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Topics of Measure Theory on Infinite Dimensional Spaces

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Abstract: This short review is devoted to measures on infinite dimensional spaces. We start by discussing product measures and projective techniques. Special attention is paid to measures on linear spaces, and in particular to Gaussian measures. Transformation properties of measures are considered, as well as fundamental results concerning the support of the measure.

Keywords: measure; infinite dimensional space; nuclear space; projective limit

1. Introduction

We present here a brief introduction to the subject of measures on infinite dimensional spaces. The author's background is mathematical physics and quantum field theory, and that is likely to be reflected in the text, but an hopefully successful effort was made to produce a review of interest to a broader audience. We have references [1–3] as our main inspiration. Obviously, some important topics are not dealt with, and others are discussed from a particular perspective, instead of another. Notably, we do not discuss the perspective of abstract Wiener spaces, emerging from the works of Gross and others [4–7]. Instead, we approach measures in general linear spaces from the projective perspective (see below).

For the sake of completeness, we include in Section 2 fundamental notions and definitions from measure theory, with particular attention to the issue of σ-additivity. We start by considering in Section 3 the infinite product of a family of probability measures. In Section 4 we consider projective techniques, which play an important role in applications (see e.g., [8,9] for applications to gauge theories and gravity). Sections 5 to 7 are devoted to measures on infinite dimensional linear spaces. In Section 6 results concerning the support of the measure are presented, which partly justify, in this context, the interest of nuclear spaces and their (topological) duals. The particular case of Gaussian measures is considered in Section 7. There are of course several possible approaches to the issue of measures in infinite dimensional linear spaces, and to Gaussian measures in particular, including the well known and widely used framework of Abstract Wiener Spaces or other approaches working directly with Banach spaces (see, e.g., [10–12]). We follow here the approach of Ref. [1], taking advantage of the facts that the algebraic dual of any linear space is a projective limit (of finite dimensional spaces) and that any consistent family of measures defines a measure on the projective limit. In Section 8 we present the main definitions and some fundamental results concerning transformation properties of measures, discussing briefly quasi-invariance and ergodicity. Finally, in Section 9 we consider in particular measures on the space of tempered distributions.

Generally speaking, and except when explicitly stated otherwise, we consider only finite (normalized) measures. (A notable exception is the Lebesgue measure on $\mathbb{R}^n$, to which we refer occasionally.)

2. Measure Space

We review in this section some fundamental aspects of measure theory, focusing (although not exclusively) on finite measures. A very good presentation of these subjects can be found in [13–17].

Definition 1. *Given a set M, a family $\mathcal{F}$ of subsets of M is said to be a (finite) algebra if it is closed under the operations of taking the complement and finite unions, i.e., if $B \in \mathcal{F}$ implies $B^c \in \mathcal{F}$ and $B_1 \in \mathcal{F}, \ldots, B_n \in \mathcal{F}$ implies $\cup_i B_i \in \mathcal{F}$.*

Definition 2. *A non-negative real function μ on an algebra $\mathcal{F}$ is said to be a measure if for any finite set of mutually disjoint elements $B_1, B_2, \ldots, B_n$ of $\mathcal{F}$ ($B_i \cap B_j = \emptyset$ for $i \neq j$) the following additivity condition is satisfied:*

$$\mu\left(\cup_i B_i\right) = \sum_i \mu\left(B_i\right). \tag{1}$$

Particularly important is the notion of measures on σ-algebras, in which case the measure is required to satisfy the so-called σ-additivity condition.

Definition 3. *Given a set M, a family $\mathcal{B}$ of subsets of M is said to be a σ-algebra if it is closed under complements and countable unions, i.e., $B \in \mathcal{B}$ implies $B^c \in \mathcal{B}$ and $B_i \in \mathcal{B}$, $i \in \mathbb{N}$, implies $\cup_i^\infty B_i \in \mathcal{B}$. The pair $(M, \mathcal{B})$ is called a measurable space and the elements of $\mathcal{B}$ are called measurable sets.*

It is obvious that for any measurable space $(M, \mathcal{B})$ the σ-algebra $\mathcal{B}$ contains M and the empty set, and it is also closed under countable intersections. Another operation of interest in a σ-algebra (or finite algebra) is the symmetric difference of sets $A \triangle B := (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

Definition 4. *Given a measurable space $(M, \mathcal{B})$, a function $\mu : \mathcal{B} \to [0, \infty]$, with $\mu(\emptyset) = 0$, is said to be a measure if it satisfies the σ-additivity property, i.e., if for any sequence of mutually disjoint measurable sets $\{B_i\}_{i \in \mathbb{N}}$ one has*

$$\mu\left(\cup_i^\infty B_i\right) = \sum_i^\infty \mu\left(B_i\right), \tag{2}$$

where the right hand side denotes either the sum of the series or infinity, in case the sum does not converge. The structure $(M, \mathcal{B}, \mu)$ is called a measure space. The measure is said to be finite if $\mu(M) < \infty$, and normalized if $\mu(M) = 1$, in which case $(M, \mathcal{B}, \mu)$ is said to be a probability space.

An important property following from σ-additivity is the following.

Theorem 1. *Let μ be a σ-additive finite measure and $B_1 \supset B_2 \supset \ldots$ a decreasing sequence of measurable sets. Then*

$$\mu\left(\cap_n B_n\right) = \lim_{n \to \infty} \mu\left(B_n\right). \tag{3}$$

also,

$$\mu\left(\cup_n A_n\right) = \lim_{n \to \infty} \mu\left(A_n\right), \tag{4}$$

for any increasing sequence $A_1 \subset A_2 \subset \ldots$ of measurable sets.

Let us consider the problem of the extension of measures on finite algebras to (σ-additive) measures on σ-algebras. Note first that given any family $\mathcal{A}$ of subsets of a set M there is a minimal σ-algebra containing $\mathcal{A}$. We will denote this σ-algebra by $\mathbb{B}(\mathcal{A})$, the σ-algebra generated by $\mathcal{A}$.

Theorem 2 (Hopf [18])**.** *A finite measure μ on an algebra $\mathcal{F}$ can be extended to σ-additive finite measure on the σ-algebra $\mathbb{B}(\mathcal{F})$ if and only if for any given decreasing sequence $B_1 \supset B_2 \supset \ldots$ of elements of $\mathcal{F}$ the condition $\lim_{n \to \infty} \mu\left(B_n\right) > 0$ implies $\cap_n B_n \neq \emptyset$.*

Theorem 3. *If it exists, the extension of a finite measure on $\mathcal{F}$ to a σ-additive finite measure on $\mathbb{B}(\mathcal{F})$ is unique.*

Among non-finite measures, so-called σ-finite measures are particularly important.

Definition 5. *A measure is said to be σ-finite if the measure space M is a countable union of mutually disjoint measurable sets, each of which with finite measure.*

The Lebesgue measure on $\mathbb{R}^n$ is of course σ-additive and σ-finite.

Definition 6. *Let (M, τ) be a topological space, τ being the family of open sets. The σ-algebra $\mathbb{B}(\tau)$ generated by open sets is called a Borel σ-algebra. The measurable space $\big(M, \mathbb{B}(\tau)\big)$ is said to be Borel (with respect to τ). A measure on $\big(M, \mathbb{B}(\tau)\big)$ is called a Borel measure.*

Except when explicitly said otherwise, $\mathbb{R}^n$ and $\mathbb{C}^n$ are considered to be equipped with the usual topology and corresponding Borel σ-algebra.

Definition 7. *A Borel measure μ is said to be regular if for any Borel set B one has:*

$$\begin{aligned} \mu(B) &= \inf\{\mu(O) \mid O \supset B,\ O \text{ open}\} \\ &= \sup\{\mu(K) \mid K \subset B,\ K \text{ compact and Borel}\}. \end{aligned} \tag{5}$$

Proposition 1. *Any Borel measure on a separable and complete metric space is regular.*

Definition 8. *Let $(M, \mathcal{B}, \mu)$ be a measure space and $\mathcal{N}_\mu := \{B \in \mathcal{B} \mid \mu(B) = 0\}$ the family of zero measure sets. Two sets $B_1, B_2 \in \mathcal{B}$ are said to be equivalent modulo zero measure sets, $B_1 \sim B_2$, if and only if $B_1 \triangle B_2 \in \mathcal{N}_\mu$.*

The family $\mathcal{N}_\mu$ of zero measure sets is an ideal on the ring of measurable sets $\mathcal{B}$ defined by the operations $\triangle$ and $\cap$, and therefore the quotient $\mathcal{B}/\mathcal{N}_\mu$ is also a ring. It is straightforward to check that the measure is well defined on $\mathcal{B}/\mathcal{N}_\mu$. From the strict measure theoretic point of view, the fundamental objects are the elements of $\mathcal{B}/\mathcal{N}_\mu$, and naturally defined transformations between measure spaces $(M, \mathcal{B}, \mu)$ and $(M', \mathcal{B}', \mu')$ are maps between the quotients $\mathcal{B}/\mathcal{N}_\mu$ and $\mathcal{B}'/\mathcal{N}_{\mu'}$.

Definition 9. *Two measure spaces $(M, \mathcal{B}, \mu)$ and $(M', \mathcal{B}', \mu')$ are said to be isomorphic if there exists a bijective transformation between $\mathcal{B}/\mathcal{N}_\mu$ and $\mathcal{B}'/\mathcal{N}_{\mu'}$, mapping μ into μ'.*

In the above sense, zero measure sets are irrelevant. (When the measure is defined in a topological space, a more restricted notion of *support of the measure* is sometimes adopted, namely the smallest closed set with full measure. We do not adhere to that definition of support.)

Definition 10. *Let $(M, \mathcal{B}, \mu)$ be a measure space. A (not necessarily measurable) subset $S \subset M$ is said to be a support of the measure if any measurable subset in its complement has zero measure, i.e., $Y \subset S^c$ and $Y \in \mathcal{B}$ implies $\mu(Y) = 0$.*

Given a measurable space $(M, \mathcal{B})$ and a subset $N \subset M$, let us consider the σ-algebra of measurable subsets of N,

$$\mathcal{B} \cap N := \{B \cap N \mid B \in \mathcal{B}\}. \tag{6}$$

If $(M, \mathcal{B}, \mu)$ is a measure space and $N \subset M$ is measurable, there is a naturally defined measure $\mu_{|N}$ on $(N, \mathcal{B} \cap N)$, by restriction of μ to $\mathcal{B} \cap N$, $\mu_{|N}(B \cap N) := \mu(B \cap N)$, $\forall B \in \mathcal{B}$. The restriction of the measure is also well defined for subsets $S \subset M$ supporting the measure, even if S is not a measurable set. In this case we have $\mu_{|S}(B \cap S) = \mu(B)$. One can in fact show the following [1].

Proposition 2. *If μ is a measure on $(M, \mathcal{B})$ and S is a support of the measure then $\mu_{|S}(B \cap S) := \mu(B)$, $\forall B \in \mathcal{B}$, defines a ($\sigma$-additive) measure on $(S, \mathcal{B} \cap S)$. The measure spaces $(M, \mathcal{B}, \mu)$ and $(S, \mathcal{B} \cap S, \mu_{|S})$ are isomorphic.*

[The measure on S is well defined, since $B_1 \cap S = B_2 \cap S$ implies $(B_1 \triangle B_2) \cap S = \emptyset$, which in turns leads to $\mu(B_1 \triangle B_2) = 0$, given that S supports the measure.]

Definition 11. *A transformation $\varphi : M_1 \to M_2$ between two measurable spaces $(M_1, \mathcal{B}_1)$ and $(M_2, \mathcal{B}_2)$ is said to be measurable if $\varphi^{-1}\mathcal{B}_2 \subset \mathcal{B}_1$, i.e., if $\varphi^{-1}B \in \mathcal{B}_1$, $\forall B \in \mathcal{B}_2$, where $\varphi^{-1}B$ is the preimage of B.*

Given a measurable transformation $\varphi : M_1 \to M_2$ between measurable spaces $(M_1, \mathcal{B}_1)$ and $(M_2, \mathcal{B}_2)$, one gets a map $\tilde{\varphi} : \mathcal{B}_2 \to \mathcal{B}_1$, defined by $\tilde{\varphi}(B) = \varphi^{-1}B$. If μ is a measure on $(M_1, \mathcal{B}_1)$, the composition map $\mu \circ \tilde{\varphi}$ is therefore a measure on $(M_2, \mathcal{B}_2)$, defined by:

$$(\mu \circ \tilde{\varphi})(B) = \mu(\varphi^{-1}B), \ \forall B \in \mathcal{B}_2. \tag{7}$$

This measure is usually called the push-forward of μ with respect to φ. [Given that a measure μ on $(M, \mathcal{B})$ is in fact a function on $\mathcal{B}$, the measure $\mu \circ \tilde{\varphi}$ is actually the pull-back of μ by $\tilde{\varphi}$; we will use however the usual expression "push-forward".] Besides $\mu \circ \tilde{\varphi}$, we will use also the alternative notations $\varphi_* \mu$ and μ_φ to denote the push-forward of a measure.

Measure theory is naturally connected to integration. From this point of view, the (in general complex) measurable functions $f : M \to \mathbb{C}$ are particularly important, in a measure space $(M, \mathcal{B}, \mu)$. More precisely, the relevant objects are equivalence classes of measurable functions.

Definition 12. *Given a measure space $(M, \mathcal{B}, \mu)$, a condition $C(x)$, $x \in M$, is said to be satisfied almost everywhere if the set:*

$$\{x \in M \mid C(x) \text{ is } false\}$$

is contained in a zero measure set.

Definition 13. *Two measurable complex functions f and g on a measure space are said to be equivalent if the condition $f(x) = g(x)$ is satisfied almost everywhere.*

The set of equivalence classes of measurable functions is naturally a linear space. With a finite measure μ, the integral defines a family of norms, by:

$$\|f\|_p := \left(\int |f|^p d\mu \right)^{1/p}, \tag{8}$$

with $p \geq 1$. With the norm $\| \ \|_p$, the linear space of equivalence classes of measurable functions is denoted by $L^p(M, \mu)$. The space L^p is defined analogously for non-finite measures, considering only functions such that the integral over the whole space is finite, $\int |f|^p d\mu < \infty$. Let us recall still that in the particular case $p = 2$ the norm comes from an inner product, $(f, g) = \int f^* g d\mu$, and therefore the space $L^2(M, \mu)$ of (classes of) square integrable (complex) functions on $(M, \mathcal{B}, \mu)$ is an inner product space. In this context, the interest of σ-additive measures is rooted in the crucial fact that the L^p spaces associated with these measures are complete. Except when explicitly stated (namely when the question of σ-additivity is explicitly concerned), we will drop the qualifier "σ-additive" when referring to measures on σ-algebras.

The next result, which follows from the definition of integral, generalizes the usual change of variables.

Proposition 3. *Let $(M, \mathcal{B}, \mu)$ be a measure space, $(M', \mathcal{B}')$ a measurable space and $\varphi : M \to M'$ a measurable transformation. Consider the measure space $(M', \mathcal{B}', \mu_\varphi)$, where μ_φ denotes the push-forward with respect to φ. Then, for any μ_φ-integrable function $f : M' \to \mathbb{C}$, the function $f \circ \varphi : M \to \mathbb{C}$ is integrable with respect to μ and:*

$$\int_M (f \circ \varphi) d\mu = \int_{M'} f d\mu_\varphi. \tag{9}$$

3. Product Measures

Let $\{(M^1, \mathcal{B}^1, \mu^1), \ldots, (M^n, \mathcal{B}^n, \mu^n)\}$ be a finite set of probability spaces. Consider the Cartesian product:

$$M_n := \prod_{k=1}^{n} M^k, \tag{10}$$

the projections:

$$p_n^k : M_n \to M^k \tag{11}$$

and the σ-algebra of subsets of M_n:

$$\mathcal{B}_n := \mathbb{B}\left(\bigcup_{k=1}^{n} (p_n^k)^{-1} \mathcal{B}^k\right). \tag{12}$$

The measurable product space of the spaces $\{(M^1, \mathcal{B}^1), \ldots, (M^n, \mathcal{B}^n)\}$ is the pair $(M_n, \mathcal{B}_n)$. Note that $\mathcal{B}_n$ is the smallest σ-algebra such that all projections p_n^k are measurable.

The σ-algebra $\mathcal{B}_n$ obviously contains the Cartesian products of measurable sets $\omega^k \in \mathcal{B}^k$, $k = 1, \ldots, n$, i.e., $\mathcal{B}_n$ contains all sets of the form:

$$(\omega^1, \ldots, \omega^n) =: \prod_{k=1}^{n} \omega^k, \ \ \omega^k \in \mathcal{B}^k, \ k = 1, \ldots, n. \tag{13}$$

It is a classic result that there exists a unique probability measure μ_n in $(M_n, \mathcal{B}_n)$ such that:

$$\mu_n\left(\prod_{k=1}^{n} \omega^k\right) = \prod_{k=1}^{n} \mu^k(\omega^k), \tag{14}$$

which is called the product measure and is represented by:

$$\mu_n = \prod_{k=1}^{n} \mu^k. \tag{15}$$

Let us consider now the infinite product, not necessarily countable. As we will see immediately, the existence and uniqueness of the product measure continue to take place.

Definition 14. *Let $\{(M^\lambda, \mathcal{B}^\lambda)\}_{\lambda \in \Lambda}$ be a family of measurable spaces labeled by a set Λ and let M_Λ be the Cartesian product of the spaces M^λ, $\lambda \in \Lambda$. For each $\lambda \in \Lambda$ let p_Λ^λ be the projection from M_Λ to M^λ. The measurable product space of the family $\{(M^\lambda, \mathcal{B}^\lambda)\}_{\lambda \in \Lambda}$ is defined as the pair $(M_\Lambda, \mathcal{B}_\Lambda)$, where:*

$$\mathcal{B}_\Lambda := \mathbb{B}\left(\bigcup_{\lambda \in \Lambda} (p_\Lambda^\lambda)^{-1} \mathcal{B}^\lambda\right) \tag{16}$$

is the smallest σ-algebra such that all projections p_Λ^λ are measurable.

Consider now a family $\{(M^\lambda, \mathcal{B}^\lambda, \mu^\lambda)\}_{\lambda\in\Lambda}$ of probability spaces and let $\mathcal{L}$ be the family of finite subsets of Λ. For each $L \in \mathcal{L}$ let us consider the (finite) product probability space $(M_L, \mathcal{B}_L, \mu_L)$ defined as above, i.e.,

$$M_L = \prod_{\lambda\in L} M^\lambda, \tag{17}$$

$$\mathcal{B}_L = \mathbb{B}\left(\bigcup_{\lambda\in L}(p_L^\lambda)^{-1}\mathcal{B}^\lambda\right) \tag{18}$$

(where p_L^λ is the projection from M_L to M^λ) and,

$$\mu_L = \prod_{\lambda\in L} \mu^\lambda. \tag{19}$$

Consider still the natural measurable projections,

$$p_{L,\Lambda} : M_\Lambda \to M_L. \tag{20}$$

The following result can be found in [1].

Theorem 4. *There is a unique (σ-additive) probability measure μ_Λ in $(M_\Lambda, \mathcal{B}_\Lambda)$ such that:*

$$(p_{L,\Lambda})_* \, \mu_\Lambda = \mu_L, \ \forall L \in \mathcal{L}. \tag{21}$$

The measure defined by this theorem is called the product measure.

Example: A simple but important example of a product measure on an infinite dimensional space is the following, which generalizes the notion of product Gaussian measures in $\mathbb{R}^n$. Consider the countable family of measurable spaces $\{(M^k, \mathcal{B}^k)\}_{k\in\mathbb{N}}$, where, for each k, $(M^k, \mathcal{B}^k)$ coincides with $\mathbb{R}$ equipped with the Borel σ-algebra. The measurable product space is the space $\mathbb{R}^{\mathbb{N}}$ of all real sequences:

$$x = (x_k) = (x_1, x_2, \ldots), \tag{22}$$

equipped with the smallest σ-algebra such that all projections $x \mapsto x_k$ are measurable. Let us consider in each of the spaces $\mathbb{R}$ of the family the same Gaussian measure of covariance $\rho \in \mathbb{R}^+$, i.e.,

$$d\mu^k(x_k) = e^{-x_k^2/2\rho}\,\frac{dx_k}{\sqrt{2\pi\rho}}, \ \ \forall k \in \mathbb{N}. \tag{23}$$

According to the Theorem 4, the product measure, here denoted by μ_ρ,

$$d\mu_\rho(x) = \prod_{k=1}^{\infty} e^{-x_k^2/2\rho}\,\frac{dx_k}{\sqrt{2\pi\rho}}, \tag{24}$$

is uniquely determined by its value on the sets of the form $\prod_{k\in\mathbb{N}} \omega^k$, where only for a finite subset of $\mathbb{N}$ the Borel sets (in $\mathbb{R}$) ω^k differ from $\mathbb{R}$.

Obviously, the above example can be generalized for any infinite sequence of probability measures on $\mathbb{R}$, not necessarily identical. The correspondent of the Lebesgue measure, "$\prod_{k=1}^{\infty} dx_k$", however, does not exist, i.e., the infinite product of Lebesgue measures in $\mathbb{R}$ does not define a measure.

Given any product measure, defined by a not necessarily countable family of probability spaces, it is also trivial to determine the measure of sets of the form:

$$Z(\{\omega^\lambda\}) := \prod_{\lambda\in\Lambda} \omega^\lambda, \ \ \omega^\lambda \in \mathcal{B}^\lambda, \tag{25}$$

where only for a countable subset $\Lambda_0 = \{\lambda_i\}_{i\in\mathbb{N}} \subset \Lambda$ the sets ω^λ differ from M^λ. Since it is a typical argument in measure theory, we present it next in some detail. Let us start by showing that the sets (25) are measurable. Consider the finite subsets of Λ_0, $L_n := \{\lambda_1, \ldots, \lambda_n\}$, $n \in \mathbb{N}$, and let Z_n be the sets defined as in (25), but where ω^{λ_k} ($\lambda_k \in \Lambda_0$) is replaced by M^{λ_k} for $k > n$. It is clear that:

$$Z_n = p^{-1}_{L_n,\Lambda} \prod_{k=1}^{n} \omega^{\lambda_k}, \tag{26}$$

and it follows that Z_n is measurable. Since $Z_{n'} \subset Z_n$ for $n' > n$, the sets $\{Z_n\}_{n\in\mathbb{N}}$ form a decreasing sequence of measurable sets. The intersection $\cap_{\mathbb{N}} Z_n$ is therefore measurable, since $\mathcal{B}_\Lambda$ is a σ-algebra. But $Z\big(\{\omega^\lambda\}\big)$ coincides precisely with $\cap_{\mathbb{N}} Z_n$. Invoking the σ-additivity of the measure we then get from theorem 1 and (21):

$$\begin{aligned}
\mu_\Lambda\Big(Z\big(\{\omega^\lambda\}\big)\Big) &= \lim_{n\to\infty}\mu_\Lambda(Z_n) \\
&= \lim_{n\to\infty}\mu_\Lambda\left(p^{-1}_{L_n,\Lambda} \prod_{k=1}^{n} \omega^{\lambda_k}\right) \\
&= \lim_{n\to\infty}\mu_{L_n}\left(\prod_{k=1}^{n} \omega^{\lambda_k}\right) \\
&= \lim_{n\to\infty} \prod_{k=1}^{n} \mu^{\lambda_k}(\omega^{\lambda_k}).
\end{aligned} \tag{27}$$

4. Projective Limits

We present in this section the notion of measurable projective limit space.

Let us start by recalling that a set $\mathcal{L}$ is said to be partially ordered if it is equipped with a partial order relation, i.e., there is a binary relation "$\geq$" such that:

(1) (reflexivity) $L \geq L$, $\forall L \in \mathcal{L}$
(2) (transitivity) $L \geq L'$ and $L' \geq L'' \Rightarrow L \geq L''$
(3) (anti-symmetry) $L \geq L'$ and $L' \geq L \Rightarrow L = L'$.

Recall still that a set $\mathcal{L}$, partially ordered with respect to the partial order relation "$\geq$", is said to be directed if $\forall L', L'' \in \mathcal{L}$ there exists $L \in \mathcal{L}$ such that $L \geq L'$ and $L \geq L''$.

Definition 15. *Let $\mathcal{L}$ be a directed set and $\{M_L\}_{L\in\mathcal{L}}$ a family of sets labeled by $\mathcal{L}$. Suppose that for each pair L', L such that $L' \geq L$ there are surjective maps:*

$$p_{L,L'} : \; M_{L'} \to M_L \tag{28}$$

satisfying:

$$p_{L,L'} \circ p_{L',L''} = p_{L,L''}, \text{ for } L'' \geq L' \geq L. \tag{29}$$

The family $\{M_L, p_{L,L'}\}_{L,L'\in\mathcal{L}}$ of sets M_L and maps $p_{L,L'}$ is called a projective family.

>From now on, the maps $p_{L,L'}$ of the projective family will be called projections. Let us consider the Cartesian product of the sets M_L:

$$M_{\mathcal{L}} := \prod_{L\in\mathcal{L}} M_L, \tag{30}$$

and denote its generic element by $(x_L)_{L\in\mathcal{L}}$, $x_L \in M_L$.

Definition 16. *The projective limit of the family $\{M_L, p_{L,L'}\}_{L,L'\in\mathcal{L}}$ is the subset M_∞ of the Cartesian product $M_{\mathcal{L}}$ defined by:*

$$M_\infty := \Big\{(x_L)_{L\in\mathcal{L}} \in M_{\mathcal{L}} \mid \; L' \geq L \Rightarrow p_{L,L'}\, x_{L'} = x_L\Big\}. \tag{31}$$

The projective limit is therefore formed by consistent families of elements $x_L \in M_L$, in the sense that x_L is defined by $x_{L'}$, for $L' \geq L$.

Definition 17. *A family* $\{(M_L, \mathcal{B}_L), p_{L,L'}\}_{L,L' \in \mathcal{L}}$ *is said to be a measurable projective family if each pair* $(M_L, \mathcal{B}_L)$ *is a measurable space and if* $\{M_L, p_{L,L'}\}$ *is a projective family such that all projections* $p_{L,L'}$ *are measurable.*

Given a measurable projective family, the structure of measurable space in the projective limit M_∞ is defined as follows. Let $\mathcal{B}_\mathcal{L}$ be the product σ-algebra defined in the previous section, i.e., $\mathcal{B}_\mathcal{L}$ is the smallest σ-algebra of subsets of the product space $M_\mathcal{L}$ such that all the projections from $M_\mathcal{L}$ to M_L are measurable (note that, with respect to the product, the spaces M_L play here the role of the spaces M^λ of the previous section). Let us consider the σ-algebra $\mathcal{B}_\infty$ of subsets of M_∞ given by:

$$\mathcal{B}_\infty := \mathcal{B}_\mathcal{L} \cap M_\infty = \{B \cap M_\infty \mid B \in \mathcal{B}_\mathcal{L}\}. \tag{32}$$

The family $\mathcal{B}_\infty$ is closed under countable unions, since:

$$\bigcup_n \Big(B_n \bigcap M_\infty\Big) = \Big(\bigcup_n B_n\Big) \bigcap M_\infty. \tag{33}$$

Let us also show that $\mathcal{B}_\infty$ is closed under the operation of taking the complement, i.e., that $M_\infty \backslash (B \cap M_\infty) \in \mathcal{B}_\infty$, $\forall B \in \mathcal{B}_\mathcal{L}$. Taking M_∞ and $B \cap M_\infty$ as subsets of $M_\mathcal{L}$ we get:

$$\begin{aligned} M_\infty \backslash (B \cap M_\infty) &= M_\infty \cap \big(M_\mathcal{L} \backslash (B \cap M_\infty)\big) \\ &= M_\infty \cap \big((M_\mathcal{L} \backslash B) \cup (M_\mathcal{L} \backslash M_\infty)\big) \\ &= M_\infty \cap \ (M_\mathcal{L} \backslash B), \end{aligned}$$

which proves the statement, since $M_\mathcal{L} \backslash B \in \mathcal{B}_\mathcal{L}$. It follows that $\mathcal{B}_\infty$ as defined above is indeed a σ-algebra.

Definition 18. *The pair* $(M_\infty, \mathcal{B}_\infty)$ *is called the measurable projective limit of the measurable projective family* $\{(M_L, \mathcal{B}_L), p_{L,L'}\}_{L,L' \in \mathcal{L}}$.

Let π_L be the projection from $M_\mathcal{L}$ to M_L and p_L the restriction of π_L to M_∞, i.e.,

$$p_L = \pi_L \circ i_\infty, \tag{34}$$

where i_∞ is the inclusion of M_∞ in $M_\mathcal{L}$. Since the maps π_L and i_∞ are measurable by construction, p_L is measurable $\forall L \in \mathcal{L}$. The consistency conditions that define M_∞ are equivalent to:

$$p_L = p_{L,L'} \circ p_{L'}\,, \ \ \forall L, L' : L' \geq L, \tag{35}$$

which in particular shows that:

$$\mathcal{F}_\infty := \bigcup_{L \in \mathcal{L}} p_L^{-1} \mathcal{B}_L \tag{36}$$

is an algebra. The algebra $\mathcal{F}_\infty$ is formed by all the sets of the type $p_L^{-1} B_L$, $B_L \in \mathcal{B}_L$, $L \in \mathcal{L}$, which are called cylindrical sets. One can further show that:

$$\mathcal{B}_\infty = \mathbb{B}(\mathcal{F}_\infty), \tag{37}$$

and it follows that $\mathcal{B}_\infty$ is the smallest σ-algebra such that all projections $p_L : M_\infty \to M_L$ are measurable [1].

Suppose now that we are given a measure μ on $(M_\infty, \mathcal{B}_\infty)$. The push-forward:

$$\mu_L := (p_L)_* \, \mu \tag{38}$$

of μ by p_L is a measure on $(M_L, \mathcal{B}_L)$. Explicitly:

$$\mu_L(B_L) = \mu(p_L^{-1} B_L), \ \ \forall B_L \in \mathcal{B}_L. \tag{39}$$

From (35) it follows that the family of measures $\{\mu_L\}_{L \in \mathcal{L}}$ satisfy the self-consistency conditions:

$$\mu_L = (p_{L,L'})_* \, \mu_{L'}\,, \ \ \forall L, L' : L' \geq L. \tag{40}$$

The problem of introducing a measure on a projective limit space is the inverse problem, i.e., we look to define a measure on $(M_\infty, \mathcal{B}_\infty)$ starting from a self-consistent family of measures $\{\mu_L\}$.

Note that given a self-consistent family $\{\mu_L\}$ one can always define, by means of (39), an additive measure μ, called cylindrical, in $\mathcal{F}_\infty$. So, the problem consists in the extension of additive measures on $\mathcal{F}_\infty$ to σ-additive measures on $\mathbb{B}(\mathcal{F}_\infty)$. An important case where the cylindrical measure can be extended to a σ-additive measure is that of the product measure of probability measures, discussed in the previous section. In fact, the product space can be seen as the projective limit of the family of finite products. In general, the existence of measure on $(M_\infty, \mathcal{B}_\infty)$ depends on topological conditions on the projective family. Another particularly interesting situation where the extension is ensured is the following [1,8].

Definition 19. *A projective family $\{M_L, p_{L,L'}\}_{L,L' \in \mathcal{L}}$ of compact Hausdorff spaces is said to be a compact Hausdorff family if all the projections $p_{L,L'}$ are continuous.*

One can show that the projective limit of a compact Hausdorff family is a compact Hausdorff space, with respect to the topology induced from the Tychonov topology (in the product space (30)) [19].

Theorem 5. *Let $\{M_L, p_{L,L'}\}_{L,L' \in \mathcal{L}}$ be a compact Hausdorff projective family. Any self-consistent family of regular Borel probability measures $\{\mu_L\}_{L \in \mathcal{L}}$ in the family of spaces $\{M_L\}_{L \in \mathcal{L}}$ defines a regular Borel probability measure on the projective limit M_∞.*

Let us conclude this section with the notion of cylindrical functions and a typical application of σ-additivity, analogous to the result (27) of the previous section. Let us suppose then that we are given a measure μ on $(M_\infty, \mathcal{B}_\infty)$ and let $\{\mu_L\}$ be the corresponding self-consistent family of measures in the spaces M_L. Given an integrable function F on M_{L_0}, one gets by pull-back an integrable function $F \circ p_{L_0}$ on M_∞. Functions of this type are called cylindrical and they are the simplest integrable functions on M_∞. From Proposition 3 we get:

$$\int_{M_\infty} (F \circ p_{L_0}) d\mu = \int_{M_{L_0}} F d\mu_{L_0}. \tag{41}$$

As a typical example of the construction of a non-cylindrical measurable set whose measure is trivially determined, let us consider a countable subset $\mathcal{L}_0$ of $\mathcal{L}$, i.e., $\mathcal{L}_0 = \{L_1, L_2, \ldots\}$, with $L_{n+1} \geq L_n$ and let $\{B_n \in \mathcal{B}_{L_n}\}_{n \in \mathbb{N}}$ be a sequence such that:

$$p^{-1}_{L_n, L_{n+1}} B_n \subset B_{n+1}. \tag{42}$$

It is then clear from (35) that:

$$p^{-1}_{L_n} B_n \subset p^{-1}_{L_{n+1}} B_{n+1}, \tag{43}$$

and $\{p_{L_n}^{-1}B_n\}$ is therefore an increasing sequence of cylindrical sets. The union of the sets $p_{L_n}^{-1}B_n$ is a measurable set which is in general non-cylindrical (it may be cylindrical if all sets $p_{L_n}^{-1}B_n$ coincide after some order). From Theorem 1 one therefore gets:

$$\mu\left(\cup_n p_{L_n}^{-1}B_n\right) = \lim_{n\to\infty} \mu_{L_n}(B_n). \tag{44}$$

Example: The space known as the Bohr compactification of the line admits a projective characterization as follows (see [20,21] for details). For arbitrary $n \in \mathbb{N}$, let us consider sets $\gamma = \{k_1, \ldots, k_n\}$ of real numbers $k_1, \ldots, k_n$, such that the condition:

$$\sum_{i=1}^{n} m_i k_i = 0, \quad m_i \in \mathbb{Z}, \tag{45}$$

can only be satisfied with $m_i = 0, \forall i$. (These are of course sets of linearly independent real numbers, with respect to the field of rationals.) For each set $\gamma = \{k_1, \ldots, k_n\}$, let us consider the subgroup of $\mathbb{R}$ freely generated by γ:

$$G_\gamma := \left\{\sum_{i=1}^{n} m_i k_i,\ m_i \in \mathbb{Z}\right\}. \tag{46}$$

Let now T denote the group of unitaries in the complex plane, and for each γ consider the group $\mathbb{R}_\gamma$ of all group morphisms from G_γ to T,

$$\mathbb{R}_\gamma := \mathrm{Hom}[G_\gamma, T]. \tag{47}$$

It can be checked that this family of spaces $\mathbb{R}_\gamma$ is a (compact Hausdorff) projective family, and that the projective limit of this family is the set of all, not necessarily continuous, group morphisms from $\mathbb{R}$ to T. This coincides of course, with the dual group of the discrete group $\mathbb{R}$, which is one of the known characterizations of the Bohr compactification of the line. Let us denote this space by $\bar{\mathbb{R}} \equiv \mathrm{Hom}[\mathbb{R}, T]$. Being a (commutative) group, $\bar{\mathbb{R}}$ is naturally equipped with the Haar measure. From the above discussion, and in particular from Theorem 5, it follows that the Haar measure is fully determined by the family of measures obtained by push-forward, with respect to the projections:

$$p_\gamma : \bar{\mathbb{R}} \to \mathbb{R}_\gamma, \ \ \bar{x} \mapsto \bar{x}_{|\gamma}, \tag{48}$$

where $\bar{x}_{|\gamma}$ denotes the restriction of $\bar{x}$ to the subgroup G_γ. Because each G_γ is freely generated, it follows that each space $\mathbb{R}_\gamma$ is homeomorphic to a n-torus T^n, where n is the cardinality of the set $\gamma = \{k_1, \ldots, k_n\}$. Furthermore, one can check that the push-forward with respect to the projections (48) produces precisely the Haar measure on the corresponding torus T^n, $\forall\gamma$. Thus, the measure space $\bar{\mathbb{R}}$ with corresponding Haar measure can be seen as the projective limit of a projective family of finite dimensional tori, each of which equipped with the natural Haar measure.

5. Measures on Linear Spaces

The infinite dimensional real linear space where a measure can be defined in the most natural way is the algebraic dual of some linear space. We will start by showing that, given any real linear space E, its algebraic dual E^a is a projective limit.

Let then E be a real linear space and let us denote by $\mathcal{L}$ the set of all finite dimensional linear subspaces $L \subset E$. The set $\mathcal{L}$ is directed when equipped with the partial order relation "$\geq$":

$$L \geq L' \text{ if and only if } L \supset L', \tag{49}$$

Let us consider the family $\{L^a\}_{L\in\mathcal{L}}$ of all spaces dual to subspaces $L \in \mathcal{L}$. For each pair L, L' such that $L' \geq L$ let $p_{L,L'} : L'^a \to L^a$ be the linear transformation such that each element of L'^a is mapped to its restriction to L. The transformations $p_{L,L'}$ are surjective, since any linear functional on L can be extended to a linear functional on $L' \supset L$, and the following conditions are satisfied:

$$p_{L,L''} = p_{L,L'} \circ p_{L',L''} \;\text{ for }\; L'' \geq L' \geq L. \tag{50}$$

It follows that $\{L^a, p_{L,L'}\}_{L,L' \in \mathcal{L}}$ is a projective family of linear spaces. Let E_∞ be the corresponding projective limit. It is clear that E_∞ is a linear subspace of the direct product of all spaces L^a, since the projections $p_{L,L'}$ are linear. Let ϕ be a generic element of E^a and $\phi_{|L}$ its restriction to L. Given that, for $L' \geq L$, $\phi_{|L}$ coincides with the restriction of $\phi_{|L'}$ to the subspace L, one gets a linear injective map:

$$\varpi : E^a \to E_\infty \tag{51}$$

$$\phi \mapsto \left(\phi_{|L}\right)_{L \in \mathcal{L}}. \tag{52}$$

On the other hand, the consistency conditions that define E_∞ ensure that any element of E_∞ defines a linear functional on E. So, the map ϖ is also surjective, and therefore establishes an isomorphism between the linear spaces E^a and E_∞.

Let us consider the measurable projective family $\left\{(L^a, \mathcal{B}_L), p_{L,L'}\right\}$, where $\mathcal{B}_L$ is the Borel σ-algebra in L^a (recall that L^a is finite dimensional $\forall L$). Let $(E_\infty, \mathcal{B}_\infty)$ be the measurable projective limit of this family and define:

$$\mathcal{B}_{E^a} := \varpi^{-1}\mathcal{B}_\infty. \tag{53}$$

The measurable spaces $(E_\infty, \mathcal{B}_\infty)$ and $(E^a, \mathcal{B}_{E^a})$ are therefore isomorphic, and we will make no distinction between them. The σ-algebra $\mathcal{B}_{E^a}$ is the smallest σ-algebra such that all the real functions:

$$E^a \ni \phi \xmapsto{\xi} \phi(\xi),\; \xi \in E \tag{54}$$

are measurable, i.e.,

$$\mathcal{B}_{E^a} = \mathbb{B}\Bigg(\bigcup_{\xi \in E} \xi^{-1}\mathcal{B}(\mathbb{R})\Bigg), \tag{55}$$

where $\xi^{-1}\mathcal{B}(\mathbb{R})$ denotes the family of inverse images of Borel sets of $\mathbb{R}$ by the map (54).

The fundamental result concerning the existence of measures on infinite dimensional real linear spaces is the following [1].

Theorem 6. *Any self-consistent family of finite Borel measures μ_L on the subspaces $L^a \subset E^a$ defines a (σ-additive) finite measure on $(E^a, \mathcal{B}_{E^a})$.*

The above result can be presented in a different way, invoking Bochner's classical theorem.

Definition 20. *Let E be a real linear space and μ a finite measure on $(E^a, \mathcal{B}_{E^a})$ (if E is finite dimensional, then $E \cong E^a \cong \mathbb{R}^n$, $\mathcal{B}_{E^a}$ is the Borel σ-algebra in $\mathbb{R}^n$ and μ is a Borel measure). The Fourier transform, or characteristic function, of the measure is the (in general complex) function on E given by:*

$$E \ni \xi \longmapsto \int_{E^a} e^{i\phi(\xi)} d\mu(\phi). \tag{56}$$

Definition 21. *A complex function χ on a real linear space E is said to be of the positive type if $\sum_{k,l=1}^m c_k \bar{c}_l \chi(\xi_k - \xi_l) \geq 0$, $\forall m \in \mathbb{N}$, $c_1, \ldots, c_m \in \mathbb{C}$ and $\xi_1, \ldots, \xi_m \in E$.*

Theorem 7 (Bochner)**.** *A complex function χ on $\mathbb{R}^n$ is the Fourier transform of a finite Borel measure on $\mathbb{R}^n$ if and only if it is continuous and of positive type. The measure is normalized if and only if $\chi(0) = 1$.*

Bochner's theorem is generalizable to the infinite dimensional situation as follows.

Theorem 8. *Let χ be a complex function on an infinite dimensional real linear space E. The function χ is the Fourier transform of a finite measure on $(E^a, \mathcal{B}_{E^a})$ if and only if it is of the positive type and continuous on every finite dimensional subspace. The measure is normalized if and only if $\chi(0) = 1$.*

This result can be proved using Theorem 6 and Bochner's theorem. We present next the essential arguments. The fact that the Fourier transform of a measure μ on $(E^a, \mathcal{B}_{E^a})$ is necessarily of the positive type on E is a consequence of:

$$\int_{E^a} \Big|\sum_k^m c_k e^{i\phi(\xi_k)}\Big|^2 d\mu(\phi) \geq 0. \tag{57}$$

From (9) and (38) one can see that the restriction of the Fourier transform of μ to a finite dimensional subspace $L \subset E$ coincides with:

$$\int_{L^a} e^{i\phi_L(\xi)} d\mu_L(\phi_L),\ \xi \in L, \tag{58}$$

and it is therefore the Fourier transform of μ_L, hence continuous. Conversely, a function χ of the positive type on E defines, by restriction, a family $\{\chi_L\}$ of positive type functions on the subspaces L:

$$\chi_L := \chi_{|L},\ \forall L. \tag{59}$$

If χ is continuous in each L one then have well-defined measures on L^a, whose self-consistency is ensured by (59).

To conclude this section, note that for the existence of a measure on E^a, Theorem 8 requires only continuity of the characteristic function on the finite dimensional subspaces. Analogously to the situation in finite dimensions, one can expect that a smoother Fourier transform will produce a measure supported in proper subspaces of E^a. The support of the measure is indeed related to continuity properties of the Fourier transform [1–3,22,23]. As an extreme example of this relation, consider the weakest possible topology in E, having only the empty set and E itself as open sets. The only continuous functions in this topology are the constant functions, and it should be clear that a measure with constant Fourier transform is a Dirac-like measure, supported on the null element of E^a. In the next section we will discuss two important cases where the characteristic functions are continuous with respect to a weaker topology than the one defined by continuity in finite dimensional subspaces. In these cases, the measure is supported in a proper (infinite dimensional) subspace of E^a, which is equivalent to give a measure on that subspace, by Proposition 2 of Section 2.

6. Minlos' Theorem

In this section we consider the relation between continuity of the characteristic function and the support of the corresponding measure, for two situations of interest.

In the first case the characteristic function is continuous in a nuclear topology. In the second case the characteristic function is continuous with respect to fixed inner product.

Let us start by recalling that any family of norms $\{\|\ \|_\alpha\}_{\alpha\in\Gamma}$ in a linear space E defines a locally convex topology (see, e.g., [24], where the more general case of semi-norms is also considered). In fact, one can take as basis of the topology the finite intersections of sets of the form:

$$V(\alpha, n) = \big\{\xi \in E \mid\ \|\xi\|_\alpha < 1/n\big\},\ \alpha \in \Gamma,\ n \in \mathbb{N}. \tag{60}$$

Also, any family of norms in the same space E is partially ordered by the natural order relation $\|\ \|_{\alpha'} \geq \|\ \|_\alpha$ if and only if $\|\xi\|_{\alpha'} \geq \|\xi\|_\alpha,\ \forall \xi \in E$. For typical applications, it is sufficient to consider the case where the topology is defined by a countable and ordered family of norms, i.e., we consider sequences of norms $\{\|\ \|_k\}_{k\in\mathbb{N}}$ such that $\|\ \|_k \geq \|\ \|_l$, for $k > l$. (In this case the corresponding topology is actually metrizable, see, e.g., [14].)

For the current application, we restrict attention further to the situation where the ordered sequence of norms $\{\| \ \|_k\}_{k\in\mathbb{N}}$ is associated with a sequence of inner products $\{\langle\ ,\ \rangle_k\}_{k\in\mathbb{N}}$, $\|\cdot\|_k = \sqrt{\langle\cdot,\cdot\rangle_k}$, $\forall k \in \mathbb{N}$. With this set-up, let $\mathcal{H}_k$ be the completion of E with respect to the inner product $\langle\ ,\ \rangle_k$. For $k > l$ we have $\mathcal{H}_k \subset \mathcal{H}_l$, since the topology defined by $\| \ \|_k$ is stronger. One can show that the topological linear space E defined in this way is complete if and only if $E = \bigcap_{k=1}^{\infty} \mathcal{H}_k$ [14].

Definition 22. *An operator H on a separable Hilbert space $(\mathcal{H}, (\ ,\ ))$ is said to be a Hilbert-Schmidt operator if given an (in fact any) orthonormal basis $\{e_k\}$ we have:*

$$\sum_{k=1}^{\infty}(He_k, He_k) \ < \ \infty. \tag{61}$$

We next define the notion of nuclear space, following [1]. (Note however that [1] considers a more general notion, admitting non-countable families of semi-norms, associated with degenerate inner products.)

Definition 23. *Let $E = \bigcap_{k=1}^{\infty} \mathcal{H}_k$ be a complete linear space, with respect to the topology defined by an ordered sequence of norms $\{\| \ \|_k\}_{k\in\mathbb{N}}$ associated with a sequence of inner products $\{\langle\ ,\ \rangle_k\}_{k\in\mathbb{N}}$. The space E is said to be nuclear if $\forall l$ there is $k > l$ and an Hilbert-Schmidt operator H on $\mathcal{H}_k$ such that $\langle\xi, \eta\rangle_l = \langle H\xi, H\eta\rangle_k$, $\forall \xi, \eta \in \mathcal{H}_k$.*

The most common examples of nuclear spaces are the following.

Example 1: Consider the space $\mathcal{S}$ of rapidly decreasing real sequences $y = (y_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} n^k y_n = 0$, $\forall k \in \mathbb{N}$, with inner products:

$$\langle y, z\rangle_k = \sum_{n=1}^{\infty} n^{2k} y_n z_n, \ k \in \mathbb{N}. \tag{62}$$

For any k, the operator H on $\mathcal{H}_k$ (the completion of $\mathcal{S}$ by means of $\langle\ ,\ \rangle_k$) defined by $(Hy)_n = y_n/n$ is obviously Hilbert-Schmidt. On the other hand, it is clear that $\langle\xi, \eta\rangle_k = \langle H\xi, H\eta\rangle_{k+1}$, and it follows that $\mathcal{S}$ is nuclear.

Example 2: The real Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of C^∞-functions f on $\mathbb{R}^d$ such that:

$$\sup_{x\in\mathbb{R}^d}\left| x_1^{k_1}\ldots x_d^{k_d}\frac{\partial^{j_1}}{\partial x_1^{j_1}}\cdots\frac{\partial^{j_d}}{\partial x_d^{j_d}} f(x)\right| < \infty,\ \forall k_1,\ldots,k_d, j_1,\ldots,j_d \in \mathbb{N} \tag{63}$$

is a nuclear space for an appropriate sequence of inner products, whose topology coincides with the topology defined by the system of norms (63) [1,22] (see also [25] for more information on the Schwartz space).

We present next the classical Bochner-Minlos theorem (whose proof can be found, e.g., in [1]), which partially justifies the relevance of nuclear spaces in measure theory. According to this result, a characteristic function which is continuous in a nuclear space E is equivalent to a measure on the topological dual of E. Note that a linear functional ϕ on a space of the type $E = \cap_{k=1}^{\infty}\mathcal{H}_k$ is continuous if and only if it is continuous with respect to (any) one of the inner products $\langle, \rangle_k$ [14]. Equivalently, ϕ belongs to the topological dual E' if and only if $\exists k$ such that $\phi \in \mathcal{H}_{-k}$, where $\mathcal{H}_{-k}$ denotes the (Hilbert space) dual of $\mathcal{H}_k$. So, the topological dual of a space $E = \cap_{k=1}^{\infty}\mathcal{H}_k$ is a union of Hilbert spaces, $E' = \cup_{k=1}^{\infty}\mathcal{H}_{-k}$, where $\mathcal{H}_{-l} \subset \mathcal{H}_{-k}$, for $k > l$. In the case of the space $\mathcal{S}$ of example 1, the dual $\mathcal{S}'$ can be seen as the linear space of real sequences $x = (x_n)$ for which there exists $k \in \mathbb{N}$ such that:

$$\sum_{n=1}^{\infty} n^{-2k} x_n^2 < \infty. \tag{64}$$

In the case of the space $\mathcal{S}(\mathbb{R}^d)$ of example 2, the dual is the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions, which includes "Dirac delta functions" and derivatives thereof (see, e.g., [22]).

Theorem 9 (Bochner-Minlos)**.** *Let E be a real nuclear space and μ a measure on $(E^a, \mathcal{B}_{E^a})$. If the characteristic function of the measure is continuous in the nuclear topology, then the measure is supported on the topological dual $E' \subset E^a$. So, a function of the positive type and continuous on a nuclear space E defines a measure on $(E', \mathcal{B}_{E'})$, where $\mathcal{B}_{E'} := \mathcal{B}_{E^a} \cap E'$ is the smallest σ-algebra such that all functions on E' of the type $\phi \mapsto \phi(\xi)$, $\xi \in E$, are measurable.*

Measure theory in $\mathcal{S}'(\mathbb{R}^d)$ plays a distinguished role in applications. The following result establishes the relation between the σ-algebra $\mathcal{B}_{\mathcal{S}'(\mathbb{R}^d)}$ and the strong topology in $\mathcal{S}'(\mathbb{R}^d)$ [26] (see also [23]). Recall that the strong topology in $\mathcal{S}'(\mathbb{R}^d)$ is generated by the family of semi-norms $\{\rho_A \mid A \subset \mathcal{S}(\mathbb{R}^d) \text{ and bounded}\}$, with $\rho_A(\phi) = \sup_{\xi \in A} |\phi(\xi)|$, $\phi \in \mathcal{S}'(\mathbb{R}^d)$.

Lemma 1. *The σ-algebra $\mathcal{B}_{\mathcal{S}'(\mathbb{R}^d)}$ generated by the functions $\phi \mapsto \phi(\xi)$, $\phi \in \mathcal{S}'(\mathbb{R}^d)$, $\xi \in \mathcal{S}(\mathbb{R}^d)$, coincides with the Borel σ-algebra associated with the strong topology in $\mathcal{S}'(\mathbb{R}^d)$.*

Corollary 1. *A continuous function of the positive type on $\mathcal{S}(\mathbb{R}^d)$ is equivalent to a Borel measure on $\mathcal{S}'(\mathbb{R}^d)$.*

In particular situations, namely for Gaussian measures, the characteristic function is continuous in a topology defined by a single inner product. In that case Minlos' theorem applies. Minlos' theorem is presented in the literature in several different ways (see, e.g., [1,22,23,27]), being most commonly formulated for the case of nuclear spaces. We start by presenting a more general version, following [1], considering next the nuclear space case.

Theorem 10 (Minlos)**.** *Let E be a real linear space, $E_1 \subset E$ a subspace, $(\,,\,)$ a inner product in E and $(\,,\,)_1$ a inner product in E_1 such that the corresponding topology in E_1 is stronger than the one induced from $\big(E, (\,,\,)\big)$. Let $\overline{(E_1, (\,,\,)_1)}$ be the completion of E_1 with respect to $(\,,\,)_1$. Let H be a Hilbert-Schmidt operator on $\overline{(E_1, (\,,\,)_1)}$ such that:*

$$(\xi, \eta) = (H\xi, H\eta)_1, \ \ \forall \xi, \eta \in \overline{(E_1, (\,,\,)_1)}. \tag{65}$$

Then, a characteristic function χ on E continuous with respect to $(\,,\,)$, defines a measure supported on the subspace of E^a of those functionals whose restriction to E_1 is continuous with respect to $(\,,\,)_1$.

In the case of a nuclear space E, let us suppose that the characteristic function χ is continuous with respect to one of the inner products $\langle\,,\,\rangle_{k_0}$ of the family $\{\langle\,,\,\rangle_k\}_{k\in\mathbb{N}}$ that defines the topology of E. By the very definition of nuclear space, there exist $k_1 > k_0$ and a Hilbert-Schmidt operator H such that $\langle\cdot,\cdot\rangle_{k_0} = \langle H\cdot, H\cdot\rangle_{k_1}$. The measure is therefore supported in $\mathcal{H}_{-k_1}$, the dual of the completion $\mathcal{H}_{k_1}$ of E with respect to $\langle\,,\,\rangle_{k_1}$. More generally we have the following

Corollary 2. *Let E be a real nuclear space and $(\,,\,)$, $(\,,\,)_1$ two inner products in E such that the corresponding topologies are weaker than the nuclear topology. Assume that the $(\,,\,)$-topology is weaker than the $(\,,\,)_1$-topology. Let $\overline{(E, (\,,\,)_1)}$ be the completion of E with respect to $(\,,\,)_1$ and H a Hilbert-Schmidt operator on $\overline{(E, (\,,\,)_1)}$ such that:*

$$(\xi, \eta) = (H\xi, H\eta)_1, \ \ \forall \xi, \eta \in \overline{(E, (\,,\,)_1)}. \tag{66}$$

Then, a characteristic function χ on E continuous with respect to $(\,,\,)$, defines a measure supported on the subspace of E' of the functionals which are continuous with respect to $(\,,\,)_1$.

7. Gaussian Measures

In this section we consider Gaussian measures on infinite dimensional real linear spaces, following [1–3]. In this approach, and following the lines of Section 5, we start with a characteristic function—determined in this case by an inner product—in a linear space E, thus defining the measure initially on the algebraic dual E^a. As already mentioned in the Introduction, in other approaches [7,11,12], Gaussian measures are defined directly on topological vector spaces. The two perspectives are nevertheless equivalent: the algebraic dual E^a is simply the "universal home" for Gaussian measures associated with inner products defined in E. The space where the measure is actually supported is at the end determined by the inner product itself, regardless of what space one initially considers the measure to be defined in.

As in finite dimensions, Gaussian measures are associated with inner products, defining the measure's covariance. (Note that positive semi-definite bilinear forms also give rise to measures, with the peculiarity that the measure degenerates into a Dirac measure along the null directions. We shall not consider that generalization.)

The fact that the Fourier transform of a Gaussian function (centered at zero) is also Gaussian allows one to define Gaussian measures on $\mathbb{R}^n$ as follows.

Definition 24. *Let $C = (C_{ij})$ be a $n \times n$ positive definite symmetric matrix. The Gaussian measure μ_C on $\mathbb{R}^n$ of covariance C is the Borel measure whose Fourier transform is:*

$$\chi_C(y_1, \ldots, y_n) = \exp\left(-\tfrac{1}{2}\textstyle\sum_{i,j} C_{ij} y_i y_j\right). \tag{67}$$

Using the Lebesgue measure $d^n x$, the Gaussian measure of covariance C is given by:

$$d\mu_C(x_1, \ldots, x_n) = (2\pi)^{-n/2} (\det C)^{-1/2} \exp\left(-\tfrac{1}{2}\textstyle\sum_{i,j} C^{-1}_{ij} x_i x_j\right) d^n x. \tag{68}$$

A positive definite symmetric matrix is equivalent to an inner product, and therefore Gaussian measures on $\mathbb{R}^n$ are determined by inner products. One can define Gaussian measures on infinite dimensional spaces in exactly the same way.

Definition 25. *Let E be an infinite dimensional real linear space and $(\,,\,)$ an inner product in E. The measure on $(E^a, \mathcal{B}_{E^a})$ with Fourier transform $\chi(\xi) = e^{-(\xi,\xi)/2}$, $\xi \in E$, is called a Gaussian measure, of covariance $(\,,\,)$.*

The existence and uniqueness of the measure are ensured by Theorem 8 of Section 5. The following characterization of Gaussian measures, sometimes taken as definition, is crucial.

Theorem 11. *A measure μ on $(E^a, \mathcal{B}_{E^a})$ is Gaussian if and only if the push-forward μ_ξ of μ by the map:*

$$E^a \ni \phi \longmapsto \phi(\xi) \in \mathbb{R} \tag{69}$$

is a Gaussian measure on $\mathbb{R}$, $\forall \xi \in E$.

The theorem is easily proved. Note first that for any Gaussian measure μ on E^a, the push-forward μ_ξ is a Gaussian measure on $\mathbb{R}$ of covariance (ξ, ξ). Conversely, let μ be a measure on E^a such that μ_ξ is a Gaussian measure on $\mathbb{R}$, $\forall \xi$. Let c_ξ be the covariance of μ_ξ. The Fourier transform χ of the measure μ is then:

$$\chi(\xi) = e^{-c_\xi/2},\ \forall \xi, \tag{70}$$

where:

$$c_\xi = \int_{\mathbb{R}} x^2 d\mu_\xi(x) = \int_{E^a} \big(\phi(\xi)\big)^2 d\mu(\phi). \tag{71}$$

On the other hand, it is clear that:

$$(\xi_1, \xi_2) := \int_{E^a} \phi(\xi_1)\phi(\xi_2) d\mu(\phi)\,, \;\; \xi_1, \xi_2 \in E \tag{72}$$

defines an inner product, thus proving that $\chi(\xi)$ is of the required form $\chi(\xi) = \exp\bigl(-(\xi,\xi)/2\bigr)$.

Expression (72) for the moments of the Gaussian measure of covariance $(\,,)$ is easily generalized. The result is the well-known Wick's theorem (see [3]). If $\xi_1, \ldots, \xi_{2n+1}$ is an odd set of elements of E then:

$$\int_{E^a} \phi(\xi_1) \ldots \phi(\xi_{2n+1}) d\mu = 0. \tag{73}$$

If on the other hand $\xi_1, \ldots, \xi_{2n}$ is an even set of elements of E then:

$$\int_{E^a} \phi(\xi_1) \ldots \phi(\xi_{2n}) d\mu = \sum_{\text{pairs}} (\xi_{i_1}, \xi_{j_1}) \ldots (\xi_{i_n}, \xi_{j_n}), \tag{74}$$

where $\sum_{\text{pairs}}$ stands for the sum over all possible ways of pairing the $2n$ labels $1, \ldots, 2n$ into n pairs.

Let us note the following. Independently of the linear space E where the covariance $(\,,)$ is originally defined, a characteristic function of the type $\chi(\xi) = e^{-(\xi,\xi)/2}$ is always obviously extendable to the Hilbert space completion $\mathcal{H}$ of E. So, the inner product $(\,,)$, taken as a covariance in the Hilbert space $\mathcal{H}$, defines a Gaussian measure on $\bigl(\mathcal{H}^a, \mathcal{B}_{\mathcal{H}^a}\bigr)$, where $\mathcal{H}^a$ is the algebraic dual of $\mathcal{H}$ and:

$$\mathcal{B}_{\mathcal{H}^a} = \mathbb{B}\Biggl(\bigcup_{\xi \in \mathcal{H}} \xi^{-1} \mathcal{B}(\mathbb{R})\Biggr). \tag{75}$$

One can show that the natural map from $\mathcal{H}^a$ to E^a (defined by the restriction to E of the elements of $\mathcal{H}^a$) is an isomorphism of measure spaces. (From Proposition 3, the push-forward of the measure on $\mathcal{H}^a$ is the Gaussian measure on E^a of covariance $(\,,)$, and it follows that $\mathcal{H}^a \subset E^a$ is a support of the Gaussian measure on E^a. To be precise, this map is not strictly measurable, but it establishes an isomorphism between the families of measurable sets modulo zero measure sets, which maps the measure on $\mathcal{H}^a$ to the measure on E^a.) Thus, whenever necessary, one can always assume that the covariance of a Gaussian measure is defined in a Hilbert space.

Example 1: As in the example of Section 3, let us consider the space $\mathbb{R}^{\mathbb{N}}$ of real sequences and the measures μ_ρ, given by the product of an infinite sequence of identical Gaussian measures on $\mathbb{R}$, each of covariance ρ. Let $\mathbb{R}^{\mathbb{N}}_{\mathrm{c}} \subset \mathbb{R}^{\mathbb{N}}$ be the linear space of those sequences that are zero after some order, i.e.,

$$\mathbb{R}^{\mathbb{N}}_{\mathrm{c}} := \{(x_n) \mid \exists N_x \in \mathbb{N} \;\text{ such that }\; x_n = 0 \;\text{ for }\; n > N_x\}. \tag{76}$$

The space $\mathbb{R}^{\mathbb{N}}$ is naturally seen as the algebraic dual of $\mathbb{R}^{\mathbb{N}}_{\mathrm{c}}$, with the action:

$$x(y) = \sum_n x_n y_n, \;\; x \in \mathbb{R}^{\mathbb{N}}, \; y \in \mathbb{R}^{\mathbb{N}}_{\mathrm{c}}, \tag{77}$$

and it is clear that the product σ-algebra in $\mathbb{R}^{\mathbb{N}}$ coincides with σ-algebra associated with the interpretation of $\mathbb{R}^{\mathbb{N}}$ as a projective limit. The Fourier transform of the measure μ_ρ is easily seen to be:

$$\chi_\rho(y) := \int_{\mathbb{R}^{\mathbb{N}}} e^{ix(y)} d\mu_\rho(x) = e^{-\frac{1}{2}\rho \sum_n y_n^2}, \;\; \forall y \in \mathbb{R}^{\mathbb{N}}_{\mathrm{c}}. \tag{78}$$

So, the product measure μ_ρ coincides with the Gaussian measure associated with the inner product:

$$\langle y', y \rangle_\rho := \rho \sum_{n=1}^{\infty} y'_n y_n, \tag{79}$$

which we assume to be defined on the real Hilbert space ℓ^2 of square summable sequences. Consider now the space $\mathcal{S}$ of rapidly decreasing sequences (Example 1, Section 6). Like $\mathbb{R}^{\mathbb{N}}_c$, $\mathcal{S}$ is dense in ℓ^2 with respect to the topology defined by $\langle\,,\rangle_\rho$ (which is in fact the natural ℓ^2 topology). Moreover, the restriction of χ_ρ to $\mathcal{S}$ is continuous in the nuclear topology, since the latter is stronger than the topology induced in $\mathcal{S}$ from the ℓ^2-norm. It then follows from the Bochner-Minlos theorem that the measure μ_ρ is supported on the topological dual $\mathcal{S}'$ of the nuclear space $\mathcal{S}$, for any value of ρ. Furthermore, Minlos' theorem allows us to find proper subspaces of $\mathcal{S}'$ that still support the measure. Let us now describe this application of Theorem 10. Let then $a = (a_n)$ be an element of ℓ^2 such that $1 \geq a_n > 0$, $\forall n$ and let $(\,,)_a$ be the inner product in $\mathbb{R}^{\mathbb{N}}_c$ given by:

$$(y', y)_a = \sum_n \frac{y'_n y_n}{a_n^2}. \tag{80}$$

It is clear that the $(\,,)_a$-topology is stronger than the ℓ^2 topology in $\mathbb{R}^{\mathbb{N}}_c$. Let H_a be the operator on $\mathbb{R}^{\mathbb{N}}_c$ defined by:

$$(H_a y)_n := a_n y_n. \tag{81}$$

The operator H_a is clearly Hilbert-Schmidt with respect to $(\,,)_a$, and we have $\langle y', y\rangle_1 = (H_a y', H_a y)_a$. Then, using the usual characterization of continuous functionals on a Hilbert space, it follows from Theorem 10 that the measure μ_ρ is supported on the subspace of $\mathbb{R}^{\mathbb{N}}$ of sequences x such that:

$$\sum_{n=1}^{\infty} a_n^2 x_n^2 < \infty. \tag{82}$$

The subspace defined by (82) is $H_a^{-1}\ell^2$, i.e., the space of sequences $x = (x_n)$ of the form $x_n = z_n/a_n$, with $z = (z_n) \in \ell^2$. [Since $a \in \ell^2$, one could be tempted to conclude that the measure is supported on the space ℓ^∞ of bounded sequences, but that is not the case. It is true that the intersection $\bigcap_{a\in\ell^2} H_a^{-1}\ell^2$ of all the spaces $H_a^{-1}\ell^2$ coincides with ℓ^∞, but in fact the space ℓ^∞ is contained in a zero measure set. There is no contradiction with σ-additivity, since the intersection is not countable.]

Let us remark that given any Gaussian measure μ of covariance $(\,,)$ in a (real, infinite dimensional and separable) Hilbert space $\mathcal{H}$, it is always possible to construct an isomorphism (of measure spaces) mapping the given measure to the Gaussian measure on $\mathbb{R}^{\mathbb{N}}$ of the example above, with $\rho = 1$ [1,3]. This can be understood as follows. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis in $\mathcal{H}$ and consider the map $\theta : \mathcal{H}^a \to \mathbb{R}^{\mathbb{N}}$ defined by:

$$\theta(\phi) = \big(\phi(\xi_n)\big). \tag{83}$$

Let $\theta_*\,\mu$ be the measure on $\mathbb{R}^{\mathbb{N}}$ obtained by push-forward of μ. We then have (see Proposition 3):

$$\int_{\mathbb{R}^{\mathbb{N}}} e^{i\sum_n y_n x_n} d(\theta_*\,\mu) = \int_{\mathcal{H}^a} e^{i\phi\left(\sum_n y_n \xi_n\right)} d\mu, \quad \forall (y_n) \in \mathbb{R}^{\mathbb{N}}_c. \tag{84}$$

Given that:

$$\int_{\mathcal{H}^a} e^{i\phi\left(\sum_n y_n \xi_n\right)} d\mu = e^{-\frac{1}{2}\sum_n y_n^2}, \tag{85}$$

it follows that $\theta_*\,\mu$ coincides with the Gaussian measure of the above example, with $\rho = 1$.

Example 2: An important family of Gaussian measures on $\mathcal{S}'(\mathbb{R}^d)$ is defined by the following family of inner products:

$$\langle f, g\rangle_m = \int f(m^2 - \Delta)^{-1} g\, d^d x, \quad f,\ g \in S(\mathbb{R}^d), \tag{86}$$

where $m \in \mathbb{R}^+$ and Δ is the Laplacian operator. These measures are relevant e.g., in quantum theory and in certain stochastic processes.

We conclude this section with a variant of Minlos' theorem tailored for Gaussian measures, following immediately from Corollary 2, Section 6.

Corollary 3. *Let E be a real nuclear space and $(\,,\,)$, $(\,,\,)_1$ two inner products in E such that the corresponding topologies are weaker than the nuclear topology. Assume that the $(\,,\,)$-topology is weaker than the $(\,,\,)_1$-topology. Let $\overline{(E,(\,,\,)_1)}$ be the completion of E with respect to $(\,,\,)_1$ and H a Hilbert-Schmidt operator on $\overline{(E,(\,,\,)_1)}$ such that:*

$$(\xi,\eta) = (H\xi, H\eta)_1, \;\; \forall \xi,\eta \in \overline{(E,(\,,\,)_1)}. \tag{87}$$

Then the Gaussian measure of covariance $(\,,\,)$ is supported on the subspace of E' of the functionals which are continuous with respect to $(\,,\,)_1$.

Another version of this result, closer to the quantum field theory literature [23,27], is the following.

Corollary 4. *Let E be a real nuclear space, $(\,,\,)$ a continuous inner product in E and $\mathcal{H}$ the completion of E with respect to $(\,,\,)$. Let be H be an injective Hilbert-Schmidt operator on $\mathcal{H}$ such that $E \subset H\mathcal{H}$ and $H^{-1} : E \to \mathcal{H}$ is continuous. Denote by $(\,,\,)_1$ the inner product in E defined by $(f,g)_1 = (H^{-1}f, H^{-1}g)$. Then the Gaussian measure of covariance $(\,,\,)$ is supported on the subspace of E' of the functionals which are continuous with respect to $(\,,\,)_1$.*

These two versions are related as follows. Let H be a Hilbert-Schmidt operator on $\mathcal{H}$, on the conditions of Corollary 4. The image $H\mathcal{H}$, equipped with the inner product $(\,,\,)_1$, coincides with the $(\,,\,)_1$-completion of E. Since $H : \mathcal{H} \to H\mathcal{H}$ is unitary and $H : H\mathcal{H} \to H\mathcal{H}$ is well defined, it follows that H is Hilbert-Schmidt on $H\mathcal{H}$. On the other hand we have that:

$$(f,g) = (Hf, Hg)_1, \tag{88}$$

which shows that the $(\,,\,)$-topology is weaker than the $(\,,\,)_1$-topology. Finally, since $(\,,\,)$ is continuous, the continuity of $H^{-1} : E \to \mathcal{H}$ implies that $(\,,\,)_1$ is also continuous, and therefore both topologies defined by $(\,,\,)$ and $(\,,\,)_1$ are weaker than the nuclear topology. All conditions of Corollary 3 are thus satisfied.

8. Quasi-invariance and Ergodicity

We present in this section some concepts relevant to the study of transformation properties of measures. The notions of quasi-invariance and ergodicity are presented, together with two important results concerning Gaussian measures. We start by reviewing general notions [13], illustrated with straightforward examples.

Definition 26. *Let μ_1 and μ_2 be two measures on the same measurable space $(M,\mathcal{B})$. The measure μ_1 is said to be absolutely continuous with respect to μ_2, and we write $\mu_1 < \mu_2$, if $\mu_2(B) = 0 \Rightarrow \mu_1(B) = 0$.*

As an example, consider the measures μ_0 and ν_0 on $\mathbb{R}$, where μ_0 is the Lebesgue measure and ν_0 is the measure supported on the interval $I = [0,1]$ defined by $\nu_0(B) = \mu_0(B \cap I)$, for any Borel set $B \subset \mathbb{R}$. It is clear that $\nu_0 < \mu_0$, whereas it is not true that $\mu_0 < \nu_0$. On the other hand we have for instance the measure μ_{Cantor} defined by the Cantor function, which is supported on the Cantor set (see, e.g., [13]). The Cantor set has Lebesgue measure zero, and therefore the measures μ_0 and μ_{Cantor} are supported on disjoint sets.

Definition 27. *Two measures μ_1 and μ_2 on the same measurable space $(M,\mathcal{B})$ are said to be mutually singular, and we write $\mu_1 \perp \mu_2$, if there exists a measurable set $B \in \mathcal{B}$ such that $\mu_1(B) = 0$ and $\mu_2(B^c) = 0$, where B^c is the complement of B.*

Theorem 12 (Radon-Nikodym)**.** *Let $(M, \mathcal{B})$ be a measurable space and μ_1 and μ_2 two σ-finite measures. The measure μ_1 is absolutely continuous with respect to μ_2 if and only if there is a real non-negative measurable function $f =: d\mu_1/d\mu_2$ on M such that $d\mu_1 = f d\mu_2$, i.e., $\mu_1(B) = \int_B f\, d\mu_2$, $\forall B \in \mathcal{B}$.*

The function $d\mu_1/d\mu_2$ in the previous theorem is said to be the Radon-Nikodym derivative.

Definition 28. *Two measures μ_1 and μ_2 on the same measurable space are said to be mutually absolutely continuous, or equivalent, and we write $\mu_1 \sim \mu_2$, if $\mu_1 < \mu_2$ and $\mu_2 < \mu_1$, i.e., if $\mu_2(B) = 0$ if and only if $\mu_1(B) = 0$.*

The measures μ_0 and ν_0 above are not equivalent. The Gaussian measure $e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$, for instance, is equivalent to the Lebesgue measure.

The next result establishes sufficient and necessary conditions for the equivalence of two Gaussian measures (centered at the null element) [1,3].

Theorem 13. *Let E be a real infinite dimensional linear space, $(\ ,\ )$ and $(\ ,\ )_1$ two inner products and μ, μ_1 the corresponding Gaussian measures. The measures are equivalent if and only if the inner product $(\ ,\ )_1$ can be written in the form $(f, g)_1 = (f, Ag)$, where A is a linear operator defined on the $(\ ,\ )$-completion of E such that:*

(1) A is bounded, positive and with bounded inverse;
(2) $A - \mathbf{1}$ is Hilbert-Schimdt.

Definition 29. *Let $(M, \mathcal{B}, \mu)$ be a measure space, $\varphi : M \to M$ a measurable transformation and μ_φ the push-forward of μ. The measure μ is said to be invariant under the action of φ, or φ-invariant, if $\mu_\varphi = \mu$. If G is a group of measurable transformations such that μ is invariant for each and every element of G, we say that μ is G-invariant.*

As an example, consider the action of $\mathbb{R}$ on itself, by translations:

$$x \mapsto x + y, \ \ \forall x \in \mathbb{R}, \tag{89}$$

where $y \in \mathbb{R}$. Modulo a multiplicative constant, the Lebesgue measure is the only (σ-finite) measure on $\mathbb{R}$ which is invariant under the action of translations (89). This is a particular case of the well-known Haar theorem, which establishes the existence and uniqueness (modulo multiplicative constants) of (regular Borel) invariant measures on locally compact groups.

The situation is radically different in the case of infinite dimensional linear spaces. The following argument [28] shows for instance that there are no (non-trivial) translation invariant σ-finite Borel measures in infinite dimensional separable Banach spaces. Let us suppose then that such an invariant measure exists, and it does not assign an infinite measure to all open balls. It follows that there is an open ball of radius R with finite measure. Since the space is infinite dimensional, one can find an infinite sequence of disjoint open balls, of radius $r < R$, all contained in the first ball. Since by hypothesis the measure is invariant under all translations, all balls of radius r have the same measure. It follows that this measure is necessarily zero, since all the balls are contained in the same set, which has finite measure. Finally, since the space is separable, it can be covered by a countable set of open balls of radius r, all of them with zero measure. It is therefore proved that the whole space has zero measure, in contradiction with the hypothesis. There are, of course, non σ-finite invariant measures, e.g., the counting measure which assigns measure 1 to each and every point of the space. There are also σ-finite measures on infinite dimensional spaces which are invariant under a restricted set of translations.

Given a group of measurable transformations G on a space M, every G-invariant measure μ defines a unitary representation U of G in $L^2(M,\mu)$, by:

$$\big(U(\varphi)\psi\big)(x) = \psi(\varphi^{-1}x), \;\; \psi \in L^2(M,\mu), \; \varphi \in G. \tag{90}$$

One can still construct unitary representations of G using measures that are not strictly invariant, but instead satisfy a weaker condition known as quasi-invariance.

Definition 30. *Let $(M,\mathcal{B},\mu)$ be a measure space, $\varphi : M \to M$ a measurable transformation, and let μ_φ denote the push-forward of μ by φ. The measure μ is said to be quasi-invariant under the action of φ, or φ-quasi-invariant, if $\mu_\varphi \sim \mu$. If G is a group of transformations such that μ is quasi-invariant for all elements of G we say that μ is G-quasi-invariant.*

Regarding the group of translations in $\mathbb{R}$ (or $\mathbb{R}^n$), one can show that any two quasi-invariant measures are equivalent, and therefore equivalent to the Lebesgue measure. More generally, when considering continuous transitive actions of a locally compact group G on a space M, there is a unique equivalence class of quasi-invariant measures [29].

Proposition 4. *Let G be a group of measurable transformations on $(M,\mathcal{B})$ and μ a G-quasi-invariant measure. The following expression defines a unitary representation U of G in $L^2(M,\mu)$:*

$$\big(U(\varphi)\psi\big)(x) = \left(\frac{d\mu_\varphi}{d\mu}(x)\right)^{1/2} \psi(\varphi^{-1}x), \tag{91}$$

where μ_φ denotes push-forward of μ by $\varphi \in G$.

Going back to the examples above, one can see that the measure $e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}$ is quasi-invariant under the action (89), and thus defines a unitary representation of translations:

$$\big(U(y)\psi\big)(x) = e^{-y^2/2+yx}\psi(x-y). \tag{92}$$

On the contrary, the measure ν_0, supported on the interval $[0,1]$, is not quasi-invariant and cannot possibly provide a unitary representation.

Concerning the existence of translation quasi-invariant measures on infinite dimensional spaces, those are not available either, in most cases of interest. In particular, one can show the following. In infinite dimensional locally convex topological linear spaces there are no (non-trivial) translation quasi-invariant (i.e., quasi-invariant under all translations) σ-finite Borel measures (see [1,28,30] and references therein). Typically, one finds situations of quasi-invariance under a subgroup of the group of all translations (like in Theorem 17 below).

We review next some concepts and results from ergodic theory, following [1,31,32]. Only finite measures are considered.

Definition 31 (Ergodicity)**.** *Let $(M,\mathcal{B},\mu)$ be a probability space, where the measure μ is G-quasi-invariant with respect to a group G of measurable transformations. The measure is said to be G-ergodic if, for $B \in \mathcal{B}$, the condition:*

$$\mu(B \triangle \varphi B) = 0, \; \forall \varphi \in G,$$

implies $\mu(B) = 0$ or $\mu(B) = 1$.

In favorable cases of continuous actions in certain topological spaces, G-ergodic measures are supported in a single orbit of G (see [29]). In general we have the following [1].

Theorem 14. *Let μ be a G-quasi-invariant probability measure in a measurable space $(M, \mathcal{B})$. The measure is G-ergodic if and only if for every $B \in \mathcal{B}$ with $\mu(B) > 0$, there exists a countable set $\{\varphi_k\}_{k\in\mathbb{N}}$ of elements of G such that $\mu\left(\bigcup_{k=1}^{\infty} \varphi_k B\right) = 1$.*

Theorem 15. *Let μ_1 and μ_2 be two G-ergodic measures on the same measurable space. Then $\mu_1 \sim \mu_2$ or $\mu_1 \perp \mu_2$. If in particular μ_1 and μ_2 are G-invariant (and normalized) then $\mu_1 = \mu_2$ or $\mu_1 \perp \mu_2$.*

The following result establishes also necessary and sufficient conditions for ergodicity.

Theorem 16. *Let μ be a G-quasi-invariant probability measure. The measure is G-ergodic if and only if the only G-invariant measurable (real) functions are constant, i.e., if and only if the condition:*

$$f(x) = f(\varphi x) \;\; (almost\ everywhere) \;\; \forall \varphi \in G$$

implies:

$$f(x) = constant \;\; (almost\ everywhere).$$

This last result can be proven with the following arguments [1,13,32]. Suppose that μ is G-ergodic. Given any invariant real function, the inverse image of any Borel set satisfies (93), and it is therefore proven that ergodicity implies that invariant functions are constant almost everywhere. Conversely, if a set B satisfies (93), then its characteristic function χ_B (equal to 1 for $x \in B$ and 0 for $x \notin B$) is invariant, and the second condition on the theorem implies $\mu(B) = 0$ or $\mu(B) = 1$.

For Gaussian measures the following important theorem holds [1]. (Essentially, point 1 of Theorem 17 is what is usually known as the Cameron-Martin theorem. The discussion following Theorem 17, as well as the content of Lemma 2 below, provide in fact illustrations of that theorem.)

Theorem 17. *Let $(\,,\,)$ be an inner product in a real linear space E and μ the corresponding Gaussian measure on E^a. Let E^* be the subspace of E^a of those functionals that are continuous with respect to the topology defined by $(\,,\,)$, and X a subspace of E^a, considered as a subgroup of the group of translations in E^a. Then:*

(1) the measure μ is X-quasi-invariant if and only if $X \subset E^*$,
(2) the measure μ is X-ergodic if and only if X is dense in E^*.

The following simplified arguments illustrate point 1 of the theorem. Consider the Gaussian measure on $\mathbb{R}^n$:

$$d\mu(x) = \prod_{j=1}^{n} e^{-x_j^2/2} \frac{dx_j}{\sqrt{2\pi}} \tag{93}$$

and its translation with respect to $y \in \mathbb{R}^n$. The Radon-Nikodym derivative is:

$$\frac{d\mu(x-y)}{d\mu(x)} = \exp\left(\sum_{j=1}^{n} x_j y_j\right) \exp\left(-\frac{1}{2}\sum_{j=1}^{n} y_j^2\right). \tag{94}$$

When considering the limit $n \to \infty$, which corresponds to a measure on $\mathbb{R}^{\mathbb{N}}$, one can see that the derivative vanishes unless $y = (y_j)_{j\in\mathbb{N}}$ is an element of ℓ^2. Note that the condition $y \in \ell^2$ is actually sufficient for equivalence of the measures, since in that case $\exp\left(\sum_{j=1}^{n} x_j y_j\right)$ defines an integrable function on the limit $n \to \infty$, with respect to the measure (93). When, on the other hand, one considers translations by more general elements of $\mathbb{R}^{\mathbb{N}}$, one obtains two (quasi-invariant with respect to ℓ^2) mutually singular measures.

9. Gaussian Measures on $\mathcal{S}'(\mathbb{R}^d)$

To conclude, we consider the particular, but important case of measures on the space of distributions $\mathcal{S}'(\mathbb{R}^d)$ (equipped with the Borel σ-algebra associated with the strong topology–see Lemma 1, Section 6).

Given $g \in S(\mathbb{R}^d)$ one can naturally define an element of $\mathcal{S}'(\mathbb{R}^d)$, by:

$$h \mapsto \int gh\, d^d x, \ \ \forall h \in S(\mathbb{R}^d). \tag{95}$$

We will continue to denote that element by g, even if considered as an element of $\mathcal{S}'(\mathbb{R}^d)$. The inclusion of $\mathcal{S}(\mathbb{R}^d)$ in $\mathcal{S}'(\mathbb{R}^d)$ defined by (95) induces an action of $\mathcal{S}(\mathbb{R}^d)$ in $\mathcal{S}'(\mathbb{R}^d)$, as a subgroup of the group of translations. Explicitly, given $g \in S(\mathbb{R}^d)$ we get a measurable transformation in $\mathcal{S}'(\mathbb{R}^d)$:

$$\phi \mapsto \phi + g, \ \ \forall \phi \in \mathcal{S}'(\mathbb{R}^d). \tag{96}$$

Let us say in advance that there are quasi-invariant normalized Borel measures, with respect to the action of $\mathcal{S}(\mathbb{R}^d)$ (96). These measures will simply be called $\mathcal{S}$-quasi-invariant measures.

Let then μ be a $\mathcal{S}$-quasi-invariant measure. From Proposition 4, we then have a unitary representation of (the commutative group) $\mathcal{S}(\mathbb{R}^d)$ in $L^2(\mathcal{S}'(\mathbb{R}^d), \mu)$:

$$\big(\mathcal{V}(g)\psi\big)(\phi) = \Big(\frac{d\mu_g}{d\mu}(\phi)\Big)^{1/2} \psi(\phi - g)\,, \ \ g \in \mathcal{S}(\mathbb{R}^d), \tag{97}$$

where μ_g denotes the push-forward of μ with respect to the map (96).

On the other hand, as is typically the case in infinite dimensions, there are no Borel measures on $\mathcal{S}'(\mathbb{R}^d)$ which remain quasi-invariant under the transitive action of all translations, i.e., with respect to the natural action of $\mathcal{S}'(\mathbb{R}^d)$ (seen as a group) on itself [1,2]. [Just like in the discussion at the end of the previous section, this immediately leads to the existence of non-equivalent $\mathcal{S}$-quasi-invariant measures. In fact, given a $\mathcal{S}$-quasi-invariant measure μ, it is obvious that the push-forward $\mu_{\phi_0}(\phi) = \mu(\phi - \phi_0)$ defined by any $\phi_0 \in \mathcal{S}'(\mathbb{R}^d)$ is also a $\mathcal{S}$-quasi-invariant measure, and there is $\phi_0\big(\notin \mathcal{S}(\mathbb{R}^d)\big)$ such that the two measures are not equivalent.]

The simplest examples of $\mathcal{S}$-quasi-invariant measures are Gaussian measures, which we now consider. In order to simplify the discussion, we impose very strong conditions on the measures' covariance. Let then C be a linear continuous bijective operator on $\mathcal{S}(\mathbb{R}^d)$, with continuous inverse. We say that C is a covariance operator C if it is bounded, self-adjoint and strictly positive in $L^2(\mathbb{R}^d)$ and if C^{-1}, considered as a densely defined operator on $L^2(\mathbb{R}^d)$, is (essentially) self-adjoint and positive. It is then obvious that the bilinear form:

$$\langle f, g \rangle_C := \int f\, Cg\, d^d x, \ \ f, g \in S(\mathbb{R}^d), \tag{98}$$

in $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ is symmetric, positive and non-degenerate, thus defining an inner product $\langle\,,\rangle_C$ in the real linear space $\mathcal{S}(\mathbb{R}^d)$. A covariance operator C therefore defines a Gaussian measure, which is supported in $\mathcal{S}'(\mathbb{R}^d)$, since the $L^2(\mathbb{R}^d)$-continuity of C ensures that the topology defined by the inner product $\langle\,,\rangle_C$ is weaker than the nuclear topology. We will say also that C is the measure's covariance, with the understanding that we are referring to an inner product of the type (98).

Using Theorem 17, one can easily check that these measures are $\mathcal{S}$-quasi-invariant and $\mathcal{S}$-ergodic. In fact, from the required properties of the operator C one can write:

$$\int gh\, d^d x = \int (C^{-1}g)(Ch)\, d^d x = \langle C^{-1}g, h\rangle_C, \ \ \forall g, h \in \mathcal{S}(\mathbb{R}^d), \tag{99}$$

from what follows that the functionals on $\mathcal{S}(\mathbb{R}^d)$ defined by (95) are continuous with respect to the $\langle\,,\rangle_C$-topology. Also, the inclusion of $\mathcal{S}(\mathbb{R}^d)$ in the dual $\mathcal{S}'(\mathbb{R}^d)$ is dense with respect to the $\langle\,,\rangle_C$-topology, since $C^{-1}(\mathcal{S}(\mathbb{R}^d)) = \mathcal{S}(\mathbb{R}^d)$. The conditions of Theorem 17 are therefore satisfied.

In the case of Gaussian measures, the Radon-Nikodym derivative appearing in (97) is easily determined, generalizing the correspondent result in finite dimension:

Lemma 2. *Let C be a covariance operator on $\mathcal{S}(\mathbb{R}^d)$ and μ the corresponding measure on $\mathcal{S}'(\mathbb{R}^d)$. Then:*

$$\frac{d\mu(\phi - g)}{d\mu(\phi)} = e^{-\frac{1}{2}\int g\, C^{-1} g\, d^d x}\, e^{\phi(C^{-1}g)}, \quad \forall g \in \mathcal{S}(\mathbb{R}^d). \tag{100}$$

We present next a result [26] applicable to the important situation of measures that remain invariant under $\mathbb{R}^d$-translations. This result characterizes the support of the measure in terms of the local behavior of typical distributions. To formulate it we need to consider the kernel $\mathcal{C}$ of a covariance C, defined by:

$$\int f\, C g\, d^d x =: \int d^d x\, d^d x' f(x) \mathcal{C}(x, x') g(x'), \quad \forall f, g \in \mathcal{S}(\mathbb{R}^d). \tag{101}$$

In general, the kernel of the covariance is a distribution on $\mathbb{R}^d \times \mathbb{R}^d$. The corresponding measure is invariant under $\mathbb{R}^d$-translations if and only if $\mathcal{C}(x, x') = \mathcal{C}(x - x')$. Let us further recall that a signed measure on a measurable space M is a function on the σ-algebra of M of the form:

$$B \mapsto \int_B F d\nu, \tag{102}$$

where ν is a measure on M and F is an integrable function. In particular, an (Lebesgue) integrable function on an open set $U \subset \mathbb{R}^d$ defines a signed measure on U. We will also say that a distribution $\phi \in \mathcal{S}'(\mathbb{R}^d)$ is a signed measure in $U \subset \mathbb{R}^d$ if there exists a measure ν on U and an integrable function F such that $\phi(f) = \int_U f F d\nu$, for any function $f \in \mathcal{S}(\mathbb{R}^d)$ supported in U. Then [26]:

Proposition 5. *Let μ be a Gaussian measure on $S'(\mathbb{R}^d)$, invariant with respect to $\mathbb{R}^d$-translations and such that the kernel $\mathcal{C}$ of the covariance is not a continuous function. Then the support of μ is such that for μ-almost every distribution $\phi \in \mathcal{S}'(\mathbb{R}^d)$ there is no (non-empty) open set $U \subset \mathbb{R}^d$ on which ϕ can be seen as a signed measure.*

Example 1: Let us consider the so-called white noise measures, defined by a covariance proportional to the identity operator , $C = \sigma\mathbf{1}$, where $\sigma \in \mathbb{R}^+$. Since the covariance is a scalar, these measures are invariant under $\mathbb{R}^d$-translations, with covariance kernel $\mathcal{C}(x) = \sigma\delta(x)$, where δ is the evaluation distribution at $x = 0$, i.e. $\delta(f) = f(0)$, $\forall f \in \mathcal{S}(\mathbb{R}^d)$. It follows from the previous proposition that distributions that can be seen as signed measures on some open set do not contribute to the measure. One concludes also immediately, from Theorem 13, that white noise measures are not equivalent to each other, for $\sigma \neq \sigma'$. Furthermore, from Theorem 17 it follows that the measures are $\mathcal{S}$-ergodic for any σ, and one concludes from Theorem 15 that the measures are in fact mutually singular, for $\sigma \neq \sigma'$.

Example 2: Let us consider again the measures of Example 2, Section 7, defined by the covariance operators:

$$C_m := (m^2 - \Delta)^{-1}, \tag{103}$$

where $m \in \mathbb{R}^+$ and Δ is the Laplacian operator. The kernel of C_m is easily found to be:

$$\mathcal{C}_m(x) = \frac{1}{(2\pi)^d} \int d^d p\, \frac{e^{ipx}}{m^2 + p^2}. \tag{104}$$

The case $d = 1$ ($m \neq 0$) corresponds to the path integral for the quantum harmonic oscillator. (The particular case $d = 1$, $m = 1$ corresponds to the Ornstein-Uhlenbeck measure.) For $d > 1$ we find measures associated with the path integral formulation of quantum field theory. For $m = 0$ we get the well-known Wiener measure. (The case $m = 0$, $d < 3$, requires special care, since the integral (104) diverges in the region $p \approx 0$. An appropriate modification leads to the so-called conditional Wiener measure.) It is well known that these measures are supported on continuous functions for $d = 1$ and on distributions for $d \geq 2$ (see, e.g., [22,23]). In $d = 1$ this result comes from the fact that $(m^2 + p^2)^{-1}$ is integrable, with Fourier transform (104) proportional to $\frac{1}{m}e^{-m|x|}$. In this situation the test functions in $\mathcal{S}(\mathbb{R})$ can be replaced by "delta functions", and it makes sense to talk about the two point correlation function $\mathcal{C}_m(x, x')$, which is proportional to $\frac{1}{m}e^{-m|x-x'|}$.

Example 3: In the canonical approach to the quantization of real scalar field theories in $d+1$ dimensions one looks for representations of the Weyl relations:

$$\mathcal{V}(g)\,\mathcal{U}(f) = e^{i\int fg\, d^dx}\,\mathcal{U}(f)\mathcal{V}(g), \tag{105}$$

where f and g belong to $\mathcal{S}(\mathbb{R}^d)$. What is actually meant by this is a pair $(\mathcal{U}, \mathcal{V})$ of (strongly continuous) unitary representations of the group $\mathcal{S}(\mathbb{R}^d)$, satisfying (105). Any $\mathcal{S}$-quasi-invariant measure μ on $\mathcal{S}'(\mathbb{R}^d)$ produces such a representation. In fact, one just needs to consider the Hilbert space $L^2(\mathcal{S}'(\mathbb{R}^d), \mu)$, a unitary representation $\mathcal{V}$ like in (97) and a second unitary representation $\mathcal{U}$ simply defined by:

$$\big(\mathcal{U}(f)\psi\big)(\phi) = e^{-i\phi(f)}\,\psi(\phi). \tag{106}$$

Note that whereas the unitary representation $\mathcal{U}$ is obviously well defined for any measure, the construction of $\mathcal{V}$ depends critically on the $\mathcal{S}$-quasi-invariance of the measure. It is moreover required that the combined action of $\mathcal{U}$ and $\mathcal{V}$ be irreducible, which can in turn be seen to be equivalent to S-ergodicity of the measure. Any Gaussian measure therefore satisfies all these criteria. However, contrary to the situation in finite dimensions, to produce a physically meaningful quantization of a given field theory, the measure must satisfy additional conditions, typically in order to achieve a proper quantum treatment of the dynamics, and/or symmetries. For instance, the canonical formulation of the free quantum scalar field of mass m (see, e.g., [33] for details) is uniquely associated with the Gaussian measure of covariance:

$$\mathbf{C}_m := \frac{1}{2}(m^2 - \Delta)^{-1/2}. \tag{107}$$

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