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Article

Banach Subspaces of Continuous Functions Possessing Schauder Bases

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Abstract: In this article, Müntz spaces $M_{\Lambda,C}$ of continuous functions supplied with the absolute maximum norm are considered. An existence of Schauder bases in Müntz spaces $M_{\Lambda,C}$ is investigated. Moreover, Fourier series approximation of functions in Müntz spaces $M_{\Lambda,C}$ is studied.

Keywords: Banach space; Müntz space; isomorphism; Schauder basis; Fourier series

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1. Introduction

The domain of mathematics concerning topology and geometry of topological vector spaces is one of the important branches of functional analysis (see, for example, [1–4]). Particularly, a great part of it consists in investigations of bases in Banach spaces (see, for example, [1,5–11] and references therein). Many open problems remain for concrete classes of Banach spaces.

Among them Müntz spaces $M_{\Lambda,C}$ play very important role and there also remain unsolved problems (see [12–16] and references therein). They are provided as completions of the linear span over the real field $\mathbf{R}$ or the complex field $\mathbf{C}$ of monomials t^λ with $\lambda \in \Lambda$ on the unit segment $[0,1]$ by the absolute maximum norm, where $\Lambda \subset [0,\infty)$, $t \in [0,1]$. It was K. Weierstrass [17,18] who in 1885 had proven his theorem about polynomial approximations of continuous functions on the segment. But the space of continuous functions also possesses the algebraic structure. Later on in 1914 C. Müntz [19] considered generalizations to spaces which did not have such algebraic structure anymore.

There was a problem about an existence in them bases [8,20]. Further a result was for lacunary Müntz spaces which satisfy the restriction $\underline{\lim}_{n\to\infty}\lambda_{n+1}/\lambda_n > 1$ with the countable set Λ, but in general this problem remained unsolved [15,16]. For Müntz spaces of L_p functions with $1 < p < \infty$ this problem was investigated in [21]. It is worth to mention that the monomials t^λ with $\lambda \in \Lambda$ generally do not form a Schauder basis of the Müntz space $M_{\Lambda,C}$.

In this article results of investigations of the author on this problem are presented.

In Section 2 a Fourier analysis in Müntz spaces $M_{\Lambda,C}$ of continuous functions on the unit segment supplied with the absolute maximum norm is studied. For this purpose auxiliary Lemma 2 and Theorem 3 are proved. They are utilized for reducing consideration to a subclass of Müntz spaces $M_{\Lambda,C}$ up to isomorphisms of Banach spaces such that a domain Λ is contained in the set of positive integers $\mathbf{N}$. It is proved that for Müntz spaces subjected to the Müntz and gap conditions their functions belong to Weil-Nagy's class (about this class of functions see, for example, [22]). Then the theorem about existence of Schauder bases in Müntz spaces $M_{\Lambda,C}$ under the Müntz condition and the gap condition is proven.

All main results of this paper are obtained for the first time.

2. Müntz Spaces $M_{\Lambda,C}$

Henceforth the notations and definitions from [15,21] are used.

Definition 1. *Let Λ be an increasing sequence in the set $(0,\infty)$.*

The completion of the linear space containing all monomials at^{λ} with $a \in \mathbf{F}$ and $\lambda \in \Lambda$ and $t \in [\alpha,\beta]$ relative to the absolute maximum norm:

$$\|f(t)\|_{C[\alpha,\beta]} := \sup_{t\in[\alpha,\beta]} |f(t)|$$

is denoted by $M_{\Lambda,C}[\alpha,\beta]$, where $0 \le \alpha < \beta < \infty$, where the symbol $\mathbf{F}$ stands for $\mathbf{R}$ or $\mathbf{C}$. Particularly, for $[\alpha,\beta] = [0,1]$ it is also shortly written $M_{\Lambda,C}$. We consider also its subspace:

$$M^0_{\Lambda,C}[0,1] := \{f : f \in M_{\Lambda,C}[0,1];\ f(0) = f(1)\}$$

of 1-periodic functions.

Henceforth it is supposed that the set Λ satisfies the gap condition:

$$\inf_k \{\lambda_{k+1} - \lambda_k\} =: \alpha_0 > 0 \tag{1}$$

and the Müntz condition:

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} =: \alpha_1 < \infty. \tag{2}$$

Lemma 1 and Theorem 1, which are proved below, deal with isomorphisms of Müntz spaces $M_{\Lambda,C}$. Utilizing these results reduces our consideration to a subclass of Müntz spaces $M_{\Lambda,C}$ where a set Λ is contained in the set of natural numbers $\mathbf{N}$.

Lemma 1. *The Müntz spaces $M_{\Lambda,C}$ and $M_{\Xi\cup(\alpha\Lambda+\beta),C}$ are isomorphic for every $\beta \ge 0$ and $\alpha > 0$ and a finite subset Ξ in $(0,\infty)$.*

Proof. The set Λ is infinite with $\lim_n \lambda_n = \infty$. By virtue of Theorem 9.1.6 [15] Müntz space $M_{\Lambda,C}$ contains a complemented isomorphic copy of $c_0(\mathbf{F})$. Therefore, $M_{\Lambda,C}$ and $M_{\Xi\cup\Lambda,C}$ are isomorphic.

The isomorphism of $M_{\alpha\Lambda,C}$ with $M_{\Lambda,C}$ follows from the equality:

$$\sup_{t\in[0,1]} |f(t)| = \sup_{t\in[0,1]} |f(t^{\alpha})|$$

for each continuous function $f : [0,1] \to \mathbf{F}$, since the mapping $t \mapsto t^{\alpha}$ is a diffeomorphism of the segment $[0,1]$ onto itself. Taking $\Lambda_1 = \Lambda \cup \{\frac{\beta}{\alpha}\}$ and then $\alpha\Lambda_1$ we infer that $M_{\Lambda,C}$ and $M_{\alpha\Lambda+\beta,C}$ are isomorphic. $\square$

Theorem 1. *Suppose that increasing sequences $\Lambda = (\lambda_n : n \in \mathbf{N})$ and $\Upsilon = (v_n : n \in \mathbf{N})$ of positive numbers satisfy the Müntz and gap conditions and $\lambda_n \le v_n$ for each n. If $\sup_n(v_n - \lambda_n) = \delta$, where $\delta < (8\sum_{n=1}^{\infty} \lambda_n^{-1})^{-1}$, then $M_{\Lambda,C}$ and $M_{\Upsilon,C}$ are isomorphic as Banach spaces.*

Proof. There are isometric linear embeddings of $M_{\Lambda,C}$ and $M_{\Upsilon,C}$ into $M_{\Lambda\cup\Upsilon,C}$. Consider a sequence of sets $\Upsilon_k = (v_{k,n} : n \in \mathbf{N}) \subset \Lambda \cup \Upsilon$. Properties of the sequences $(\Upsilon_k : k = 0,1,2,3,...)$ include: $v_{k,n} \in \{\lambda_n, v_n\}$ for each $k = 0,1,2,...$ and $n = 1,2,...$, where $\Upsilon_0 = \Lambda$; $v_{k,n} \le v_{k+1,n}$ for each $k = 0,1,2,...$

and $n = 1, 2, ...$; $\Delta_{k+1,n}$ is an enumeration of the non-zero numbers of the form $\delta_{k+1,j} := v_{k+1,j} - v_{k,j}$ by elimination of zero terms, also $(m(k+1) : k)$ is a monotone increasing sequence with:

$$m(k+1) := \min\{n : v_n - v_{k+1,n} \neq 0; \& \ \forall l < n \ v_l = v_{k+1,l}\}.$$

For more details see (1–4) in the proof of Theorem 1 in [21].

For each $f \in M_{\Upsilon_k,C}$ we consider the power series $f_1(t) = \sum_{n=1}^{\infty} a_n t^{v_{k+1,n}}$, where the power series expansion $f(t) = \sum_{n=1}^{\infty} a_n t^{v_{k,n}}$ converges for each $0 \leq t < 1$, since f is analytic on $[0, 1)$ (see [14,15]). Then we infer that:

$$f(t^2) - f_1(t^2) = \sum_{n=1}^{\infty} a_n t^{v_{k,n}} u_n(t) \text{ with } u_n(t) := t^{v_{k,n}} - t^{v_{k,n}+2\Delta_{k+1,n}}$$

so that $u_n(t)$ is a monotone decreasing sequence in n and hence:

$$|f(t^2) - f_1(t^2)| \leq 2|u_{m(k+1)}(t)||f(t)|$$

according to Dirichlet's criterium for each $0 \leq t < 1$. Therefore, the function $f_1(t)$ has a continuous extension onto $[0, 1]$ and:

$$\|f - f_1\|_{C([0,1],\mathbf{F})} \leq 4\|f\|_{C([0,1],\mathbf{F})} \Delta_{k+1,m(k+1)}/\lambda_{m(k+1)},$$

since the mapping $t \mapsto t^2$ is an order preserving diffeomorphism of $[0, 1]$ onto itself. Thus the series $\sum_{l=1}^{\infty} a_n t^{v_{k+1,n}}$ converges on $[0, 1)$. Analogously to each $g_1 \in M_{\Upsilon_{k+1},C}$ there corresponds $g \in M_{\Upsilon_{k+1},C}$ which is continuous on $[0, 1]$.

This implies that there exists a linear isomorphism T_k of $M_{\Upsilon_k,C}$ with $M_{\Upsilon_{k+1},C}$ so that $\|T_k - I\| \leq 4\Delta_{k+1,m(k+1)}/\lambda_{m(k+1)}$, $T_k : M_{\Upsilon_k,C} \to M_{\Upsilon_{k+1},C}$. Take the sequence of operators $S_n := T_n T_{n-1} ... T_0 : M_{\Lambda,C} \to M_{\Upsilon_{n+1},C} \subset M_{\Lambda\cup\Upsilon,C}$. The space $M_{\Lambda\cup\Upsilon,C}$ is complete and the sequence $\{S_n : n\}$ operator norm converges to an operator $S : M_{\Lambda,C} \to M_{\Lambda\cup\Upsilon,C}$ so that $\|S - I\| < 1$, since δ satisfies the conditions of this theorem and:

$$\sum_{k=0}^{\infty} \Delta_{k+1,m(k+1)}/\lambda_{m(k+1)} \leq \delta \sum_{n=1}^{\infty} \lambda_n^{-1} < 1/8,$$

where I denotes the identity operator. Therefore, the operator S is invertible. From the conditions on Υ_k it follows that $S(M_{\Lambda,C}) = M_{\Upsilon,C}$. $\square$

Remark 1. *Next we recall necessary definitions and notations of the Fourier approximation. Then the auxiliary Proposition 1 about the weak L_1-space $L_{1,w}[0,1]$ is given. This proposition is used to prove Theorem 2 about the property that functions in a Müntz space satisfying the Müntz and gap conditions belong to Weil-Nagy's class. For this purpose in the space of continuous functions is considered its subspace:*

$$C_0([\alpha, \alpha+1], \mathbf{F}) := \{f : f \in C([\alpha, \alpha+1], \mathbf{F}), f(\alpha) = f(\alpha+1)\}$$

of 1*-periodic functions.*

Let $Q = (q_{n,k})$ be a lower triangular infinite matrix with matrix elements $q_{n,k}$ having values in the field $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$ so that $q_{n,k} = 0$ for each $k > n$, where k, n are nonnegative integers. To each 1-periodic function $f : \mathbf{R} \to \mathbf{F}$ in the space $L_1([a, a+1], \mathbf{F})$ is counterposed a trigonometric polynomial:

$$(1) \quad U_n(f, x, Q) := \frac{a_0}{2} q_{n,0} + \sum_{k=1}^{n} q_{n,k}(a_k \cos(2\pi kx) + b_k \sin(2\pi kx)),$$

where $a_k = a_k(f)$ and $b = b_k(f)$ are the Fourier coefficients of a function $f(x)$, whilst on $\mathbf{R}$ the Lebesgue measure is considered.

For measurable 1-periodic functions h and g their convolution is defined whenever it exists:

$$(2)\quad (h*g)(x) := 2\int_a^{a+1} h(x-t)g(t)dt.$$

The approximation methods by trigonometric polynomials use integral operators provided with the help of the convolution. We recall it briefly (for more details see [22–25]). We consider summation methods in the space of continuous periodic functions. Putting the kernel of the operator U_n to be:

$$(3)\quad U_n(x,Q) := \frac{q_{n,0}}{2} + \sum_{k=1}^{n} q_{n,k}\cos(2\pi kx)$$

one gets:

$$(4)\quad U_n(f,x,Q) = (f*U_n(\cdot,Q))(x) = (U_n(\cdot,Q)*f)(x).$$

The norms of these operators are well-known:

$$(5)\quad \mathsf{L}_n(Q) := \sup_{\|f\|_C=1} \|U_n(f,x,Q)\|_C = 2\|U_n(x,Q)\|_{L_1} = 2\int_a^{a+1} |U_n(t,Q)|dt,$$

where $\|*\|_C$ and $\|*\|_{L_1}$ denote norms on Banach spaces $C([a,a+1],\mathbf{F})$ and $L_1([a,a+1],\mathbf{F})$ respectively, while $a\in\mathbf{R}$ is a marked real number. These numbers $\mathsf{L}_n(Q)$ are called Lebesgue constants of a summation method (see also [22,23]).

Henceforth, we consider spaces of real-valued functions if something other will not be specified, since an existence of a Schauder basis in the Müntz space over the real field $\mathbf{R}$ implies its existence in the corresponding Müntz space over the complex field $\mathbf{C}$.

Definition 2. *For a function $f\in L_1(\alpha,\alpha+1)$ by $S[f]$ or $S(f,x)$ is denoted its Fourier series with coefficients $a_k = a_k(f)$ and $b_k = b_k(f)$:*

$$\rho_n(f,x) := f(x) - S_{n-1}(f,x)$$

is the approximation precision of f by the Fourier series $S(f,x)$, where:

$$S_n(f,x) := \frac{a_0}{2} + \sum_{k=1}^{n}(a_k\cos(2\pi kx) + b_k\sin(2\pi kx))$$

is the partial Fourier sum approximating a Lebesgue integrable 1-periodic function f on $(0,1)$.

If the following function:

$$D^{\psi}_{\beta} f := f^{\psi}_{\beta} := \sum_{k=1}^{\infty}[a_k(f)\cos(2\pi kx+\beta\pi/2) + b_k(f)\sin(2\pi kx+\beta\pi/2)]/\psi(k)$$

belongs to the space $L(\alpha,\alpha+1)$ of all Lebesgue integrable (summable) functions on $(\alpha,\alpha+1)$, then f^{ψ}_{β} is called the Weil (ψ,β) derivative of f, where $(\psi(k):\ k)$ is a sequence of non-zero numbers in $\mathbf{F}$ and β is a real parameter.

Let for a Banach space $\mathcal{N}$ of some functions on $[a,a+1]$:

$$C^{\psi}_{\beta}\mathcal{N}[a,a+1] := \{f\in\mathcal{N}:\ \exists f^{\psi}_{\beta}\in C_0[a,a+1]\}$$

(see in more details Notation 2 and Definition 2 in [21]).

In particular, let $C^{\psi}_{\beta}M[a, a+1]$ (or $C^{\psi}_{\beta}[a, a+1]$ for short) be the space of all continuous 1-periodic functions f having a continuous Weil derivative f^{ψ}_{β}, $f^{\psi}_{\beta} \in C_0[0,1]$ and considered relative to the absolute maximum norm and such that:

$$(1) \quad \|f^{\psi}_{\beta}\|_{C[0,1]} := \max\{|f^{\psi}_{\beta}(t)| : \ t \in [0,1]\} < \infty.$$

Particularly, for $\psi(k) = k^{-r}$ there is the Weil-Nagy class:

$$W^r_{\beta,C} = W^r_{\beta}C[a, a+1] := \{f : f \in C_0[a, a+1], \exists f^{\psi}_{\beta} \in C_0[a, a+1]\}.$$

Then let:

$$\mathcal{E}_n(X) := \sup\{\|\rho_n(f; x)\|_{C[a,a+1]} : f \in X\},$$

where $\rho_n(f, x)$ is described at the beginning of these Definitions 2:

$$E_n(f) := \inf\{\|f - T_{n-1}\|_{C[a,a+1]} : T_{n-1} \in \mathcal{T}_{2n-1}\},$$

$$E_n(X) := \sup\{E_n(f) : \ f \in X\},$$

where $E_n(f)$ is given just above, while T_{n-1} and $\mathcal{T}_{2n-1}$ are described just below, where a set X is contained in $C_0[a, a+1]$:

$$\mathcal{T}_{2n-1} := \{T_{n-1}(x) = \frac{c_0}{2} + \sum_{k=1}^{n-1}(c_k \cos(2\pi kx) + d_k \sin(2\pi kx)); \ c_k, d_k \in \mathbf{R}\}$$

denotes the family of all trigonometric polynomials T_{n-1} of degree not greater than $n-1$ (see the definitions in more details in [22]).

The family of all Lebesgue measurable functions $f : (a, b) \to \mathbf{R}$ satisfying the condition:

$$\|f\|_{L_{s,w}(a,b)} := \sup_{y>0}(y^s \mu\{t : \ t \in (a, b), \ |f(t)| \geq y\})^{1/s} < \infty$$

is called the weak L_s space and denoted by $L_{s,w}(a, b)$, where μ notates the Lebesgue measure on the real field $\mathbf{R}$, $0 < s < \infty$, $(a, b) \subset \mathbf{R}$ (see, for example, §9.5 in [26], §IX.4 in [27]).

By F is denoted the set of all pairs (ψ, β), for which:

$$\mathcal{D}_{\psi,\beta}(x) := \sum_{k=1}^{\infty} \psi(k) \cos(2\pi kx + \beta\pi/2)$$

is the Fourier series of some function belonging to $L_1[0,1]$. Then F_1 denotes the family of all positive sequences $(\psi(k) : \ k \in \mathbf{N})$ tending to zero with $\Delta_2\psi(k) := \psi(k-1) - 2\psi(k) + \psi(k+1) \geq 0$ for each k so that the series:

$$\sum_{k=1}^{\infty} \frac{\psi(k)}{k} < \infty$$

converges.

Proposition 1. *Suppose that an increasing sequence $\Lambda = \{\lambda_n : n\}$ of natural numbers satisfies the Müntz condition. If $f \in M_{\Lambda,C}[0,1]$, then $df(x)/dx \in L_{1,w}(0,1)$.*

Proof. The **proof** is similar to that of Proposition 1 in [21] with the following modifications. Consider any $f \in M_{\Lambda,C}[0,1]$. From [14] (or see Theorem 6.2.3 and Corollary 6.2.4 in [15]) it follows that f is analytic on the unit open disk $\dot{B}_1(0)$ in $\mathbf{C}$ with center at zero and the series:

$$f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}$$

converges on $\dot{B}_1(0)$, where $a_n \in \mathbf{F}$ is an expansion coefficient for each $n \in \mathbf{N}$.

Using the Riemann integral we have that:

$$(1) \quad f(x) - f(0) = \int_0^x f'(t)dt \text{ for each } 0 \leq x < 1 \text{ and}$$

$$(2) \quad \lim_{x \uparrow 1} \int_0^x f'(t)dt = f(1) - f(0)$$

due to Newton-Leibnitz' formula (see, for example, §II.2.6 in [28]), since $f(x)$ is continuous on $[0,1]$.

By virtue of the uniqueness theorem for holomorphic functions (see, for example, II.2.22 in [29]), applied to the considered case, if a nonconstant holomorphic function g on $\dot{B}_1(0)$ has a set $E(g) = \{x : g(x) = 0, 0 \leq x < 1\}$ of zeros in $[0,1)$, then either $E(g)$ is finite or infinite with the unique limit point 1. Then we take a linear function $s(x) = \alpha + \beta x$ with real coefficients α and β, put $u(x) = f(x) + s(x)$ and choose α and β so that $u(0) = u(1) = 0$.

On the other hand, $\min \Lambda = \lambda_1 > 0$ and hence f is nonconstant. The case $du(x)/dx = const$ is trivial. So there remains the variant when $du(x)/dx$ is nonconstant. Denote by x_n zeros in $[0,1)$ of $du(t)/dt$ of odd order so that $x_{n+1} > x_n$ for each $n \in \mathbf{N}$. Therefore:

$$(3) \quad \int_{x_n}^{x_{n+1}} u'(t)dt \int_{x_{n+1}}^{x_{n+2}} u'(\tau)d\tau < 0$$

for each $n \in \mathbf{N}$ according to Theorem II.2.6.10 in [28]. If $\{x_n : n\}$ is a finite set, then from Formulas (1) and (2) it follows that $u' \in L_1[0,1]$ and hence $u' \in L_{1,w}[0,1]$.

Consider now the case when the set $\{x_n : n\}$ is infinite. We take a convex connected domain V such that V is canonically closed, $V = cl(Int(V))$, $[0,1] \subset V$, $|z - 0.5| \leq 1/2$ for each $z \in V$, $x + \eta \in V$ for each $3/4 < x < 1$ and $|\eta| \leq 1 - x$, $\eta \in \mathbf{C}$, where $cl(A)$ and $Int(A)$ denote the closure and the interior of a set A in the complex field $\mathbf{C}$. According to Cauchy's formula $21(5)$ in [29]:

$$f'(z) = \frac{1}{2\pi i} \int_{\omega} \frac{f(\xi)}{(\xi - z)^2} d\xi$$

for each $z \in Int(V)$, where ω is a rectifiable path encompassing once a point z in the positive direction so that $\omega \subset V$, for example, a circle with center at z. A set V can be taken as the disc $\{u \in \mathbf{C} : |u - 1/2| \leq 1/2\}$. For each $3/4 < x < 1$ a circle can be chosen with center at x and of radius $0 < r < 1 - x$ with $r \uparrow (1 - x)$ while $x \uparrow 1$. Using the homotopy theorem and the continuity of f on the compact disc V one can take simply the circle $\omega = \partial V = \{u \in \mathbf{C} : |u - 1/2| = 1/2\}$. Since $\max_{z \in V} |f(z)| =: G < \infty$ due to the Weierstrass theorem (see Vol. 1, Part III, Ch. 1, §12 in [28]), then from the estimate of the Cauchy integral (see Ch. II, §7, subsection 24 in [29]) it follows that:

$$(4) \quad |f'(x)| \leq G/(2\pi(1 - x))$$

for each $3/4 < x < 1$, hence $f'(x) \in L_{1,w}(3/4, 1)$ and consequently $u'(t) \in L_{1,w}(3/4, 1)$. Therefore, from (4) we infer that:

$$\sup_{y > 0} y\mu\{t : t \in [0,1), |f'(t)| \geq y\} \leq$$

$$\sup_{y>0} y\mu\{t:\ t\in[0,3/4),\ |f'(t)|\geq y\} + \sup_{y>0} y\mu\{t:\ t\in[3/4,1),\ |f'(t)|\geq y\} < \infty,$$

where μ denotes the Lebesgue measure on $[0,1]$. The latter means that $df(x)/dx \in L_{1,w}(0,1)$. $\square$

Theorem 2. *Let an increasing sequence $\Lambda = \{\lambda_n :\ n\} \subset \mathbf{N}$ of natural numbers satisfy the Müntz condition and let $f \in M_{\Lambda,C}[0,1]$. Then for each $0<\gamma<1$ there exists $\beta = \beta(\gamma) \in \mathbf{R}$ so that $v_f \in W^{\gamma}_{\beta,C}[0,1]$, where $v_f(t) = f(t) + (f(0) - f(1))t$ for each $t \in [0,1)$ and v_f is 1-periodic on $\mathbf{R}$.*

Proof. We have that $f(0) = 0$, since $\lambda_n \geq 1$ for each n. Therefore, we consider $v(t) = v_f(t) = f(t) - f(1)t$ on $[0,1) := \{t : 0 \leq t < 1\}$ and take its 1-periodic extension v on $\mathbf{R}$.

According to Proposition 1.7.2 [22] (or see [23]) a function h belongs to $W^{\gamma}_{\beta}L_{\infty}(l, l+1)$ if and only if there exists a function $\phi = \phi_{h,\gamma,\beta}$ which is 1-periodic on $\mathbf{R}$ and Lebesgue integrable on $[0,1]$ such that:

$$(1)\quad h(x) = \frac{a_0(h)}{2} + (\phi * \mathcal{D}_{\psi,\beta})(x),$$

where $a_0(h) = 2\int_0^1 h(t)dt$.

We take a sequence $U_n(t,Q)$ given by (3) in Remark 1 or see Formula (6) in [21] so that:

$$\lim_m q_{m,k} = 1 \text{ for each } k \text{ and } \sup_m \mathsf{L}_m(Q) < \infty \text{ and } \sup_{m,k} |q_{m,k}| < \infty$$

and write for short $U_n(t)$ instead of $U_n(t,Q)$. Under these conditions the limit exists:

$$(2)\quad \lim_n (v * U_n)(x) = v(x)$$

in $L_\infty(0,1)$ norm for each $v \in L_\infty((0,1),\mathbf{F})$ according to Chapters 2 and 3 in [22] (see also [23,30]).

Put $\theta(k) = k^{\gamma-1}$ for all $k \in \mathbf{N}$. Then for $\beta = \beta(\gamma) = 1-\gamma$ we get that $\mathcal{D}_{\theta,-\beta}(x) \in L_\infty(0,1)$ (see the proof of Theorem 2 in [21]).

With the help of Proposition 1 and Formula (2) we define the function $s(x)$ such that:

$$(3)\quad s(x) = \lim_{\eta\downarrow 0}\lim_n \eta^{-1}\int_0^{\eta} ((\mathcal{D}_{\theta,-\beta} * U_n) * v')(x-t)dt.$$

By virtue of the weak Young inequality (see Theorem 9.5.1 in [26], §IX.4 in [27]) and Proposition 1 this function s is in $L_\infty(0,1)$.

In view of Formula I(10.1) in [22] if $(\psi_1,\bar{\beta}_1) \leq (\psi_2,\bar{\beta}_2)$, then $S[(y^{\psi_1}_{\bar{\beta}_1})^{\psi_2/\psi_1}_{\bar{\beta}_2-\bar{\beta}_1}] = S[y^{\psi_2}_{\bar{\beta}_2}]$, where $y \in L^{\psi_2}_{\bar{\beta}_2}$. Therefore $\phi_{v,\gamma,\beta} = s$ and $D^{\psi}_{\beta}v = s$ according to Formula (1), where $\psi(k) = k^{-\gamma}$ for each $k = 1,2,3,\ldots$. Thus $v \in W^{\gamma}_{\beta}L_\infty(0,1)$. For δ such that $0<\gamma<\delta<1$ similarly $v \in W^{\delta}_{\beta(\delta),C}[0,1]$. On the other hand, v is analytic on $(0,1)$, 1-periodic and continuous on $[0,1)$, consequently, s is analytic on $(0,1)$ and 1-periodic. Therefore, from the latter and Formulas (1)–(3) it follows that $\|s\|_{L_\infty(0,1)} = \|s\|_{C[0,1]}$ and hence $v \in W^{\gamma}_{\beta,C}[0,1]$. $\square$

Lemma 2. *Let Λ be an increasing sequence of natural numbers satisfying the Müntz condition. Define the subset Y of the unit ball of $C([0,1])$:*

$Y = \{v:\ v$ is $1-$periodic and $\exists\, f \in M_{\Lambda,C}$ such that:
$\forall t \in [0,1)\ v(t) = f(t) + (f(0)-f(1))t$ and $\|v\|_{C([0,1])} \leq 1\}$.

Then for each $0<\gamma<1$ a positive constant $\omega = \omega(\gamma)$ exists so that:

$$(1)\quad E_n(Y) \leq \mathcal{E}_n(Y) \leq \omega n^{-\gamma}\ln n$$

for each natural number $n \in \mathbf{N}$

Proof. Let $f \in M_{\Lambda,C}$ and put $v(t) = f(t) + (f(0) - f(1))t$ for all $t \in [0,1)$. Suppose that the 1-periodic extension v of $v(t)$ belongs to Y and let $0 < \gamma < 1$. By Theorem 2 it follows that $v \in W^{\gamma}_{\beta,C}[0,1]$.

Then estimate (1) follows from Theorems 3.12.3 and 3.12.3′ in [22]. □

Lemma 3. *If $\psi \in \mathsf{F}_1$ and $(\psi, \beta) \in \mathsf{F}$ (see at the end of Definitions 2), then $C^{\psi,0}_{\beta}[0,1] := \{f \in C^{\psi}_{\beta}[0,1] : \int_0^1 f(t)dt = 0\}$ is the Banach space relative to the norm given by the formula:*

$$(1) \quad \|f\|_{C^{\psi}_{\beta}[0,1]} := \|f^{\psi}_{\beta}\|_{C[0,1]}.$$

Proof. Using the notation of Definitions 2 (see the notation 2 in [21]) we have that $C^{\psi}_{\beta}[0,1]$ is the $\mathbf{F}$-linear space and hence $C^{\psi,0}_{\beta}[0,1]$ is such also as the kernel of the linear functional $\phi(f) := \int_0^1 f(t)dt$, since each $f \in C^{\psi}_{\beta}[0,1]$ is integrable. Therefore, the assertion of this lemma follows from Propositions I.8.1 and I.8.3 [22], since each $f \in C^{\psi}_{\beta}[0,1]$ has the convolution representation:

$$(2) \quad f(x) = \frac{a_0(f)}{2} + 2\int_0^1 f^{\psi}_{\beta}(x-t)\mathcal{D}_{\psi,\beta}(t)dt$$

for each $x \in [0,1]$, but $a_0(f) = 0$ for each $f \in C^{\psi,0}_{\beta}[0,1]$, while the convolution $h * u$ is continuous for each $h \in C[0,1]$ and $u \in L_1[0,1]$ so that

$$\|h * u\|_{C[0,1]} \leq 2\|h\|_{C[0,1]}\|u\|_{L_1[0,1]},$$

where $\mathcal{D}_{\psi,\beta}$ is given in Definition 2. □

Theorem 3. *If an increasing sequence Λ of positive numbers satisfies the Müntz condition and the gap condition, then the Müntz space $M_{\Lambda,C}[0,1]$ has a Schauder basis.*

Proof. By virtue of Lemma 1 and Theorem 1 it is sufficient to prove an existence of a Schauder basis in the Müntz space $M_{\Lambda,C}$ for $\Lambda \subset \mathbf{N}$. According to Definition 1 and the proof of Lemma 1 the Banach spaces $M^0_{\Lambda,C}$ and $M_{\Lambda,C}$ are isomorphic.

The functional:

$$(1) \quad \phi(h) := \int_0^1 h(\tau)d\tau$$

is continuous on $C^{\psi}_{\beta}[0,1]$, where ψ and β satisfy conditions of Lemma 3. Then $coker(\phi) = \mathbf{F}$. Therefore, $C^{\psi}_{\beta}[0,1] = \mathbf{F} \oplus C^{\psi,0}_{\beta}[0,1]$.

In view of Theorem 6.2.3 and Corollary 6.2.4 [15] each function $g \in M_{\Lambda,C}[0,1]$ has an analytic extension on $\dot{B}_1(0)$ and hence:

$$(2) \quad g(z) = \sum_{n=1}^{\infty} c_n z^{\lambda_n} = \sum_{k=1}^{\infty} p_k u_k(z)$$

are the convergent series on the unit open disk $\dot{B}_1(0)$ in $\mathbf{C}$ with center at zero, where $\Lambda \subset \mathbf{N}$ and $c_n = c_n(g) \in \mathbf{N}$, $p_n = p_n(g) = c_1 + ... + c_n$, $u_1(z) := z^{\lambda_1}$, $u_{n+1}(z) := z^{\lambda_{n+1}} - z^{\lambda_n}$ for each $n = 1, 2, ...$

Take the finite dimensional subspace $X_n := span_{\mathbf{R}}(u_1, ..., u_n)$ in $X := M^0_{\Lambda,C}$, where $n \in \mathbf{N}$. Due to Lemma 1 the Banach space $X \ominus X_n$ exists and is isomorphic with $M_{\Lambda,C}$.

Consider the trigonometric polynomials $U_m(f, x, Q)$ for $f \in X$, where $m = 1, 2, ...$ (see Formula (6) in [21] and Remark 1 above). Put $Y_{K,n}$ to be the completion in $C[0,1]$ of the linear span $span_{\mathbf{R}}(U_m(f, x, Q) : (m, f) \in K)$, where $K \subset \mathbf{N} \times (X \ominus X_n)$, $m \in \mathbf{N}$, $f \in (X \ominus X_n)$.

There exists a countable subset $\{f_n : n \in \mathbf{N}\}$ in X such that $f_n = \mathcal{D}_{\psi,\beta} * g_n$ with $g_n \in L(0,1)$ for each $n \in \mathbf{N}$ and so that $span_{\mathbf{R}}\{f_n : n \in \mathbf{N}\}$ is dense in X, since X is separable. From Formulas

(1) and (2) and Theorem 2 and Lemmas 2 and 3 we infer that a countable set K and a sufficiently large natural number n_0 exist so that the Banach space Y_{K,n_0} is isomorphic with $(X \ominus X_{n_0})$ and $Y_{K,n_0}|_{(0,1)} \subset W^{\gamma}_{\beta,C}(0,1)$, where $0 < \gamma < 1$ and $\beta = 1 - \gamma$. Thus the Banach space Y_{K,n_0} is the $C[0,1]$ completion of the real linear span of a countable family $(s_l : l \in \mathbf{N})$ of trigonometric polynomials s_l.

Without loss of generality this family can be refined by induction such that s_l is linearly independent of $s_1, ..., s_{l-1}$ over $\mathbf{F}$ for each $l \in \mathbf{N}$. With the help of transpositions in the sequence $\{s_l : l \in \mathbf{N}\}$, the normalization and the Gaussian exclusion algorithm we construct a sequence $\{r_l : l \in \mathbf{N}\}$ of trigonometric polynomials which are finite real linear combinations of the initial trigonometric polynomials $\{s_l : l \in \mathbf{N}\}$ and which satisfy the conditions:

(3) $\|r_l\|_{C[0,1]} = 1$ for each l;

(4) the infinite matrix having l-th row of the form $..., a_{l,k}, b_{l,k}, a_{l,k+1}, b_{l,k+1}, ...$ for each $l \in \mathbf{N}$ is upper trapezoidal (step), where:

$$r_l(x) = \frac{a_{l,0}}{2} + \sum_{k=m(l)}^{n(l)} [a_{l,k}\cos(2\pi kx) + b_{l,k}\sin(2\pi kx)]$$

with $a^2_{l,m(l)} + b^2_{l,m(l)} > 0$ and $a^2_{l,n(l)} + b^2_{l,n(l)} > 0$, where $1 \le m(l) \le n(l)$, $deg(r_l) = n(l)$, or $r_1(x) = \frac{a_{1,0}}{2}$ when $deg(r_1) = 0$; $a_{l,k}, b_{l,k} \in \mathbf{R}$ for each $l \in \mathbf{N}$ and $0 \le k \in \mathbf{Z}$.

Then as X and Y in Proposition 2 of [21] we take $X = C[0,1]$ and $Y = Y_{K,n_0}$. In view of the aforementioned proposition and Lemma 1 a Schauder basis exists in Y_{K,n_0} and hence also in $M_{\Lambda,C}[0,1]$. $\square$

3. Conclusions

The results as described above are utilizable for further studies of mapping approximations, Banach space geometry within mathematical analysis and functional analysis and certainly in their diverse applications. Among them it are worth mentioning measure theory and stochastic processes in Banach spaces, approaches scrutinizing periodic or almost periodic function perturbations [31], of distortions in high-frequency pulse acoustic signals [32].

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