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Quasirecognition by Prime Graph of the Groups ${}^2D_{2n}(q)$ Where $q < 10^5$

Hossein Moradi [1], Mohammad Reza Darafsheh [2,*] and Ali Iranmanesh [1]

[1] Department of Mathematics, Tarbiat Modares University, P.O. Box 14115-137, Tehran, Iran; moradi61hh@yahoo.com (H.M); iranmanesh@modares.ac.ir (A.I.)

[2] School of mathematics, statistics and computer, College of science, University of Tehran, P.O. Box 14155-6455, Tehran, Iran

* Correspondence: darafsheh@ut.ac.ir; Tel.: +9-8912-148-0610

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Abstract: Let G be a finite group. The prime graph $\Gamma(G)$ of G is defined as follows: The set of vertices of $\Gamma(G)$ is the set of prime divisors of $|G|$ and two distinct vertices p and p' are connected in $\Gamma(G)$, whenever G contains an element of order pp'. A non-abelian simple group P is called recognizable by prime graph if for any finite group G with $\Gamma(G) = \Gamma(P)$, G has a composition factor isomorphic to P. It is been proved that finite simple groups ${}^2D_n(q)$, where $n \neq 4k$, are quasirecognizable by prime graph. Now in this paper we discuss the quasirecognizability by prime graph of the simple groups ${}^2D_{2k}(q)$, where $k \geq 9$ and q is a prime power less than 10^5.

Keywords: prime graph; simple group; orthogonal groups; quasirecognition

1. Introduction

Let G be a finite group. By $\pi_e(G)$ we denote the set of element orders of G. For an integer n we define $\pi(n)$ as the set of prime divisors of n and we set $\pi(G)$ for $\pi(|G|)$. The prime graph of the Gruenberg-Kegel graph of G is denoted by $\Gamma(G)$ and it is a graph with vertex set $\pi(G)$ in which two distinct vertices p and q are adjacent if and only if $pq \in \pi(G)$, and in this case we will write $p \sim q$.

A subset of vertices of $\Gamma(G)$ is called an independent subset of $\Gamma(G)$ if its vertices are pairwise nonadjacent. Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise nonadjacent in $\Gamma(G)$. We also denote by $t(2, G)$ the maximal number of vertices in the independent sets of $\Gamma(G)$ containing 2. A finite nonabelian simple group P is called quasirecognizable by prime graph if every finite group G with $\Gamma(G) = \Gamma(P)$ has a composition factor isomorphic to P. P is called recognizable by prime graph if $\Gamma(G) = \Gamma(P)$ implies $G \cong P$. In addition, finite group P is considered to be recognizable by a set of element orders if the equality $\pi_e(G) = \pi_e(P)$, for each finite group G implies that $G \cong P$. A finite simple nonabelian group P is considered to be quasirecognizable by the set of element orders if each finite group H with $\pi_e(H) = \pi_e(P)$ has a composition factor isomorphic to P. If a finite simple group is quasirecognizable (recognizable) by prime graph, then it is quasirecognizable (recognizable) by set of element orders, but the inverse is not true necessarily and proving by prime graph is more difficult.

Hagie determined finite groups H satisfying $\Gamma(H) = \Gamma(S)$, where S is a sporadic simple group [1]. In [2,3], finite groups with the same prime graph as $\Gamma(PSL(2, q))$, where q is a prime power, are determined. Quasirecognizability by prime graph of groups $G_2(3^{2n+1})$ and ${}^2B_2(2^{2n+1})$ has been proved in [4]. In [5–7], finite groups with the same prime graphs as $\Gamma(L_n(2))$, $\Gamma(U_n(2))$, $\Gamma(D_n(2))$, $\Gamma({}^2D_n(2))$ and $\Gamma({}^2D_{2k}(3))$ are obtained. In addition, in [8], it is proved that if p is a prime less than 1000, for suitable n, the finite simple groups $L_n(p)$ and $U_n(p)$ are quasirecognizable by prime graph. Now as the main result of this paper, we prove the following theorem:

Main Theorem. *The finite simple group* $^2D_{2k}(q)$*, where* $k \geq 9$ *and* $q < 10^5$*, is quasirecognizable by prime graph.*

Throughout this paper, all groups are finite and by a simple group we mean a nonabelian simple group. All further unexplained notations are standard and the reader is referred to [9].

2. Preliminary Results

Lemma 1 ([10] Theorem 1)**.** *Let* G *be a finite group with* $t(G) \geq 3$ *and* $t(2, G) \geq 2$*. Then the following hold:*

1. *there exists a finite nonabelian simple group* S *such that*

$$S \leq \bar{G} = G/K \leq Aut(S)$$

for the maximal normal soluble subgroup K *of* G*.*

2. *for every independent subset* ρ *of* $\pi(G)$ *with* $|\rho| \geq 3$ *at most one prime in* ρ *divides the product* $|K| \cdot |\bar{G}/S|$*. In particular,* $t(S) \geq t(G) - 1$*.*
3. *one of the following holds:*
   - (a) *every prime* $r \in \pi(G)$ *nonadjacent to* 2 *in* $\Gamma(G)$ *does not divide the product* $|K| \cdot |\bar{G}/S|$*; in particular,* $t(2, S) \geq t(2, G)$*;*
   - (b) *there exists a prime* $r \in \pi(K)$ *nonadjacent to* 2 *in* $\Gamma(G)$*; in this case* $t(G) = 3$*,* $t(2, G) = 2$*, and* $S \cong A_7$ *or* $L_2(q)$ *for some odd* q*.*

Remark 1. *In Lemma 1, for every odd prime* $p \in \pi(S)$*, we have* $t(p, S) \geq t(p, G) - 1$*.*

If q is a natural number, r is an odd prime and $(q, r) = 1$, then by $e(r, q)$ we denote the smallest natural number m such that $q^m \equiv 1 \pmod r$. Given an odd q, put $e(2, q) = 1$ if $q \equiv 1 \pmod 4$ and put $e(2, q) = 2$ if $q \equiv -1 \pmod 4$. Using Fermat's little theorem we can see that if r is an odd prime such that $r \mid (q^n - 1)$, then $e(r, q) \mid n$.

Lemma 2 ([11] Proposition 2.5)**.** *Let* $G = D^{\varepsilon}_n(q)$*, where* q *is power of prime* p*. Define*

$$\eta(m) = \begin{cases} m, & \text{if } m \text{ is odd;} \\ m/2, & \text{otherwise.} \end{cases}$$

Suppose r, s *are odd primes and* $r, s \in \pi(D^{\varepsilon}_n(q)) \setminus \{p\}$*. Put* $k = e(r, q)$*,* $l = e(s, q)$*, and* $1 \leq \eta(k) \leq \eta(l)$*. Then* r *and* s *are non-adjacent if and only if* $2\eta(k) + 2\eta(l) > 2n - (1 - \varepsilon(-1)^{k+l})$ *and* k *and* l *satisfy the following condition:*

$$\frac{l}{k} \text{ is not an odd integer,}$$

and if $\varepsilon = +$*, then the chain of equalities*

$$n = l = 2\eta(l) = 2\eta(k) = 2k$$

is not true.

Lemma 3 ([11] Proposition 2.3)**.** *Let* G *be one of the simple groups of Lie type,* $B_n(q)$ *or* $C_n(q)$*, over a field of characteristic* p*. Define*

$$\eta(m) = \begin{cases} m, & \text{if } m \text{ is odd;} \\ m/2, & \text{otherwise.} \end{cases}$$

Let r, s be odd primes with $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r, q)$ and $l = e(s, q)$, and suppose that $1 \leq \eta(k) \leq \eta(l)$. Then r and s are non-adjacent if and only if $\eta(k) + \eta(l) > n$, and k, l satisfy:

$$\frac{l}{k} \text{ is not an odd natural number.}$$

Lemma 4 ([12] Proposition 2.1)**.** *Let $G = L_n(q)$, where q is a power of prime p. Let r and s be odd primes and $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r, q)$ and $l = e(s, q)$ and assume that $2 \leq k \leq l$. Then r and s are nonadjacent if and only if $k + l > n$ and k does not divide l.*

Lemma 5 ([12] Proposition 2.2)**.** *Let $G = U_n(q)$, where q is a power of prime p. Define*

$$\nu(m) = \begin{cases} m, & m \equiv 0 \pmod 4; \\ m/2, & m \equiv 2 \pmod 4; \\ 2m, & m \equiv 1 \pmod 2. \end{cases}$$

Let r and s be odd primes and r, $s \in \pi(G) \setminus \{p\}$. Put $k = e(r, q)$ and $l = e(s, q)$ and suppose that $2 \leq \nu(k) \leq \nu(l)$. Then r and s are nonadjacent if and only if $\nu(k) + \nu(l) > n$ and $\nu(k)$ does not divide $\nu(l)$.

For Lemmas 2 and 5, simultaneously, we define the following function:

$$\nu'(m) = \begin{cases} m, & \varepsilon = +; \\ \nu(m), & \varepsilon = -. \end{cases}$$

which we will use in the proofs. We note that a prime r with $e(r, q) = m$ is called a primitive prime divisor of $q^m - 1$ (obviously, $q^m - 1$ can have more than one primitive prime divisors).

Lemma 6. *(Zsigmondy's theorem) [13] Let p be a prime and let n be a positive integer. Then one of the following holds:*

1. *there is a primitive prime p' for $p^n - 1$, that is , $p' \mid (p^n - 1)$ but $p' \not| (p^m - 1)$, for every $1 \leq m < n$,*
2. *$p = 2$, $n = 1$ or 6,*
3. *p is a Mersenne prime and $n = 2$.*

3. Proof of the Main Theorem

Throughout this section, we suppose that $D := {}^2D_{2k}(p^\alpha)$ where $k \geq 9$, $2 < p^\alpha < 10^5$ and G is a finite group such that $\Gamma(G) = \Gamma(D)$. We denote a primitive prime divisor of $q^i - 1$ by r_i and a primitive prime divisor of $q'^i - 1$ by r'_i, where $q' \neq q$.

By ([12] Tables 4, 6 and 8), we deduce that $t(D) \geq 14$ and $t(2, D) \geq 2$. Therefore, $t(G) \geq 14$ and $t(2, G) \geq 2$. Now by Lemma 1, it follows that there exists a finite nonabelian simple group S such that

$$S \leq \bar{G} := G/K \leq \mathrm{Aut}(S),$$

where K is the maximal normal solvable subgroup of G. In addition, $t(S) \geq t(G) - 1$ and $t(2, S) \geq t(2, G)$ by Lemma 1. Therefore, $t(S) \geq 13$ and $t(2, S) \geq 2$. On the other hand, by ([12] Tables 2 and 9), if S is isomorphic to a sporadic or an exceptional simple group of Lie type, then $t(S) \leq 12$. This implies that S is not isomorphic to any sporadic or any exceptional simple groups of Lie type.

In the sequel, we consider each possibility for S.

Lemma 7. *S is not isomorphic to any alternating group.*

Proof. Suppose that $S \cong A_m$, where $m \geq 5$. Since $t(S) \geq 13$, Lemma 1, we get that $m \geq 61$ and so $\{47, 59\} \subseteq \pi(S)$.

Case 1. Let $\{47, 59\} \not\subseteq \pi(q^2 - 1)$, where $q = p^\alpha < 10^5$. Therefore, we get that $e(47, q) \geq 23$ or $e(59, q) \geq 29$. $\{47, 59\} \subseteq \pi(S)$, so G contains an element $a \in A_m$, such that $e(a, q) \geq 23$, which implies that $n \geq 24$. Now we have $\min\{t(47, G), t(59, G)\} \geq 19$. Hence, according to Remark 1, $\min\{t(47, S), t(59, S)\} \geq 18$ in S. On the other hand, 47 is not connected to the prime numbers in the interval $[m - 46, m]$ in the prime graph of A_m and similarly, 59 is not connected to the prime numbers in the interval $[m - 58, m]$, in the prime graph of A_m. However, these intervals contain at most 16 prime numbers, and this implies that $t(S) < t(G) - 1$, is a contradiction.

Case 2. Let $\{47, 59\} \subseteq \pi(q^2 - 1)$, where $q = p^\alpha < 10^5$. Using GAP, we get that:

$q \in A := \{11093, 21713, 27259, 28201, 38351, 38821, 39293, 44839, 55931, 61007, 66553, 93811, 99829\}$.

First let $q \in A \setminus \{21713\}$. Then we have $e(23, q) \geq 11$ and so similarly to Case 1, we get that $t(23, G) - 1 > t(23, S)$, is a contradiction.

Let $q = 21713$. Since $e(19, q) = 18$, again similarly to Case 1, we get a contradiction. □

Lemma 8. *If S is isomorphic to a classical simple group of Lie type over a field of characteristic p, then $S \cong D$.*

Proof. Let S be a nonabelian simple group of Lie type over $\mathrm{GF}(q')$, $q' = p^\beta$. By the hypothesis,

$$S \leq \bar{G} := G/N \leq \mathrm{Aut}(S),$$

where N is the maximal normal solvable subgroup of G. In the sequel, we denote by r_i a primitive prime divisor of $q^i - 1$ and by r'_i a primitive prime divisor of $q'^i - 1$. We remark that $\{p, r_{2n}\} \subseteq \pi(S)$ and $|\rho(p, G) \cap \pi(S)| \geq 3$ by Lemma 1.

Now we consider the following cases:

Case 1. Let $r_{2n-2} \in \pi(S)$. In addition, let p_1 and p_2 be two primitive prime divisors of $p^{(2n-2)\alpha} - 1$ and $p^{2n\alpha} - 1$, respectively. So we may assume that p_1 and p_2 are r_{2n-2} and r_{2n}, respectively. This implies that $\{r_{2n-2}, r_{2n}\} \subseteq \pi(S)$. Thus r_{2n-2} is a primitive prime divisor of $q'^s - 1$ and r_{2n} is a primitive prime divisor of $q'^t - 1$, where $s = e(r_{2n-2}, p^\beta)$ and $t = e(r_{2n}, p^\beta)$. It follows that $(2n - 2)\alpha \mid s\beta$ and $2n\alpha \mid t\beta$. On the other hand, using Zsigmondy's theorem, we conclude that $t\beta \leq 2n\alpha$ and so $t\beta = 2n\alpha$. Furthermore, since $2n < 2(2n - 2)$, we have $s\beta = (2n - 2)\alpha$ and $s < t$.

Now we consider each possibility for S, separately. If $\rho(p, S) = \{r'_i \mid i \in I\} \cup \{p\}$, then using the results in [12], each $r'_j \in \pi(S)$, where $j \notin I$ is adjacent to p in $\Gamma(S)$.

Subcase 1.1. Let $S \cong L_m(q')$. By [[12] Proposition 2.6], we see that each prime divisor of $|L_m(q')|$ is adjacent to p, except r'_m and r'_{m-1}. Hence $\rho(p, S) = \{p, r'_{m-1}, r'_m\}$. Therefore, p_1 and p_2 are some primitive prime divisors of $q'^m - 1$ and $q'^{m-1} - 1$. Since $s < t$, we conclude that $m = t$ and $m - 1 = s$. Hence $2n\alpha = m\beta$ and $(2n - 2)\alpha = (m - 1)\beta$. Consequently, we get that $\beta = 2\alpha$ and $m = n$, that is $S \cong L_n(p^{2\alpha})$. Then S has a maximal torus of order $(p^{2n\alpha} - 1)/((p^{2\alpha} - 1)(n, p^{2\alpha} - 1))$, say T. Obviously, $r_n, r_{2n} \in \pi(T)$. Therefore, $r_n \sim r_{2n}$ in $\Gamma(L_n(p^{2\alpha}))$, whereas $r_n \not\sim r_{2n}$ in $\Gamma(G)$, by Lemma 2, which is a contradiction.

Subcase 1.2. Let $S \cong U_m(q')$. If $m = 3$, then $\rho(p, S) = \{p, r'_1 \neq 2, r'_6\}$ and so $s = 1$ and $t = 6$. Hence $(2n - 2)\alpha = \beta$ and $2n\alpha = 6\beta$. Therefore $n = 6/5$, is a contradiction.

If $m \equiv 0 \pmod 4$, then $\rho(p, S) = \{p, r'_{2m-2}, r'_m\}$ and so $s = m$ and $t = 2m - 2$. Hence $(2n - 2)\alpha = m\beta$ and $2n\alpha = (2m - 2)\beta$. Then $n = (2m - 2)/(m - 2)$ and so $n = 3$, is a contradiction.

If $m \equiv 3 \pmod 4$, then Therefore, $s = (m - 1)/2$ and $t = 2m$. Hence $(2n - 2)\alpha = (m - 1)\beta/2$ and $2n\alpha = 2m\beta$. Now easy computation shows that it is impossible.

If $m \equiv 1, 2 \pmod 4$, then similarly to the above discussion, we get a contradiction.

Subcase 1.3. Let $S \cong B_m(q')$ or $C_m(q')$. Since $t(p, S) \geq 3$, using [[12] Table 4], we get that m is odd. In this case, $\rho(p, S) = \{p, r'_m, r'_{2m}\}$. Hence $s = m$ and $t = 2m$ and so $(2n - 2)\alpha = m\beta$ and $2n\alpha = 2m\beta$, which implies that $n = 2$, which is a contradiction. Let $S \cong {}^2D_m(q')$, where m is odd. Since $\rho(p, S) = \{p, r'_{2m-2}, r'_{2m}\}$, we conclude that $(2n - 2)\alpha = (2m - 2)\beta$ and $2n\alpha = 2m\beta$ and so $m = n$, which is impossible, since n is even.

Similarly, we can prove that $S \not\cong {}^2D_m(q')$, where m is even and $S \not\cong D_m(q')$.

Case 2. Let $r_{2n-2} \notin \pi(S)$. Hence $r_{n-1} \in \pi(S)$. Let p_1 and p_2 be as r_{n-1} and r_{2n}, respectively. Therefore r_{n-1} and r_{2n} are primitive prime divisors of $q'^s - 1$ and $q'^t - 1$, respectively, where $s = e(r_{n-1}, p^\beta)$ and $t = e(r_{2n}, p^\beta)$. Now we conclude that $(n-1)\alpha \mid s\beta$ and $2n\alpha \mid t\beta$. On the other hand, using Zsigmondy's theorem, we conclude that $t\beta \leq 2n\alpha$ and so $t\beta = 2n\alpha$. If $s\beta > (n-1)\alpha$, then using Zsigmondy's theorem, we conclude that $s\beta = (2n-2)\alpha$, which implies that $r_{2n-2} \in \pi(S)$, which is a contradiction. Hence we suppose that $s\beta = (n-1)\alpha$.

Subcase 2.1. Let $S \cong L_m(q')$, where $q' = p^\beta$. We know that $\rho(p, S) = \{p, r'_{m-1}, r'_m\}$. Hence $t = m$, $s = m-1$, $2n\alpha = m\beta$, and $(n-1)\alpha = (m-1)\beta$. These equations imply that $m = 2 - 2/(n+1)$, which is impossible.

Subcase 2.2. Let $S \cong {}^2D_m(q')$, where m is odd. We note that $\rho(p, S) = \{p, r'_{2m-2}, r'_{2m}\}$ and so $(n-1)\alpha = (2m-2)\beta$ and $2n\alpha = 2m\beta$ and so $m = 2 - 2/(n+1)$, which is impossible.

Subcase 2.3. Let $S \cong {}^2D_m(q')$, where m is even. Since $\rho(p, S) = \{p, r'_{m-1}, r'_{2m-2}, r'_{2m}\}$, we get that $2n\alpha = 2m\beta$ and $(n-1)\alpha = (2m-2)\beta$ or $(n-1)\alpha = (m-1)\beta$. If $(n-1)\alpha = (2m-2)\beta$, then we get that $m = 2 - 2/(n+1)$, which is impossible. Hence $(n-1)\alpha = (m-1)\beta$, which implies that $m = n$ and $\alpha = \beta$, and so $S \cong D$, which is a contradiction, since $r_{2n-2} \notin \pi(S)$.

We can use a similar proof for groups $U_m(q')$, $B_m(q')$, $C_m(q')$ and $D_m(q')$ and get a contradiction. We omit the proof for convenience. □

Lemma 9. *If S is isomorphic to a classical simple group of Lie type over a field of characteristic $p' \neq p$, then $S \ncong D$.*

Proof. Let S be isomorphic to a classical simple group of Lie type over a field with q' elements, where $q' = p'^\beta$. Using [[12] Table 4], $t(p', S) \leq 4$ and so Lemma 1 implies that $t(p', G) \leq 5$. On the other hand, by Lemma 2, we deduce that if $r \in \pi(G) \setminus \{r_1, r_2, r_3, r_4, r_6\}$, then $t(r, G) > 5$. Hence $p' \in \{r_1, r_2, r_3, r_4, r_6\}$ and so $p' \mid (q^2+1)(q^6-1)$.

Consider $r'_3 \in \pi(S) \subseteq \pi(G)$ and $3 = e(r'_3, q') \leq e(r'_3, p')$. By Lemmas 2, 3, 4 and 5 we get that for each classical simple group of Lie type S, we have $t(r'_3, S) \leq 6$. On the other hand, using Remark 1, we have $t(r'_3, G) \leq t(r'_3, S) + 1$ and so $t(r'_3, G) \leq 7$. We note that $r'_3 \in \pi(S) \subseteq \pi(G) = \pi({}^2D_{2k}(q))$. Hence, by Lemma 2, it follows that $e(r'_3, q) \leq 10$. Since $t(S) \geq 13$, we conclude that $r'_i \in \pi(S)$, where $2 \leq i \leq 10$ and we have a similar argument for r'_i, $2 \leq i \leq 10$ and $e(r'_i, q) \leq 2i + 4$.

Hence, according to the above discussion, if $p' \in \pi((q^2+1)(q^6-1))$, then the following condition holds:

$$\text{If } r'_i \in \pi(p'^i - 1), \text{ then } e(r'_i, q) \leq 2i + 4, \text{ where } 2 \leq i \leq 10.$$

Using GAP, we get that the above condition holds only for $q = 54251$, where $p' = 2$. Since $t(S) \geq 13$, we conclude that $r'_{13} \in \pi(S)$. If $p' = 2$ and $q = 54251$, then $r'_{13} = 8191$ and so $e(8191, q) = 1365$, which contradicts Remark 1. Therefore, by the above argument, we get that S is not isomorphic to any classical simple group of Lie type over a field of characteristic $p' \neq p$. □

Using the Classification Theorem of finite simple groups and Lemmas 7–9, we get that the finite simple group ${}^2D_{2k}(q)$, where $k \geq 9$ and $q < 10^5$ is quasirecognizable by prime graph.

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References

1. Hagie, M. The prime graph of a sporadic simple group. *Commun. Algebra* **2003**, *31*, 4405–4424.
2. Khosravi, B. n-Recognition by prime graph of the simple group $PSL(2, q)$. *J. Algebra Appl.* **2008**, *7*, 735–748.

3. Khosravi, B.; Khosravi, B.; Khosravi, B. 2-Recognizability of $PSL(2, p^2)$ by the prime graph. *Sib. Math. J.* **2008**, *49*, 749–757.
4. Zhang, Q.; Shi, W.; Shen, R. Quasirecognition by prime graph of the simple groups $G_2(q)$ and ${}^2B_2(q)$. *J. Algebra Appl.* **2011**, *10*, 309–317.
5. Khosravi, B.; Moradi, H. Quasirecognition by prime graph of finite simple groups $L_n(2)$ and $U_n(2)$. *Acta. Math. Hung.* **2011**, *132*, 140–153.
6. Khosravi, B.; Moradi, H. Quasirecognition by prime graph of some orthogonal groups over the binary field. *J. Algebra Appl.* **2012**, *11*, 1250056.
7. Khosravi, B.; Moradi, H. Quasirecognition by prime graph of finite simple groups ${}^2D_n(3)$. *Int. J. Group Theory* **2014**, *3*, 47–56.
8. Mahmoudifar, A.; Khosravi, B. On quasirecognition by prime graph of the simple groups $A_n^+(p)$ and $A_n^-(p)$. *J. Algebra Appl.* **2015**, *14*, 1550006.
9. Conway, J.H.; Curtis, R.T.; Norton, S.P.; Parker, R.A.; Wilson, R.A. *Atlas of Finite Groups*; Oxford University Press: Oxford, UK, 1985.
10. Vasil'ev, A.V.; Gorshkov, I.B. On recognition of finite groups with connected prime graph. *Sib. Math. J.* **2009**, *50*, 233–238.
11. Vasil'ev, A.V.; Vdovin, E.P. Cocliques of maximal size in the prime graph of a finite simple group. *Algebra Logic* **2011**, *50*, 291–322.
12. Vasil'ev, A.V.;Vdovin, E.P. An adjacency criterion for the prime graph of a finite simple group. *Algebra Logic* **2005**, *44*, 381–406.
13. Zsigmondy, K. Zur theorie der potenzreste. *Monatsh. Math. Phys.* **1892**, 3, 265–284.