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Some Results on $\mathcal{S}$-*Contractions* of Type *E*

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Abstract: In this manuscript, we consider the compositions of simulation functions and *E*-contraction in the setting of a complete metric space. We investigate the existence and uniqueness of a fixed point for this composite form. We give some illustrative examples and provide an application.

1. Introduction and Preliminaries

In the last decades, the renowned metric fixed point results of Banach [1] has been improved, extended, and generalized in several ways, see e.g., [2–21]. We first mention that the notion of *E*-contraction, defined by Fulga and Proca [10,11], is one of the interesting approach to improve the Banach mapping contraction. Another interesting fixed point result was given by Khojasteh et al. [19] via the newly defined notion, *simulation function*. Both results generalize and extend the basic results on the theory of metric fixed point. In this paper, we combine these two interesting notions and get some interesting results in this direction.

We recollect some basic notions as well as the fundamental definitions to provide a self-contained manuscript. For more details about the tools and notations, we refer e.g., [19,20]. We shall use the letters $\mathbb{R}, \mathbb{R}^+, \mathbb{N}$ for the reals, nonnegative real numbers, and natural numbers, accordingly. Moreover, we employ the symbols $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\} = [0, \infty)$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 1 (See [19]). *A function* $\sigma : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ *is called* simulation function *if it checks the following conditions:*

(sf_1) $\sigma(0,0) = 0$;
(sf_2) $\sigma(t,s) < s - t$ *for all* $t, s \in \mathbb{R}^+$;
(sf_3) *if* $\{t_n\}, \{s_n\}$ *in* $(0, \infty)$ *are two sequences such that* $\lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n > 0$, *then*

$$\limsup_{n \to \infty} \sigma(t_n, s_n) < 0. \tag{1}$$

We will denote by $\mathcal{S}$ the family of all simulation functions. It is clear, due to the axiom (sf_2), that

$$\sigma(t,t) < 0 \text{ for all } t > 0. \tag{2}$$

Let Φ be the class of continuous functions $\phi : [0, \infty) \to [0, \infty)$ which satisfies the condition

$$\phi(x) = 0 \text{ if, and only if, } x = 0.$$

Example 1 (See e.g., [2,19,20]). *Let $\phi_i \in \Phi$ for $i = 1, 2, 3$ and $\sigma_j : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ for $j = 1, 2, 3, 4, 5, 6$. Each of the functions defined below is an example of simulation functions.*

(i) $\sigma_1(t, s) = \phi_1(s) - \phi_2(t)$ *for all* $t, s \geq 0$ *where* $\phi_1(t) < t \leq \phi_2(t)$ *for all* $t > 0$.

(ii) $\sigma_2(t, s) = s - \phi_3(s) - t$ *for all* $t, s \geq 0$.

(iii) $\sigma_3(t, s) = s - \int_0^t \mu(u) du$ *for all* $t, s \geq 0$, *where* $\mu : [0, \infty) \to [0, \infty)$ *is a function such that* $\int_0^\varepsilon \mu(u) du$ *exists and* $\int_0^\varepsilon \mu(u) du > \varepsilon$, *for each* $\varepsilon > 0$.

(iv) $\sigma_4(t, s) = f(s) - t$ *for all* $t, s \geq 0$, $t, s \geq 0$, *where the function* $f : [0, \infty) \to [0, \infty)$ *is upper semi-continuous and such that* $f(t) < t$ *for all* $t > 0$ *and* $f(0) = 0$.

(v) $\sigma_5(t, s) = s - \dfrac{g(t, s)}{h(t, s)}$ *for all* $t, s \geq 0$, *where* $g, h : [0, \infty)^2 \to (0, \infty)$ *are two continuous functions with respect to each variable such that* $g(t, s) > h(t, s)$ *for all* $t, s > 0$.

(vi) $\sigma_6(t, s) = s\, \eta(s) - t$ *for all* $t, s \geq 0$, *where* $\eta : [0, \infty) \to [0, 1)$ *is a function with the property* $\limsup_{t \to r^+} \eta(t) < 1$ *for all* $r > 0$

Let $(\mathcal{M}, d)$ be a metric space and $\sigma \in \mathcal{S}$ be a simulation function. We say that a function $S : \mathcal{M} \to \mathcal{M}$ is $\mathcal{S}$-contraction with respect to σ [19], if the inequality

$$\sigma(d(Sp, Sq), d(p, q)) \geq 0 \qquad \text{for all } p, q \in \mathcal{M}, \tag{3}$$

is attained.

Remark 1. *From* (σ_2), *we find that*

$$d(Sp, Sq) \neq d(p, q) \text{ for all distinct } p, q \in \mathcal{M}, \tag{4}$$

which infers that, whenever S is a $\mathcal{S}$-contraction, S cannot be an isometry. As a result, the fixed point of a $\mathcal{S}$-contraction S (if there exists) is necessarily unique.

Theorem 1 ([19]). *For every $\mathcal{S}$-contraction on a complete metric space there exists exactly one fixed point. In fact, every Picard sequence converges and its limit is the unique fixed point.*

2. Main Results

We state now our main results. For this purpose, we start by defining a new type of $\mathcal{S}$-contraction.

Definition 2. *A self-mapping S defined on a complete metric space $(\mathcal{M}, d)$ is a $\mathcal{S}$-contraction of type E with respect to σ if there exists $\sigma \in \mathcal{S}$ such that*

$$\sigma(d(Sp, Sq), E(p, q)) \geq 0 \qquad \text{for all } p, q \in \mathcal{M}, \tag{5}$$

where,

$$E(p, q) = d(p, q) + |d(p, Sp) - d(q, Sq)|. \tag{6}$$

We denote by $\mathcal{C}_E(\mathcal{M})$ the set of $\mathcal{S}$-contractions of type E with respect to σ defined on $\mathcal{M}$.

We now present the results regarding the existence of a fixed point, since the uniqueness follows from Remark 1.

Theorem 2. *There exists a fixed point for every $S \in \mathcal{C}_E(\mathcal{M})$*

Proof. Given an arbitrary $p_0 \in \mathcal{M}$ we consider the constructive sequence $\{p_n\} \subset \mathcal{M}$ which is defined by $u_{n+1} = Su_n = S^n u_0$ for all $n \in \mathbb{N}$.

We shall assume that $p_{n+1} \neq p_n$ for all $n \in \mathbb{N}$. Indeed, in the opposite case, where $p_{n_0} = p_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then $Sp_{n_0} = p_{n_0}$. This completes the proof and we conclude that the given point p_{n_0} turns to be a fixed point of S.

Consequently, $d\,(p_{n+1}, p_n) > 0$ and from Equation (5), it follows, for all $n \geq 1$, that

$$
\begin{aligned}
0 \quad &\leq \sigma(d(Sp_n, Sp_{n-1}), E(p_n, p_{n-1})) \\
&= \sigma(d(p_{n+1}, p_n), E(p_n, p_{n-1})) \\
&< E(p_n, p_{n-1}) - d(p_{n+1}, p_n).
\end{aligned}
\tag{7}
$$

In conclusion, for all $n = 1, 2, \ldots$, we have

$$
d(p_n, p_{n+1}) < E(p_n, p_{n-1}).
\tag{8}
$$

To understand the inequality Equation (8), we consider two cases. For the first case, we suppose that $d(p_n, p_{n+1}) \geq d(p_{n-1}, p_n)$. In this case, the inequality Equation (8), becomes

$$
d(p_n, p_{n+1}) \quad < d(p_{n-1}, p_n) - d(p_{n-1}, p_n) + d(p_n, p_{n+1}) = d(p_n, p_{n+1}),
$$

a contradiction. Thus, the following case occurs

$$
d(p_n, p_{n+1}) < d(p_{n-1}, p_n) \text{ for all } n = 1, 2, \ldots.
\tag{9}
$$

Accordingly, we deduce that the sequence $\{d(p_n, p_{n-1})\}$ is non-increasing and bounded below by 0. Hence, the sequence $\{d(p_n, p_{n-1})\}$ converges to some $d^* \geq 0$. In the same time, we note that

$$
\lim_{n \to \infty} E(p_n, p_{n-1}) = \lim_{n \to \infty} (2d(p_{n-1}, p_n) - d(p_n, p_{n+1})) = d^*.
\tag{10}
$$

We claim that

$$
d^* = \lim_{n \to \infty} d(p_n, p_{n-1}) = 0.
\tag{11}
$$

Suppose, on the contrary that $d^* > 0$. Then, letting $t_n = d(p_n, p_{n+1})$ and $s_n = E(p_n, p_{n-1})$ we get from Equation (5) and (sf_3), that

$$
0 \leq \limsup_{n \to \infty} \sigma(d(p_{n+1}, p_n), E(p_n, p_{n-1})) = \limsup_{n \to \infty} \sigma(t_n, s_n)) < 0
\tag{12}
$$

This contradiction shows that $d^* = 0$. We will now show that the sequence $\{p_n\}$ is Cauchy. Suppose, on the contrary, that $\{p_n\}$ is not a Cauchy sequence. Then there exists a real positive number $\varepsilon > 0$ and sequences $\alpha(n), \beta(n)$ of natural numbers such that $\alpha(n) > \beta(n) > n$ and

$$
d(p_{\alpha(n)}, p_{\beta(n)}) \geq \varepsilon, \quad d(p_{\alpha(n)-1}, p_{\beta(n)}) < \varepsilon \quad (\forall)\, n \in \mathbb{N}.
$$

So, from triangle inequality,

$$
\begin{aligned}
\varepsilon \quad &\leq \quad d\left(p_{\alpha(n)}, p_{\beta(n)}\right) \leq d\left(p_{\alpha(n)}, p_{\alpha(n)-1}\right) + d\left(p_{\alpha(n)-1}, p_{\beta(n)}\right) \\
&< \quad d\left(p_{\alpha(n)}, p_{\alpha(n)-1}\right) + \varepsilon
\end{aligned}
$$

and by Equation (11) we get

$$
\lim_{n \to \infty} d\left(p_{\alpha(n)}, p_{\beta(n)}\right) = \varepsilon
\tag{13}
$$

On the other hand, it is easy to see that

$$\left| d\left(p_{\alpha(n)-1}, p_{\beta(n)-1} \right) - d\left(p_{\alpha(n)}, p_{\beta(n)} \right) \right| \;\leq\; d\left(p_{\beta(n)}, p_{\alpha(n)-1} \right) + $$
$$+ d\left(p_{\beta(n)-1}, p_{\beta(n)} \right)$$

and from Equation (11) respectively Equation (13)

$$\lim_{n\to\infty} d\left(p_{\alpha(n)-1}, p_{\beta(n)-1} \right) = \varepsilon. \tag{14}$$

Moreover from Equations (6), (11) and (13), we have

$$
\begin{aligned}
\lim_{n\to\infty} E(p_{\alpha(n)-1}, p_{\beta(n)-1}) &= \lim_{n\to\infty} \Big(d(p_{\alpha(n)-1}, p_{\beta(n)-1}) + \\
&\quad + \left| d(p_{\alpha(n)-1}, Sp_{\alpha(n)-1}) - d(p_{\beta(n)-1}, Sp_{\beta(n)-1}) \right| \Big) \\
&= \lim_{n\to\infty} \Big(d(p_{\alpha(n)-1}, p_{\beta(n)-1}) + \\
&\quad + \left| d(p_{\alpha(n)-1}, p_{\alpha(n)}) - d(p_{\beta(n)-1}, p_{\beta(n)}) \right| \Big) \\
&= \varepsilon.
\end{aligned}
\tag{15}
$$

Letting $t_n = d(p_{\alpha(n)}, p_{\beta(n)})$ and $s_n = E(p_{\alpha(n-1)}, p_{\beta(n-1)})$ we have $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = \varepsilon$ and combing with (sf_3),

$$
\begin{aligned}
0 &\leq \limsup_{n\to\infty} \sigma(d(Sp_{\alpha(n)-1}, Sp_{\beta(n)-1}), E(p_{\alpha(n)-1}, p_{\beta(n)-1}) \\
&= \limsup_{n\to\infty} \sigma(d(p_{\alpha(n)}, p_{\beta(n)}), E(p_{\alpha(n)-1}, p_{\beta(n)-1}) \\
&= \limsup_{n\to\infty} \sigma(t_n, s_n) < 0.
\end{aligned}
\tag{16}
$$

This contradiction proves that $\varepsilon = 0$ and thus the sequence $\{p_n\}$ is Cauchy. On the account of the completeness of $(\mathcal{M}, d)$ there exists a point $p^* \in \mathcal{M}$ such that

$$\lim_{n\to\infty} p_n = p^* = 0. \tag{17}$$

We must prove now that $p^* = Sp^*$. Arguing by contradiction, we will assume that $d(p^*, Sp^*) > 0$. By the property (sf_2), for $r \in \mathbb{N}$ sufficiently large, we have

$$
\begin{aligned}
0 &\leq \sigma(d(Sp_r, Sp^*), E(p_r, p^*)) \\
&= \sigma(d(p_{r+1}, Sp^*), E(p_r, p^*)) \\
&< E(p_r, p^*) - d(p_{r+1}, Sp^*).
\end{aligned}
\tag{18}
$$

Considering the sequences $t_r^* = d(p_{r+1}, Sp^*)$ respectivelly $s_r^* = E(p_r, p^*) = d(p_r, p^*) + |d(p_r, Sp_r) - d(p^*, Sp^*)|$ we find that

$$\lim_{r\to\infty} t_r^* = \lim_{r\to\infty} s_r^* = d(p^*, Sp^*) > 0, \tag{19}$$

which implies together with Equation (18)

$$0 \leq \limsup_{r\to\infty} \sigma(d(Sp_r, Sp^*), E(p_r, p^*) < 0. \tag{20}$$

a contradiction. Thus, we have $d(p^*, Sp^*) = 0$, i.e., $Sp^* = p^*$. $\quad\square$

Examples

Example 2. *Let the set* $\mathcal{M} = \left[0, \frac{5}{3}\right] \cup \{2\}$ *and* $d(p,q) = |p-q|,\ d\ :\ \mathcal{M}\ \to\ \mathcal{M}.$ *Suppose that* $\sigma : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *is defined as* $\sigma(s,t) = \frac{s}{2} - t$ *and hence* $\sigma \in \mathcal{S}.$ *Define a map* $S\ :\ \mathcal{M} \to \mathcal{M}$ *as follows*

$$S(p) = \begin{cases} 1 & \text{if } p \in \left[0, \frac{5}{3}\right], \\ \frac{1}{3} & \text{if } p = 2. \end{cases}$$

Notice that for $p = 2$ *and* $q = \frac{5}{3}$, *we have*

$$d\left(2, \frac{5}{3}\right) = \frac{1}{3},\ d\left(S2, S\frac{5}{3}\right) = \frac{2}{3}$$

and for these values there is no $k_1 \in [0,1)$ *such that*

$$d\left(S2, S\frac{5}{3}\right) = \frac{2}{3} \le k_1 \frac{1}{3} = k_1 d\left(2, \frac{5}{3}\right)$$

Thus, the mapping S is not a contraction.

On the other hand, it is $\mathcal{S}$-*contraction of type E. For the proof of our claim, we need to consider two distinct cases:*

Case 1. $q = 2, p < 1.$ *Then, we find that*

$$d(p,2) = 2 - p, d(p, Sp) = 1 - p \text{ and } d(2, S2) = \left|2 - \frac{1}{3}\right| = \frac{5}{3} \text{ and } d(S2, Sp) = \frac{2}{3}.$$

Since

$$E(p,2) = 2 - p + \left|1 - p - \frac{5}{3}\right| = 2 - p + \frac{3p+2}{3} = \frac{8}{3},$$

we have that

$$\sigma(d(Sp, S2), E(p,2)) = \frac{E(p,2)}{2} - d(Sp, S2) = \frac{8}{6} - \frac{2}{3} = \frac{2}{3} > 0.$$

Case 2. If $q = 2, p \ge 1$ *then*

$$d(p,2) = 2 - p \text{ and } d(p, Sp) = p - 1 \text{ and } d(2, S2) = \left|2 - \frac{1}{3}\right| = \frac{5}{3} \text{ and } d(S2, Sp) = \frac{2}{3}.$$

As a result, we have

$$E(p,2) = 2 - p + \left|p - 1 - \frac{5}{3}\right| = 2 - p + \frac{8 - 3p}{3} = \frac{14 - 6p}{3}$$

and also

$$\sigma(d(Sp, S2), E(p,2)) = \frac{E(p,2)}{2} - d(Sp, S2) = \frac{14 - 6p}{6} - \frac{2}{3} = \frac{5 - 3p}{3} \ge 0.$$

We deduce that S is a $\mathcal{S}$-*contraction of type E. Further, all conditions of Theorem 2 are fulfilled and* $p = 1$ *is a fixed point of S. Finally, we mention that the uniqueness of the fixed point follows from the Remark 1.*

Example 3. *Let* $\mathcal{M} = \{1,3,4,5\}$ *and* $d\ :\ \mathcal{M} \to \mathcal{M}, d(p,q) = |p-q|.$ *We define the function* $S\ :\ \mathcal{M} \to \mathcal{M}$ *as,* $S1 = S3 = S4 = 3, S5 = 1,$ *and set* $\sigma(t,s) = \frac{1}{2}s - t.$

One can easily get that

$$d(3,4) = d(4,5) = 1, \ d(3,5) = d(1,3) = 2, \ d(1,4) = 3, \ d(1,5) = 4,$$
$$d(S3, S4) = d(S1, S3) = d(S1, S4) = 0, \ d(S3, S5) = d(S4, S5) = d(S1, S5) = 2.$$

Moreover, we have

$$E(1,4) = E(1,3) = E(4,5) = 4 \text{ and } E(1,5) = E(3,5) = 6 \text{ and } E(3,4) = 2$$

Firstly, we claim that S is not a contraction. Indeed, for $p = 4$ and $q = 5$, we could not find a real constant $k_2 \in [0,1)$ such that $d(T4, T5) = 2 \le k_2 d(4,5)$ is satisfied. So, S is not a contraction.

Now, we shall show that S is a $\mathcal{S}$-contraction of type E. For this purpose, we examine all possible cases:

For $p = 1, q = 3$, we have $\sigma \left(d(S1, S3), E(1,3) \right) = \sigma(0,4) = \frac{4}{2} - 0 = 2$.
For $p = 1, q = 4$, we find $\sigma \left(d(S1, S4), E(1,4) \right) = \sigma(0,4) = \frac{4}{2} - 0 = 2$.
For $p = 1, q = 5$, we get $\sigma \left(d(S1, S5), E(1,5) \right) = \sigma(2,6) = \frac{6}{2} - 2 = 1$.
For $p = 3, q = 4$ we obtain $\sigma \left(d(S3, S4), E(3,4) \right) = \sigma(0,2) = \frac{2}{2} - 0 = 1$.
For $p = 3, q = 5$ we observe $\sigma \left(d(S3, S5), E(3,5) \right) = \sigma(2,6) = \frac{6}{2} - 2 = 1$.

As a last case, for $p = 4, q = 5$, we derive $\sigma \left(d(S4, S5), E(4,5) \right) = \sigma(2,4) = \frac{4}{2} - 2 = 0$. Evidently, we conclude that $S \in \mathcal{C}_E(\mathcal{M})$.

In addition, all conditions of Theorem 2 are attained and $p = 3$ is a fixed point of S. As in the above example, the uniqueness results from the Remark 1.

Example 4. *Let $\mathcal{M} = [0, \frac{1}{2}] \cup \{\frac{3}{4}\}$ and define $d(p,q) = \begin{cases} \max\{p,q\} & \text{if } p \ne q \\ \\ 0 & \text{otherwise.} \end{cases}$*

We consider the following self-mapping $S(p) = \begin{cases} \frac{p}{p+1} & \text{if } p \in \left[0, \frac{1}{4}\right) \cup \left(\frac{1}{4}, 1\right] \\ \\ \frac{1}{2} & \text{if } p = \frac{3}{4}. \end{cases}$

We claim that S is a $\mathcal{S}$-contraction of type E for $\sigma(t,s) = \frac{s}{s+1} - t$.

Case 1. For $0 \le q \le p \le \frac{1}{2}$, we have

$$d(p,q) = \max\{p,q\} = p \text{ and } d(Sp, Sq) = \max\left\{\frac{p}{p+1}, \frac{q}{q+1}\right\} = \frac{p}{p+1},$$

$$d(p, Tp) = \max\left\{p, \frac{p}{p+1}\right\} = p \text{ and } d(q, Tq) = q.$$

So, we have

$$E(p,q) = p + |p - q| = 2p - q.$$

and

$$\begin{aligned} \sigma \left(d(Sp, Sq), E(p,q) \right) &= \frac{E(p,q)}{1 + E(p,q)} - d(Sp, Sq) \\ \\ &= \frac{2p - q}{1 + 2p - q} - \frac{p}{p+1} \\ \\ &= \frac{p - q}{(2p - q + 1)(p+1)} > 0. \end{aligned}$$

It is clear that the above observation is also valid for $0 \le p \le q \le \frac{1}{2}$.

Case 2. For $0 \le p \le \frac{1}{2}$, and $q = \frac{3}{4}$, we have

$$d(p,q) = \max\{p,q\} = \frac{3}{4} \text{ and } d(Sp, Sq) = \max\left\{\frac{p}{p+1}, \frac{1}{2}\right\} = \frac{1}{2}$$
$$d(p, Sp) = \max\left\{p, \frac{p}{p+1}\right\} = p \text{ and } d(q, Sq) = \max\left\{\frac{3}{4}, \frac{1}{2}\right\} = \frac{3}{4}.$$

So,

$$E(p,q) = \frac{3}{4} + \left| p - \frac{3}{4} \right| = \frac{3}{2} - p.$$

Consequently,

$$\sigma\left(\frac{1}{2}, \frac{3}{2} - p \right) = \frac{\frac{3}{2} - p}{\frac{5}{2} - p} - \frac{1}{2} = \frac{1 - 2p}{5 - 2p} > 0$$

The case, for $0 \leq q \leq \frac{1}{2}$, and $p = \frac{3}{4}$, is the analog of Case 2.

In any case, we observe that $S \in \mathcal{C}_E(\mathcal{M})$. This completes the proof. Therefore, we conclude that S has a unique fixed point, namely, $x = 0$. On the account of Remark 1, the fixed point of S is unique.

3. Consequences and Application

In this section we give one corollary and we consider an application of the main result in which the solution for an integral equation can be described.

Corollary 1. *Let $S : \mathcal{M} \to \mathcal{M}$ be a function on a complete metric space $(\mathcal{M}, d)$. If there exist $\mu_1, \mu_2 \in \Phi$ with $\mu_1(s) < s \leq \mu_2(s)$ for all $s > 0$, such that for all $p, q \in \mathcal{M}$, the following inequality is fulfilled*

$$\mu_2(d(Sp, Sq)) \leq \mu_1(E(p,q)),$$

where, $E(p,q) = d(p,q) + |d(p, Sp) - d(q, Sq)|$. Then, S has exactly one fixed point.

Proof. It is enough to take $\sigma(t,s) = \sigma_1(t,s)$ in Example 1 and apply Theorem 2. □

It is clear that by choosing simulation function σ from Example 1 and applying Theorem 2, we get further corollaries, as we got Corollary 1. So, we skip this list of corollaries by using the analogy.

Let $\mathcal{M} = C(I, \mathbb{R})$ be the set of all continuous functions on $I = [0,1]$ equipped with a metric $d(x,y) = \|x - y\| = \sup\{|x(s) - y(s)| : s \in I\}$, for all $x, y \in \mathcal{M}$. Then $(\mathcal{M}, d)$ forms a complete metric space. We investigate the integral equation

$$x(s) = \xi(s) + \int_0^1 K(s,u)\eta(u, x(u))du, \quad s \in [0,1], \tag{21}$$

where the functions $\eta : I \times \mathbb{R} \to \mathbb{R}$ and $\xi : I \to \mathbb{R}$ are continuous and $K : I \times I \to \mathbb{R}_0^+$ is a function such that $K(s, \cdot) \in L^1(I)$ for all $s \in [0,1]$. We handle the map $S : \mathcal{M} \to \mathcal{M}$ which is defined by

$$S(x)(s) = \xi(s) + \int_0^1 K(s,u)\eta(u, x(u))du, \quad s \in [0,1]. \tag{22}$$

Theorem 3. *The Equation (21) has a unique solution in $\mathcal{M}$, if the following conditions are fulfilled:*

$(a1)$　*there exists $\mu \in \Phi$ with $\mu(s) < s$ for all $s > 0$ satisfying*

$$0 \leq |\eta(u, x_1(u)) - \eta(u, x_2(u))| \leq \mu\left(|x_1(u) - x_1(u)| + \left| |x_1(u) - S(x_1)(u)| - |x_2(u) - S(x_2)(u)| \right| \right),$$

　　for all $u \in I$ and for all $x_1, x_2 \in \mathcal{M}$.
$(a2)$　*followed by the inequality assumed*

$$\sup_{s \in I} \int_0^1 K(s,u)du \leq 1.$$

Proof. Note that any fixed point of Equation (21) is a solution of the integral Equation (21). On account of $(a1)$ and $(a2)$, we find that

$$\left| S(x_1)(s) - S(x_2)(s) \right| = \left| \int_0^1 K(s,u)[\eta(u, x_1(u)) - \eta(u, x_2(u))]du \right|$$

$$\leq \int_0^1 K(s,u) \left| \eta(u, x_1(u)) - \eta(u, x_2(u)) \right| du$$

$$\leq \int_0^1 K(s,u)\mu \left(|x_1(u) - x_2(u)| + \left| |x_1(u) - S(x_1)(u)| - |x_2(u) - S(x_2)(u)| \right| \right) du$$

$$\leq \mu \left(E(x_1, x_2) \right),$$

where, $E(x_1, x_2) = \|x_1 - x_2\| + \left| \|x_1 - Sx_1\| - \|x_2 - Sx_2\| \right|$. Hence, we derive that

$$\|Sx_1 - Sx_2\| \leq \mu(\|x_1 - x_2\| + \left| \|x_1 - Sx_1\| - \|x_2 - Sx_2\| \right|).$$

Therefore we have

$$\sigma(d(Sx_1, Sx_2), E(x_1, x_2)) = \mu(E(x_1, x_2)) - d(Sx_1, Sx_2) \geq 0.$$

This implies that all the conditions of Corollary 1 and hence Theorem 2 are satisfied. Thus, the operator S has a unique fixed point which is the solution of the integral Equation (21) in $\mathcal{M}$. □

Example 5. *As an example of Theorem 3, we consider the next integral equation*

$$x(s) = \frac{1}{1+s^4} + \frac{1}{3}\int_0^1 \frac{u\sin 2u}{12(1+s^2)}\frac{|x|}{1+|x|}du, \quad s \in [0,1], \tag{23}$$

This equation is obtained from Equation (21) by choosing

$$\xi(s) = \frac{1}{1+s^4}, \quad K(s,u) = \frac{u}{2(1+s^2)}, \text{ and } \eta(s,x) = \frac{|x|\sin 2s}{6(1+|x|)}$$

Let S be a self-mapping, defined as

$$S(x)(s) = \xi(s) + \int_0^1 K(s,u)\eta(u, x(u))du, \quad s \in [0,1]. \tag{24}$$

By letting $\mu(t) = \frac{s}{2}$, we get that

$$|\eta(s, x_1) - \eta(s, x_2)| = \left| \frac{\sin 2s}{6}\frac{|x_1|}{1+|x_1|} - \frac{\sin 2s}{6}\frac{|x_2|}{1+|x_2|} \right|$$

$$\leq \frac{1}{6}|x_1 - x_2| \leq \mu \left(|x_1 - x_2| + \left| |x_1 - x_2| - |x_1 - x_2| \right| \right) = \mu(E(x_1, x_2)).$$

On the other hand,

$$\sup_{s\in I}\int_0^1 K(s,u)du = \sup_{s\in I}\int_0^1 \frac{u}{2(1+s^2)}du = \frac{1}{4} \leq 1.$$

Hence, we conclude that the integral Equation (23) has exactly one solution in $C(I, \mathbb{R})$.

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