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Fixed Point Theorems for Almost $\mathcal{Z}$-Contractions with an Application

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Abstract: In this paper, we investigate the existence and uniqueness of a fixed point of almost contractions via simulation functions in metric spaces. Moreover, some examples and an application to integral equations are given to support availability of the obtained results.

Keywords: fixed point; simulation function; almost contraction; complete metric space

1. Introduction and Preliminaries

In 1922, Banach [1] initiated studies of metrical fixed points by using contractive mappings in a complete metric space. Since then, fixed point theory has been a focus of attention because of its application potential in mathematical analysis and other disciplines. In particular, Berinde [2,3] extended the class of contractive mappings, introducing the notion of almost contractions as follows.

Definition 1. *Let (X, d) be a metric space. A self mapping T on X is called an almost contraction if there are constants $\lambda \in (0, 1)$ and $\theta \geq 0$ such that*

$$d(Tx, Ty) \leq \lambda d(x, y) + \theta d(y, Tx), \quad \text{for all } x, y \in X.$$

Berinde [2] then proved that every almost contraction mapping defined on a complete metric space has at least one fixed point. Subsequently, Babu et al. [4] demonstrated that almost-contraction-type mappings have a unique fixed point under conditions that present the notion of B-almost contraction. See [5,6] for fixed point theory.

Definition 2. *Let (X, d) be a metric space. A self mapping T on X is called an B-almost contraction if there are constants $\lambda \in (0, 1)$ and $\theta \geq 0$ such that*

$$d(Tx, Ty) \leq \lambda d(x, y) + \theta N(x, y)$$

for all $x, y \in X$, where

$$N(x, y) = min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Very recently, Khojasteh et al. [7] presented the notion of $\mathcal{Z}$-contractions involving a new class of mappings, namely simulation functions to prove the following theorem.

Theorem 1. *Let (X, d) be a complete metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect to a function ζ satisfying certain conditions, that is,*

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0$$

for all $x, y \in X$. Then, T has a unique fixed point, and, for every initial point $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to this fixed point.

In this study, by combining the ideas in [4] and [7], we define almost $\mathcal{Z}$-contractions and prove the existence of fixed points for these operators. Moreover, some examples and an application to integral equations are given to support the availability of the obtained results.

Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a function. Consider the following conditions:

(ζ_1) $\zeta(0, 0) = 0$.

(ζ_2) $\zeta(t, s) < s - t$ for all $t, s > 0$.

(ζ_3) If $(t_n), (s_n)$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0. \tag{1}$$

(ζ_4) If $(t_n), (s_n)$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then Equation (1) is satisfied.

If the function ζ satisfies the conditions (ζ_1)–(ζ_3), we say that ζ is a simulation function according to the sense of Khojasteh et al. [7]. If it satisfies (ζ_2) and (ζ_3), it is a simulation function according to the sense of Argoubi et al. [8] and if it satisfies (ζ_1), (ζ_2), and (ζ_4), then it is a simulation function according to the sense of Roldan Lopez de Hierro et al. [9].

For the sake of openness, we consider the following definition.

Definition 3. *A simulation function is a function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the conditions (ζ_2) and (ζ_4).*

Let $\mathcal{Z}$ be the family of all simulation functions $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$.

Example 1. *Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a function defined by*

$$\zeta(t, s) = \begin{cases} 1 & \text{if } s = t \text{ or } (s, t) = (0, 0) \\ 2(s - t) & \text{if } s < t \\ \lambda s - t & \text{otherwise} \end{cases}$$

where $\lambda \in (0, 1)$. It is easy to see that $\zeta \in \mathcal{Z}$, but ζ is not a simulation function in the sense of Khojasteh et al. [7], Argoubi et al. [8], or Roldan Lopez de Hierro et al. [9].

2. Main Results

Firstly, we present the following definition which will be used in our main results.

Definition 4. *Let (X, d) be a metric space and $\zeta \in \mathcal{Z}$. We say that $T : X \to X$ is an almost $\mathcal{Z}$-contraction if there is a constant $\theta \geq 0$ such that*

$$\zeta(d(Tx, Ty), d(x, y) + \theta N(x, y)) \geq 0 \tag{2}$$

for all $x, y \in X$, where $N(x, y)$ is defined as in Definition 2.

Remark 1. *If T is an almost $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, then*

$$d(Tx, Ty) < d(x, y) + \theta N(x, y) \tag{3}$$

for all $x, y \in X$.

The following lemma shows us that a fixed point of an almost $\mathcal{Z}$-contraction is unique.

Lemma 1. *If an almost $\mathcal{Z}$-contraction has a fixed point in a metric space, then it is unique.*

Proof. Let (X, d) be a metric space and $T : X \to X$ be an almost $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. Suppose that there are two distinct fixed points $u, v \in X$ of the mapping T. Then, $d(u, v) > 0$. Therefore, it follows from Equation (2) and (ζ_2) that

$$\begin{aligned}
0 &\leq \zeta\left(d(Tu, Tv), d(u, v) + \theta N(u, v)\right) \\
&= \zeta\left(d(Tu, Tv), d(u, v) + \theta \min\{d(u, Tu), d(v, Tv), d(u, Tv), d(v, Tu)\}\right) \\
&= \zeta(d(u, v), d(u, v)) \\
&< d(u, v) - d(u, v) = 0
\end{aligned}$$

which is a contradiction. Thus, the fixed point of T in X is unique. □

Our main result is as follows.

Theorem 2. *Let (X, d) be a complete metric space and $T : X \to X$ be an almost $\mathcal{Z}$-contraction with respect to a function $\zeta \in \mathcal{Z}$. Then, T has a unique fixed point, and, for every initial point $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to this fixed point.*

Proof. Take $x_0 \in X$ and consider the Picard sequence $\{x_n = T^n x_0 = T x_{n-1}\}_{n \geq 0}$. If $x_{n_0} = x_{n_0+1}$ for some n_0, then x_{n_0} is a fixed point of T. Hence, for the rest of the proof, we assume that $d(x_n, x_{n+1}) > 0$ for all $n \geq 0$.

We shall divide the proof into three steps. The first step is to prove that

$$\lim_{n \to \infty} d\left(x_n, x_{n+1}\right) = 0. \tag{4}$$

Since

$$\begin{aligned}
N(x_{n-1}, x_n) &= min\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\} \\
&= min\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n)\} = 0
\end{aligned}$$

using Equation (2), for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
0 &\leq \zeta(d(Tx_{n-1}, Tx_n), d(x_{n-1}, x_n) + \theta N(x_{n-1}, x_n)) \\
&= \zeta(d(x_n, x_{n+1}), d(x_{n-1}, x_n)) \\
&< d(x_{n-1}, x_n) - d(x_n, x_{n+1}). \qquad (5)
\end{aligned}$$

It follows from the above inequality that

$$0 < d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$.

Therefore, the sequence $\{d(x_n, x_{n+1})\}$ is decreasing, so $r \geq 0$ such that $\lim_{n \to \infty} d\left(x_n, x_{n+1}\right) = r$. Assume that $r > 0$.

Take the sequences $\{t_n\}$ and $\{s_n\}$ as $t_n = d(x_n, x_{n+1})$ and $s_n = d(x_{n-1}, x_n)$. Since $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = r$ and $t_n < s_n$ for all n, by the axiom (ζ_4) and Equation (5), we deduce

$$0 \leq \limsup_{n\to\infty} \zeta\left((d(x_n, x_{n+1}), d(x_{n-1}, x_n)\right) < 0$$

which is a contradiction. Thus, $r = 0$, that is, Equation (4) holds.

As a second step, we show that the sequence $\{x_n\}$ is bounded. On the contrary, assume that $\{x_n\}$ is not bounded. Then there is a subsequence $\{x_{n_k}\}$ such that $n_1 = 1$ and for each $k \in \mathbb{N}$, n_{k+1} is the minimum integer greater than n_k such that

$$d(x_{n_{k+1}}, x_{n_k}) > 1$$

and

$$d(x_m, x_{n_k}) \leq 1 \text{ for } n_k \leq m \leq n_{k+1} - 1.$$

By the triangular inequality, we have

$$\begin{aligned} 1 &< d(x_{n_{k+1}}, x_{n_k}) \leq d(x_{n_{k+1}}, x_{n_{k+1}-1}) + d(x_{n_{k+1}-1}, x_{n_k}) \\ &\leq d(x_{n_{k+1}}, x_{n_{k+1}-1}) + 1. \end{aligned}$$

Letting $k \to \infty$ in the last equation and using Equation (4), we obtain

$$\lim_{k\to\infty} d(x_{n_{k+1}}, x_{n_k}) = 1. \tag{6}$$

Since T is an almost $\mathcal{Z}$-contraction, we can deduce that $d(x_{n_{k+1}}, x_{n_k}) \leq d(x_{n_{k+1}-1}, x_{n_k-1})$. Hence, using the triangular inequality, we obtain

$$\begin{aligned} 1 &< d(x_{n_{k+1}}, x_{n_k}) \leq d(x_{n_{k+1}-1}, x_{n_k-1}) \\ &\leq d(x_{n_{k+1}-1}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}) \\ &\leq 1 + d(x_{n_k}, x_{n_k-1}). \end{aligned}$$

Taking limit as $k \to \infty$ on both sides of the above inequality and using Equation (4), we deduce

$$\lim_{k\to\infty} d(x_{n_{k+1}-1}, x_{n_k-1}) = 1. \tag{7}$$

Since

$$N(x_{n_{k+1}-1}, x_{n_k-1}) = min\{d(x_{n_{k+1}-1}, x_{n_{k+1}}), d(x_{n_k-1}, x_{n_k}), d(x_{n_{k+1}-1}, x_{n_k}), d(x_{n_k-1}, x_{n_{k+1}})\}$$

letting $k \to \infty$ and using Equation (4), we obtain

$$\lim_{k\to\infty} N(x_{n_{k+1}-1}, x_{n_k-1}) = 0. \tag{8}$$

By Equation (2), we have

$$\begin{aligned} 0 &\leq \zeta\left((d(Tx_{n_{k+1}-1}, Tx_{n_k-1}), d(x_{n_{k+1}-1}, x_{n_k-1}) + \theta N(x_{n_{k+1}-1}, x_{n_k-1})\right) \\ &< d(x_{n_{k+1}-1}, x_{n_k-1}) + \theta N(x_{n_{k+1}-1}, x_{n_k-1}) - d(x_{n_{k+1}}, x_{n_k}) \end{aligned} \tag{9}$$

which implies that

$$d(x_{n_{k+1}}, x_{n_k}) < d(x_{n_{k+1}-1}, x_{n_k-1}) + \theta N(x_{n_{k+1}-1}, x_{n_k-1}).$$

If we choose the sequences $\{t_k\}$ and $\{s_k\}$ as $t_k = d(x_{n_{k+1}}, x_{n_k})$ and $s_k = d(x_{n_{k+1}-1}, x_{n_k-1}) + \theta N(x_{n_{k+1}-1}, x_{n_k-1})$, then $t_k < s_k$ for all k. Moreover, taking into account Equations (6)–(8), $\lim_{k\to\infty} t_k = \lim_{k\to\infty} s_k = 1$. Thus, we can apply the axiom (ζ_4) to these sequences, that is,

$$\limsup_{n\to\infty} \zeta\left((d(x_{n_{k+1}}, x_{n_k}), d(x_{n_{k+1}-1}, x_{n_k-1}) + \theta N(x_{n_{k+1}-1}, x_{n_k-1}))\right) < 0$$

which contradicts Equation (9). This contradiction proves that $\{x_n\}$ is a bounded sequence. Now, we claim that this sequence is a Cauchy sequence. Consider the sequence $\{C_n\} \subset [0, \infty)$ given by

$$C_n = \sup\{d(x_i, x_j) : i, j \geq n\}.$$

It is clear that $\{C_n\}$ is a positive decreasing sequence; hence, there is some $C \geq 0$ such that $\lim_{n\to\infty} C_n = C$. If $C > 0$, then, by definition of C_n, for every $k \in \mathbb{N}$, n_k and m_k exist such that $m_k > n_k \geq k$ and

$$C_k - \frac{1}{k} < d(x_{m_k}, x_{n_k}) \leq C_k.$$

Thus,

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = C. \tag{10}$$

Using Equation (2) and the triangular inequality, we have

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k-1}, x_{n_k-1}) \\ &\leq d(x_{m_k-1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}). \end{aligned}$$

Taking $k \to \infty$ and using Equations (4) and (10), we obtain

$$\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) = C. \tag{11}$$

Since T is an almost $\mathcal{Z}$-contraction, we can deduce that

$$d(x_{m_k}, x_{n_k}) < d(x_{m_k-1}, x_{n_k-1}) + \theta N(x_{m_k-1}, x_{n_k-1}). \tag{12}$$

Additionally, with the aid of Equation (4), we have

$$\lim_{k\to\infty} N(x_{m_k-1}, x_{n_k-1}) = 0. \tag{13}$$

Taking the sequences $\{t_k = d(x_{m_k}, x_{n_k})\}$ and $\{s_k = d(x_{m_k-1}, x_{n_k-1}) + \theta N(x_{m_k-1}, x_{n_k-1})\}$, and considering Equations (10)–(13), $\lim_{k\to\infty} t_k = \lim_{k\to\infty} s_k = C$ and $t_k < s_k$ for all k. Then, by Equation (2) and (ζ_4), we obtain

$$0 \leq \limsup_{n\to\infty} \zeta\left((d(x_{m_k}, x_{n_k}), d(x_{m_k-1}, x_{n_k-1}) + \theta N(x_{m_k-1}, x_{n_k-1})\right) < 0$$

which is a contradiction and so $C = 0$. That is, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there is a $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

As a final step, we shall show that the point u is a fixed point of T. Suppose that $Tu \neq u$. Then $d(u, Tu) > 0$. By Equation (2), (ζ_2), and (ζ_4), we obtain

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \zeta\left(d(Tx_n, Tu), d(x_n, u) + \theta N(x_n, u)\right) \\ &\leq \limsup_{n\to\infty} [d(x_n, u) + \theta N(x_n, u) - d(x_{n+1}, Tu)] \\ &= -d(u, Tu) \end{aligned}$$

which implies that $d(u, Tu) = 0$, that is, u is a fixed point of T. The uniqueness of the fixed point follows from Lemma 1. $\square$

The following example shows that Theorem 2 is a proper generalization of Theorem 1.

Example 2. *Let $X = [0, 1]$ be endowed with the usual metric. Define a mapping $T : X \to X$ as $Tx = 1 - x$ for all $x \in X$. Then, T is not a $\mathcal{Z}$-contraction with respect to ζ_λ where for all $t, s \in [0, \infty)$*

$$\zeta(t, s) = \lambda s - t, \qquad \lambda \in [0, 1) .$$

Indeed, for all $x \neq y$, we have

$$\begin{aligned} \zeta\left(d(Tx, Ty), d(x, y)\right) &= \lambda\,|x - y| - |1 - x - (1 - y)| \\ &= \lambda\,|x - y| - |x - y| \\ &< |x - y| - |x - y| = 0. \end{aligned}$$

Now, we show that T is an almost $\mathcal{Z}$-contraction with respect to ζ_λ. For an arbitrary $x, y \in X$, since

$$\begin{aligned} N(x, y) &= \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\ &= \min\{\mid 2x - 1 \mid, \mid 2y - 1 \mid, \mid x + y - 1 \mid\}, \end{aligned}$$

we deduce that

$$\begin{aligned} \zeta\left(d(Tx, Ty), d(x, y) + \theta N(x, y)\right) &= \lambda[d(x, y) + \theta N(x, y)] - d(Tx, Ty) \\ &= \lambda[\mid x - y \mid + \theta \min\{\mid 2x - 1 \mid, \mid 2y - 1 \mid, \mid x + y - 1 \mid\}] \\ &\quad - \mid x - y \mid . \end{aligned}$$

Thus, we get two cases:

Case I: *If $x = y$, then*

$$\zeta\left(d(Tx, Ty), d(x, y) + \theta N(x, y)\right) = \lambda\theta \mid 2x - 1 \mid \geq 0.$$

Case II: *Without loss of generality, assume that $x > y$. Then*

$$\zeta\left(d(Tx, Ty), d(x, y) + \theta N(x, y)\right) = \lambda \mid x - y \mid + \lambda\theta \mid 2y - 1 \mid - \mid x - y \mid .$$

If we especially choose $\lambda = \frac{1}{2}$ and $\theta = 10$, then we get

$$\zeta\left(d(Tx, Ty), d(x, y) + \theta N(x, y)\right) = 5 \mid 2y - 1 \mid - \frac{1}{2} \mid x - y \mid \geq 0.$$

Therefore, all of the conditions of Theorem 2 are satisfied; hence, T has a unique fixed point $u = \frac{1}{2} \in X$.

If we take $\zeta(t, s) = \lambda s - t$ and $\theta = 0$ in Theorem 2, then we have the following result.

Corollary 1 ([1])**.** *Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping such that*

$$d(Tx, Ty) \leq \lambda d(x, y)$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Thus, T has a unique fixed point.

If we take $\zeta(t, s) = \phi(s) - t$ and $\theta = 0$ in Theorem 2, then we have the following result.

Corollary 2 ([10]). *Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping satisfying*

$$d(Tx, Ty) \leq \phi(d(x, y))$$

for all $x, y \in X$ where $\phi : [0, \infty) \to [0, \infty)$ is an upper semi-continuous function with $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$. Thus, T has a unique fixed point.

3. An Application

Consider the following integral equation:

$$p(r) = q(r) + \theta \int_a^b H(r, z) f(z, p(z)) dz, \ r \in I = [a, b] \tag{14}$$

where $q : I \to \mathbb{R}$, $H : I \times I \to \mathbb{R}$, $f : I \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

In this section, we prove the existence of a unique solution of the integral Equation (14) that belongs to $X := C(I, \mathbb{R})$ by using the results of the previous section.

Define a mapping $T : X \to X$ by

$$Tp(r) := q(r) + \theta \int_a^b H(r, z) f(z, p(z)) dz, \ r \in I = [a, b].$$

Thus, the existence of a unique solution of Equation (14) is equivalent to the existence of a unique fixed point of T.

Meanwhile, X is endowed with the metric d, which is defined by

$$d(p, q) = \sup_{r \in I} |p(r) - q(r)|.$$

f is a complete metric space.

We will analyze Equation (14) under the following assumptions:

(a) $|\theta| \leq 1$.

(b) $\sup_{r \in I} \int_a^b H(r, z)\, dz \leq \frac{1}{b-a}$.

(c) for all $p, q \in \mathbb{R}$,

$$|f(z, p) - f(z, q)| \leq \phi(|p - q|)$$

where $\phi : [0, \infty) \to [0, \infty)$ is a nondecreasing upper semi-continuous function with $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$.

Theorem 3. *Under the assumptions (a)–(c), the integral Equation (14) has a unique solution in X.*

Proof. On account of our suppositions, for all $r \in I$, we deduce that

$$\begin{aligned} d(Tp_1, Tp_2) &= \sup_{r \in I} |Tp_1(r) - Tp_2(r)| \\ &= \sup_{r \in I} \left| q(r) + \theta \int_a^b H(r, z) f(z, p_1(z)) dz - q(r) + \theta \int_a^b H(r, z) f(z, p_2(z)) dz \right| \\ &= |\theta| \sup_{r \in I} \left| \int_a^b H(r, z) \left[f(z, p_1(z)) - f(z, p_2(z)) \right] dz \right| \\ &\leq |\theta| \sup_{r \in I} \left\{ \int_a^b H(r, z)\, dz \int_a^b |f(z, p_1(z)) - f(z, p_2(z))|\, dz \right\} \end{aligned}$$

$$
\begin{aligned}
&= |\theta| \sup_{r\in I} \left\{ \int_a^b H(r,z)\, dz \right\} \left\{ \int_a^b |f(z,p_1(z)) - f(z,p_2(z))|\, dz \right\} \\
&\le |\theta| \frac{1}{b-a} \left\{ \int_a^b \phi(|p_1(z) - p_2(z)|) dz \right\} \\
&\le |\theta| \frac{1}{b-a} \int_a^b \phi(d(p_1,p_2)) dz \\
&= |\theta|\, \phi(d(p_1,p_2)) \le \phi(d(p_1,p_2)).
\end{aligned}
$$

Hence, all of the conditions of Corollary 2 are fulfilled. This means that T has a unique fixed point; that is, Equation (14) has a unique solution in X. □

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