{a}^{(m-1)}] = [\tilde{a}^{(m)}]$. Then we want to claim $\eta_{m-1} = \eta_m = \eta_{m+1} = \dots = 0$. Since $\tilde{a}^{(m-1)} \stackrel{\Omega}{=} \tilde{a}^{(m)}$, we have

$$\eta_{m-1} = d(\tilde{a}^{(m-1)}, \tilde{a}^{(m)}) = 0.$$

Using the first condition of Definition 14, we also have

$$\begin{aligned} 0 &= d(\tilde{T}(\tilde{a}^{(m-1)}), \tilde{T}(\tilde{a}^{(m)})) = d(\tilde{T}(\tilde{T}^{m-1}(\tilde{a})), \tilde{T}(\tilde{T}^m(\tilde{a}))) \\ &= d(\tilde{T}^m(\tilde{a}), \tilde{T}^{m+1}(\tilde{a})) = d(\tilde{a}^{(m)}, \tilde{a}^{(m+1)}) = \eta_m, \end{aligned}$$

which says that $\tilde{a}^{(m)} \stackrel{\Omega}{=} \tilde{a}^{(m+1)}$, i.e., $[\tilde{a}^{(m)}] = [\tilde{a}^{(m+1)}]$. Using the similar argument, we can obtain $\eta_{m+1} = 0$ and $[\tilde{a}^{(m+1)}] = [\tilde{a}^{(m+2)}]$. Therefore the sequence $\{\eta_n\}_{n=1}^{\infty}$ is decreasing to zero.

- Suppose that $[\tilde{a}^{(m+1)}] \neq [\tilde{a}^{(m)}]$ for all $m \geq 1$. Since the sequence $\{\eta_n\}_{n=1}^{\infty}$ is decreasing, we assume that $\eta_n \downarrow \epsilon > 0$, i.e., $\eta_n \geq \epsilon > 0$ for all n . Then there exists $\delta > 0$ such that $\epsilon \leq \eta_m < \epsilon + \delta$ for some m , i.e.,

$$\epsilon \leq d(\tilde{a}^{(m)}, \tilde{a}^{(m+1)}) < \epsilon + \delta.$$

By the second condition of Definition 14, we have

$$\begin{aligned} \eta_{m+1} &= d(\tilde{a}^{(m+1)}, \tilde{a}^{(m+2)}) = d(\tilde{T}^{m+1}(\tilde{a}), \tilde{T}^{m+2}(\tilde{a})) \\ &= d(\tilde{T}(\tilde{a}^{(m)}), \tilde{T}(\tilde{a}^{(m+1)})) < \epsilon, \end{aligned}$$

which contradicts $\eta_{m+1} \geq \epsilon$. Therefore we must have $\eta_n \downarrow 0$.

This completes the proof. \square

Theorem 3. (Near Fixed Point Theorem) Let $(\mathcal{F}_{cc}(\mathbb{R}), d)$ be a complete near metric space of fuzzy numbers with the null set Ω , and let $\tilde{T} : (\mathcal{F}_{cc}(\mathbb{R}), d) \rightarrow (\mathcal{F}_{cc}(\mathbb{R}), d)$ be a weakly uniformly strict metric contraction on $\mathcal{F}_{cc}(\mathbb{R})$. Then \tilde{T} has a near fixed point satisfying $\tilde{T}(\tilde{a}) \stackrel{\Omega}{=} \tilde{a}$. Moreover, the near fixed point \tilde{a} is obtained by the limit

$$d\left(\tilde{T}^n(\tilde{a}^{(0)}), \tilde{a}\right) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for some } \tilde{a}^{(0)}.$$

Assume further that d satisfies the null equality. Then we also have the following properties.

- The uniqueness is in the sense that there is a unique equivalence class $[\tilde{a}]$ such that any $\tilde{a}^\circ \notin [\tilde{a}]$ cannot be a near fixed point.
- Each point $\tilde{a}^\circ \in [\tilde{a}]$ is also a near fixed point of \tilde{T} satisfying $\tilde{T}(\tilde{a}^\circ) \stackrel{\Omega}{=} \tilde{a}^\circ$ and $[\tilde{a}^\circ] = [\tilde{a}]$.
- If \tilde{a}° is a near fixed point of \tilde{T} , then $\tilde{a}^\circ \in [\tilde{a}]$, i.e., $[\tilde{a}^\circ] = [\tilde{a}]$. Equivalently, if \tilde{a} and \tilde{a}° are the near fixed points of \tilde{T} , then $\tilde{a} \stackrel{\Omega}{=} \tilde{a}^\circ$.

Proof. According to Theorem 2 and Remark 1, we just need to claim that if \tilde{T} is a weakly uniformly strict metric contraction, then $\{\tilde{T}^n(\tilde{a}^{(0)})\}_{n=1}^\infty \equiv \{\tilde{a}^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence for $\tilde{a}^{(0)} \in \mathcal{F}_{cc}(\mathbb{R})$. Suppose that $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ such that, given any N , there exist $m, n \geq N$ satisfying $d(\tilde{a}^{(m)}, \tilde{a}^{(n)}) > 2\epsilon$. Since \tilde{T} is a weakly uniformly strict metric contraction on $\mathcal{F}_{cc}(\mathbb{R})$, for $\tilde{a} \not\stackrel{\Omega}{=} \tilde{b}$, there exists $\delta > 0$ such that

$$\epsilon \leq d(\tilde{a}, \tilde{b}) < \epsilon + \delta \text{ implies } d(\tilde{T}(\tilde{a}), \tilde{T}(\tilde{b})) < \epsilon.$$

Let $\delta' = \min\{\delta, \epsilon\}$. For $\tilde{a} \not\stackrel{\Omega}{=} \tilde{b}$, we are going to claim

$$\epsilon \leq d(\tilde{a}, \tilde{b}) < \epsilon + \delta' \text{ implies } d(\tilde{T}(\tilde{a}), \tilde{T}(\tilde{b})) < \epsilon. \tag{17}$$

Indeed, if $\delta' = \delta$, then it is done, and if $\delta' = \epsilon$, i.e., $\epsilon < \delta$, then $\epsilon + \delta' = \epsilon + \epsilon < \epsilon + \delta$. This proves the statement (17).

Let $\eta_n = d(\tilde{a}^{(n)}, \tilde{a}^{(n+1)})$. Since the sequence $\{\eta_n\}_{n=1}^\infty$ is decreasing to zero by Lemma 1, we can find N such that $\eta_N < \delta'/3$. For $n > m \geq N$, we have

$$d(\tilde{a}^{(m)}, \tilde{a}^{(n)}) > 2\epsilon \geq \epsilon + \delta', \tag{18}$$

which says that $\tilde{a}^{(m)} \not\stackrel{\Omega}{=} \tilde{a}^{(n)}$. Since the sequence $\{\eta_n\}_{n=1}^\infty$ is decreasing by Lemma 1 again, we obtain

$$d(\tilde{a}^{(m)}, \tilde{a}^{(m+1)}) = \eta_m \leq \eta_N < \frac{\delta'}{3} \leq \frac{\epsilon}{3} < \epsilon. \tag{19}$$

For j with $m < j \leq n$, using the triangle inequality, we also have

$$d(\tilde{a}^{(m)}, \tilde{a}^{(j+1)}) \leq d(\tilde{a}^{(m)}, \tilde{a}^{(j)}) + d(\tilde{a}^{(j)}, \tilde{a}^{(j+1)}). \tag{20}$$

We want to show that there exists j with $m < j \leq n$ such that $\tilde{a}^{(m)} \not\stackrel{\Omega}{=} \tilde{a}^{(j)}$ and

$$\epsilon + \frac{2\delta'}{3} < d(\tilde{a}^{(m)}, \tilde{a}^{(j)}) < \epsilon + \delta'. \tag{21}$$

Let $\gamma_j = d(\tilde{a}^{(m)}, \tilde{a}^{(j)})$ for $j = m + 1, \dots, n$. Then (18) and (19) says that $\gamma_{m+1} < \epsilon$ and $\gamma_n > \epsilon + \delta'$. Let j_0 be an index such that

$$j_0 = \max \left\{ j \in [m + 1, n] : \gamma_j \leq \epsilon + \frac{2\delta'}{3} \right\}.$$

Then we see that $j_0 < n$, since $\gamma_n > \epsilon + \delta'$. By the definition of j_0 , we also see that $j_0 + 1 \leq n$ and

$$\gamma_{j_0+1} > \epsilon + \frac{2\delta'}{3}, \tag{22}$$

which also says that $\tilde{a}^{(m)} \stackrel{\Omega}{\neq} \tilde{a}^{(j_0+1)}$; otherwise, $\gamma_{j_0+1} = 0$ that is a contradiction. Therefore, from (22), expression (21) will be sound if we can show that $\gamma_{j_0+1} < \epsilon + \delta'$. Suppose that this is not true, i.e., $\gamma_{j_0+1} \geq \epsilon + \delta'$. We also see that $\gamma_{j_0} \leq \epsilon + \frac{2\delta'}{3}$. Since $\{\eta_n\}_{n=1}^\infty$ is decreasing, from (19) and (20), we have

$$\frac{\delta'}{3} > \eta_N \geq \eta_{j_0} = d(\tilde{a}^{(j_0)}, \tilde{a}^{(j_0+1)}) \geq \gamma_{j_0+1} - \gamma_{j_0} \geq \epsilon + \delta' - \epsilon - \frac{2\delta'}{3} = \frac{\delta'}{3}.$$

This contradiction says that (21) is sound. Since $\tilde{a}^{(m)} \stackrel{\Omega}{\neq} \tilde{a}^{(j)}$, using (17), we see that (21) implies

$$d(\tilde{a}^{(m+1)}, \tilde{a}^{(j+1)}) = d(\tilde{T}(\tilde{a}^{(m)}), \tilde{T}(\tilde{a}^{(j)})) < \epsilon. \tag{23}$$

Therefore, using the triangle inequality, we obtain

$$\begin{aligned} & d(\tilde{a}^{(m)}, \tilde{a}^{(j)}) \\ & \leq d(\tilde{a}^{(m)}, \tilde{a}^{(m+1)}) + d(\tilde{a}^{(m+1)}, \tilde{a}^{(j+1)}) + d(\tilde{a}^{(j+1)}, \tilde{a}^{(j)}) \\ & < \eta_m + \epsilon + \eta_j \text{ (by (23))} \\ & \leq \eta_m + \epsilon + \eta_m < \frac{\delta'}{3} + \epsilon + \frac{\delta'}{3} \text{ (by (19))} \\ & = \epsilon + \frac{2\delta'}{3}, \end{aligned}$$

which contradicts (21). This contradiction says that every sequence $\{\tilde{T}^n(\tilde{a})\}_{n=1}^\infty = \{\tilde{a}^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence. This completes the proof. \square

7. Near Fixed Point Theorems in Near Banach Space of Fuzzy Numbers

Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a near Banach space of fuzzy numbers. In this section, we shall study the near fixed point in $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$.

Definition 15. Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a near pseudo-seminormed space of fuzzy numbers. A fuzzy-number-valued function $\tilde{T} : (\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|) \rightarrow (\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ is called a norm contraction on $\mathcal{F}_{cc}(\mathbb{R})$ if and only if there is a real number $0 < \alpha < 1$ such that

$$\|\tilde{T}(\tilde{a}) \ominus \tilde{T}(\tilde{b})\| \leq \alpha \cdot \|\tilde{a} \ominus \tilde{b}\|$$

for any $\tilde{a}, \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$.

Theorem 4. Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a near Banach space of fuzzy numbers with the null set Ω such that $\|\cdot\|$ satisfies the null equality. Suppose that the fuzzy-number-valued function $\tilde{T} : (\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|) \rightarrow (\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ is a norm contraction on $\mathcal{F}_{cc}(\mathbb{R})$. Then \tilde{T} has a near fixed point $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ satisfying $\tilde{T}(\tilde{a}) \stackrel{\Omega}{=} \tilde{a}$. Moreover, the near fixed point \tilde{a} is obtained by the limit

$$\| \tilde{a} \ominus \tilde{a}^{(n)} \| = \| \tilde{a}^{(n)} \ominus \tilde{a} \| \rightarrow 0 \text{ as } n \rightarrow \infty$$

in which the sequence $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ is generated according to (14). We also have the following properties.

- The uniqueness is in the sense that there is a unique equivalence class $[\tilde{a}]$ such that any $\tilde{a}^\circ \notin [\tilde{a}]$ cannot be a near fixed point.
- Each point $\tilde{a}^\circ \in [\tilde{a}]$ is also a near fixed point of \tilde{T} satisfying $\tilde{T}(\tilde{a}^\circ) \stackrel{\Omega}{=} \tilde{a}^\circ$ and $[\tilde{a}^\circ] = [\tilde{a}]$.
- If \tilde{a}° is a near fixed point of \tilde{T} , then $\tilde{a}^\circ \in [\tilde{a}]$, i.e., $[\tilde{a}^\circ] = [\tilde{a}]$. Equivalently, if \tilde{a} and \tilde{a}° are the near fixed points of \tilde{T} , then $\tilde{a} \stackrel{\Omega}{=} \tilde{a}^\circ$.

Proof. Given any initial element $\tilde{a}^{(0)} \in \mathcal{F}_{cc}(\mathbb{R})$, we are going to show that $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence. Since \tilde{T} is a norm contraction on $\mathcal{F}_{cc}(\mathbb{R})$, we have

$$\begin{aligned} \| \tilde{a}^{(m+1)} \ominus \tilde{a}^{(m)} \| &= \| \tilde{T}(\tilde{a}^{(m)}) \ominus \tilde{T}(\tilde{a}^{(m-1)}) \| \leq \alpha \| \tilde{a}^{(m)} \ominus \tilde{a}^{(m-1)} \| \\ &= \alpha \| \tilde{T}(\tilde{a}^{(m-1)}) \ominus \tilde{T}(\tilde{a}^{(m-2)}) \| \leq \alpha^2 \| \tilde{a}^{(m-1)} \ominus \tilde{a}^{(m-2)} \| \\ &\leq \dots \leq \alpha^m \| \tilde{a}^{(1)} \ominus \tilde{a}^{(0)} \|. \end{aligned}$$

For $n < m$, using Proposition 3, we obtain

$$\begin{aligned} \| \tilde{a}^{(m)} \ominus \tilde{a}^{(n)} \| &\leq \| \tilde{a}^{(m)} \ominus \tilde{a}^{(m-1)} \| + \| \tilde{a}^{(m-1)} \ominus \tilde{a}^{(m-2)} \| + \dots + \| \tilde{a}^{(n+1)} \ominus \tilde{a}^{(n)} \| \\ &\leq (\alpha^m + \alpha^{m+1} + \dots + \alpha^n) \cdot \| \tilde{a}^{(1)} \ominus \tilde{a}^{(0)} \| \\ &= \alpha^n \cdot \frac{1 - \alpha^{m-n}}{1 - \alpha} \cdot \| \tilde{a}^{(1)} \ominus \tilde{a}^{(0)} \|. \end{aligned}$$

Since $0 < \alpha < 1$, we have $1 - \alpha^{m-n} < 1$ in the numerator, which says that

$$\| \tilde{a}^{(m)} \ominus \tilde{a}^{(n)} \| \leq \frac{\alpha^n}{1 - \alpha} \cdot \| \tilde{a}^{(1)} \ominus \tilde{a}^{(0)} \| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This proves that $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence. Since $\mathcal{F}_{cc}(\mathbb{R})$ is complete, there exists $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ such that

$$\| \tilde{a} \ominus \tilde{a}^{(n)} \| = \| \tilde{a}^{(n)} \ominus \tilde{a} \| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

We are going to show that any point $\tilde{a}^\circ \in [\tilde{a}]$ is a near fixed point. Now we have $\tilde{a}^\circ \oplus \tilde{\omega}^{(1)} = \tilde{a} \oplus \tilde{\omega}^{(2)}$ for some $\tilde{\omega}^{(1)}, \tilde{\omega}^{(2)} \in \Omega$. Using the triangle inequality and the fact of norm contraction on $\mathcal{F}_{cc}(\mathbb{R})$, we have

$$\begin{aligned} \| \tilde{a}^\circ \ominus \tilde{T}(\tilde{a}^\circ) \| &= \| (\tilde{a}^\circ \oplus \tilde{\omega}^{(1)}) \ominus \tilde{T}(\tilde{a}^\circ) \| \text{ (since } \|\cdot\| \text{ satisfies the null equality)} \\ &\leq \| (\tilde{a}^\circ \oplus \tilde{\omega}^{(1)}) \ominus \tilde{a}^{(m)} \| + \| \tilde{a}^{(m)} \ominus \tilde{T}(\tilde{a}^\circ) \| \text{ (using Proposition 3)} \\ &= \| (\tilde{a}^\circ \oplus \tilde{\omega}^{(1)}) \ominus \tilde{a}^{(m)} \| + \| \tilde{T}(\tilde{a}^{(m-1)}) \ominus \tilde{T}(\tilde{a}^\circ) \| \\ &\leq \| (\tilde{a}^\circ \oplus \tilde{\omega}^{(1)}) \ominus \tilde{a}^{(m)} \| + \alpha \| \tilde{a}^{(m-1)} \ominus \tilde{a}^\circ \| \\ &= \| (\tilde{a}^\circ \oplus \tilde{\omega}^{(1)}) \ominus \tilde{a}^{(m)} \| + \alpha \| \tilde{a}^{(m-1)} \ominus \tilde{a}^\circ \oplus (-\tilde{\omega}^{(1)}) \| \\ &\text{(since } -\tilde{\omega}^{(1)} \in \Omega \text{ and } \|\cdot\| \text{ satisfies the null equality)} \\ &= \| (\tilde{a}^\circ \oplus \tilde{\omega}^{(1)}) \ominus \tilde{a}^{(m)} \| + \alpha \| \tilde{a}^{(m-1)} \ominus (\tilde{a}^\circ \oplus \tilde{\omega}^{(1)}) \| \text{ (using Proposition 1)} \\ &= \| (\tilde{a} \oplus \tilde{\omega}^{(2)}) \ominus \tilde{a}^{(m)} \| + \alpha \| \tilde{a}^{(m-1)} \ominus (\tilde{a} \oplus \tilde{\omega}^{(2)}) \| \\ &= \| \tilde{a} \ominus \tilde{a}^{(m)} \| + \alpha \| \tilde{a}^{(m-1)} \ominus \tilde{a} \| \\ &\text{(using } -\tilde{\omega}^{(2)} \in \Omega, \text{ the null equality and Proposition 1),} \end{aligned}$$

which implies $\| \tilde{a}^\circ \ominus \tilde{T}(\tilde{a}^\circ) \| = 0$ as $m \rightarrow \infty$. We conclude that $\tilde{T}(\tilde{a}^\circ) \stackrel{\Omega}{=} \tilde{a}^\circ$ for any point $\tilde{a}^\circ \in [\tilde{a}]$ by part (ii) of Proposition 4.

Now assume that there is another near fixed point \tilde{a}° of \tilde{T} with $\tilde{a}^\circ \notin [\tilde{a}]$, i.e., $\tilde{a}^\circ \stackrel{\Omega}{=} \tilde{T}(\tilde{a}^\circ)$. Then

$$\tilde{a}^\circ \oplus \tilde{\omega}^{(1)} = \tilde{T}(\tilde{a}^\circ) \oplus \tilde{\omega}^{(2)} \text{ and } \tilde{a} \oplus \tilde{\omega}^{(3)} = \tilde{T}(\tilde{a}) \oplus \tilde{\omega}^{(4)}$$

for some $\tilde{\omega}_i \in \Omega, i = 1, \dots, 4$. Since \tilde{T} is a norm contraction on $\mathcal{F}_{cc}(\mathbb{R})$ and $\| \cdot \|$ satisfies the null equality, we obtain

$$\begin{aligned} \| \tilde{a}^\circ \ominus \tilde{a} \| &= \| (\tilde{a}^\circ \oplus \tilde{\omega}^{(1)}) \ominus (\tilde{a} \oplus \tilde{\omega}^{(3)}) \| \\ &\quad \text{(using } -\tilde{\omega}^{(3)} \in \Omega, \text{ the null equality and Proposition 1)} \\ &= \| (\tilde{T}(\tilde{a}^\circ) \oplus \tilde{\omega}^{(2)}) \ominus (\tilde{T}(\tilde{a}) \oplus \tilde{\omega}^{(4)}) \| = \| \tilde{T}(\tilde{a}^\circ) \ominus \tilde{T}(\tilde{a}) \| \\ &\quad \text{(using } -\tilde{\omega}^{(4)} \in \Omega, \text{ the null equality and Proposition 1)} \\ &\leq \alpha \| \tilde{a}^\circ \ominus \tilde{a} \|. \end{aligned}$$

Since $0 < \alpha < 1$, we conclude that $\| \tilde{a}^\circ \ominus \tilde{a} \| = 0$, i.e., $\tilde{a}^\circ \stackrel{\Omega}{=} \tilde{a}$, which contradicts $\tilde{a}^\circ \notin [\tilde{a}]$. Therefore, any $\tilde{a}^\circ \notin [\tilde{a}]$ cannot be the near fixed point. Equivalently, if \tilde{a}° is a near fixed point of \tilde{T} , then $\tilde{a}^\circ \in [\tilde{a}]$. This completes the proof. \square

Definition 16. Let $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ be a near pseudo-normed space of fuzzy numbers. A fuzzy-number-valued function $\tilde{T} : (\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|) \rightarrow (\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ is called a weakly strict norm contraction on $\mathcal{F}_{cc}(\mathbb{R})$ if and only if the following conditions are satisfied:

- $\tilde{a} \stackrel{\Omega}{=} \tilde{b}$, i.e., $[\tilde{a}] = [\tilde{b}]$ implies $\| \tilde{T}(\tilde{a}) \ominus \tilde{T}(\tilde{b}) \| = 0$.
- $\tilde{a} \not\stackrel{\Omega}{=} \tilde{b}$, i.e., $[\tilde{a}] \neq [\tilde{b}]$ implies $\| \tilde{T}(\tilde{a}) \ominus \tilde{T}(\tilde{b}) \| < \| \tilde{a} \ominus \tilde{b} \|$.

By part (ii) of Proposition 4, we see that if $\tilde{a} \not\stackrel{\Omega}{=} \tilde{b}$, then $\| \tilde{a} \ominus \tilde{b} \| \neq 0$, which says that the weakly strict norm contraction is well-defined. In other words, $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ should be assumed to be a near pseudo-normed space of fuzzy numbers rather than pseudo-seminormed space of fuzzy numbers. We further assume that $\| \cdot \|$ satisfies the null super-inequality and null condition. Part (iii) of Proposition 4 says that if \tilde{T} is a norm contraction on $\mathcal{F}_{cc}(\mathbb{R})$, then it is also a weakly strict norm contraction on $\mathcal{F}_{cc}(\mathbb{R})$.

Theorem 5. Let $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ be a near Banach space of fuzzy numbers with the null set Ω . Suppose that $\| \cdot \|$ satisfies the null super-inequality and null condition, and that the fuzzy-number-valued function $\tilde{T} : (\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|) \rightarrow (\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ is a weakly strict norm contraction on $\mathcal{F}_{cc}(\mathbb{R})$. If $\{ \tilde{T}^n(\tilde{a}^{(0)}) \}_{n=1}^\infty$ forms a Cauchy sequence for some $\tilde{a}^{(0)} \in \mathcal{F}_{cc}(\mathbb{R})$, then \tilde{T} has a near fixed point $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ satisfying $\tilde{T}(\tilde{a}) \stackrel{\Omega}{=} \tilde{a}$. Moreover, the near fixed point \tilde{a} is obtained by the limit

$$\| \tilde{T}^n(\tilde{a}^{(0)}) \ominus \tilde{a} \| = \| \tilde{a} \ominus \tilde{T}^n(\tilde{a}^{(0)}) \| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Assume further that $\| \cdot \|$ satisfies the null equality. Then we also have the following properties.

- The uniqueness is in the sense that there is a unique equivalence class $[\tilde{a}]$ such that any $\tilde{a}^\circ \notin [\tilde{a}]$ cannot be a near fixed point.
- Each point $\tilde{a}^\circ \in [\tilde{a}]$ is also a near fixed point of \tilde{T} satisfying $\tilde{T}(\tilde{a}^\circ) \stackrel{\Omega}{=} \tilde{a}^\circ$ and $[\tilde{a}^\circ] = [\tilde{a}]$.
- If \tilde{a}° is a near fixed point of \tilde{T} , then $\tilde{a}^\circ \in [\tilde{a}]$, i.e., $[\tilde{a}^\circ] = [\tilde{a}]$. Equivalently, if \tilde{a} and \tilde{a}° are the near fixed points of \tilde{T} , then $\tilde{a} \stackrel{\Omega}{=} \tilde{a}^\circ$.

Proof. Since $\{\tilde{T}^n(\tilde{a}^{(0)})\}_{n=1}^\infty$ is a Cauchy sequence, the completeness says that there exists $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ such that

$$\|\tilde{T}^n(\tilde{a}^{(0)}) \ominus \tilde{a}\| = \|\tilde{a} \ominus \tilde{T}^n(\tilde{a}^{(0)})\| \rightarrow 0.$$

Therefore, given any $\epsilon > 0$, there exists an integer N such that $\|\tilde{T}^n(\tilde{a}^{(0)}) \ominus \tilde{a}\| < \epsilon$ for $n \geq N$. We consider the following two cases.

- Suppose that $\tilde{T}^n(\tilde{a}^{(0)}) \stackrel{\Omega}{=} \tilde{a}$. Since \tilde{T} is a weakly strict norm contraction on $\mathcal{F}_{cc}(\mathbb{R})$, it follows that

$$\|\tilde{T}^{n+1}(\tilde{a}^{(0)}) \ominus \tilde{T}(\tilde{a})\| = 0 < \epsilon.$$

by part (iii) of Proposition 4.

- Suppose that $\tilde{T}^n(\tilde{a}^{(0)}) \stackrel{\Omega}{\neq} \tilde{a}$. Since \tilde{T} is a weakly strict norm contraction on $\mathcal{F}_{cc}(\mathbb{R})$, we have

$$\|\tilde{T}^{n+1}(\tilde{a}^{(0)}) \ominus \tilde{T}(\tilde{a})\| < \|\tilde{T}^n(\tilde{a}^{(0)}) \ominus \tilde{a}\| < \epsilon \text{ for } n \geq N.$$

The above two cases say that $\|\tilde{T}^{n+1}(\tilde{a}^{(0)}) \ominus \tilde{T}(\tilde{a})\| \rightarrow 0$. Using Proposition 3, we obtain

$$\|\tilde{a} \ominus \tilde{T}(\tilde{a})\| \leq \|\tilde{a} \ominus \tilde{T}^{n+1}(\tilde{a}^{(0)})\| + \|\tilde{T}^{n+1}(\tilde{a}^{(0)}) \ominus \tilde{T}(\tilde{a})\| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

which says that $\|\tilde{a} \ominus \tilde{T}(\tilde{a})\| = 0$, i.e., $\tilde{T}(\tilde{a}) \stackrel{\Omega}{=} \tilde{a}$ by part (ii) of Proposition 4. This shows that \tilde{a} is a near fixed point.

Assume that $\|\cdot\|$ satisfies the null equality. We are going to claim that each point $\tilde{a}^\circ \in [\tilde{a}]$ is also a near fixed point of \tilde{T} . Since $\tilde{a}^\circ \stackrel{\Omega}{=} \tilde{a}$, we have $\tilde{a}^\circ \oplus \tilde{\omega}^{(1)} = \tilde{a} \oplus \tilde{\omega}^{(2)}$ for some $\tilde{\omega}^{(1)}, \tilde{\omega}^{(2)} \in \Omega$. Then, using the null equality for $\|\cdot\|$, we obtain

$$\begin{aligned} \|\tilde{T}^n(\tilde{a}^{(0)}) \ominus \tilde{a}^\circ\| &= \|\tilde{a}^\circ \ominus \tilde{T}^n(\tilde{a}^{(0)})\| = \|(\tilde{a}^\circ \oplus \tilde{\omega}^{(1)}) \ominus \tilde{T}^n(\tilde{a}^{(0)})\| = \|(\tilde{a} \oplus \tilde{\omega}^{(2)}) \ominus \tilde{T}^n(\tilde{a}^{(0)})\| \\ &= \|\tilde{a} \ominus \tilde{T}^n(\tilde{a}^{(0)})\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Using the above argument, we can also obtain $\|\tilde{T}^{n+1}(\tilde{a}^{(0)}) \ominus \tilde{T}(\tilde{a}^\circ)\| \rightarrow 0$ as $n \rightarrow \infty$. Using Proposition 3, we have

$$\|\tilde{a}^\circ \ominus \tilde{T}(\tilde{a}^\circ)\| \leq \|\tilde{a}^\circ \ominus \tilde{T}^{n+1}(\tilde{a}^{(0)})\| + \|\tilde{T}^{n+1}(\tilde{a}^{(0)}) \ominus \tilde{T}(\tilde{a}^\circ)\| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

which says that $\|\tilde{a}^\circ \ominus \tilde{T}(\tilde{a}^\circ)\| = 0$. Therefore we conclude that $\tilde{T}(\tilde{a}^\circ) \stackrel{\Omega}{=} \tilde{a}^\circ$ for any point $\tilde{a}^\circ \in [\tilde{a}]$ by part (ii) of Proposition 4.

Suppose that $\tilde{a}^\circ \notin [\tilde{a}]$ is another near fixed point of \tilde{T} . Then $\tilde{T}(\tilde{a}^\circ) \stackrel{\Omega}{=} \tilde{a}^\circ$ and $[\tilde{a}^\circ] \neq [\tilde{a}]$, i.e., $\tilde{a} \stackrel{\Omega}{\neq} \tilde{a}^\circ$. Then $\tilde{T}(\tilde{a}) \oplus \tilde{\omega}^{(1)} = \tilde{a} \oplus \tilde{\omega}^{(2)}$ and $\tilde{T}(\tilde{a}^\circ) \oplus \tilde{\omega}^{(3)} = \tilde{a}^\circ \oplus \tilde{\omega}^{(4)}$, where $\tilde{\omega}_i \in \Omega$ for $i = 1, 2, 3, 4$. Therefore we obtain

$$\begin{aligned} \|\tilde{a} \ominus \tilde{a}^\circ\| &= \|(\tilde{a} \oplus \tilde{\omega}^{(2)}) \ominus (\tilde{a}^\circ \oplus \tilde{\omega}^{(4)})\| \\ &\quad \text{(using } -\tilde{\omega}^{(4)} \in \Omega, \text{ the null equality and Proposition 1)} \\ &= \|(\tilde{T}(\tilde{a}) \oplus \tilde{\omega}^{(1)}) \ominus (\tilde{T}(\tilde{a}^\circ) \oplus \tilde{\omega}^{(3)})\| = \|\tilde{T}(\tilde{a}) \ominus \tilde{T}(\tilde{a}^\circ)\| \\ &\quad \text{(using } -\tilde{\omega}^{(3)} \in \Omega, \text{ the null equality and Proposition 1)} \\ &< \|\tilde{a} \ominus \tilde{a}^\circ\| \text{ (since } \tilde{a} \stackrel{\Omega}{\neq} \tilde{a}^\circ \text{ and } \tilde{T} \text{ is a weakly strict norm contraction).} \end{aligned}$$

This contradiction says that \tilde{a}° cannot be a near fixed point of \tilde{T} . Equivalently, if \tilde{a}° is a near fixed point of \tilde{T} , then $\tilde{a}^\circ \in [\tilde{a}]$. This completes the proof. \square

Now we consider another fixed point theorem based on the concept of weakly uniformly strict norm contraction which was proposed by Meir and Keeler [21].

Definition 17. Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a near pseudo-normed space of fuzzy numbers with the null set Ω . A fuzzy-number-valued function $\tilde{T} : (\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|) \rightarrow (\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ is called a weakly uniformly strict norm contraction on $\mathcal{F}_{cc}(\mathbb{R})$ if and only if the following conditions are satisfied:

- for $\tilde{a} \stackrel{\Omega}{=} \tilde{b}$, i.e., $[\tilde{a}] = [\tilde{b}]$, $\|\tilde{T}(\tilde{a}) \ominus \tilde{T}(\tilde{b})\| = 0$;
- for $\tilde{a} \not\stackrel{\Omega}{=} \tilde{b}$, i.e., $[\tilde{a}] \neq [\tilde{b}]$, given any $\epsilon > 0$, there exists $\delta > 0$ such that $\epsilon \leq \|\tilde{a} \ominus \tilde{b}\| < \epsilon + \delta$ implies $\|\tilde{T}(\tilde{a}) \ominus \tilde{T}(\tilde{b})\| < \epsilon$.

By part (ii) of Proposition 4, we see that if $\tilde{a} \not\stackrel{\Omega}{=} \tilde{b}$, then $\|\tilde{a} \ominus \tilde{b}\| \neq 0$, which says that the weakly uniformly strict norm contraction is well-defined. In other words, $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ should be assumed to be a near pseudo-normed space of fuzzy numbers rather than pseudo-seminormed space of fuzzy numbers.

Remark 2. We observe that if \tilde{T} is a weakly uniformly strict norm contraction on $\mathcal{F}_{cc}(\mathbb{R})$, then \tilde{T} is also a weakly strict norm contraction on $\mathcal{F}_{cc}(\mathbb{R})$.

Lemma 2. Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a near pseudo-normed space of fuzzy numbers with the null set Ω , and let $\tilde{T} : (\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|) \rightarrow (\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a weakly uniformly strict norm contraction on $\mathcal{F}_{cc}(\mathbb{R})$. Then the sequence $\{\|\tilde{T}^n(\tilde{a}) \ominus \tilde{T}^{n+1}(\tilde{a})\|\}_{n=1}^{\infty}$ is decreasing to zero for any $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$.

Proof. For convenience, we write $\tilde{T}^n(\tilde{a}) = \tilde{a}^{(n)}$ for all n . Let $c_n = \|\tilde{a}^{(n)} \ominus \tilde{a}^{(n+1)}\|$.

- Suppose that $[\tilde{a}^{(n-1)}] \neq [\tilde{a}^{(n)}]$. By Remark 2, we have

$$c_n = \|\tilde{a}^{(n)} \ominus \tilde{a}^{(n+1)}\| = \|\tilde{T}^n(\tilde{a}) \ominus \tilde{T}^{n+1}(\tilde{a})\| < \|\tilde{T}^{n-1}(\tilde{a}) \ominus \tilde{T}^n(\tilde{a})\| = \|\tilde{a}^{(n-1)} \ominus \tilde{a}^{(n)}\| = c_{n-1}.$$

- Suppose that $[\tilde{a}^{(n-1)}] = [\tilde{a}^{(n)}]$. Then, by the first condition of Definition 17,

$$c_n = \|\tilde{T}^n(\tilde{a}) \ominus \tilde{T}^{n+1}(\tilde{a})\| = \|\tilde{T}(\tilde{a}^{(n-1)}) \ominus \tilde{T}(\tilde{a}^{(n)})\| = 0 < c_{n-1}.$$

The above two cases say that the sequence $\{c_n\}_{n=1}^{\infty}$ is decreasing. We consider the following cases.

- Let m be the first index in the sequence $\{\tilde{a}^{(n)}\}_{n=1}^{\infty}$ such that $[\tilde{a}^{(m-1)}] = [\tilde{a}^{(m)}]$. Then we want to claim $c_{m-1} = c_m = c_{m+1} = \dots = 0$. Since $\tilde{a}^{(m-1)} \stackrel{\Omega}{=} \tilde{a}^{(m)}$, we have

$$c_{m-1} = \|\tilde{a}^{(m-1)} \ominus \tilde{a}^{(m)}\| = 0.$$

Using the first condition of Definition 17, we also have

$$0 = \|\tilde{T}(\tilde{a}^{(m-1)}) \ominus \tilde{T}(\tilde{a}^{(m)})\| = \|\tilde{T}^m(\tilde{a}) \ominus \tilde{T}^{m+1}(\tilde{a})\| = \|\tilde{a}^{(m)} \ominus \tilde{a}^{(m+1)}\| = c_m,$$

which says that $\tilde{a}^{(m)} \stackrel{\Omega}{=} \tilde{a}^{(m+1)}$, i.e., $[\tilde{a}^{(m)}] = [\tilde{a}^{(m+1)}]$. Using the similar arguments, we can obtain $c_{m+1} = 0$ and $[\tilde{a}^{(m+1)}] = [\tilde{a}^{(m+2)}]$. Therefore the sequence $\{c_n\}_{n=1}^{\infty}$ is decreasing to zero.

- Suppose that $[\tilde{a}^{(m+1)}] \neq [\tilde{a}^{(m)}]$ for all $m \geq 1$. Since the sequence $\{c_n\}_{n=1}^{\infty}$ is decreasing, we assume that $c_n \downarrow \epsilon > 0$, i.e., $c_n \geq \epsilon > 0$ for all n . There exists $\delta > 0$ such that $\epsilon \leq c_m < \epsilon + \delta$ for some m , i.e.,

$$\epsilon \leq \|\tilde{a}^{(m)} \ominus \tilde{a}^{(m+1)}\| < \epsilon + \delta.$$

By the second condition of Definition 17, we have

$$c_{m+1} = \|\tilde{a}^{(m+1)} \ominus \tilde{a}^{(m+2)}\| = \|\tilde{T}^{m+1}(\tilde{a}) \ominus \tilde{T}^{m+2}(\tilde{a})\| = \|\tilde{T}(\tilde{a}^{(m)}) \ominus \tilde{T}(\tilde{a}^{(m+1)})\| < \epsilon,$$

which contradicts $c_{m+1} \geq \epsilon$. Therefore we must have $\epsilon = 0$.

This completes the proof. \square

Theorem 6. Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a near Banach space of fuzzy numbers with the null set Ω . Suppose that $\|\cdot\|$ satisfies the null super-inequality, and that the fuzzy-number-valued function $\tilde{T} : (\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|) \rightarrow (\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ is a weakly uniformly strict norm contraction on $\mathcal{F}_{cc}(\mathbb{R})$. Then \tilde{T} has a near fixed point satisfying $\tilde{T}(\tilde{a}) \stackrel{\Omega}{=} \tilde{a}$. Moreover, the near fixed point \tilde{a} is obtained by the limit

$$\|\tilde{T}^n(\tilde{a}^{(0)}) \ominus \tilde{a}\| = \|\tilde{a} \ominus \tilde{T}^n(\tilde{a}^{(0)})\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Assume further that $\|\cdot\|$ satisfies the null equality. Then we also have the following properties.

- The uniqueness is in the sense that there is a unique equivalence class $[\tilde{a}]$ such that any $\tilde{a}^\circ \notin [\tilde{a}]$ cannot be a near fixed point.
- Each point $\tilde{a}^\circ \in [\tilde{a}]$ is also a near fixed point of \tilde{T} satisfying $\tilde{T}(\tilde{a}^\circ) \stackrel{\Omega}{=} \tilde{a}^\circ$ and $[\tilde{a}^\circ] = [\tilde{a}]$.
- If \tilde{a}° is a near fixed point of \tilde{T} , then $\tilde{a}^\circ \in [\tilde{a}]$, i.e., $[\tilde{a}^\circ] = [\tilde{a}]$. Equivalently, if \tilde{a} and \tilde{a}° are the near fixed points of \tilde{T} , then $\tilde{a} \stackrel{\Omega}{=} \tilde{a}^\circ$.

Proof. According to Theorem 5 and Remark 2, we just need to claim that if \tilde{T} is a weakly uniformly strict norm contraction, then $\{\tilde{T}^n(\tilde{a}^{(0)})\}_{n=1}^\infty \equiv \{\tilde{a}^{(n)}\}_{n=1}^\infty$ forms a Cauchy sequence. Suppose that $\{\tilde{a}^{(n)}\}_{n=1}^\infty$ is not a Cauchy sequence. Then there exists $2\epsilon > 0$ such that, given any N , there exist $n > m \geq N$ satisfying $\|\tilde{a}^{(m)} \ominus \tilde{a}^{(n)}\| > 2\epsilon$. Since \tilde{T} is a weakly uniformly strict norm contraction on $\mathcal{F}_{cc}(\mathbb{R})$, for $\tilde{a} \not\stackrel{\Omega}{=} \tilde{b}$, there exists $\delta > 0$ such that

$$\epsilon \leq \|\tilde{a} \ominus \tilde{b}\| < \epsilon + \delta \text{ implies } \|\tilde{T}(\tilde{a}) \ominus \tilde{T}(\tilde{b})\| < \epsilon.$$

Let $\delta' = \min\{\delta, \epsilon\}$. For $\tilde{a} \not\stackrel{\Omega}{=} \tilde{b}$, we are going to claim

$$\epsilon \leq \|\tilde{a} \ominus \tilde{b}\| < \epsilon + \delta' \text{ implies } \|\tilde{T}(\tilde{a}) \ominus \tilde{T}(\tilde{b})\| < \epsilon. \tag{24}$$

Indeed, if $\delta' = \delta$ then it is done, and if $\delta' = \epsilon$, i.e., $\epsilon < \delta$, then $\epsilon + \delta' = \epsilon + \epsilon < \epsilon + \delta$.

Let $c_n = \|\tilde{a}^{(n)} \ominus \tilde{a}^{(n+1)}\|$. Since the sequence $\{c_n\}_{n=1}^\infty$ is decreasing to zero by Lemma 2, we can find N such that $c_N < \delta'/3$. For $n > m \geq N$, we have

$$\|\tilde{a}^{(m)} \ominus \tilde{a}^{(n)}\| > 2\epsilon \geq \epsilon + \delta', \tag{25}$$

which implicitly says that $\tilde{a}^{(m)} \not\stackrel{\Omega}{=} \tilde{a}^{(n)}$. Since the sequence $\{c_n\}_{n=1}^\infty$ is decreasing by Lemma 2 again, we obtain

$$\|\tilde{a}^{(m)} \ominus \tilde{a}^{(m+1)}\| = c_m \leq c_N < \frac{\delta'}{3} \leq \frac{\epsilon}{3} < \epsilon. \tag{26}$$

For j with $m < j \leq n$, using Proposition 3, we have

$$\|\tilde{a}^{(m)} \ominus \tilde{a}^{(j+1)}\| \leq \|\tilde{a}^{(m)} \ominus \tilde{a}^{(j)}\| + \|\tilde{a}^{(j)} \ominus \tilde{a}^{(j+1)}\|. \tag{27}$$

We want to show that there exists j with $m < j \leq n$ such that $\tilde{a}^{(m)} \not\stackrel{\Omega}{=} \tilde{a}^{(j)}$ and

$$\epsilon + \frac{2\delta'}{3} < \|\tilde{a}^{(m)} \ominus \tilde{a}^{(j)}\| < \epsilon + \delta'. \tag{28}$$

Let $\gamma_j = \|\tilde{a}^{(m)} \ominus \tilde{a}^{(j)}\|$ for $j = m + 1, \dots, n$. Then (25) and (26) says that $\gamma_{m+1} < \epsilon$ and $\gamma_n > \epsilon + \delta'$. Let j_0 be an index such that

$$j_0 = \max \left\{ j \in [m + 1, n] : \gamma_j \leq \epsilon + \frac{2\delta'}{3} \right\}.$$

Then we see that $j_0 < n$, since $\gamma_n > \epsilon + \delta'$. By the definition of j_0 , we also see that $j_0 + 1 \leq n$ and $\gamma_{j_0+1} > \epsilon + \frac{2\delta'}{3}$, which also says that $\tilde{a}^{(m)} \not\stackrel{\Omega}{=} \tilde{a}^{(j_0+1)}$. Therefore expression (28) will be sound if we can show that

$$\epsilon + \frac{2\delta'}{3} < \gamma_{j_0+1} < \epsilon + \delta'.$$

Suppose that this is not true, i.e., $\gamma_{j_0+1} \geq \epsilon + \delta'$. From (27), we have

$$\frac{\delta'}{3} > c_N \geq c_{j_0} = \| \tilde{a}^{(j_0)} \ominus \tilde{a}^{(j_0+1)} \| \geq \gamma_{j_0+1} - \gamma_{j_0} \geq \epsilon + \delta' - \epsilon - \frac{2\delta'}{3} = \frac{\delta'}{3}.$$

This contradiction says that (28) is sound. Since $\tilde{a}^{(m)} \not\stackrel{\Omega}{=} \tilde{a}^{(j)}$, using (24), we see that (28) implies

$$\| \tilde{a}^{(m+1)} \ominus \tilde{a}^{(j+1)} \| = \| \tilde{T}(\tilde{a}^{(m)}) \ominus \tilde{T}(\tilde{a}^{(j)}) \| < \epsilon. \tag{29}$$

Therefore we obtain

$$\begin{aligned} \| \tilde{a}^{(m)} \ominus \tilde{a}^{(j)} \| &\leq \| \tilde{a}^{(m)} \ominus \tilde{a}^{(m+1)} \| + \| \tilde{a}^{(m+1)} \ominus \tilde{a}^{(j+1)} \| + \| \tilde{a}^{(j+1)} \ominus \tilde{a}^{(j)} \| \quad (\text{by Proposition 3}) \\ &< c_m + \epsilon + c_j \quad (\text{by (29)}) \\ &< \frac{\delta'}{3} + \epsilon + \frac{\delta'}{3} = \epsilon + \frac{2\delta'}{3}, \end{aligned}$$

which contradicts (28). This contradiction says that the sequence $\{\tilde{T}^n(\tilde{a})\}_{n=1}^\infty \equiv \{\tilde{a}^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence, and the proof is complete. \square

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