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A Numerical Method for Solving a Class of Nonlinear Second Order Fractional Volterra Integro-Differential Type of Singularly Perturbed Problems

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Abstract: In this paper, we study a class of fractional nonlinear second order Volterra integro-differential type of singularly perturbed problems with fractional order. We divide the problem into two subproblems. The first subproblem is the reduced problem when $\epsilon = 0$. The second subproblem is fractional Volterra integro-differential problem. We use the finite difference method to solve the first problem and the reproducing kernel method to solve the second problem. In addition, we use the Padé approximation. The results show that the proposed analytical method can achieve excellent results in predicting the solutions of such problems. Theoretical results are presented. Numerical results are presented to show the efficiency of the proposed method.

Keywords: singularly perturbed volterra integro-differential; caputo fractional derivative; nonlinear boundary value problem

1. Introduction

Volterra integral equations are considered as a type of integral equations. In 1913, Volterra published the first book that talked about Volterra integral equations. In 1884, Volterra began working on integral equations, but his important study was in 1896. However, the name Volterra integral equation was first called by Lalesco in 1908. Volterra integral equations have many applications in science and engineering such as elasticity, semi-conductors, scattering theory, seismology, heat conduction, metallurgy, fluid flow, chemical reactions, population dynamics, and spread of epidemics [1].

Volterra integral equations have growingly been recognized as useful tools for problems in science and engineering. In [2], they proposed and examined a spectral Jacobi-collocation approximation for fractional order integro-differential equations. Ray et al. [3], used the Legendre wavelet method to find the solutions for a system of nonlinear Volterra integro-differential equations. In [4], they used Laguerre polynomials and the collocation method to solve the pantograph-type Volterra integro-differential equations under some initial conditions. Yang et al. [5], discussed the blow-up of Volterra integro-differential equations with a dissipative linear term to show the differences of the solutions. In [6], they solved a non-linear system of higher order Volterra integro-differential equations using the Single Term Walsh Series (STWS) method. Also in [7], they solved the fractional Fredholm-Volterra integro-differential equations by the fractional-order functions based on the Bernoulli polynomials. We also indicate the interested reader to [8–16] for more research works on Volterra integro-differential equations.

In 1904, A German physicist called Ludwig Prandtl was revolutionized fluid dynamics. He noted that the influence of friction is experienced only very near an object moving through a fluid. In [16], he presented the idea of the boundary layer and its significance for drag and streamlining. In his

paper, Ludwig Prandtl assumed that the impact of friction was to cause the fluid instantly adjacent to the surface to stick to the surface. This boundary-layer notion had been the basis stone for the new fluid dynamics. Schlichting was one of the most famous books on boundary layer theory [17]. The scientific justification of boundary layer theory gave us more general hypothesis to determine asymptotic expansions of the solutions to the complete equations of the motion. Singular perturbation problem was the result of reduced the problem which was then solved by the method of matched asymptotic expansions. In 1946, Friedrichs and Wasow were the first time used the expression “singular perturbation” [18].

The differential equations of the singularly perturbed problem indicated the study of a group of differential equations including an asymptotically small parameter. The singularly perturbed problem is very important to both applied and pure mathematicians, physicists and engineers because of the fact that the solutions exhibit some interesting behavior. For example, the boundary layer, interior layer, and resonance phenomena [19].

There are a lot of applications of the singularly perturbed problem such as the nonlinear problems of plates and shells by means of the singular perturbation method [20]. Petar discussed typical applications of singular perturbation techniques to control problems in the last fifteen years [21]. Kokotovic et al., showed results on singular perturbations surveyed as a tool for model order reduction and separation of time scales in control system design [22], Ghorbel and Spong, reviewed results of integral manifolds of singularly perturbed non-linear differential equations and outlined the basic elements of the integral manifold method in the context of control system design [23]. Fridman, studied the H_∞ control problem for an affine nonlinear singularly perturbed system [24]. Fridman, studied the infinite horizon nonlinear quadratic optimal control problem for a singularly perturbed system [25]. Several other techniques to solve such problems are presented in [26–31].

In this paper, we consider the following class of fractional nonlinear second order Volterra integro-differential type of singularly perturbed problems of the form

$$\epsilon D^\alpha y + u(x, y)y' + \int_0^x K(x, t)v(t, y)dt = f(x), x \in (0, 1), 1 < \alpha \leq 2 \tag{1}$$

subject to

$$y(0) = y_0, y(1) = y_1 \tag{2}$$

where $\epsilon > 0$ is a small positive parameter, y_0 and y_1 are constants, and $K(x, t)$ and $f(x)$ are smooth functions. The derivative which we use in this paper is in the Caputo sense. We organize our paper as follows. In Section 2, we present some preliminaries and the reproducing kernel (RKM) method which we use in this paper. In Section 3, we present some analytical results. In Section 4, we present the proposed method. Some numerical results are presented in Section 5. We end this paper by conclusions which presented in Section 6.

2. Reproducing Kernel Method

In this section, we present some preliminaries and RKM which we use in this paper.

Definition 1. Let $\alpha > 0$ and x be a positive real number. Then, the Riemann-Liouville fractional integral of order α is given by

$$I^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x y(\zeta)(x - \zeta)^{\alpha-1} d\zeta$$

where Γ is the Euler gamma function.

Some of its properties are given as follows:

- $I^0 y(x) = y(x)$,
- $I^\alpha (ay(x) + by(x)) = a I^\alpha y(x) + b I^\alpha y(x)$ where a and b are constants.
- If $y(x)$ is continuous on $[0, \infty)$, then $I^\alpha I^\beta y(x) = I^\beta I^\alpha y(x) = I^{\alpha+\beta} y(x)$ for positive real numbers α and β .

Definition 2. Let $\alpha > 0$ and x be a positive real number. Then, the Caputo derivative of order α is given by

$$D^\alpha y(x) = \left\{ \begin{array}{ll} \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{y^{(n)}(\zeta)}{(x-\zeta)^{\alpha-n+1}} d\zeta, & n-1 < \alpha < n \in \mathbb{N}, \\ y^{(\alpha)}(x), & \alpha \in \mathbb{N} \end{array} \right\}$$

Some of its properties are given as follows:

- $D^\alpha x^\zeta = \frac{\Gamma(\zeta+1)}{\Gamma(\zeta-\alpha+1)} x^{\zeta-\alpha}$ where ζ is a real number greater than -1 ,
- $D^\alpha c = 0$ where c is constant,
- $D^\alpha (ay(x) + by(x)) = a D^\alpha y(x) + b D^\alpha y(x)$ where a and b are constants,
- For $\alpha > 0$ and x is a positive real number, $D^\alpha I^\alpha y(x) = y(x)$,
- $I^\alpha D^\alpha y(x) = y(x) - \sum_{i=0}^{n-1} \frac{(D^{\alpha-i+1} y(0)) x^{\alpha-i+1}}{\Gamma(\alpha-i)}$, $n-1 < \alpha \leq n \in \mathbb{N}$.

For more details, see [32,33].

Definition 3. Let E be a nonempty abstract set. A function $M : E \times E \rightarrow C$ is a reproducing kernel of the Hilbert space H if and only if

- $M(., x) \in H$ for all $x \in E$,
- $(\phi(.), M(., x)) = \phi(x)$ for all $x \in E$ and $\phi \in H$.

The second condition is called the reproducing property and a Hilbert space which possesses a reproducing kernel is called a reproducing kernel Hilbert space.

Consider the second order nonlinear fractional equation of the form

$$D^\alpha y + g(x, y)y' = 0, \quad x \in [0, 1], 1 < \alpha \leq 2 \tag{3}$$

subject to

$$y(0) = \theta, y(1) = \phi \tag{4}$$

where θ and ϕ are constants. First, we study the linear case when $g(y) = a(x)$. To homogenize the initial condition, we assume $u = y - \phi x - \theta(1 - x)$. Thus, Equations (3) and (4) can be rewritten as

$$D^\alpha u + a(x)u' = (-\phi + \theta) a(x) = h(x), \quad x \in [0, 1], 0 < \alpha \leq 1 \tag{5}$$

subject to

$$u(0) = 0, u(1) = 0. \tag{6}$$

In order to solve the linear Equations (5) and (6), we construct the kernel Hilbert spaces $W_2^1[0, 1]$ and $W_2^3[0, 1]$ in which every function satisfies the initial condition (6). Let

$$W_2^1[0, 1] = \{u(s) : u \text{ is absolutely continuous real value function, } u' \in L^2[0, 1]\}.$$

The inner product in $W_2^1[0, 1]$ is defined as

$$(u(y), v(y))_{W_2^1[0,1]} = u(0)v(0) + \int_0^1 u'(y)v'(y)dy,$$

and the norm $\|u\|_{W_2^1[0,1]}$ is given by

$$\|u\|_{W_2^1[0,1]} = \sqrt{(u(y), u(y))_{W_2^1[0,1]}}$$

where $u, v \in W_2^1[0, 1]$.

Theorem 1. *The space $W_2^1[0, 1]$ is a reproducing kernel Hilbert space, i.e.; there exists $R(s, y) \in W_2^1[0, 1]$ and its second partial derivative with respect to y exists such that for any $u \in W_2^1[0, 1]$ and each fixed $y, s \in [0, 1]$, we have*

$$(u(y), R(s, y))_{W_2^1[0,1]} = u(s)$$

In this case, $R(s, y)$ is given by

$$R(s, y) = \begin{cases} 1 + y, & y \leq s \\ 1 + s, & y > s \end{cases}$$

Proof. Using the integration by parts, one can get

$$\begin{aligned} (u(y), R(s, y))_{W_2^1[0,1]} &= u(0)R(s, 0) + \int_0^1 u'(y) \frac{\partial R}{\partial y}(s, y)dy \\ &= u(0)R(s, 0) + u(1) \frac{\partial R}{\partial y}(s, 1) - u(0) \frac{\partial R}{\partial y}(s, 0) - \int_0^1 u(y) \frac{\partial^2 R}{\partial y^2}(s, y)dy. \end{aligned}$$

Since $R(s, y)$ is a reproducing kernel of $W_2^1[0, 1]$,

$$(u(y), R(s, y))_{W_2^1[0,1]} = u(s)$$

which implies that

$$-\frac{\partial^2 R}{\partial y^2}(s, y) = \delta(y - s), \tag{7}$$

$$R(s, 0) - \frac{\partial R}{\partial y}(s, 0) = 0, \tag{8}$$

and

$$\frac{\partial R}{\partial y}(s, 1) = 0. \tag{9}$$

Since the characteristic equation of $-\frac{\partial^2 R}{\partial y^2}(s, y) = \delta(y - s)$ is $\lambda^2 = 0$ and its characteristic value is $\lambda = 0$ with 2 multiplicity roots, we write $R(s, y)$ as

$$R(s, y) = \begin{cases} c_0(s) + c_1(s)y, & y \leq s \\ d_0(s) + d_1(s)y, & y > s \end{cases}.$$

Since $\frac{\partial^2 R}{\partial y^2}(s, y) = -\delta(y - s)$, we have

$$R(s, s + 0) - R(s, s + 0) = 0, \tag{10}$$

$$\frac{\partial R}{\partial y}(s, s + 0) - \frac{\partial R}{\partial y}(s, s + 0) = -1. \tag{11}$$

Using Conditions (8)–(11), we get the following system of equations

$$\begin{aligned} c_0(s) - c_1(s) &= 0, \\ d_1(s) &= 0, \\ c_0(s) + c_1(s) s &= d_0(s) + d_1(s) s, \\ d_1(s) - c_1(s) &= -1, \end{aligned} \tag{12}$$

which implies that

$$c_0(s) = 1, c_1(s) = 1, d_0(s) = 1 + s, d_1(s) = 0.$$

Next, we study the space $W_2^3[0, 1]$. Let

$$\begin{aligned} W_2^3[0, 1] &= \{f(s) : f \text{ is absolutely continuous real value functions,} \\ &f, f', f'', f''' \in L^2[0, 1], f(0) = 0, f(1) = 0\}. \end{aligned}$$

The inner product in $W_2^3[0, 1]$ is defined as

$$(u(y), v(y))_{W_2^3[0,1]} = u(0)v(0) + u'(0)v'(0) + u(1)v(1) + u'(1)v'(1) + \int_0^1 u^{(3)}(y)v^{(3)}(y)dy,$$

and the norm $\|u\|_{W_2^3[0,1]}$ is given by

$$\|u\|_{W_2^3[0,1]} = \sqrt{(u(y), u(y))_{W_2^3[0,1]}}$$

where $u, v \in W_2^3[0, 1]$. □

Theorem 2. *The space $W_2^3[0, 1]$ is a reproducing kernel Hilbert space, i.e.; there exists $K(s, y) \in W_2^3[0, 1]$ which has its six partial derivative with respect to y such that for any $u \in W_2^3[0, 1]$ and each fixed $y, s \in [0, 1]$, we have*

$$(u(y), K(s, y))_{W_2^3[0,1]} = u(s).$$

In this case, $K(s, y)$ is given by

$$K(s, y) = \left\{ \begin{array}{ll} \sum_{i=0}^5 c_i(s)y^i, & y \leq s \\ \sum_{i=0}^5 d_i(s)y^i, & y > s \end{array} \right\}$$

where

$$\begin{aligned} c_0 &= 0, c_1 = 0, c_2 = \frac{1}{120}(5s^4 - 111s^2 - 10s^3 - s^5), \\ c_3 &= 0, c_4 = -\frac{s}{24}, c_5 = \frac{1}{120}(1 + s^5), \\ d_0 &= \frac{s^5}{120}, d_1 = -\frac{s^4}{24}, d_2 = \frac{1}{120}(5s^4 - 111s^2 - s^5), d_3 = -\frac{s^2}{12}, \\ d_4 &= -\frac{s^2}{12}, d_5 = 0, d_6 = \frac{s^2}{120}. \end{aligned}$$

Proof. Using integration by parts, one can get

$$\begin{aligned} (u(y), K(s, y))_{W_2^3[0,1]} &= u(0)K(s, 0) + u(1)K(s, 1) + u'(0)K_y(s, 0) + u'(1)K_y(s, 1) \\ &\quad + u''(1)K_{yyy}(s, 1) - u''(0)K_{yyy}(s, 0) - u'(1)\frac{\partial^4 K}{\partial y^4}(s, 1) + u'(0)\frac{\partial^4 K}{\partial y^4}(s, 0) \\ &\quad + u(1)\frac{\partial^5 K}{\partial y^5}(s, 1) - u(0)\frac{\partial^5 K}{\partial y^5}(s, 0) + \int_0^1 u(y)\frac{\partial^6 K}{\partial y^6}(s, y)dy. \end{aligned}$$

Since $u(y)$ and $K(s, y) \in W_2^3[0, 1]$,

$$u(0) = 0, u(1) = 0$$

and

$$K(s, 0) = 0, K(s, 1) = 0. \tag{13}$$

Thus,

$$\begin{aligned} (u(y), K(s, y))_{W_2^3[0,1]} &= u'(0)K_y(s, 0) + u'(1)K_y(s, 1) + u''(1)K_{yyy}(s, 1) - u''(0)K_{yyy}(s, 0) \\ &\quad - u'(1)\frac{\partial^4 K}{\partial y^4}(s, 1) + u'(0)\frac{\partial^4 K}{\partial y^4}(s, 0) + \int_0^1 u(y)\frac{\partial^6 K}{\partial y^6}(s, y)dy. \end{aligned}$$

Since $K(s, y)$ is a reproducing kernel of $W_2^3[0, 1]$,

$$(u(y), K(s, y))_{W_2^3[0,1]} = u(s)$$

which implies that

$$\frac{\partial^6 K}{\partial y^6}(s, y) = \delta(y - s) \tag{14}$$

where δ is the dirac-delta function and

$$K(s, 1) + \frac{\partial^5 K}{\partial y^5}(s, 0) = 0, \tag{15}$$

$$K_y(s, 1) - \frac{\partial^4 K}{\partial y^4}(s, 1) = 0, \tag{16}$$

$$K_{yyy}(s, 1) = 0, \tag{17}$$

$$K_{yyy}(s, 0) = 0. \tag{18}$$

Since the characteristic equation of $\frac{\partial^6 K}{\partial y^6}(s, y) = \delta(s - y)$ is $\lambda^6 = 0$ and its characteristic value is $\lambda = 0$ with 6 multiplicity roots, we write $K(s, y)$ as

$$K(s, y) = \left\{ \begin{array}{ll} \sum_{i=0}^5 c_i(s)y^i, & y \leq s \\ \sum_{i=0}^5 d_i(s)y^i, & y > s \end{array} \right\}.$$

Since $\frac{\partial^5 K}{\partial y^5}(s, y) = \delta(s - y)$, we have

$$\frac{\partial^m K}{\partial y^m}(s, s + 0) = \frac{\partial^m K}{\partial y^m}(s, s - 0), \quad m = 0, 1, 2, 3, 4. \tag{19}$$

On the other hand, integrating $\frac{\partial^5 K}{\partial y^5}(s, y) = \delta(s - y)$ from $s - \epsilon$ to $s + \epsilon$ with respect to y and letting $\epsilon \rightarrow 0$ to get

$$\frac{\partial^5 K}{\partial y^5}(s, s + 0) - \frac{\partial^5 K}{\partial y^5}(s, s - 0) = -1. \tag{20}$$

Using the Conditions 13 and 15–20, we get the following system of equations

$$\begin{aligned} c_0(s) &= 0, c_1(s) = 0, c_3(s) = 0, \\ 6d_3(s) + 24d_4(s) + 60d_5(s) &= 0, \sum_{i=0}^5 d_i(s) + 120d_5(s) = 0, \end{aligned}$$

$$\sum_{i=0}^3 c_i(s)s^i = \sum_{i=0}^3 d_i(s)s^i,$$

$$\sum_{i=1}^5 c_i(s)s^i = \sum_{i=1}^5 d_i(s)s^i,$$

$$\sum_{i=1}^5 ic_i(s)s^{i-1} = \sum_{i=1}^5 id_i(s)s^{i-1},$$

$$\sum_{i=2}^5 i(i-1)c_i(s)s^{i-2} = \sum_{i=1}^5 i(i-1)d_i(s)s^{i-2},$$

$$\sum_{i=3}^5 i(i-1)(i-2)c_i(s)s^{i-3} = \sum_{i=3}^5 i(i-1)(i-2)d_i(s)s^{i-3},$$

$$\sum_{i=4}^5 i(i-1)(i-2)(i-3)c_i(s)s^{i-4} = \sum_{i=4}^5 i(i-1)(i-2)(i-3)d_i(s)s^{i-4},$$

$$5!d_5(s) - 5!c_5(s) = -1.$$

We solved the last system using Mathematica to get

$$\begin{aligned} c_0 &= 0, c_1 = 0, c_2 = \frac{1}{120}(5s^4 - 111s^2 - 10s^3 - s^5), \\ c_3 &= 0, c_4 = -\frac{s}{24}, c_5 = \frac{1}{120}(1 + s^5), \\ d_0 &= \frac{s^5}{120}, d_1 = -\frac{s^4}{24}, d_2 = \frac{1}{120}(5s^4 - 111s^2 - s^5), d_3 = -\frac{s^2}{12}, \\ d_4 &= -\frac{s^2}{12}, d_5 = 0, d_5 = \frac{s^2}{120}. \end{aligned}$$

Now, we present how to solve Equations (5) and (6). Let

$$\sigma_i(s) = R(s_i, s)$$

for $i = 1, 2, \dots$ where $\{s_i\}_{i=1}^\infty$ is dense on $[0, 1]$. It is clear that $L : W_2^3[0, 1] \rightarrow W_2^1[0, 1]$ is bounded linear operator. Let

$$\psi_i(s) = L^* \sigma_i(s)$$

where $L(\sigma_i(s)) = D^\alpha \sigma_i(s) + a(s)\sigma_i(s)$ and L^* is the adjoint operator of L . Using Gram-Schmidt orthonormalization to generate orthonormal set of functions $\{\bar{\psi}_i(s)\}_{i=1}^\infty$ where

$$\bar{\psi}_i(s) = \sum_{j=1}^i \alpha_{ij} \psi_j(s) \tag{21}$$

and α_{ij} are coefficients of Gram-Schmidt orthonormalization. In the next theorem, we show the existence of the solution of Equations (5) and (6). \square

Theorem 3. *If $\{s_i\}_{i=1}^\infty$ is dense on $[0, 1]$, then*

$$u(s) = \sum_{i=1}^\infty \sum_{j=1}^i \alpha_{ij} h(s_j) \bar{\psi}_i(s). \tag{22}$$

Proof. First, we want to prove that $\{\psi_i(s)\}_{i=1}^\infty$ is complete system of $W_2^3[0, 1]$ and $\psi_i(s) = L(K(s, s_i))$. It is clear that $\psi_i(s) \in W_2^3[0, 1]$ for $i = 1, 2, \dots$. Simple calculations imply that

$$\begin{aligned} \psi_i(s) &= L^* \sigma_i(s) = (L^* \sigma_i(s), K(s, y))_{W_2^3[0,1]} \\ &= (\sigma_i(s), L(K(s, y)))_{W_2^3[0,1]} = L(K(s, s_i)). \end{aligned}$$

For each fixed $u(s) \in W_2^3[0, 1]$, let

$$(u(s), \psi_i(s))_{W_2^3[0,1]} = 0, \quad i = 1, 2, \dots$$

Then

$$\begin{aligned} (u(s), \psi_i(s))_{W_2^3[0,1]} &= (u(s), L^* \sigma_i(s))_{W_2^3[0,1]} \\ &= (Lf(s), \sigma_i(s))_{W_2^3[0,1]} \\ &= Lu(s_i) = 0. \end{aligned}$$

Since $\{s_i\}_{i=1}^\infty$ is dense on $[0, 1]$, $Lu(s) = 0$. Since L^{-1} exists, $u(s) = 0$. Thus, $\{\psi_i(s)\}_{i=1}^\infty$ is the complete system of $W_2^3[0, 1]$.

Second, we prove Equation (22). Simple calculations imply that

$$\begin{aligned}
 u(s) &= \sum_{i=1}^{\infty} (u(s), \bar{\psi}_i(s))_{W_2^3[0,1]} \bar{\psi}_i(s) \\
 &= \sum_{i=1}^{\infty} \sum_{j=1}^i \alpha_{ij} (u(s), L^*(K(s, s_j)))_{W_2^3[0,1]} \bar{\psi}_i(s) \\
 &= \sum_{i=1}^{\infty} \sum_{j=1}^i \alpha_{ij} (Lf(s), K(s, s_j))_{W_2^3[0,1]} \bar{\psi}_i(s) \\
 &= \sum_{i=1}^{\infty} \sum_{j=1}^i \alpha_{ij} (c, K(s, s_j))_{W_2^3[0,1]} \bar{\psi}_i(s) \\
 &= \sum_{i=1}^{\infty} \sum_{j=1}^i \alpha_{ij} h(s_j) \bar{\psi}_i(s).
 \end{aligned}$$

Let the approximate solution of Equations (5) and (6) be given by

$$u_N(s) = \sum_{i=1}^N \sum_{j=1}^i \alpha_{ij} h(s_j) \bar{\psi}_i(s). \tag{23}$$

In the next theorem, we show the uniform convergence of the $\left\{ \frac{d^m f_N(s)}{ds^m} \right\}_{N=1}^{\infty}$ to $\frac{df(s)}{ds}$ for $m = 0, 1, 2$. □

Theorem 4. If $u(s)$ and $u_N(s)$ are given as in (22) and (23), then $\left\{ \frac{d^m u_N(s)}{ds^m} \right\}_{N=1}^{\infty}$ converges uniformly to $\frac{d^m u(s)}{ds^m}$ for $m = 0, 1, 2$.

Proof. First, we prove the theorem for $m = 0$. For any $s \in [0, 1]$,

$$\begin{aligned}
 \|u(s) - u_N(s)\|_{W_2^3[0,1]}^2 &= (u(s) - u_N(s), u(s) - u_N(s))_{W_2^3[0,1]} \\
 &= \sum_{i=N+1}^{\infty} ((u(s), \bar{\psi}_i(s))_{W_2^3[0,1]} \bar{\psi}_i(s), (u(s), \bar{\psi}_i(s))_{W_2^3[0,1]} \bar{\psi}_i(s))_{W_2^3[0,1]} \\
 &= \sum_{i=N+1}^{\infty} (u(s), \bar{\psi}_i(s))_{W_2^3[0,1]}^2.
 \end{aligned}$$

Thus,

$$\text{Sup}_{s \in [0,1]} \|u(s) - u_N(s)\|_{W_2^3[0,1]}^2 = \text{Sup}_{s \in [0,1]} \sum_{i=N+1}^{\infty} (u(s), \bar{\psi}_i(s))_{W_2^3[0,1]}^2.$$

From Theorem (4), one can see that $\sum_{i=1}^{\infty} (u(s), \bar{\psi}_i(s))_{W_2^3[0,1]} \bar{\psi}_i(s)$ converges uniformly to $u(s)$. Thus,

$$\text{Lim}_{N \rightarrow \infty} \text{Sup}_{s \in [0,1]} \|u(s) - u_N(s)\|_{W_2^3[0,1]} = 0$$

which implies that $\{u_N(s)\}_{N=1}^{\infty}$ converges uniformly to $u(s)$.

Second, we prove the uniform convergence for $m = 1, 2$. Since $\frac{d^m K(s,y)}{ds^m}$ is bounded function on $[0, 1] \times [0, 1]$,

$$\left\| \frac{d^m K(s,y)}{ds^m} \right\|_{W_2^3[0,1]} \leq \chi_m, \quad m = 1.$$

Thus, for any $s \in [0, 1]$,

$$\begin{aligned} \left| u^{(m)}(s) - u_N^{(m)}(s) \right| &= \left| \left(u(s) - u_N(s), \frac{d^m K(s, y)}{ds^m} \right)_{W_2^3[0,1]} \right| \\ &\leq \| u(s) - u_N(s) \|_{W_2^3[0,1]} \left\| \frac{d^m K(s, y)}{ds^m} \right\|_{W_2^3[0,1]} \\ &\leq \chi_m \| u(s) - u_N(s) \|_{W_2^3[0,1]} \\ &\leq \chi_m \operatorname{Sup}_{s \in [0,1]} \| u(s) - u_N(s) \|_{W_2^3[0,1]}. \end{aligned}$$

Hence,

$$\operatorname{Sup}_{s \in [0,1]} \left\| u^{(m)}(s) - u_N^{(m)}(s) \right\|_{W_2^3[0,1]} \leq \chi_m \operatorname{Sup}_{s \in [0,1]} \| u(s) - u_N(s) \|_{W_2^3[0,1]}$$

which implies that

$$\lim_{N \rightarrow \infty} \operatorname{Sup}_{s \in [0,1]} \left\| u^{(m)}(s) - u_N^{(m)}(s) \right\|_{W_2^3[0,1]} = 0.$$

Therefore, $\left\{ \frac{d^m u_N(s)}{ds^m} \right\}_{N=1}^\infty$ converges uniformly to $\frac{d^m u(s)}{ds^m}$ for $m = 1, 2$. Now, we discuss how to solve Equations (3) and (4). Let $\mathcal{L}(y(x)) = D^\alpha y(x)$ and $N(y(x)) = g(x, y)y'$ are the linear and nonlinear parts of Problem (3), respectively. We construct the homotopy as follows:

$$H(y, \lambda) = \mathcal{L}(y(x)) + \lambda N(y(x)) = 0 \tag{24}$$

where $\lambda \in [0, 1]$ is an embedding parameter. If $\lambda = 0$, we get a linear equation

$$D^\alpha y(x) = 0$$

which implies that $y(x) = \theta$. If $\lambda = 1$, we turn out to be Problem (3). Following the Homotopy Perturbation method [34], we expand the solution in term of the Homotopy parameter λ as

$$y = y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3 + \dots \tag{25}$$

Substitute Equation (25) into Equation (24) and equating the coefficients of the identical powers of λ to get the following system

$$\begin{aligned} \lambda^0 &: D^\alpha y_0(x) = 0, y_0(0) = \theta, \\ \lambda^1 &: D^\alpha y_1(x) = -N\left(\sum_{i=0}^\infty \lambda^i y_i(x)\right) |_{\lambda=0}, y_1(0) = 0, \\ \lambda^2 &: D^\alpha y_2(x) = -\frac{dN\left(\sum_{i=0}^\infty \lambda^i y_i(x)\right)}{d\lambda} |_{\lambda=0}, y_2(0) = 0, \\ \lambda^3 &: D^\alpha y_3(x) = -\frac{d^2N\left(\sum_{i=0}^\infty \lambda^i y_i(x)\right)}{d\lambda^2} |_{\lambda=0}, y_3(0) = 0, \\ &\vdots \\ \lambda^k &: D^\alpha y_k(x) = -\frac{d^{k-1}N\left(\sum_{i=0}^\infty \lambda^i y_i(x)\right)}{d\lambda^{k-1}} |_{\lambda=0}, y_k(0) = 0. \end{aligned}$$

To solve the above equations, we use the RKM which described above and we obtain

$$y_k(x) = \sum_{i=1}^{\infty} \sum_{j=1}^i \alpha_{ij} h_k(x_j) \bar{\psi}_i(s), k = 0, 1, \dots \tag{26}$$

where

$$\begin{aligned} h_0(x) &= 0 \\ h_1(x) &= -N\left(\sum_{i=0}^{\infty} \lambda^i y_i(x)\right) |_{\lambda=0} \\ &\vdots \\ h_k(x) &= -\frac{d^{k-1} N\left(\sum_{i=0}^{\infty} \lambda^i y_i(x)\right)}{d\lambda^{k-1}} |_{\lambda=0}, k > 1. \end{aligned}$$

From Equation (26), it is easy to see the solution to Equations (3) and (4) is given by

$$y(x) = \sum_{k=0}^{\infty} y_k(x) = \sum_{k=0}^{\infty} \left(\sum_{i=1}^{\infty} \sum_{j=1}^i \alpha_{ij} h_k(x_j) \bar{\psi}_i(x) \right). \tag{27}$$

We approximate the solution of Equations (3) and (4) by

$$y_{n,m}(x) = \sum_{k=0}^m \left(\sum_{i=1}^n \sum_{j=1}^i \alpha_{ij} h_k(x_j) \bar{\psi}_i(x) \right). \tag{28}$$

□

3. Analytical Results

In this section, we present the maximum principle, the stability theorem, and the uniqueness theorem. Firstly, Equations (1) and (2) is transformed into an equivalent problem as follows:

$$Py : -\epsilon D^\alpha y + u(x, y)y' + \int_0^x K(x, t)v(t, y)dt = f(x), x \in (0, 1), 0 < \alpha \leq 1, \tag{29}$$

$$y(0) = y_0, y(1) = y_1. \tag{30}$$

The following conditions are needed in order to guarantee that Equations (1) and (2) do not have the turning-point problem

$$-k_2 \geq u(x, y) \geq -k_1, \tag{31}$$

$$0 \geq v(x, y) \geq -k_3, \tag{32}$$

$$K(x, t) \geq k_4 \geq 0, \tag{33}$$

for all $x \in [0, 1]$, where k_1, k_2, k_3 , and k_4 are positive constants and $y \in C^2(0, 1) \cup C[0, 1]$.

Lemma 1. [35] Let $y \in C^2[0, 1]$ attains its minimum at $x_0 \in (0, 1)$. Then, $y'(x_0) \leq 0$ and $D^\alpha y(x_0) \geq 0$ for $1 < \alpha \leq 2$.

Theorem 5. (Maximum Principle). Consider the initial value problem (1) and (2) with conditions (31) and (33). Assume that $Py \geq 0$ and $y(0) \geq 0$. then, $y(1) \geq 0$ in $[0, 1]$.

Proof. Assume that the conclusion is false, then $\phi(x) < 0$ for some $x \in [0, 1]$. Then, $y(x)$ has a local minimum at x_0 for some $x_0 \in (0, 1]$. Simple calculations and using Lemma (1) imply that

$$Py(x_0) = -\epsilon D^\alpha y(x_0) + u(x_0, y)y'(x_0) + \int_0^{x_0} K(x_0, t)v(t, y)dt \leq 0.$$

This is a contradiction. Therefore, $y(x) \geq 0$ in $[0, 1]$. \square

In the next theorem, the stability result is presented.

Theorem 6. (stability result). Consider Equations (1) and (2) with conditions $u = u(x)$ and $v = v(x)$. If $y(x)$ is a smooth function, then

$$\|y\| = \max \{|y(x)| : x \in [0, 1]\} \leq 2\zeta \max \left\{ |y_0|, |y_1|, \max_{x \in [0,1]} |Py| \right\}$$

where $\zeta = 1 + \frac{1}{k_2}$.

Proof. Following [36], let

$$K_0 = \max \left\{ |y_0|, |y_1|, \max_{x \in [0,1]} |Py| \right\} = \max \left\{ |y_0|, |y_1|, \max_{x \in [0,1]} |f(x)| \right\}$$

and

$$s^\pm(x) = 2\zeta K_0 \left(1 - \frac{x}{2}\right) \pm y(x), x \in [0, 1].$$

Then,

$$\begin{aligned} Ps^\pm(x) &= -\epsilon D^\alpha \left(2\zeta K_0 \left(1 - \frac{x}{2}\right) \pm y(x)\right) + u(x) \left(2\zeta K_0 \left(1 - \frac{x}{2}\right) \pm y(x)\right)' + \int_0^x K(x, t)v(t)dt \\ &= -\zeta K_0 u(x) \pm Py(x) > K_0 \pm Py(x) \geq 0 \end{aligned}$$

for all $x \in [0, 1]$. Also,

$$s^\pm(0) = 2\zeta K_0 \pm y_0 > K_0 \pm y_0 \geq 0, x \in [0, 1]$$

and

$$s^\pm(1) = \zeta K_0 \pm y_1 > K_0 \pm y_1 \geq 0, x \in [0, 1].$$

From Theorem (5), we can see that $s^\pm(x) \geq 0$ for all $x \in [0, 1]$. Therefore,

$$\|y\| = \max \{|y(x)| : x \in [0, 1]\} \leq 2\zeta \max \left\{ |y_0|, |y_1|, \max_{x \in [0,1]} |Py| \right\}.$$

\square

Theorem 7. (Uniqueness Theorem). Consider Equations (1) and (2) under the conditions (31)–(33) with conditions $u = u(x)$ and $v = v(x)$. If y_1 and y_2 are two solutions to Equations (1) and (2), then $y_1(x) = y_2(x)$ for all $x \in [0, 1]$.

Proof. Let $w(x) = y_1(x) - y_2(x)$. Then,

$$Pw = 0, w(0) = 0, w(1) = 0$$

$$P(-w) = 0, -w(0) = 0, -w(1) = 0.$$

Using Theorem (5), it follows that $w(x) \geq 0$ and $w(x) \leq 0$ for all $x \in [0, 1]$ which imply that $y_1(x) = y_2(x)$ for all $x \in [0, 1]$. \square

4. Solution Method

Consider the following of class of nonlinear second order fractional nonlinear Volterra integro-differential type of singularly perturbed problems of the form

$$-\epsilon D^\alpha y + u(x, y)y' + \int_0^x K(x, t)v(t, y)dt = f(x), x \in (0, 1), 1 < \alpha \leq 2$$

subject to

$$y(0) = y_0, y(1) = y_1$$

where $\epsilon > 0$ is a small positive parameter, y_0 and y_1 are constant, and $K(x, t)$ and $f(x)$ are smooth functions. To solve Equations (1) and (2), we use the following steps.

Step 1: A reduced *subproblem* is obtained by setting $\epsilon = 0$ in Equation (1) to get

$$u(x, y_1)y_1' + \int_0^x K(x, t)v(t, y_1)dt = f(x), x \in [0, 1]. \tag{34}$$

On most of the interval, the solution of Equation (34) behaves like the solution of Equations (1) and (2). However, there is small interval around $x = 0$ in which the solution of problem (1) and (2) does not agree with the solution of problem (1) and (2). To handle this situation, the boundary layer correction *subproblem* is introduced in step 2.

Step 2: Choose $x = \epsilon^{\frac{1}{\alpha-1}}s$ to get

$$\begin{aligned} D^\alpha y(x) &= \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{1-\alpha} y''(t) dt \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^{\epsilon^{\frac{1}{\alpha-1}}s} (\epsilon^{\frac{1}{\alpha-1}}s - t)^{1-\alpha} y''(t) dt \\ &= \frac{\epsilon^{\frac{1-\alpha}{\alpha-1}}}{\Gamma(1-\alpha)} \int_0^{\epsilon^{\frac{1}{\alpha-1}}s} (s - \frac{t}{\epsilon^{\frac{1}{\alpha-1}}})^{-\alpha} y''(t) dt. \end{aligned}$$

Let $r = \frac{t}{\epsilon^{\frac{1}{\alpha-1}}}$. Then, $dt = \epsilon^{\frac{1}{\alpha-1}} dr$ and

$$\begin{aligned} \frac{dy}{dt} &= \frac{dy}{dr} \frac{dr}{dt} = \frac{1}{\epsilon^{\frac{1}{\alpha-1}}} \frac{dy}{dr} \\ \frac{d^2y}{dt^2} &= \frac{d\left(\frac{dy}{dt}\right)}{dr} \frac{dr}{dt} = \left(\frac{1}{\epsilon^{\frac{1}{\alpha-1}}}\right)^2 \frac{dy}{dr}. \end{aligned}$$

Thus,

$$\begin{aligned}
 D^\alpha y(x) &= \frac{\epsilon^{\frac{1-\alpha}{\alpha-1}}}{\Gamma(1-\alpha)} \int_0^s (s-r)^{-\alpha} \frac{1}{\left(\epsilon^{\frac{1}{\alpha-1}}\right)^2} \frac{dy}{dr} \epsilon^{\frac{1}{\alpha-1}} dr \\
 &= \frac{\epsilon^{\frac{-\alpha}{\alpha-1}}}{\Gamma(1-\alpha)} \int_0^s (s-r)^{-\alpha} \frac{dy}{dr} dr \\
 &= \epsilon^{\frac{-\alpha}{\alpha-1}} D^\alpha y(s).
 \end{aligned}
 \tag{35}$$

Hence, Equation (1) becomes

$$-\epsilon \epsilon^{\frac{-\alpha}{\alpha-1}} D^\alpha y(s) + \frac{1}{\epsilon^{\frac{1}{\alpha-1}}} u(\epsilon^{\frac{1}{\alpha-1}} s, y) \frac{dy}{ds} + \int_0^{\epsilon^{\frac{1}{\alpha-1}} s} K(\epsilon^{\frac{1}{\alpha-1}} s, t) v(t, y) dt = f(\epsilon^{\frac{1}{\alpha-1}} s)
 \tag{36}$$

or

$$-D^\alpha y + u(\epsilon^{\frac{1}{\alpha-1}} s, y) \frac{dy}{ds} + \epsilon^{\frac{1}{\alpha-1}} \int_0^{\epsilon^{\frac{1}{\alpha-1}} s} K(\epsilon^{\frac{1}{\alpha-1}} s, t) v(t, y) dt = \epsilon^{\frac{1}{\alpha-1}} f(\epsilon^{\frac{1}{\alpha-1}} s).
 \tag{37}$$

Setting $\epsilon = 0$ in Equation (36) implies that

$$-D^\alpha y(s) + u(0, y) \frac{dy}{ds} = 0.
 \tag{38}$$

Since the solution of the reduced problem in step 1 does not satisfy the initial condition at $x = 0$, then the solution of the above equation should satisfy it. This means, its solution has the form $y_1(0) + y_2(x)$. Substitute

$$y(x) = y_1(0) + y_2(x)$$

in Equation (38) to get the boundary layer correction equation

$$-D^\alpha y_2(s) + u(0, y_1(0) + y_2(s)) \frac{dy_2}{ds} = 0.
 \tag{39}$$

The solution of Equation (1) will be expressed in the form as

$$y(x) = y_1(x) + y_2\left(\frac{x}{\epsilon^{\frac{1}{\alpha-1}}}\right)
 \tag{40}$$

and the initial condition must be satisfied by expression (40). When $x = 0$, the condition will be

$$y_0 = y(0) = y_1(0) + y_2(0)$$

or

$$y_2(0) = y_0 - y_1(0).
 \tag{41}$$

The solution of Equations (1) and (2) can be produced using the RKM as described in the previous section. More details can be found in [36–39].

5. Numerical Results

In this section, we present two of our examples to show the efficiency of the proposed method.

Example 1: Consider the following problem

$$-\epsilon D^{\frac{3}{2}}y(x) - 2y'(x) - \int_0^x e^{y(t)} dt = x^2 - 2x - \frac{2}{x-2}, \quad 0 \leq x \leq 1, \quad 0 < \epsilon \ll 1, \tag{42}$$

subject to

$$y(0) = 0, y(1) = 0. \tag{43}$$

When $\epsilon = 0$, we get

$$-2y'(x) - \int_0^x e^{y(t)} dt = x^2 - 2x - \frac{2}{x-2}, y(1) = 0. \tag{44}$$

We discretized the interval $[0, 1]$ by $x_i = ih, h = \frac{1}{n}, n \in N$. Let $y_k \approx y(x_k)$ for $k = 0 : n$. Using the backward finite difference method to approximate $y'(x_k)$ and the trapezoidal quadrature to approximate the integral $\int_0^{x_k} e^{y(t)} dt$, we get

$$-2\frac{y_k - y_{k-1}}{h} - h \sum_{j=0}^{k-1} (e^{y_j} + e^{y_{j+1}}) = x_k^2 - 2x_k - \frac{2}{x_k - 2}, x_n = 0.$$

Thus, we get the following system

$$AY + Be^Y = F$$

where

$$A = -\frac{2}{h} \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & 0 & -1 & 1 & 0 \\ \vdots & \cdots & \cdots & 0 & -1 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 \end{pmatrix}, B = -h \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 1 & 2 & 1 & \ddots & 0 & \cdots & 0 \\ 1 & 2 & 1 & \ddots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \cdots & 2 & 1 & 0 \\ \vdots & \vdots & \vdots & \cdots & 2 & 2 & 1 \\ 2 & 2 & 2 & \cdots & 2 & 2 & 2 \end{pmatrix},$$

$$F = \begin{pmatrix} f(x_1) \\ f(x_1) \\ \vdots \\ f(x_{n-1}) \\ f(x_n) \end{pmatrix}, Y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-2} \\ y_n \end{pmatrix}.$$

Using Mathematica, one can see that the solution of the above system for $n = 12$ is given in Figure 1. Using the change of variable $x = \epsilon^2 s$, we get

$$-D^{\frac{3}{2}}y_2(s) - 2\frac{dy_2}{ds} = 0$$

subject to

$$y_2(0) = y_0 - y_1(0) = -0.694147, y_2'(0) = \theta.$$

Using the RKM, we get

$$y_2(s) \approx -0.694147 + \frac{2\theta s}{3} - \frac{2s^2}{\sqrt{\pi}} + \frac{8\theta s^{5/2}}{5} - \frac{32\theta s^3}{9\sqrt{\pi}} + \frac{16\theta s^{7/2}}{7} - \frac{64\theta s^4}{15\sqrt{\pi}}.$$

Using the Pade' approximation of order [2, 2], we have $\theta = -0.693147$. In Figures 2–4, we plot the approximate solution for $\epsilon = 0.0001, 0.00001, \text{ and } 0.000001$, respectively. Let

$$Error(x) = \left| -\epsilon D^{\frac{3}{2}}y(x) - 2y'(x) - \int_0^x e^{y(t)} dt - \left(x^2 - 2x - \frac{2}{x-2} \right) \right|.$$

In Table 1, we present the error for $x = 0, 0.1, \dots, 1$ for $\epsilon = 0.0001, 0.00001, \text{ and } 0.000001$. In Table 2, we present the computational time when $\epsilon = 0.0001, 0.00001, \text{ and } 0.000001$.

Table 1. The error for $\epsilon = 0.0001, 0.00001, \text{ and } 0.000001$.

| x | $\epsilon = 0.0001$ | $\epsilon = 0.00001$ | $\epsilon = 0.000001$ |
|-----|-----------------------|-----------------------|-----------------------|
| 0 | 2.1×10^{-2} | 3.6×10^{-3} | 2.8×10^{-6} |
| 0.1 | 1.9×10^{-3} | 5.4×10^{-4} | 2.3×10^{-7} |
| 0.2 | 2.3×10^{-5} | 2.4×10^{-7} | 2.1×10^{-8} |
| 0.3 | 3.1×10^{-6} | 1.4×10^{-7} | 1.9×10^{-8} |
| 0.4 | 1.8×10^{-7} | 1.8×10^{-9} | 1.5×10^{-11} |
| 0.5 | 1.5×10^{-9} | 1.3×10^{-11} | 1.1×10^{-12} |
| 0.6 | 3.4×10^{-9} | 2.2×10^{-12} | 2.1×10^{-13} |
| 0.7 | 3.3×10^{-9} | 2.3×10^{-13} | 2.2×10^{-14} |
| 0.8 | 3.1×10^{-11} | 1.4×10^{-13} | 1.7×10^{-14} |
| 0.9 | 1.0×10^{-13} | 1.5×10^{-14} | 1.4×10^{-15} |
| 1 | 2.1×10^{-15} | 2.2×10^{-16} | 1.1×10^{-16} |

Table 2. Computational time.

| ϵ | Computational time |
|------------|--------------------|
| 0.0001 | 3.4S |
| 0.00001 | 4.6S |
| 0.000001 | 6.1S |

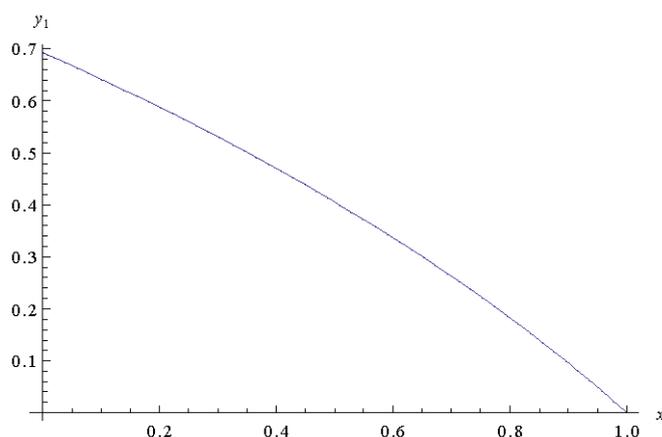


Figure 1. The approximate solution y_1 .

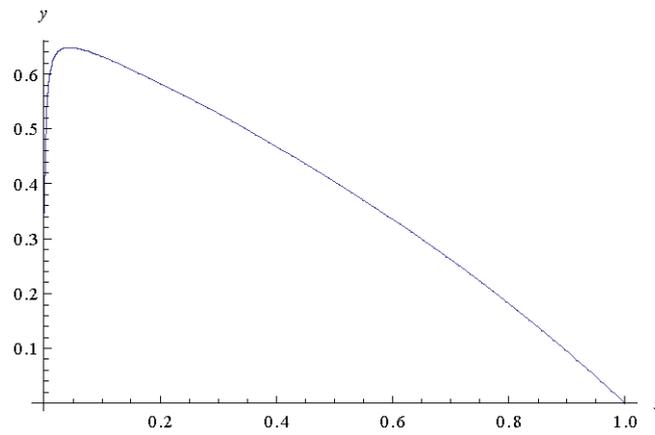


Figure 2. The approximate solution y for $\epsilon = 0.0001$.

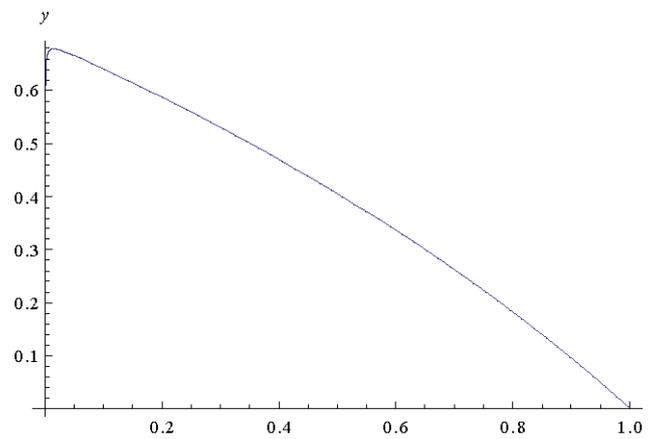


Figure 3. The approximate solution y for $\epsilon = 0.00001$.

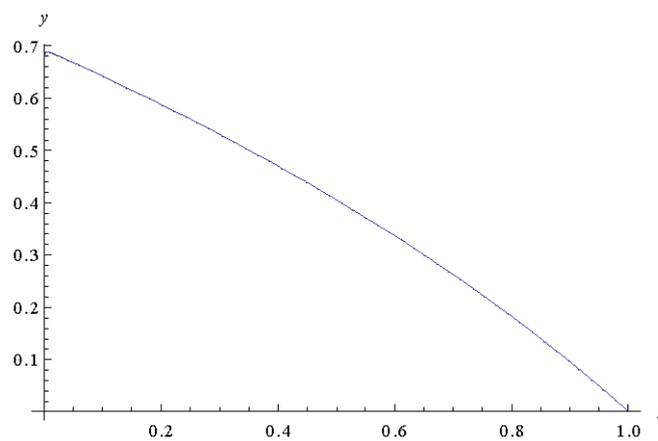


Figure 4. The approximate solution y for $\epsilon = 0.000001$.

Example 2: Consider the following problem

$$-\epsilon D^{\frac{3}{2}}y(x) - yy' - \int_0^x (x-t)^2 y^2(t) dt = f(x), \quad 0 \leq x \leq 1, \quad 0 < \epsilon \ll 1 \tag{45}$$

subject to

$$y(0) = -1, y(1) = 6 \tag{46}$$

where

$$f(x) = -5 - x - \frac{25}{3}x^3 - \frac{5}{6}x^4 - \frac{x^5}{30}.$$

When $\epsilon = 0$, we get

$$-yy' - \int_0^x (x-t)y^2(t)dt = f(x), y(1) = 6. \tag{47}$$

We discretized the interval $[0, 1]$ by $x_i = ih, h = \frac{1}{n}, n \in N$. Let $y_k \approx y(x_k)$ for $k = 0 : n$. Using the backward finite difference method to approximate $y'(x_k)$ and the trapezoidal quadrature to approximate the integral $\int_0^{x_k} (x-t)y^2(t)dt$, we get

$$-y_k \frac{y_k - y_{k-1}}{h} - \frac{h}{2} \sum_{j=0}^{k-1} \left((x_k - x_{j+1})y_{j+1}^2 + (x_k - x_j)y_j^2 \right) = f(x_k), y_n = 6.$$

Using Mathematica, one can see that the solution of the above system for $n = 12$ is given in Figure 5. Using the change of variable $x = \epsilon^2 s$, we get

$$-D^{\frac{3}{2}}y_2(s) - (y_2(s) + 5) \frac{dy_2}{ds} = 0$$

subject to

$$y_2(0) = y_0 - y_1(0) = -6, y_2'(0) = \theta.$$

Using the RKM, we get

$$y_2(s) \approx -6 + \theta s - \frac{4\theta}{3\sqrt{\pi}}s^{\frac{3}{2}} + \frac{\theta}{2}s^2 - \frac{8\theta^2}{15\sqrt{\pi}}s^{\frac{5}{2}} - \frac{7\theta^2}{12}s^3 + \frac{7\theta^3}{48}s^4.$$

Using the Pade' approximation of order $[2, 2]$, we have $\theta = 0.0927388622769557$. In Table 3, we present the error for $x = 0, 0.1, \dots, 1$ for $\epsilon = 0.001, 0.0001, \text{ and } 0.00001$. In Table 4, we present the computational time when $\epsilon = 0.001, 0.0001, \text{ and } 0.00001$.

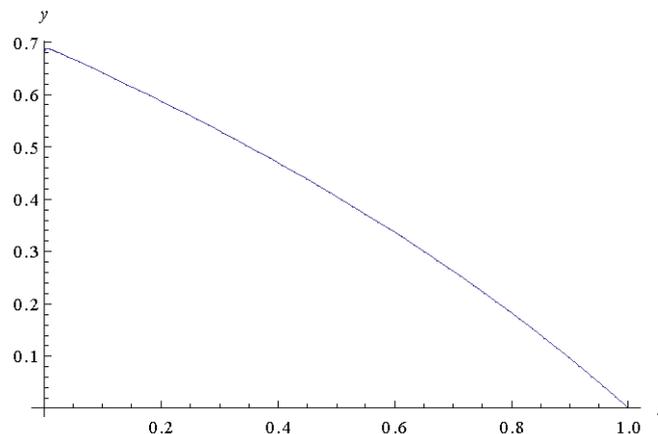


Figure 5. The approximate solution y_1 .

Table 3. The error for $\epsilon = 0.0001, 0.00001, \text{ and } 0.000001$.

| x | $\epsilon = 0.001$ | $\epsilon = 0.0001$ | $\epsilon = 0.00001$ |
|-----|-----------------------|-----------------------|-----------------------|
| 0 | 6.1×10^{-2} | 6.6×10^{-3} | 7.2×10^{-6} |
| 0.1 | 5.5×10^{-3} | 6.1×10^{-4} | 7.1×10^{-7} |
| 0.2 | 4.8×10^{-5} | 4.3×10^{-7} | 7.0×10^{-8} |
| 0.3 | 4.1×10^{-6} | 4.1×10^{-7} | 5.8×10^{-8} |
| 0.4 | 3.7×10^{-7} | 3.9×10^{-9} | 5.6×10^{-11} |
| 0.5 | 3.4×10^{-9} | 3.8×10^{-11} | 5.2×10^{-12} |
| 0.6 | 3.1×10^{-9} | 3.4×10^{-12} | 5.1×10^{-13} |
| 0.7 | 2.5×10^{-9} | 3.1×10^{-13} | 3.4×10^{-14} |
| 0.8 | 2.3×10^{-11} | 2.4×10^{-13} | 3.3×10^{-14} |
| 0.9 | 2.1×10^{-13} | 2.2×10^{-14} | 2.8×10^{-15} |
| 1 | 1.9×10^{-15} | 2.1×10^{-16} | 1.7×10^{-16} |

Table 4. Computational time.

| ϵ | Computational time |
|------------|--------------------|
| 0.001 | 3.5S |
| 0.0001 | 4.7S |
| 0.00001 | 6.2S |

Thus,

$$y(x) = y_1(x) + y_2\left(\frac{x}{e^2}\right)$$

In Figures 6–8, we plot the approximate solution for $\epsilon = 0.001, 0.0001, \text{ and } 0.00001$, respectively.

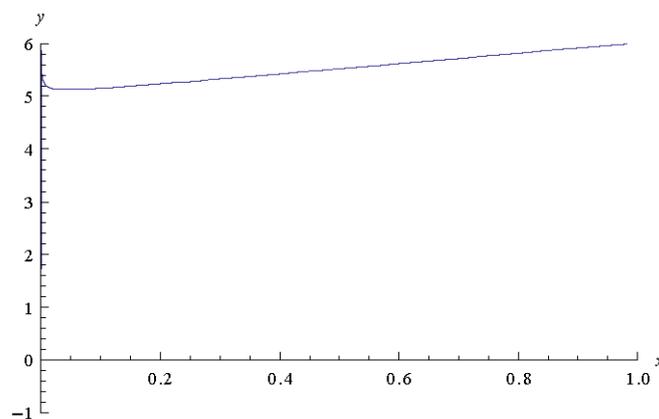


Figure 6. Approximate solution for $\epsilon = 0.001$.

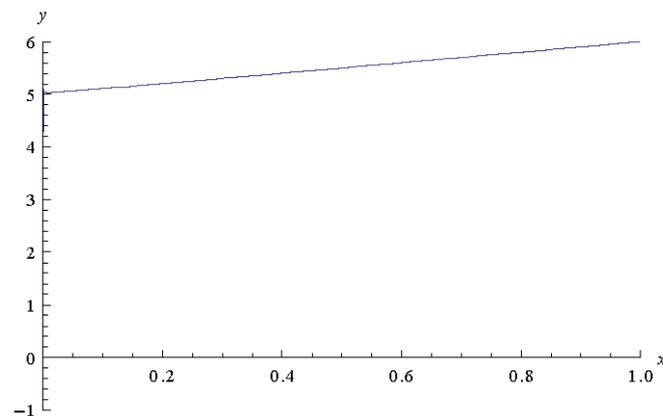


Figure 7. Approximate solution for $\epsilon = 0.0001$.

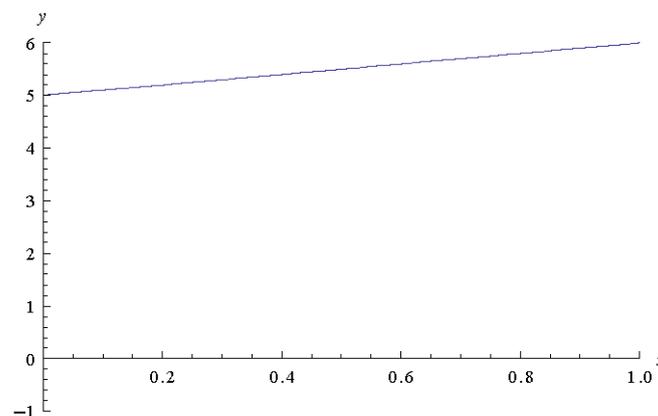


Figure 8. Approximate solution for $\epsilon = 0.00001$.

6. Conclusions

In this paper, we study a class of fractional nonlinear second order Volterra integro-differential type of singularly perturbed problems with fractional order. We divide the problem into two *subproblems*. The first *subproblem* is the reduced problem when $\epsilon = 0$. The second *subproblem* is the second order fractional Volterra integro-differential problem. We use the finite difference method to solve the first *subproblem* and the reproducing kernel method to solve the second *subproblem*. The results show that the proposed analytical method can achieve excellent results in predicting the solutions of such problems. Theoretical results are presented. Numerical results are presented to show the efficiency of the proposed method. Figures 1–8 show the efficiency of the proposed method.

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